**Discreteness of Equilibria in Incomplete
Markets with a Continuum of States**

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Abstract

We discuss the issue, raised by Mas-Colell(1991) whether the local uniqueness(relative to the L^∞ topology) may be a generic property of equilibria in incomplete markets economies with a continuum of states.

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1 Introduction.

The analysis of equilibria in incomplete markets economies with a continuum of states is still an important, open and difficult subject. The known results on existence of equilibrium use either a strong assumption on non-negativity of ex-post endowments—i.e. endowments plus real returns (see Hellwig(1996), Mas-Colell and Zame(1996) and Monteiro(1996))— or a modification of the equilibrium concept to allow for default subject to penalties(see Araujo, Monteiro and Páscoa (1996,1997)). Mas-Colell(1991) gave a robust example of continuum of states incomplete markets economies with uncountably many equilibria. In the case of the (separable) L^1 topology the equilibrium set is therefore not discrete, but in the case of the non-separable L^∞ topology discreteness is compatible with a continuum of distinct equilibria.

Our purpose is precisely to obtain results on discreteness of equilibria under the L^∞ topology. In the case where there are no trades in assets markets, the pure spot markets equilibria of the incomplete markets economy are shown to be discrete, generically, on endowments and utilities. Moreover if utilities are state-independent, discreteness is generic on endowments.

Our result on the no-assets case is interesting since this case has been studied previously in the literature; by Mas-Colell(1991) and Davila(1996) having been pointed out by Mas-Colell (1991) that "...in what concerns the indeterminacy question the trivial no assets case is not trivial at all, but it represents perfectly well what happens in the general incomplete situation". When assets are traded, we show first that, generically on endowments and assets returns matrices ex-post endowments are regular, for almost every state. In particular, this shows that the regularity assumption in Mas-Colell and Monteiro(1996) work on existence of equilibrium is actually, generic. Secondly, we establish the discreteness of equilibria with asset trades for an open dense set of endowments, provided that utility is state-independent and the set of non-regular spot endowments translated by the span of the assets returns matrices has an empty interior.

2 The model and definitions.

Consider a two periods economy $\mathcal{E} = ((w^i, u^i)_{i=1}^I, A)$ with a finite number I of agents. There is no consumption in the first period. There is a continuum

of states of nature that may occur in the second period affecting endowments, assets returns and possibly also preferences over G physical goods. We endow with a σ -algebra $\mathcal{B}$ and a non-atomic probability measure μ . Preferences are assumed to be additively separable over states and the instantaneous utility functions belongs to the space $\mathcal{U}$ of C^2 functions on G_{++} which are differentially strictly concave, strictly monotone and proper; we endow $\mathcal{U}$ with the relative topology of C^2 uniform convergence on compacta. To define a metric to this topology define for each compact set $K \subset G_{++}$ the $C^2(K)$ norm $\|a\|_K = \max_{k \in K} \{|a(k)| + |\partial a(k)| + |\partial^2 a(k)|\}$. Then if $K_n = [1/n, n]^G$ a metric for this topology is $d_{\mathcal{U}}$, where

$$d_{\mathcal{U}} : \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}, \quad d_{\mathcal{U}}(u, v) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|u - v\|_{K_n}}{1 + \|u - v\|_{K_n}}.$$

The consumer $i, 1 \leq i \leq I$ is described by an essentially bounded measurable endowment function $w^i : \Omega \rightarrow G_{++}$ and instantaneous utility map $u^i : \Omega \rightarrow \mathcal{U}$. There are J real assets traded in the first period with returns in the second period. Assets returns are described by $A = (A^1, \dots, A^J)$. The returns of asset j is given by a measurable essentially bounded function $A^j : \Omega \rightarrow G$. We denote by $\mathcal{A}$ the set of consumers characteristics:

$$\mathcal{A} = \{(w, u) \in (G_{++} \times \mathcal{U}); w \in L^{\infty}(\Omega, G_{++}) \text{ and } u \text{ is measurable}\}.$$

As in Mas-Colell (1991), we endow $\mathcal{A}$ with the complete metric $d_{\mathcal{A}}$ defined by:

$$d_{\mathcal{A}}((w', u'), (w, u)) = \|w - w'\|_{\infty} + \text{esssup}_{\Omega} d_{\mathcal{U}}(u_s, u'_s).$$

Naturally the set $\mathcal{A}^I$ is endowed with the product topology and we denote by $d_{\mathcal{A}^I}$ the metric $d((a^i, b^i)_{i=1}^I, (c^i, d^i)_{i=1}^I) = \sum_{i=1}^I d_{\mathcal{A}}((a^i, b^i), (c^i, d^i))$. Later we will also need the set of consumers characteristics and assets matrices, $\mathcal{A}_{IM} = \mathcal{A}^I \times L^{\infty}(\Omega, G^J)$. We denote by $d_{\mathcal{A}_{IM}}$ the metric

$$d_{\mathcal{A}_{IM}}((w, u, A), (w', u', A')) = d_{\mathcal{A}^I}((w', u'), (w, u)) + \|A - A'\|_{\infty}.$$

We denote by S_+^G the $G - 1$ dimensional simplex. An equilibrium for $\mathcal{E}$ is a vector $(p, q, (x^i, \theta^i)_{i=1}^I)$ such that

1. $p : \Omega \rightarrow S_+^G$ is measurable;

2. $x^i : \rightarrow \frac{G}{+}$ is measurable and $\sum_{i=1}^I x_s^i \leq \sum_{i=1}^I w_s^i$ for μ a.e. $s \in$;
3. $\theta^i \in J$ for every i and $\sum_{i=1}^I \theta^i = 0$;
4. (x^i, θ^i) maximizes

$$\int u_s^i(x_s) d\mu(s) \text{ subject to } p_s(x_s - w_s^i) \leq p_s A_s \theta^i \text{ for } \mu \text{ a.e. } s. \quad (1)$$

Notice that if $(p, q, (x^i, \theta^i)_{i=1}^I)$ is an equilibrium, then for μ a.e. s , the vector $(p_s, (x_s^i)_{i=1}^I)$ is a Walrasian equilibrium for the pure exchange economy $(w^i(s) + A(s)\theta^i, u_s^i)_{i=1}^I$ for μ a.e. $s \in$. Finally we define local uniqueness.

- Definition 1**
1. An equilibrium of the economy $\mathcal{E}$, $(p, q, (x^i, \theta^i)_{i=1}^I)$ is locally unique if there is a neighbourhood $\mathcal{W}$ of $(p, x^1, \dots, x^I)$ such that $(p, x^1, \dots, x^I)$ is the only price and equilibrium allocation of $\mathcal{E}$ that belongs to $\mathcal{W}$.
 2. The equilibrium set of $\mathcal{E}$ is discrete if every price and equilibrium allocations of $\mathcal{E}$ are locally unique.

Remark 1 The definition above corresponds exactly and respectively to the mathematical notion of isolated point and discrete set.

3 Discreteness in the no assets case.

We will consider in this section the case where $A = 0$. Let us define

$$\mathcal{N} = \{(a, b) \in (\frac{G}{++} \times \mathcal{U})^I; (a, b) \text{ is not a regular economy}\}.$$

Theorem 1 The subset of consumer characteristics $(w, u) \in \mathcal{A}^I$ such that the equilibrium set of $\mathcal{E} = ((w^i, u^i)_{i=1}^I, 0)$ is discrete contains an open and dense subset of $\mathcal{A}^I$.

Proof. Let $\mathcal{C}$ be the subset of $\mathcal{A}^I$ of economies with only isolated equilibria. Let $\mathcal{C}_{rs}$ be the subset of $\mathcal{C}$ of simple functions with regular values. We start by showing that $\mathcal{C}_{rs} \subset \text{int}(\mathcal{C})$. Take $(\bar{w}, \bar{u}) \in \mathcal{C}_{rs}$. Then we can write $(\bar{w}, \bar{u})$ as $\sum_{i=1}^L (\bar{w}_i, \bar{u}_i) \chi_{T_i}$. Since $(\bar{w}_i, \bar{u}_i)$ is regular, there exists neighbourhoods $O_i \subset$

$(\bar{C}_{++})^I$ and $V_l \subset U^I$ of $\bar{w}_l$ and $\bar{u}_l$, respectively, and $\delta > 0$ such that for any spot economy $(\bar{w}, \bar{u})$ any spot equilibrium prices p, p' satisfying $|p - p'| < \delta$ we must have $p = p'$. Let $\epsilon > 0$ be such that if $(a, b) \in (\bar{C}_{++})^I \times U^I$ satisfies $|a - \bar{w}_l| < \epsilon$ and $d_{U^I}(b, \bar{u}_l) < \epsilon$, for $l = 1, \dots, L$, then $(a, b) \in O_l \times V_l$, for $l = 1, \dots, L$. Now let $(w, u) \in \mathcal{A}^I$ be such that $d_{\mathcal{A}^I}((w, u), (\bar{w}, \bar{u})) < \epsilon$. Therefore if p_1 and p_2 are two equilibrium price functions for (w, u) such that $\|p_1 - p_2\|_\infty < \delta$ then $p_1 = p_2$. Since if equilibrium allocations are near equilibrium prices are near too we showed that $C_{rs} \subset \text{int}(C)$. We now show that C_{rs} is dense. It suffices to show that C_{rs} is dense in the set of simple functions. Let $(\hat{w}, \hat{u}) = \sum_{i=1}^L (\hat{w}_i, \hat{u}_i) \chi_{T_i}$ be any simple function in $\mathcal{A}^I$. Smale's (1974) results imply that, given $\epsilon > 0$, there exist, for each l , a regular spot economy (w_l^o, u_l^o) such that $|w_l^o - \hat{w}_l| < \epsilon$ and $d_{U^I}(u_l^o, \hat{u}_l) < \epsilon$. Then the simple function $(w^o, u^o) = \sum_{i=1}^L (w_i^o, u_i^o) \chi_{T_i}$ belongs to C_{rs} and $d_{\mathcal{A}^I}((w^o, u^o), (\hat{w}, \hat{u})) < \epsilon$ as claimed.

Remark 2 *The above theorem is in apparent contradiction with Davila(1996) result that the sets of endowments whose equilibria are determinate and indeterminate are both large. However Davila restricts his consumers endowments to a set with a constant Edgeworth box(namely $[0, 1]^2$). This set is a first category set in the set of all endowments and is therefore small for our purposes.*

4 Regularity of ex-post endowments.

In this section we allow for asset trades. We will assume that $(, \mathcal{B})$ is the Lebesgue measurable space $([0, 1], \mathcal{B}([0, 1]))$ and μ is the Lebesgue measure. Let $=$

$$\{(w, u, A) \in \mathcal{A}_{IM}; \exists \theta \in (J)^I \text{ such that } \mu(\{s \in ; (w_s^i + A_s \theta^i, u_s^i)_{i=1}^I \in \mathcal{N}\}) > 0\}$$

Theorem 2 *is a first category set of $\mathcal{A}^I \times L^\infty(, C^J)$.*

Proof. Let $_m$ be the set

$$\{(w, u, a) \in ; \exists \theta \in (J)^I, \|\theta\| \leq m, \mu(\{s \in ; (w_s^i + A_s \theta^i, u_s^i)_{i=1}^I \in \mathcal{N}\}) \geq 1/m\}$$

and notice that $= \bigcup_{m=1}^\infty _m$. We start by showing that $_m$ is closed. Let $(w^n, u^n, A^n) \in F_m$ for every $n \in \mathbb{N}$ be a sequence such that $(w^n, u^n, A^n) \rightarrow$

(w, u, A) . For each n there is a $\theta^n \in ({}^G Y, \|\theta^n\| \leq m$ such that $\mu\{s \in ; (w_s^i + A_s(\theta^n)^i, u_s^i)_{i=1}^I \in \mathcal{N}\} \geq 1/m$. Without loss of generality $\theta^n \rightarrow \theta$, $\|\theta\| \leq m$. If we define $T = \{s \in ; ((W_s^n)^i + A_s^n(\theta^n)^i)_{i=1}^I \in \mathcal{N} \text{ infinitely often} \}$ we have that $\mu(T) \geq 1/m$. It shall be clear that $(W_s^n + A_s \theta^n)_{i=1}^I \in \mathcal{N}$ for all $s \in T$. This proves that $\mathcal{N}$ is closed. We now show that $\mathcal{C}$ is dense. Let $(w, u, A) \in \mathcal{A}_{JM}$ and $\epsilon > 0$. There exist simple functions $\hat{w}$, $\hat{u}$ and $\hat{A}$ such that $d_{JM}((\hat{w}, \hat{u}), (w, u)) < \epsilon$ and $\|A - \hat{A}\|_\infty < \epsilon$. Let us write $(\hat{w}, \hat{u}, \hat{A}) = \sum_{i=1}^L (w_i, u_i, A_i) \chi_{T_i}$ and, by the isomorphism theorem (see Halmos(1974), theorem 41.C, page 173), each T_i is isomorphic to $[0, \mu(T_i)]$ and also to $([0, \mu(T_i)]^G)^I$. Then there exist $\phi_i : T_i \rightarrow B_\epsilon(0) \subset ({}^G)^I$ such that the pre-image of a null set is null. We claim that $\sum_{i=1}^L (w_i + \phi_i) \chi_{T_i} = \hat{w}$ together with $\hat{u}$ and $\hat{A}$ are as desired, i.e., $(\hat{w}, \hat{u}, \hat{A}) \in \mathcal{C}$.

To see this, let $M_i = \{x \in ({}^G_{++})^I; (x_i, \hat{u}_i)_{i=1}^I \text{ is a non regular economy}\}$ and, given $\theta \in ({}^G)^I$, we have $\mu(\{s \in T_i; (\hat{w}_i + \hat{A}_i \theta^i)_{i=1}^I \in M_i\}) = \mu(\{s \in T_i; \phi_i(s) \in M_i - \hat{w}_i - \hat{A}_i \theta^i\}) = 0$ since $\mu(M_i) = 0$. Therefore $\mu(\{s \in ; (\hat{w}_s + \hat{A}_s \theta^s, \hat{u}_s^i)_{i=1}^I \in \mathcal{N}\}) = 0$ for any $\theta \in ({}^G)^I$, that is, $(\hat{w}, \hat{u}, \hat{A}) \in \mathcal{C}$.

5 Discreteness when assets are traded.

In the previous section we showed that there is a generic set of consumers characteristics and matrices such that for each portfolio choice ex-post endowments fails to be regular is a null set. However this null set depends on the portfolio choice. This difficulty, prevents us from providing a general result if $A \neq 0$. However we give below a partial result. Three assumptions are needed.

A-1 The instantaneous utility and assets returns function $(u^1, \dots, u^I, A) : \rightarrow \mathcal{U}^I \times ({}^G)^J$ is a simple function.

A-2 There exists a short sales lower bound $v \in -{}^G_{++}$.

Suppose $T_1, \dots, T_L$ are the elements of the partition of defining the simple function $(u^1, \dots, u^I, A)$. Define then

$$\mathcal{D}_i = \{w \in ({}^G_{++})^I; (w^i, u_i^i)_{i=1}^I \text{ is a non regular spot economy}\}$$

and

$$K_l = \{(A_l \theta^1, \dots, A_l \theta^I); \theta \in (\mathbb{R}^J)^I \text{ such that } \theta^i \geq v, \sum_{i=1}^I \theta^i = 0\}.$$

A-3 $\cup_{l=1}^L (\mathcal{D}_l - K_l)$ has an empty interior.

This last assumption states that the set of non-regular endowments of the spot economy $(w^i, u_i^i)_{i=1}^I$ when translated by vectors in the span of the assets returns matrices must have an empty interior.

Theorem 3 *Under assumptions A-1, A-2 and A-3 equilibria are locally unique for a set of ex-ante endowments which contains an open and dense set in $L^\infty(\cdot, (\frac{G}{++})^I)$.*

Proof. Let

$$G = \{w \in L^\infty(\cdot, (\frac{G}{++})^I); (w_s + K_l) \cap \mathcal{D}_l = \emptyset, \text{ for a.e. } s \in T_l, 1 \leq l \leq L\}$$

and $G_{rs} = \{w \in G; w \text{ is simple}\}$. Finally define

$$V = \{w; (w, u) \in \mathcal{A}^I \text{ has a discrete equilibrium set}\}.$$

To show that $G_{rs} \subset \text{int}(V)$ suppose $w^o \in G_{rs}$, $w^o = \sum_{l=1}^L \sum_{m=1}^{M_l} w_{ml}^o \chi_{m,l}$ where, $m_l \subset T_l$, $1 \leq m \leq m_l$, $1 \leq l \leq L$. By definition we have $(w_{ml}^o + K_l) \cap \mathcal{D}_l = \emptyset$ for every m, l . Since K_l is compact, $\mathcal{D}_l - K_l$ is closed. By assumption it has an empty interior. Therefore from $w_{ml}^o \notin \mathcal{D}_l - K_l$ there exists a $\gamma > 0$ such that $\overline{B(w_{ml}^o, \gamma)} \cap (\mathcal{D}_l - K_l) = \emptyset$ for $1 \leq m \leq m_l$, $1 \leq l \leq L$. Hence $A \cap \mathcal{D}_l = \emptyset$ where $A = \overline{B(w_{ml}^o, \gamma)} + K_l$. Then, for any $x \in A$, there is a neighbourhood O_x and a $\delta_x > 0$ such that for every $y \in O_x$ and every spot equilibrium prices p, p' , $|p - p'| < \delta_x$ implies $p = p'$. The cover $\{O_x; x \in A\}$ has a finite subcover $\{O_1, \dots, O_Y\}$. Define $\delta = \min\{\delta_1, \dots, \delta_Y\}$. For any $w \in (\frac{G}{++})^I$ such that $|w - w_{ml}^o| < \delta$, we conclude from $w + (A_l \theta^1, \dots, A_l \theta^I) \in w + K_l \subset A \subset \cup_{j=1}^Y O_j$ that there is an $\epsilon > 0$ independent of w such that for any equilibrium spot prices p and p' , for which $|p - p'| < \epsilon$, we must have $p = p'$. Therefore for any $w \in L^\infty(\cdot, (\frac{G}{++})^I)$ such that $|w - w^o|_\infty < \delta$, if p and p' are equilibrium price functions in $L^\infty(\cdot, \frac{G}{++})$ for which $|p - p'|_\infty < \epsilon$ we have $p = p'$. This proves that $G_{rs} \subset \text{int}(V)$.

To show that G_{rs} is dense let, $w \in L^\infty \left(\mathcal{G}_{++}^I \right)$. Given any $\epsilon > 0$ there is a simple function $\bar{w}$ such that $|w - \bar{w}|_\infty < \epsilon/2$. By the assumption the complement of $\cup_{l=1}^L (\mathcal{D}_l - K_l)$ is open and dense. Therefore there is a function $\bar{w} \in G_{rs}$ such that $|w - \bar{w}|_\infty < \epsilon/2$. This ends the proof.

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