Nova School of Business and Economics

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General Equilibrium Essays on Securities Financing Transactions

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Abstract

This dissertation studies different aspects of Securities Financing Transactions (SFT). The first chapter addresses the impact of *repo* margins on security prices. It is shown that when the agents leveraging their positions are short sellers, prices rise as margins go up. The second chapter studies how *repo* margins are determined in a model allowing for bankruptcy. The third chapter discusses shareholders' unanimity under market incompleteness when shares can be short sold. For a firm that is perfectly competitive both in the securities market and in the SFT market (a *securities lending* market), shareholders unanimously agree on maximizing firm's present value.

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Contents

Introduction

Securities Financing Transactions (SFTs) are transactions where either securities act as collateral in a cash loan (as in repo markets) or cash is used as collateral when borrowing securities (as in securities lending markets). SFTs allow different types of agents trading in securities to access secure funding. Banks, hedgefunds and other financial institutions increase their leverage by borrowing against their assets as collateral. But leverage can also be built up by short selling securities that were obtained through SFTs (either as collateral in repo or being borrowed in the securities lending market), for a given down payment which is the SFT margin.

SFTs played a major role in the financial crisis of 2008 and in the European sovereign debts episode of 2010-2012. Since then, it has become increasingly clear that a better understanding of its components is needed. Each of the three chapters in this dissertation aims at improving our understanding of a different aspect of the Securities Financing Markets.

The first chapter contributes to the literature in the general equilibrium theory tradition that studies the impact of leverage on asset prices. One of the main insights of papers such as Geanakoplos (2010), Fostel and Geanakoplos (2012) and Fostel and Geanakoplos (2014) is that leverage makes the price of the asset that serves as collateral rise. I show that this need not be the case when assets can be short sold, that is, when the collateral can be reused by the creditor on whose hands the collateral was pledged. Asset prices increase with leverage in a long market and decrease in a short market. By *short market* I mean a market where agents that have short positions in the security are fully levered, in the sense of having built the largest possible short position given the required down payment, which is the margin that must be paid to the Central

Clearing Counterparty (CCP). This means that these agents are short selling the whole collateral that has been pledged to them in repo. Analogously, in a long market, the longs are those that are fully levered, which means that they have pledged their entire long position as collateral in repo. I show that leverage is inversely related to the margins charged by the CCP.

I consider a simple binomial economy, as it is done in many of the models in the literature, and show that when the market is short, leverage and the security price move in opposite directions. In another example, where the market is long, leverage and the price of the security move together. It all depends on how short and long positions get reduced when margins are raised. If the former were fully levered and, therefore, are reduced proportionally more in response to a higher margin, then the price of the security should go up in order to clear the market.

Several episodes during the European sovereign debt crisis suggest that raised margins made bond yields fall, as should happen in a short market. It is known that in those days there were large speculative positions being built on the public debt of Ireland, Portugal, Italy and Spain. These episodes suggest that the increase in margins hit the short positions harder than the long ones and this forced the market clearing price to rise.

The findings in this chapter may be surprising as they challenge a widespread view that margins are always procyclical. The observed 2010-2012 events suggest that spiral movements are not the rule and the theoretical analysis in this first chapter of my dissertation says that procyclical or anticyclical movements are both likely to happen depending on which side of the market is more leveraged.

Having seen what is the impact of different exogenous haircuts on security prices, the next step should be an attempt to make those haircuts endogenous. How are repo haircuts determined? I contribute to this issue by examining it in what seems to be the most natural setting to start investigating it, when repo trades are done over the counter. This is my research goal in the second chapter of this dissertation. I build on the model in Bottazzi, Luque and Páscoa (2012) to account for limited commitment involved in repo trades. Repo trades are recourse loans. Default triggers bankruptcy and is therefore a very serious event that needs to be modeled by taking into consideration the whole bankruptcy process. It is not a decision that can be taken asset by asset, comparing promised payments and collateral values. Debtors can't be assumed to be repaying the minimum of these two, contrary to what happens in non-recourse loans, as is the case in a long standing literature emerging from the work by Geanakoplos and Zame in the nineties (see Geanakoplos (1997), Geanakoplos and Zame (1997) and Geanakoplos and Zame (2014)). For the same reason, default can't be avoided by designing contracts so that collateral values never fall below promised payments, as was the case in a contemporaneous literature dating back to Kiyotaki and Moore (1997). Garnishable estates must now be set against total debts (net of credits that the defaulter may be entitled to). This creates a non-convexity in the borrower's budget set that seems to have put off previous research efforts. However, there is still an interesting discussion to be had about what determines haircut and how it does it. I provide concrete examples of bankruptcy equilibria in a simple setting and then move on to characterize the incentives agents have while setting haircuts in a more general economy (provided an equilibrium exist).

The model has a finite number of agents and my approach does not rely on the convexifying effect of large numbers that is used in models with a continuum of agents (see e.g., Araujo and Pascoa (2002) and Sabarwal (2003)). I take this approach as it makes sense to keep the number of agents finite if one wants to capture the role of *counterparty risk* in the determination of haircuts. On this issue, there are different views in the applied literature. In particular, Gorton and Metrick (2012) argues that haircuts depend both on the underlying asset and on who is the counterparty in a repo transaction but that, in times of crisis, the latter gains importance. I want the model to account for this and it is clear that the risk associated to a specific counterparty would vanish in an economy with a continuum of agents.

The loss that a lender may suffer from counterparties' default may be related to the collateral falling in value (or being sold in a fire sale) but, since the loan is recourse, the loss cannot be associated to that asset risk in a simple way. It may happen that there is no *asset risk* but the *counterparty risk* will nevertheless govern what the lender gets back, which does not have to be equal to the collateral liquidation value.

After constructing the model, I address the existence of bankruptcy equilibria in a one-security, two agents economy. Then I characterize equilibria for more general economies (leaving the question of existence temporarily open) and infer how haircuts should move by anticipating asset or counterparty risks.

In the third chapter I leave repo markets aside and look instead at the *Securities Lending Market*. These are the SFT markets where firms' shares have been traditionally borrowed and lent while repo was for a long time used predominantly for fixed income securities. My goal in this chapter is to reexamine, in a context where short sales can be done, the longstanding problem of defining the objective of the firm when markets are incomplete.

There are two reasons for disagreement among shareholders when markets are incomplete. First, ownership of the firm's shares confers access to a stream of dividends that can be used to transfer income across time and across states of nature, that is, the share provides some *hedging* or *spanning* services to the shareholder. Dividends are contingent on the production plan the firm adopts and different shareholders, having different inter-temporal and inter-state marginal rates of substitution, will generally disagree on the production plan that provides the best hedging services.

Apart from these different preferences over the *hedging effect*, there is a second reason for disagreement as even the *income effect* may be perceived differently. Under incomplete markets, there are multiple non-arbitrage deflators and, therefore, shareholders might not be coordinating on how to anticipate the impact of production changes on the share price and on their current incomes.

I extend a result from Makowski (1983) and prove that shareholders of a perfectly competitive firm will unanimously agree to maximize its present value. A perfectly competitive firm is defined as one whose equity is traded at prices taken as given by holders of the shares and lent at lending fees taken as given by lenders of the shares. Shareholders of a perfectly competitive firm not only agree on maximizing the present value of the firm but also in how to compute this value.

I characterize the correct pricing function, which corresponds to the maximum valuation across agents and conclude the chapter showing that a necessary condition for a firm to be perfectly competitive is that its shares do not trade at a *special* rebate rate in the securities lending market, that is, a rate below the *General Collateral* rate (the highest available rate across securities).

Chapter 1

Do Security Prices Rise or Fall When Margins Are Raised?

1.1 Introduction

Without rehypothecation, raised margin typically decreases the price of the collateral (houses prices for mortgage, or the price of other collateral used to asset back securities). In fact, the only leverage provided is to the long side of the collateral market, who is funding collateral purchases. Reduced leverage typically means less buying and hence a lower collateral price.

The situation is less clear, however, in the presence of rehypothecation, for example in the securities markets. Such markets are self-collateralised and allow short positions: traders that accept a security as collateral can short sell it. Since leverage is provided to both the long and the short side, the price impact of a leverage reduction can also can go either way. With the ever more prevalent counterparty clearing houses (CCPs), both the long side and the short side need to post margin and are thus affected by variation of haircut[1]: it is well documented how higher

[0] Joint work with Mário Páscoa and Jean-Marc Bottazzi.

[1] Stricly speaking, the haircut indicates, in percentage terms, how the loan falls below the collateral value, whereas the initial margin expresses the collateral as a percentage of the loan (it is therefore the inverse of

volatility (in particular volatile increase in bond yields) make CCPs (and futures exchanges) increase the initial margins charged in repo trades to both sides of the market[2], agents that pledge securities and agents that accept these as collateral for cash loans.

It has been argued, in most of the literature we review below, that the impact of higher haircuts is always to lower the price of the collateral. If this were true this would mean that raising margin is always procyclical : if the price of the collateral is already under pressure, a higher haircut pushes it further down in some vicious cycle that in the worst case can feed on itself. However, to take just an example, if we observe what happened in the European sovereign debt crisis, we see yields dropping after some margins were raised. This is more notorious for the 10 year government bonds of the following countries and for the following episodes: Ireland, 12 and 26 November 2010 and again for the six consecutive margin increases from April through June 2011; Italy, 9 Nov 2011 and July 2012; Portugal, 11 May 2011 and 29 June 2011; Spain, 20 June 2012.

There are nevertheless also some down spirals in prices[3], but this is certainly not the most frequent behavior when CCPs were raising margins. Later, in particular for Irish and Spanish bonds, as the crisis started to fade away, speculation decreased and the fall in haircuts was accompanied by a fall in yields.

There may be other factors behind such yield movement, but it does suggest that the impact of raising margin is *not always* to push the price of the collateral down. Our view is that *in securities markets* it depends on *positioning.* For this reason, as positioning changes quickly over time, one can expect the impact of margins to be more visible in short term market reactions rather than in long term secular trends. It is well-known that at various times speculative positions were quite significant. This raises the possibility that raising haircut could have had more impact on the short side. And a contributing factor of yields going down at times is shorts being the leveraged traders being forced to deleverage when margin is raised. In market parlance,

1-haircut).

[2]It should be clear that a similar analysis can be done replacing security with deliverable futures and haircut with futures initial margin.

[3]19 November 2010 and 25 March 2011 for Ireland; throughout April 2011 and on 14 June 2011 for Portugal

a security market is *long* when the longs are leveraged and is said to be *short* when it is the shorts that are more leveraged instead. We will give a more formal definition for a terminology that superficially seems to clash with market clearing (which requires longs and shorts to be matched). The emphasis is on where the leverage is when the haircut rises. If the short side gets to be reduced more than the long side (which happens when 'the market is short'), then the price of the security going up, everything else being equal, enables markets to clear.

In this paper we build a model of centrally cleared repo trades and provide examples where prices rise or fall in the wake of an increase in initial margins, depending on whether the market is short or long. Most of the literature emphasized the procyclical role of repo haircuts and the resulting propagating down-spiral. Adrian and Shin (2009) observe a strong positive correlation between leverage and balance sheet size of US financial intermediaries, inferring from it a procyclical role of leverage. A theoretical work by Brunnermeier and Pedersen (2009) notices that short selling also requires capital in the form of margin and provides an interesting result pointing out that a margins spiral arises when long margins are decreasing in prices or short margins are increasing in prices. It seems to be implicit that prices might increase with short margins and that a counter-cyclical pattern could happen if the short margins were decreasing in prices, just like the long margins.

The observed, not so infrequent, escapes of what just before looked like an implacable vicious circle logic motivated us. When, in response to a price drop and volatility, margins are increased and the price subsequently rises, is the raised margin a contributing factor to this rise or does the rise in price happen in spite of a headwind margin effect? We suggest that margin effect can sometime be supportive: higher margin also can nudge prices higher.

In the empirical literature, the analysis that Gorton and Metrick (2012) did of the 2008 financial crisis argued that the spiral interaction of repo haircuts and prices of structured securities led to a run on repo and insolvency of the US banking system. While that pro-cyclicality may have actually occurred in what was then a long market for those securities, Krishnamurty et al. (2014) downplayed its magnitude and impact on the whole financial system. These different assessments may be due, as Infante (2015) pointed out, to Gorton and Metrick (2012) having focused on bilateral trades, while in the large US tri-party market haircuts may have been changed much less. We argue for a balanced view: in a market with both longs and shorts, *leverage by itself does not have to be procyclical.*

In any case, the presumed procyclical nature of margins set by CCPs, in a less disperse way than in the over-the-counter (OTC) market, has even led to the view that CCPs might be factor of instability. This view was nevertheless met with some scepticism by European policy-makers that value CCPs' role as a firewall against the propagation of default shocks and find an increasingly centrally cleared market easier to monitor (see Constâncio V. (2012)). In the case of the European sovereign debt crisis, a lot of positioning was done by banks and hedge funds. Banks tend to use CCPs and hedge funds use prime brokers who act as clearers (also charging margin regardless of whether the trader is short or long). Also Futures contracts (where the exchange asks for initial margin regardless of position direction) were extensively used.

There is an important literature in the general equilibrium theory tradition that has studied the impact of leverage on asset prices. This is the literature that is closest related to our work. One of the main insights of the seminal papers by Geanakoplos (2010) or Fostel and Geanakoplos (2012 and 2015) is that leverage makes the price of the asset that serves as collateral rise. We show that this need not be the case if *assets can be short sold*, that is, when the collateral can be reused by the creditor on whose hands the collateral was pledged. In a nutshell: Asset prices increase with leverage in a *long* market and decrease in a *short* market. In a long market the optimists are leveraged, while it is the pessimists who are if it is *short.* To make the comparison easier, our examples are as close as possible to the framework used in those papers, in particular to the benchmark model in Fostel and Geanakoplos (2012). We have a binomial economy (the initial node is followed by two nodes) and two assets, one that can be pledged as collateral and another one that cannot. The former has stochastic returns and the latter is riskless. We can interpret the former as a risky security and the latter as cash. Agents have initial holdings of these assets at the first date and have linear utilities over second date outcomes. There is a continuum of such agents, differing only in the weights attached to the two states of nature in their utility functions.

We depart from the general equilibrium model of these authors in three aspects. First and foremost the collateral can be reused (and short sold). Second, loans are recourse (as in typically the case in repo) and default is a bankruptcy trigger but for simplicity and tractability we focus on asset price fluctuations, ignore bankruptcy and look for full repayment outcomes[4]. Third, agents have endowments of the riskless asset also at the two nodes of the second date. This departure allows us to choose second date endowments high enough so that the shorts can contemplate maximum leverage for a given level of haircut, without going bankrupt at the second date [5].

Our model is used to construct two benchmark cases for a repo market cleared through a CCP. One where the market is short and raising haircuts/initial margin leads to higher security prices (as in the sovereign debts episodes mentioned above). Another, where the market is long and raising the haircut makes that the security price go down. What determines whether the market will be long or short is how security returns vary across nodes by comparison with endowments of the riskless asset. The latter might be too low (in the state where the security has higher return) to allow for the shorts to lever up to what the haircut allows for.

The paper is quite compact, as it is intended to question well-known results through our examples. In the first section we provide a very brief motivation on the basis of observed haircut. In section 2 we describe the model of centrally cleared repo and in the following sections we present the examples, discussing in more detail a short market where prices rise in response to higher margin. That being said we cannot claim that haircuts have a stabilizing effect by themselves. First there are many markets where leverage is only offered to the long side. Also when shorts are possible, leveraged positioning could stabilise the price when margin is raised

[4] As default leads to insolvency, it is a very serious event, not so common in repo markets. In (Bottazzi, Páscoa and Ramírez (2012)), we work out how bankruptcy can be modelled (and also how the individual optimality of full repayment can be checked), but in the current paper we leave that intricate non-convex problem aside.

[5] Such second date endowments were not needed in the non-recourse model of Geanakoplos (2010), Fostel and Geanakoplos (2012) and Geanakoplos and Zame (2014), since collateral liquidation values were never lower than effective loan repayments.

only if the market would be well equipped, in the sense of having plenty of leveraged shorts willing to take profit in any excessive down price move.

1.2 Suggestive episodes

The way CCPs raise repo margins when security prices fall has been well documented and elaborated on in the literature. For European government bonds, the major clearing houses are LCH.Clearnet Ltd (London), LCH. Clearnet SA (Paris) and Cassa di Compensazione e Garanzia (CC&G) SpA (Rome). The first one tends to raise initial margin when spreads exceed 450 basis points over a 10 year AAA benchmark. The second and third use the same margins model based on several indicators. Italian governments bonds are the main collateral used in repo in Europe and are cleared by both LCH.Clearnet SA and CC&G. Irish and Portuguese bonds (together with many bonds issued by less riskier countries) are traded by CLH.Clearnet Ltd, while Spanish bonds may be cleared by both LCH branches. Several papers did a very detailed analysis of how initial margins on 10 year bonds issued by these 4 governments responded to yields changes[6]. We notice that there are several episodes where margins went up but yields fell, contrary to a long standing view that haircuts are procyclical. For the 10 year *Irish bond*, figure 1.1 shows for late 2010 and the first half of 2011 several instances when haircut rises were immediately followed by falls in yields. For example, when in response to a yields increase (by 9% on the 9th of November 2010), LCH.Clearnet Ltd. announced on the 10th of November that the initial margin would be raised on the 12th to 15%. This led to decreases in yields of 7.9% and 2.7% on the 12th and the next market date, respectively. On the 1st of April 2011, initial margins were raised to 45% and yields dropped that day (by 2.62%), the following day (by 2.32%) and very significantly by 5.9% on the third subsequent market day (the 6th, followed by smaller reductions in the next four days).

[6]We borrow Figure 1.1 on the Irish bond from Altenhofen and Lohff (2013) and Figures 1.2 and 1.3 on the other three bonds from Molteni et al. (2016). In the Appendix we present tables reporting haircuts and yields for the most relevant dates.

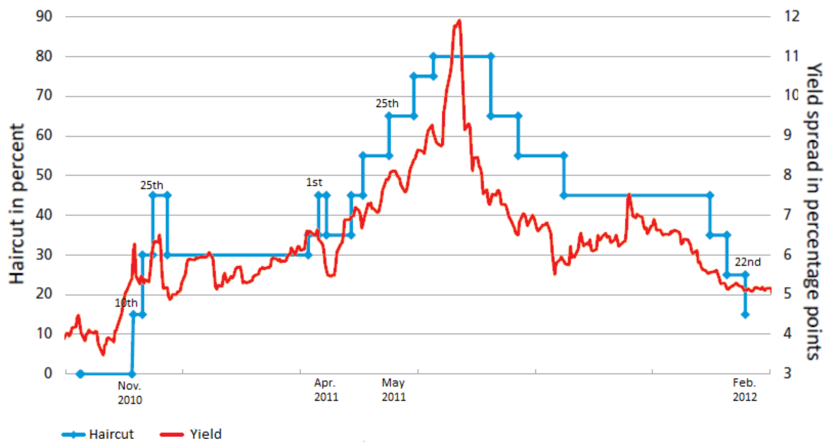


Figure 1.1: Haircuts and yield spread for 10 year **Irish** government bonds on 10 year German Government bond. Source: Molteni et al. (2016)

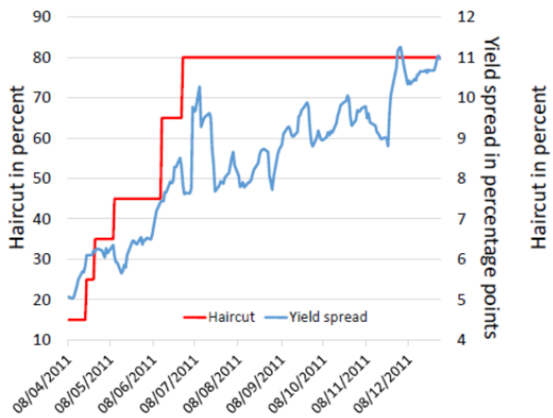


Figure 1.2: Haircuts and yield spread for 10 year **Portuguese** government bonds on 10 year German Government bond. Source: Molteni et al. (2016)

In the case of the 10 year *Portuguese bond*, when margins went up from 35% to 45% on the 11th of May 2011, the yield fell 4% that day (followed by smaller reductions in the next 3 days) and one month later, when LCH.Clearnet Ltd. announced on the 28th of June that margins were being raised to 80%, yields fell by 4% that day, by 2% the next day and by 1% the following date (with further decreases into July) .

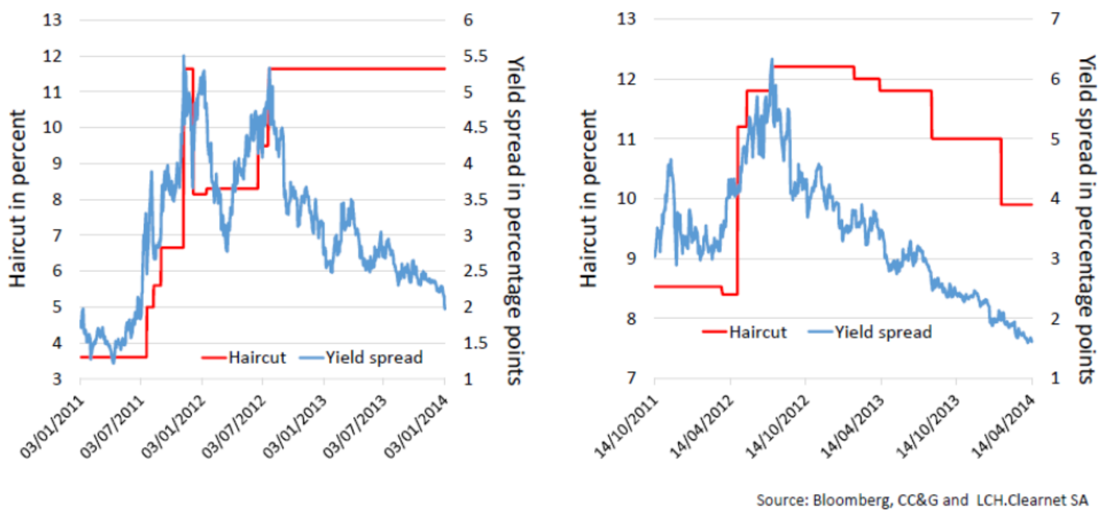


Figure 1.3: Haircuts and yield spread for 10 year **Italian** (left) and **Spanish** (right) government bonds on 10 year German Government bond. Source: Molteni et al. (2016)

For the 10 year *Italian bond*, we observe that margins rose (from 6.65% to 11.65%) on the 9th of November 2011 and yields fell by 4.5% and 6.4% on the 10th and the 11th of November, respectively. After a period of lower haircuts, margins were back at the same high level on the 23rd of July 2012 and yields fell by 1.8%, 6.2% and 1.6% on the 25th, the 26th and 27th. For the 10 year *Spanish bond*, we see that the announcement of a margins update (to 11.8%) made on the 19th of June 2012, with effect on the 21st, led to yield reductions of 2%, 5%, 2% and 3% in the announcement date and the next three dates, respectively.

Other factors may have impacted on yields. Margin increases took place at times when agents' (and credit rating agencies') perceptions of default risk may have fluctuated, affecting also the yields. Sometimes prices go down after margin being raised, as is noticeable for the Spanish bond from April to June 2012 and for the Portuguese bond throughout April 2011. However, this evolution is certainly not the rule. Even for Ireland and Portugal, where, in the first half of 2011, at a first glance, both yields and haircuts seem to have moved around upward trends, when we take a closer look, we see that there are many downward bumps in yields happening just after rises in margins.

We could have looked at other examples but this is not our focus. Our point is mostly theoretic and simplified for emphasis. It is not our purpose to try to adhere to those illustrative empirical examples closely. However, we see the above mentioned counter-cyclical episodes as *suggesting* that the speculative leveraged positions on these four sovereign bonds may be a factor explaining why prices may rise in response to higher margins. With this empirical motivation in mind we proceed to a theoretical analysis of the mechanisms that could possibly be at work and explain such a behaviour.

1.3 The Model

1.3.1 An agent's problem

We consider a binomial economy with two dates. At an initial date (date 0) there is only one node in the event tree, followed by nodes U and D at the second date.

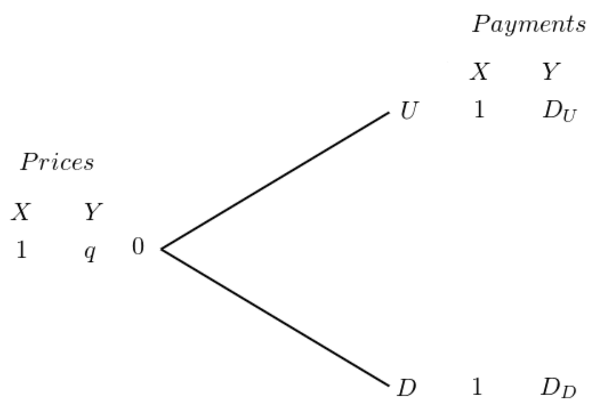


Figure 1.4: Events tree of the binomial economy.

This economy has just one good. Agents are endowed with, trade and value it only at the second date, in states U and D. We normalize its price to 1. At the first date, agents have initial holdings and trade two assets, X and Y. Asset X pays one unit of the good in each state and asset Y pays D_s units of the good in state $s \in \{U, D\}$, with $D_U < D_D$. The market for these assets opens only at date 0. The price of asset X is normalized to 1 and the price of security Y is denoted by q. There are two possible interpretations. The good can be seen as a perishable consumption good (and X as a productive resource) or we can identify the good with asset X (thought of as a durable good or money).

There is a continuum of agents indexed by $i \in (0,1)$. All agents have the same endowments: an unitary endowment of each asset at the initial node and state contingent endowments ω_s of the good at the second date. Agents only differ in the weights that their linear utilities attribute to consumption C_U^i and C_D^i in the U and D states, respectively:

$$U^i(C_U^i, C_D^i) = \gamma^i C_U^i + (1-\gamma^i) C_D^i, \tag{1.1}$$

We suppose that $\gamma : (0,1) \to (0,1)$ is a continuous and increasing function.

At date 0, agents pledge the security Y in a *repo* trade. This transaction consists in buying the security and promising to resell it at next date at a predetermined price. There is a difference between the price at which the title is bought, in the first leg of the transaction, and the price at which it is resold to its original owner, in the second leg of the transaction, at the second date. This difference is captured by the *repo rate*.

The repo trade can be seen as a secured loan: in the first leg, one agent pledges the security to get a loan that should be repaid in the second leg, at an interest rate (the repo rate) that was locked in before. Agents take positions x^i and y^i on asset X and security Y, respectively, and at the same time can pledge ψ^i units of the security in repo (in exchange for obtaining a loan with the same value $q\psi^i$) or accept θ^i units of the security as collateral (in exchange for giving a loan with the same value $q\theta^i$).

Equivalently, one agent borrows the security by accepting it as collateral and the other agent lends it by pledging it. At the second leg, the swap of securities for cash is reversed: the loan is repaid and the collateral given back to the agent that originally held it.

A non-negativity constraint applies to X, repo long and repo short positions:

$$x^i, \theta^i, \psi^i \geq 0 \tag{1.2}$$

Security positions are bounded in another way, by the interaction with repo positions. What can be pledged as collateral cannot exceed the security long position. So far, this is would be traditional collateral constraint. But we go beyond, as the *collateral can be reused* by the creditor: can be pledged in another repo trade or can be sold. Selling the collateral that is temporarily in the creditor's hands is what a *short sale* is. Naked short sales, unrelated to the amount of the security that was borrowed (accepted as collateral), are not accepted in reality and are ruled out in our model. Hence, security position can be negative y^i but short sales are bounded by the net repo position. The *box constraint* captures the two-sided interaction between security and repo positions[7]:

$$y^i + \theta^i - \psi^i \geq 0 \tag{1.3}$$

In other words, the net security title balance held by each agent (in each agent's "box") must be non negative. If the reuse of the collateral were not allowed (as in mortgages), this constraint would collapse to a plain collateral constraint of the form $y^i - \psi^i \geq 0$.

Repo trades are centrally cleared by an exchange (also known as central clearing counterparty, CCP), that charges a margin to both sides of the market, in a proportion $1 - h$ of the value of the repo trade[8]. Then, agent i budget constraint at date 0 is as follows:

$$(x^i - 1) + q(y^i - 1) + q(\theta^i - \psi^i) + (1 - h)q(\theta^i + \psi^i) \leq 0 \tag{1.4}$$

Or equivalently,

$$(x^i - 1) + q(y^i - 1) + q(2 - h)\theta^i - qh\psi^i \leq 0 \tag{1.5}$$

At the second date, agents settle their repo obligations, using their endowments and the returns from their first date security (long or short) positions. Repo loans are repaid at the repo rate ρ and, in exchange, the collateral is given back to the cash borrowers. The CCP gives back to the agents the margins it had collected, accrued of interest, at the same rate, the repo rate. The

[7] See Bottazzi, Luque and Páscoa (2012) on this constraint in a very general OTC repo model and also Duffie (1996) on similar constraints.

[8] See Bottazzi, Luque and Páscoa (2012) on OTC case and how haircuts are different in that case.

budget constraint for agent i in state s is as follows:

$$C_s^i = \omega_s + x^i + y^i D_s + (1+\rho)q(\theta^i - \psi^i) + (1+\rho)q(1-h)(\theta^i + \psi^i) \geq 0 \tag{1.6}$$

That is,

$$C_s^i = \omega_s + x^i + y^i D_s + (1+\rho)q[(2-h)\theta^i - h\psi^i] \geq 0 \tag{1.7}$$

A plan $(x^i, y^i, \theta^i, \psi^i)$ is said to be *feasible* for agent i at prices (q, ρ) if it satisfies the budget constraints (1.5) and (1.7), the box constraint (1.3) and the non-negativity restrictions (1.2). A plan that maximizes (1.1) among all plans that are feasible for i at (q, ρ) is said to *optimal* for i given (q, ρ).

1.3.2 Equilibrium

The CCP collects margins from both sides of the repo market and is supposed to pay back these margins, at the repo interest rate. To accomplish this, it should invest the margins and there is no better way than to invest also in repo, that is, to be also a repo long (accept the security as collateral and provide a loan, on its own), with a position $\Theta^{EX} \geq 0$.

The market clearing condition for the repo market is:

$$\int_0^1 \theta^i di + \Theta^{EX} = \int_0^1 \psi^i di, \tag{1.8}$$

Since the CCP's repo investment must coincide with the margins it collects from both sides of the market, that is,

$$\Theta^{EX} = (1-h) \cdot \left(\int_0^1 \theta^i di + \int_0^1 \psi^i di \right)$$

Hence, repo markets clear if

$$(2-h) \int_0^1 \theta^i di = h \int_0^1 \psi^i di, \tag{1.9}$$

The market clearing conditions for the two asset markets are:

$$\int_0^1 x^i di = 1 \text{ and } \int_0^1 y^i di = 1 \tag{1.10}$$

An **equilibrium** consists in prices (q, ρ) and an allocation of agents' plans $(x^i, y^i, \theta^i, \psi^i)_{i\in(0,1)}$ such that **(i)** each agent's plan is optimal for him at these prices and **(ii)** asset and repo markets clear.

1.3.3 Leverage

Leverage in securities market is usually measured by the ratio of the value of position to the down payment needed to build that position. In the case of a long position, the purchase of the security can be financed in the repo market - pledging the *whole* long position and obtaining a cash loan with the same value), but incurring the cost of paying the margin to the CCP, that is, spending just a fraction $1 - h$ of the security value. Leverage is given by $1/(1 - h)$. Analogously, in the case of a short position, if the agent is short selling the *whole* collateral that he has accepted, it presumes that a cash loan of the same value was given and that a margin was paid to the CCP. Again, the net cost is just the fraction $1 - h$ of the security value being shorted. So, leverage is once more given by $1/(1 - h)$. However, in both cases, this is what leverage is when the agent has decided to *lever up to full potential given the haircut.* Lower leverage would result if the long hadn't pledged the entire long position or if the short hadn't reused the entire collateral that was pledged to him.

Collateral reuse may play a role in allowing for the longs to lever up to full haircut potential. The aggregate (fully) leveraged long position, which is inversely proportional to the haircut $1-h$, may exceed what the aggregate initial holdings were for this security. Such outcome can only satisfy market clearing if short sales are done in equilibrium. However, how much leverage can be achieved depends both on how wealthy agents are and how low the haircut is.

In the examples that we present in this paper we exhibit equilibria where the longs or the shorts are levered up to full potential but not both at the same time, hence providing clear example of 'long' and 'short' markets. The side hitting that full leverage constraint - only one side does- will tell us whether the *market is long or short*, respectively. In both cases, being fully leveraged means that the security position (long or short, respectively) can be built to the most that the margin downpayment allows for (by combining the first date budget and box binding equalities). Second date endowments should not be an obstacle to such leverage. That is, in the case of a short market, state D endowments should be high enough (or margins not too low) to allow for the shorts to fully lever while still having non-negative income (not becoming insolvent) at that state, where security returns outweigh repo repayments. Similarly, in the case of a long market, state U endowments should be high enough to allow for the longs to fully lever without becoming insolvent in the state where security returns fall below the repo repayment that is due.

Finally, we note that in both examples, the collateral gets reused. In the 'short' case this is obvious but we observe also that in the 'long' case short sales are always done in equilibrium. The longs' wealth (their initial holdings of the two assets, evaluated at equilibrium prices) is high enough (by comparison with what the haircut is), so that aggregate long positions exceed the given aggregate initial holdings.

1.4 How do security prices respond to increases in leverage?

We will now show that, in our economy, whether the market is long or short is what determines how increases in leverage (reductions in margin) affect the price of the security.

We also identify the conditions that give rise to short and long markets. Short markets are likely to occur when the endowment in the state that short sellers value less (ω_D in this economy) is high relative to the endowment in the other state. When ω_U is sufficiently high relative to ω_D, a long market is more likely to occur.

1.4.1 Short market

We construct an equilibrium with a short market. There is a *marginal agent* m who is indifferent between buying or selling security Y and this type of equilibrium only occurs when ω_D is large enough so that what ends up constraining the short position of agents $i \in (m, 1)$ is not the non-negativity of C_D^i but instead the binding box constraint. The shorts sell all their endowments of assets X and Y to build up the largest possible short position in Y. Denoting a typical agent in the set $(m,1)$ by H, we find that his positions are as follow:

$$y^H = -\theta^H = -\frac{1+q}{q} \cdot \frac{1}{1-h} \tag{1.11}$$

$$\psi^H = x^H = 0 \tag{1.12}$$

$$C_U^H, C_D^H > 0$$

Equation (1.11) shows that when it is the box constraint that bounds short-sellers' positions, their positions are inversely related to the haircut, that is given by the leverage $1/(1-h)$. This is what we have termed a short market. Denote a typical agents in the set $(0,m)$ by L. This is a buyer of both X and Y and holds both long and short positions in the repo market. From market clearing in the security we see that the security long position of this agent must be the following:

$$y^L = \frac{1}{m}\left[1 + (1-m)\frac{1+q}{q} \cdot \frac{1}{1-h}\right] \tag{1.13}$$

Repo market clearing requires

$$(2-h)[(1-m)\theta^H + m\theta^L] = hm\psi^L \tag{1.14}$$

The box constraint says that $\theta^L = \psi^L - y^L$. Hence, $(2-h)[(1-m)\theta^H + m\psi^L - my^L] = hm\psi^L$.

Now, $my^L = 1 + (1-m)\theta^H$ and therefore

$$\psi^L = \frac{2-h}{1-h} \cdot \frac{1}{2m} \tag{1.15}$$

$$\theta^L = \frac{1}{m(1-h)}\left[\frac{h}{2} - (1-m)\frac{1+q}{q}\right] \tag{1.16}$$

The non-negativity of θ^L has to be checked. It holds if and only if

$$h \geq 2(1-m)\frac{1+q}{q} \tag{1.17}$$

Moreover,

$$C_U^L = 0, C_D^L > 0, x^L = \frac{1}{m} \tag{1.18}$$

Agents in $(0,m)$ are moving all their consumption from state U to state D, that is, $C_U^L = 0$.

To ensure the optimality of agents' decisions we can start writing agent i's optimization problem. We denote $E^i D = \gamma^i D_U + (1-\gamma^i) D_D$, which is agent i's expected payment of security Y. On the right of each restriction we write the corresponding Lagrange multiplier.

$$
\begin{aligned}
&\max \\
&\qquad E^i\omega + (1+q) + y^i(E^i D - q) + \theta^i q(2-h)\rho - \psi^i qh\rho \\
&s.t. \\
&\qquad x^i = (1+q) - qy^i - q(2-h)\theta^i + qh\psi^i \geq 0 \quad (\mu^i_x) \\
&\qquad y^i + \theta^i - \psi^i \geq 0 \quad (\mu^i) \\
&\qquad C^i_s = \omega_s + (1+q) + y^i(D_s - q) + \theta^i q(2-h)\rho - \psi^i qh\rho \geq 0 \quad (\lambda^i_s) \\
&\qquad \theta^i \geq 0 \quad (\nu^i_\theta), \quad \psi^i \geq 0 \quad (\nu^i_\psi)
\end{aligned}
$$

We can now write the first order conditions of the problem of agent $i \in (0,1)$:

$$
\begin{aligned}
y^i: &\qquad E^i D - q = q\mu^i_x - \mu^i - \sum_s \lambda^i_s (D_s - q) \\
\theta^i: &\qquad q(2-h)\rho = q(2-h)\mu^i_x - \mu^i - \nu^i_\theta - q(2-h)\rho \sum_s \lambda^i_s \\
\psi^i: &\qquad qh\rho = qh\mu^i_x - \mu^i + \nu^i_\psi - qh\rho \sum_s \lambda^i_s \ ,
\end{aligned}
$$

Notice that an agent may have both θ^i and ψ^i positive only if the multiplier μ^i for the box constraint is zero.

Given that X pays a null interest rate, we look for equilibria where the repo rate is zero as well[9]. For agents i, with $i > m$, the first order conditions are satisfied for $\rho = 0$ with the

[9]If X had stochastic returns, the repo rate would fall in between the implied interest rates on X, interpretable as IOR.

following multipliers:

$$\mu_x^H = \frac{q - E^H D}{q(1-h)} > 0$$

$$\mu^H = q(2-h)\mu_x^H > 0$$

$$\nu_\theta^H = 0$$

$$\nu_\psi^H = 2q(1-h)\mu_x^H$$

$$\lambda_U^H = \lambda_D^H = 0$$

When $i < m$, the multipliers that satisfy the first order conditions for $\rho = 0$ are:

$$\mu_x^L = \mu^L = 0$$

$$\nu_\theta^L = \nu_\psi^L = 0$$

$$\lambda_U^L = \frac{E^L D - q}{q - D_U} > 0$$

$$\lambda_D^L = 0$$

Shorts have positive shadow values for both the box constraint (a possession value of the collateral, due to the desire to short sell) and the non-negativity constraint on X (expressing a wish to do a "‘naked short sale"’ in X). If the former were positive and the latter zero we would expect the repo rate to be negative, below the zero interest rate on X.

From these conditions, it follows that if agent $m \in (0,1)$ is indifferent between buying or selling Y the following condition must hold:

$$E^m D = q \tag{1.19}$$

Consider an economy where $D_U = 0.70$, $D_D = 1.15$. We assume that ω_D is high enough so that $C_D^H \geq 0$, which holds if $\omega_D \geq (1 + D_D)\left(\frac{D_D - (2-h)D_U}{D_U(1-h)}\right)$, $\omega_U = 0$. Agents' beliefs are given by $\gamma^i = i$. The equilibrium price q for the security must satisfy $0 = C_U^L = 1 + q + y^L(D_U - q)$. This implies that leverage is

$$\frac{1}{1-h} = \left[\frac{m(1+q)}{q - D_U} - 1\right]\frac{q}{(1+q)(1-m)} \tag{1.20}$$

where $q = E^m D = mD_U + (1-m)D_D$. Solving for m as a function of h allows us to find the equilibrium price q of the security, for different values of h.

As leverage $\frac{1}{1-h}$ increases, the price of the security goes down. Figure 1.5 shows how *the equilibrium security price decreases with the loan-to-value ratio* h, along the black curve. Condition (1.17) should hold in equilibrium, this is the case in the region below the gray curve. Equation (1.20) is depicted by the black curve (when (1.17) holds) and its dashed continuation (when (1.17) is not met).

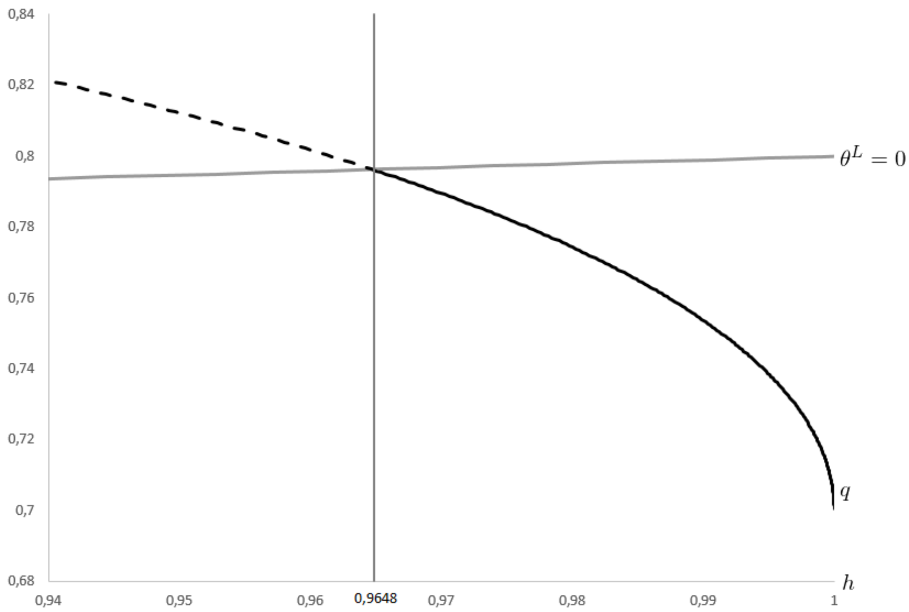
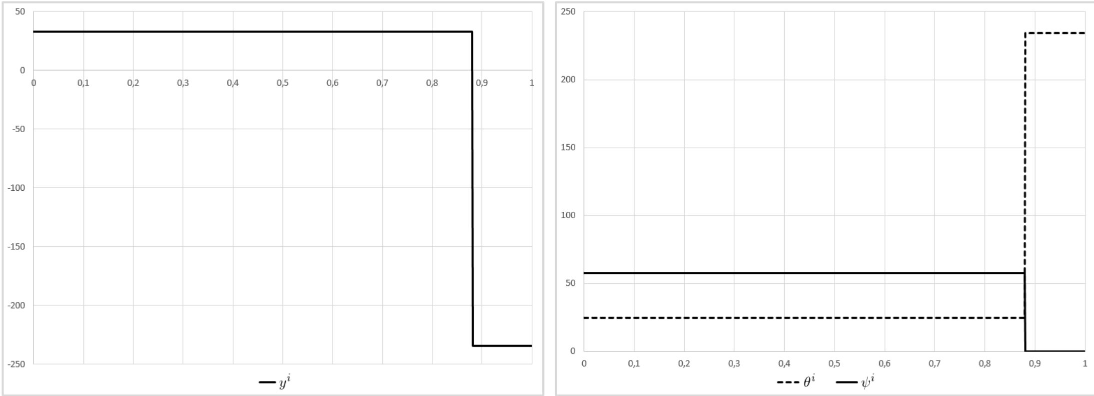


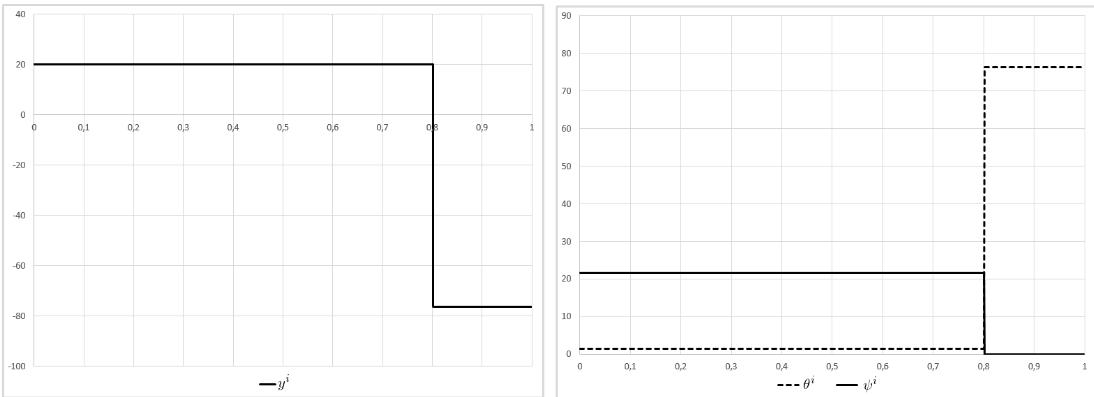
Figure 1.5: q vs. h - Short market

To gain more intuition, let us see look at the equilibria generated by two different haircuts, 1% and 3%. Figures 1.6 and 1.7 show the security and repo positions for the continuum of agents in the two equilibria.



(a) y^i (b) θ^i and ψ^i

Figure 1.6: Equilibrium corresponding to haircut of 1% ($h = 0.99$): $m = 0.881$, $q = 0.754$, $x^i = 1/m$ for $i \leq m$ and $x^i = 0$ for $i > m$ (assuming $\omega_D \geq 107.6$).



(a) y^i (b) θ^i and ψ^i

Figure 1.7: Equilibrium corresponding to haircut of 3% ($h = 0.97$): $m = 0.801$, $q = 0.789$, $x^i = 1/m$ for $i \leq m$ and $x^i = 0$ for $i > m$ (assuming $\omega_D \geq 34.7$).

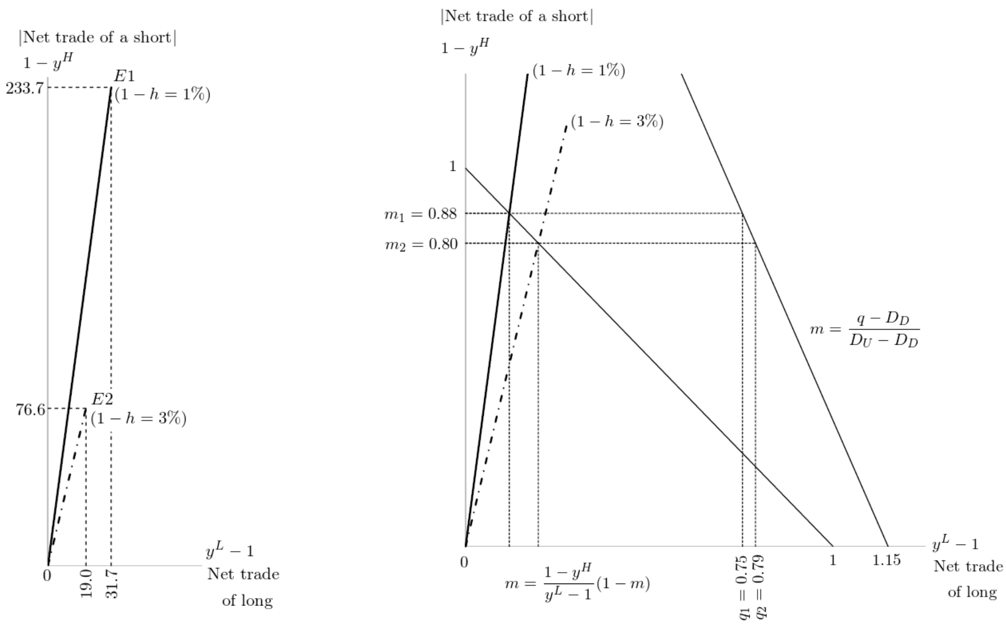
An agent above the marginal agent ($i > m$) is short selling the whole collateral that was pledged to him. An agent below the marginal agent ($i < m$) is long in the security and both long and short in repo (which is compatible with having a null shadow value for the box constraint), but with a net short repo position (pledging collateral in net terms). Asset X is bought entirely by agents below the marginal agent ($i < m$), those that haven't fully levered in Y.

Market clearing requires $my^L + (1-m)y^H = 1$. As the haircut increases, m has to change so that 1 remains a convex combination of the new positions y^L and y^H. Now, the graphs on the left-hand-side of Figures 1.6 and 1.7 show that, as the haircut is raised, the fully leveraged (individual) short position $|y^H|$ falls much more than the (individual) long position y^L. The short position y^H changes from -232.7 to -75.6 (is 32% of what it was before), while the long position y^L changes from 32.7 to 20.0 (is still 61% of what it was before). Both y^H and y^L got closer to 1, but the former much more than the latter. The weights on the convex combination must reflect that: m falls and, therefore, $q = E^m D$ rises. More formally,

$$m = \frac{1 - y^H}{y^L - y^H} = \frac{1}{1 + \frac{y^L - 1}{1 - y^H}} \tag{1.21}$$

That is, m is increasing with the ratio $\frac{1-y^H}{y^L-1}$ of the absolute value of the net trade of a short $(1 - y^H)$ to the net trade of a long $(y^L - 1)$. In a short market, this ratio falls as the haircut is raised. Figures 1.8a and 1.8b illustrate how m and q vary as this ratio changes. In the first graph we see that this ratio is the slope of a ray joining the origin to the point representing net trade positions of the two typical agents. The second graph takes an amplified look at what happens closer to the origin, where m can be found as the vertical coordinate of the point where the ray crosses the simplex[10]. Such m induces $q = D_D - (D_D - D_U)m$, read on the horizontal axis.

[10] The scales on the two axis are different, to allow for the change in m to be visible.



(a) Net trades of longs and shorts

(b) m and q determined from net trades ratio

In a long market, the opposite happens: an increase in the haircut makes the short's net trade drop proportionately less than the long's net trade, which leads to an increase in m and, therefore, a decrease in q, as will be examined next.

1.4.2 Long market

A long market is more likely to occur when ω_U is large relative to ω_D. The equilibrium is again characterized by a marginal agent $m \in (0, 1)$. The set $(m, 1)$ of agents has a typical agent denoted by H and the set $(0, m)$ has a typical agent denoted by L. Now, the agents with high valuations of the expected return of security Y have long positions on the security and are leveraged,

$$y^L = \psi^L = \frac{1+q}{q} \cdot \frac{1}{1-h} \tag{1.22}$$

$$\theta^L = x^L = 0 \tag{1.23}$$

$$C_U^L, C_D^L > 0$$

while agents with low valuations of security returns are short and not leveraged,

$$y^H = \frac{1}{1-m}\left[1 - m \cdot \frac{1+q}{q} \cdot \frac{1}{1-h}\right] \tag{1.24}$$

$$\theta^H = \frac{1}{2(1-m)} \cdot \frac{h}{1-h} \geq 0 \tag{1.25}$$

$$\psi^H = y^H + \theta^H \geq 0 \tag{1.26}$$

$$x^H = \frac{1}{1-m} \tag{1.27}$$

$$C_U^H > 0 \quad C_D^H = 0$$

The problem of the agents is the same as in the short economy and with the positions presented above, the first order conditions are satisfied when agents' positive multipliers are the following: $\mu_x^L = \frac{E^L D - q}{q(1-h)}$, $\mu^L = qh\mu_x^L$, $\nu_\theta^L = 2q(1-h)\mu_x^L$ and $\lambda_D^H = \frac{q - E^H D}{D_D - q}$.

We will consider the case when the parameters of the economy are all as before with only one exception: we interchange the values of the endowments ω_U and ω_D. Let $D_U = 0.70$, $D_D = 1.15$. We assume now $\omega_U \geq (1+D_D)\left(\frac{D_D-(2-h)D_U}{D_U(1-h)}\right)$ to ensure that $C_U^L \geq 0$. We make $\omega_D = 0$. Agents' beliefs are again given by $\gamma^i = i$. Now, $C_D^H = 0$ implies that leverage should be such that

$$\frac{1}{1-h} = \left[\frac{(1-m)(1+q)}{D_D - q} + 1\right]\frac{q}{(1+q)m} \tag{1.28}$$

where $m = \frac{D_D - q}{D_D - D_U}$. The non-negativity of $\psi^H = \frac{1}{1-m}[1 - m\frac{1+q}{q(1-h)} + \frac{h}{2(1-h)}]$ holds when the following condition is met,

$$\frac{D_D - D_U}{D_D - q}\,\frac{q}{1+q} \geq \frac{2}{2-h} \tag{1.29}$$

In Figure 1.9, condition (1.29) holds the region above the gray curve. Combining inequalities (1.28) and (1.29) we get a lower bound on h compatible with $\psi^H \geq 0$. For h above that lower bound ($h \geq 0.96189$), equation (1.28) describes how q evolves with h. The black curve in Figure 1.9 illustrates how the security price increases with the loan-to-value ratio h (and, therefore, with leverage) in the long market.

Observe that short sales are always done in equilibrium. In fact, $y^H < 0$ if and only if $m\frac{1+q}{q}\frac{1}{1-h} \geq 1$. By (1.28), we get $y^H < 0$ if and only if $\frac{(1-m)(1+q)}{D_D-q} + 1 \geq 1$, which holds trivially.

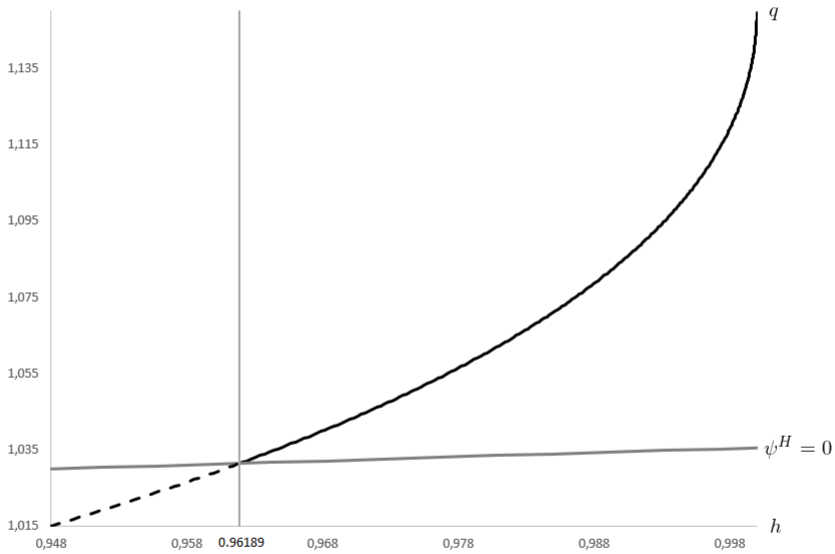


Figure 1.9: q vs. h - Long market

1.5 Concluding remarks

We adapted the binomial model in Fostel and Geanakoplos (2012 and 2015) to the securities and repo markets: the security can be pledged and short sold, repo loans are full recourse. For non-recourse loans and non-reusable collateral, leverage always makes the collateral price increase. In our self-collateralised securities market case, such implication holds only if the corresponding security market is 'long' - meaning people long the security are more leveraged than those who are short. On the contrary, if the security market is 'short', more leverage pushes the security price down. In Fostel and Geanakoplos (2012 and 2015) the asset market would always be long in our terminology as leverage is only provided to purchase the asset.

Geanakoplos (2010) had already show that a higher exogenous loan-to-value ratio leads to a higher collateral price. The binomial case allows for such ratio to be endogenously determined in a non-recourse equilibrium (so that default never occurs, as thoroughly shown by Fostel and Geanakoplos (2015)). The asset price was found to be higher in a leverage equilibrium than in a no-leverage equilibrium and, moreover, bad news affecting the asset payoff in the adverse state get reinforced by the endogenous leverage, (as shown in Fostel and Geanakoplos (2012))

We do not think the recourse/non-recourse aspect is key to drive our conclusion. On the other hand, rehypothecation is necessary to the extent that it is a precondition of even being able to short. This aside what matters is where leverage is used (on the short or long side). The adjustment mechanism goes as follows. Looking at the impact of raised margin requirements separately for agents with a long or a short position in the asset, leverage reduction will mean reduction of position for both. The price of the asset is the balancing factor that restores market clearing. If the position of a short is reduced by more than the position of a long - in what we call a 'short' market - the price of the security will go up. The reverse happens when the market is 'long' and it has been the case studied by Fostel and Geanakoplos (2012 and 2015), as this is the possible outcome for the price of a house securing a mortgage, or for the price of other collateral used to asset back securities.

Positioning (market 'long' or 'short') is the driver of whether the price of the security will go up or down when leverage is adjusted.

Appendix

1.6 10 year Bonds Yield Data.

Source: www.investing.com and www.lch.com

ITALY	Haircut	Yield	Change
Nov. 8, 2011	6.65%	6.707	-
Nov. 9, 2011	6.65%	7.286	8.63%
Nov. 10, 2011	11.65%*	6.957	-4.52%
Nov. 11, 2011	11.65%	6.512	-6.40%
Nov. 14, 2011	11.65%	6.702	2.92%

*Announced on the 9th, with impact on margin call in the morning of the 10th.

ITALY	Haircut	Yield	Change
July 23, 2012	9.5%	6.317	-
July 24, 2012	11.65%*	6.565	3.93%
July 25, 2012	11.65%	6.446	-1.81%
July 26, 2012	11.65%	6.047	-6.19%
July 27, 2012	11.65%	6.951	-1.59%
July 30, 2012	11.65%	6.014	1.06%

*Announced on the 23rd.

IRELAND	Haircut	Yield	Change
Nov. 9, 2010	≈0%	8.064	-
Nov. 10, 2010	≈0%	8.802	9.15%
Nov. 11, 2010	≈0%	9.108	3.48%
Nov. 12, 2010	15%*	8.391	-7.87%
Nov. 15, 2010	15%	8.162	-2.73%
Nov. 16, 2010	15%	8.368	2.52%

*Announced on the 10th.

IRELAND	Haircut	Yield	Change
Mar. 31, 2011	35%	10.349	-
Apr. 1, 2011	45%*	10.078	-2.62%
Apr. 4, 2011	45%	9.844	-2.32%
Apr. 5, 2011	45%	10.123	2.83%
Apr. 6, 2011	45%	9.530	-5.86%
Apr. 7, 2011	45%	9.462	-0.71%
Apr. 8, 2011	45%	9.337	-1.32%
Apr. 11, 2011	45%	9.304	-0.35%
Apr. 12, 2011	45%	9.274	-0.32%

*Announced on Apr. 1.

SPAIN	Haircut	Yield	Change
June 15, 2012	11.8%	6.870	-
June 18, 2012	11.8%	7.210	4.95%
June 19, 2012	11.8%	7.080	-1.80%
June 20, 2012	11.8%	6.730	-4.94%
June 21, 2012	11.8%	6.570	-2.38%
June 22, 2012	12.2%*	6.400	-2.59%
June 25, 2012	12.2%	6.640	3.75%
June 26, 2012	12.2%	6.690	0.75%

*Announced on June 20.

PORTUGAL	Haircut	Yield	Change
May 9, 2011	35%	9.849	-
May 10, 2011	35%	9.864	0.15%
May 11, 2011	45%*	9.464	-4.06%
May 12, 2011	45%	9.301	-1.72%
May 13, 2011	45%	9.259	-0.45%
May 16, 2011	45%	9.050	-2.26%
May 17, 2011	45%	9.157	1.18%

*Announced on May 10.

PORTUGAL	Haircut	Yield	Change
June 24, 2011	65%	11.551	-
June 27, 2011	65%	12.086	4.63%
June 28, 2011	65%	11.652	-3.59%
June 29, 2011	80%*	11.404	-2.13%
June 30, 2011	80%	11.243	-1.41%

*Announced on June 28.

Chapter 2

Determinants of Repo Haircuts and Bankruptcy

2.1 Introduction

In a repo trade, a security is pledged as collateral for a cash loan and can then by reused by the cash lender, that is, pledged in a another contract or short-sold. The reuse of the collateral makes repo trades quite different from mortgage loans where the durable good collateral stays put. The resulting leverage was studied in detail by Bottazzi, Luque and Páscoa (2012), under the assumption that agents always fulfilled their financial obligations.

Leverage played a major role in the recent financial crisis of 2008. Leading to the crisis, it was not only households that were highly indebted but also large financial institutions. These large institutions turned to the *shadow banking* system to finance themselves (see e.g. Gorton and Metrick (2010)). The repo market is a crucial part of this system. The haircut applied to the loan given in a repo trade is inversely related to how much agents can build up their positions in a security by using the repo market as a means of financing security positions. Security

[0] Joint work with Mário Páscoa and Jean-Marc Bottazzi.

and repo trades can be combined in way that allows security positions to be increased, as the security gets pledged as collateral in repo, then repledged or short sold by the creditor (then again pledged by the counterparty of the short seller and so on). How do collateral reuse and haircuts determine what leverage is? The former may be an ingredient but does not determine by itself what leverage is (and limitations on reuse do not automatically translate into targeted reductions in leverage). For long agents to lever up to haircut potential, the reuse of the collateral only becomes necessary when the wealth of these agents is high enough and the haircut is low enough that the resulting aggregate leveraged long positions exceed aggregate initial holdings of the security. It is ultimately the haircut that determines what leverage is (and it could be the shorts being leveraged instead). More recently, in the sovereign debt crisis of 2010-12, there was substantial deleverage (also of those short selling) caused by the consecutive hikes in repo margins on bonds issued by several European governments.

Given that leverage and the haircut are inversely related, it is crucial to understand how the latter is determined. The haircut is the difference between the values of the collateral and the respective cash loan, at the time when the repo trade starts. It is usually expressed as a percentage (less than or equal to 1) of the collateral value. Equivalently, the initial margin captures that difference by expressing the collateral value as a percentage (greater than or equal to 1) of the cash loan. A repo trade has a purchase leg and then a repurchase leg at a repurchase price that is locked in at the first leg. The difference between the purchase price (the cash loan) and the repurchase price is the repo interest rate, agreed upon in advance. Hence, in the absence of default, there would be no reason to charge a haircut. The haircut reflects the cash lender's perceived risk of loss in the event of the cash borrower's default.

In this article we model the limited commitment involved in repo trades. In this respect, also, there is a key difference by comparison with what happens in many (but not all) mortgages, as captured in the GE collateral literature. Repo trades are recourse loans, whereas many (but not all) mortgages are non-recourse. If an household that has signed a non-recourse mortgage decides to default, it would just surrender the house and walk away without suffering any other penalties. That is not the case in recourse loans: in the event of default, creditors can be repaid above the collateral liquidation value by forcing the bankruptcy of the faulty borrower and then becoming claimants in the partition of the borrower's estate. It may also happen that creditors end up recovering less than the collateral liquidation value, when that is the outcome from the partition of the estate among all creditors. Repo collateral is exempted from certain provisions of the US Bankruptcy Code that normally apply to pledges, in particular, the automatic stay on enforcement of collateral in the event of insolvency. That is, creditors can keep the collateral that had been pledged to them (and can sell it) but, when the bankruptcy court takes the final decisions, they may get more or less than what their claim was (the promised repayment) and this may be different from the liquidation value of the collateral.

It should be noted that when an agent goes bankrupt, it is not just the repayment of the cash borrowed in repo that is at stake. If a security happened to be pledged to this agent in repo, then this collateral will not be given back to the cash borrowers - a *"fail"* occurs as a result of bankruptcy - and the respective manufactured dividends due to the beneficial owner will not be paid also.

Default is a very serious event and needs to be modeled by taking into consideration the whole bankruptcy process. It is not a decision that can be taken asset by asset, comparing promised payments and collateral values. Debtors can't be assumed to be repaying the minimum of these two, contrary to what happens in non-recourse loans, as shown in a long standing literature emerging from the work by Geanakoplos and Zame in the nineties (see Geanakoplos (1997), Geanakoplos and Zame (1997) and Geanakoplos and Zame (2014)). For the same reason, default can't be avoided by designing contracts so that collateral values never fall below promised payments, as was the case in a contemporaneous literature dating back to Kiyotaki and Moore (1997). Garnishable estates must now be set against total debts (net of credits that the defaulter may be entitled to). This creates a non-convexity in the borrower's budget set that seems to have put off previous research efforts.

There are however interesting results that can be established, in spite of the intrinsic non-convexity of individual decision problems. We consider binomial economies, where just two states of nature, U or D, may occur after the initial node (and each of these states may be followed without uncertainty into a third date). Our paper focus on over-the-counter (OTC) repo, that is, trades that are not centrally cleared through an exchange (or central clearing counterparty, CCP), and bilateral (as opposed to tri-party where collateral selection, payment, custody and settlement are outsourced to a third-party agent). Our finite-agent model does not let us explore the convexifying effect of large numbers that has been used in continuum of agents models in several contexts, including in consumer bankruptcy problems with unsecured loans (see Araujo and Pascoa (2002) and Sabarwal (2003)). However, modeling the agents set as a continuum is not appropriate in a context of OTC repo where each trader should anticipate counterparty bankruptcy risk and choose repo haircuts accordingly[1].

For equilibrium to exist, leverage should be bounded. Here there is another important distinction between credit backed by securities and credit backed by houses or productive resources. In the latter, the aggregate supply of collateral is fixed and, therefore, under exogenous collateral margins, borrowing becomes bounded. In the former, the collateral supply is endogenous since it includes short-sales and, therefore, there are no a priori bounds on secured borrowing, even under exogenous margins. However, repo and security positions must be related in another way: the net security title balance held by each agent must be non negative. This is known as the *box constraint* and says that in to order to pledge the agent must be long in the security and in order to short-sell the agent must be long in repo (the security being pledged to him). In the one-security case, by combining the box and budget constraints we can bound secured borrowing. This would be enough to bound all sorts of leverage (long or short) in convex full commitment economies. However, in non-convex economies allowing for bankruptcy, we need to bound secured lending as well, since an equilibrium for a truncated economy (whose portfolios are assumed to be market feasible) may fail to be an equilibrium. In the multi-security case, it

[1] A bankruptcy analysis might be doable also for centrally cleared repo, but aggregate default risk should take the place of counterparty risk. The OTC case seems to be more informative and easier to relate to the applied literature on the determinants of repo haircuts.

was already known that, even in the convex full commitment setting, other constraints should be added with the purpose of bounding repo and security trades[2].

In order to gain intuition and allow for a full characterization of equilibria, we start by examining a one-security and two-agent case. In this simple case, the optimist is long in the security (short in repo) and the pessimist is short in the security (long in repo). We find equilibria where both or just the former go bankrupt (the former in the state where the security has lower returns and the latter in the other state). Then, we contemplate the multi-security and multi-agent case to see what are the determinants of haircuts. On this issue, there are different views in the applied literature. Gorton and Metrick (2012) argue that haircuts depend both on the underlying asset and on who is the counterparty in a repo transaction but that, particularly in times of crisis, the latter gains importance. In contrast, Krishnamurty et al. (2014) report little variation of haircuts across counterparties and place much more weight on the underlying asset. Infante (2015) argues that these observed differences on haircuts arise because two different markets are studied: the bilateral and tri-party repo.

The loss that a lender may suffer from counterparties' default may be related to the collateral falling in value (or being sold in a fire sale) but, since the loan is recourse, the loss cannot be associated to that asset risk in such a simple way. It may happen that there is no *asset risk* but the *counterparty risk* will nevertheless govern what the lender gets back, which does not have to be equal to the collateral liquidation value. What a secured creditor recovers in the bankruptcy process depends on what is the liquidation value of the whole estate of the defaulter and how it will be partitioned among all creditors, even though the exemption from automatic stay allows the creditor to sell the collateral while waiting for the final outcome of the bankruptcy process.

We characterize how haircuts respond to asset and counterparty risks. Suppose there are many traders in the repo market of each security, repo rates are security-specific but haircuts are specific to each pair of traders. In such competitive setting, we should expect counterparty

[2]See Bottazzi, Luque and Páscoa (2012) on bounds that result from the segregation of haircuts or the distinction between dealers and non-dealers and Bottazzi, Luque and Páscoa (2017) on bounds that follow from equity requirements in the spirit of the Basel regulation of banks.

bankruptcy risk to affect pair-specific haircuts but not the repo rate, as opposed to what happens in the two-agent example. Say state D is the state where bankruptcy may occur. Suppose an agent i is solvent in state D and is, in terms of the whole portfolio, a net creditor to a counterparty j (in state D) and the expected repayment rate of this counterparty decreases (an increased counterparty risk). Then, agent i would like to raise (lower) the haircut charged to counterparty j when accepting collateral from j, for securities whose repo repayment exceeds (falls below) the collateral value. That is, when the asset is risky from the creditor's perspective, haircuts tend to move in the same direction as the counterparty risk. But for the other securities (risky from the debtors' point of view, wary of a repo fail), haircuts move in the opposite direction.

Quite differently, in the two-agent and one-security example, counterparty risk affects the repo rate and his effect is strong enough to make bankruptcy rates decrease as the haircut increases. In a small numbers context, it is now the other direction that may become more relevant: how are overall solvency rates affected when the haircut charged in one security changes? That is why in such extreme non-competitive case, haircuts and expected repayment rates may move together, contrary to our results for the competitive case. To summarize, the way counterparty risk may impact haircuts depends on how competitive the repo market is and to understand how that impact works in the competitive case we need to couple this risk with asset risk. When faced with a rise in counterparty risk, competitive creditors tend to ask for higher haircuts for securities that exhibit an asset risk from the creditors' perspective, but lower haircuts may arise if the security involves the opposite risk (a fail rather than a default risk).

2.2 The Model

2.2.1 Fundamentals

We consider a binomial economy with three dates. At an initial date (date 0) there is only one node in the event tree, followed by nodes U and D at the second date. Each second date note has a unique successor at the third date: U^+ and D^+ are the successors of U and D, respectively.

As we will see, the third date just serves to guarantee that securities retain value at the second date, when borrowing and lending transactions are settled (and we may want to dispense with the third date in some cases, as discussed below).

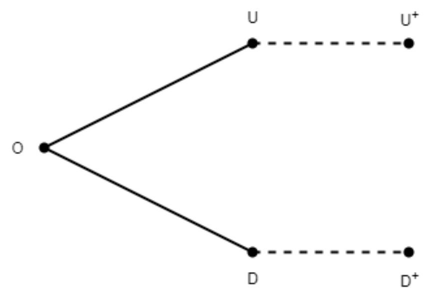


Figure 2.1: Events tree of the binomial economy.

Binomial models have been used to study the leverage cycle in economies with default on non-recourse loans (see e.g., Fostel and Geanakoplos (2012) and Fostel and Geanakoplos (2014)). Given that repo trades constitute recourse loans, we will model default as a bankruptcy process.

There is only one consumption good. Markets for this commodity open at each event. We denote the price of this good at event e by p_e. There is a finite set of $I \geq 2$ agents, indexed by i. A bundle of commodities consumed by agent i is denoted by $x^i = (x^i_0, x^i_U, x^i_D, x^i_{U+}, x^i_{D+})$. There are also F real securities indexed by f, each one being characterized by a vector of non-negative real returns $R_f = (R_{fU}, R_{fD}, R_{fU^+}, R_{fD^+})$. Given spot prices p_e, the nominal return of security f is $p_e R_{fe}$[3].

Trading of securities occurs at the first and second dates. Each agent chooses a securities portfolio $\phi^i \in \mathbb{R}^{3F}$ consisting of positions in the F securities at the initial nodes and nodes U and D. Security prices are denoted by $q \equiv (q^f_e) \in \mathbb{R}^{3F}$. Agents' endowments of commodities are $\omega^i \in \mathbb{R}^5_+$, with $\omega^i_s > 0$ in both states. Agents have initial holdings, at date 0, of each security f, $o^i_f > 0$. Preferences are described by utility functions $U^i : \mathbb{R}^5_+ \to \mathbb{R}$. For each security f, we normalize its positive net supply to be one: $\sum_i o^i_f = 1$.

[3]We could have considered nominal securities instead.

2.2.2 Repo markets

Agents can have negative positions in securities, short-sales are permitted. Short-selling, however, is not the same as issuing (which we take as given this model, having occurred prior to date 0). In order to short-sell a security, an agent must go first in the repo market and *borrow* the desired amount of securities. This is the way short-selling is actually done in reality.

Borrowing of securities actually consists in buying the security and promising to resell it to the lender, at a future date and at a predetermined price. There is a difference between the price at which a security is bought, in the first leg of the transaction, and the price at which it is resold to its original owner, in the second leg of the transaction, at a future date. This difference is captured by the *repo rate*. The highest repo rate within its class of securities is referred to as the *general collateral rate* (GC).

The borrower of a security acquires possession rights associated with the security. However, any coupon or dividend paid to the borrower during the term of the transaction is passed through to the original owner; this is called a *manufactured payment* or a *manufactured dividend*.

A repo transaction is actually a collateralized loan. If agent i buys security f from j in the first leg of the transaction, i is at the same time the borrower of the security and a lender of cash, and j is the lender of the security and borrower of cash. A haircut $(1 - h_f^{ij})$ is applied to the market value of the security to compensate the lender of funds for the risk associated to the transaction, so that the cash loan may be lower than the value of the collateral. Haircut is such that $0 < h_f^{ij} \leq 1$.

For simplicity, repo trading takes place only at date 0. We denote by z_f^{ij} agent i's repo position, with counterparty j, on security f. If $z_f^{ij} > 0$, it means that i is the lender of cash and borrower of the security and we say that he is *long* in repo. At the same time, since markets must clear, we should expect that $z_f^{ji} < 0$, which means that agent j is a borrower of cash, a lender of the security, and we say that he is *short* in repo.

Given that a haircut is applied to every repo transaction, the amount of funds that can be borrowed by pledging one unit of security f in the repo market, in a transaction with counterparties i and j, is given by $h_f^{ij} q_{f0}$. As a matter of notation, we use both h_f^{ij} and h_f^{ji} to denote the haircut applied to transactions in the repo market involving security f and counterparties i and j, regardless of which one of the agents is long in repo for security f and which one is short. Denote by ρ_f the repo rate of a loan backed by security f and let $r_f = 1 + \rho_f$.

An important feature of a repo transaction is that the borrower of the security has the right to lend it in the repo market or short-sell it in the security market. That is, the collateral can be *rehypothecated* directly or indirectly, and this process may occur many times over for the same settlement period. This is an important feature of repo markets as it it the *reuse* of the collateral that allows agents to leverage their portfolio positions or their cash loans beyond what would be possible if collateral was used only once.

There is a constraint that captures both the need to pledge collateral when borrowing cash in repo and the need to borrow a security (accept it as collateral) when short-selling it. This is the *box constraint*. This restriction states that the agent must hold a nonnegative amount of the security in his possession. That is, the sum of security position and repo trades must be nonnegative. At the initial date (the only node where repo markets are open) the box constraint for security f is:

$$\phi_{fe}^i + \sum_{j \neq i} z_f^{ij} \geq 0 \tag{2.1}$$

At second date nodes, repo markets are not open and the box constraints reduce to plain no-short-sales constraints:

$$\phi_{fU}^i \geq 0, \;\; \phi_{fD}^i \geq 0 \tag{2.2}$$

When there is more than one security, without any further assumptions on portfolio or repo positions, leverage can be unbounded (see Bottazzi, Luque and Páscoa (2012) on some institutional arrangements that bound leverage). In this paper we take the simple approach of imposing some bounds on repo positions whenever there is a need to bound leverage[4].

[4]In the one-security model in Section 2.3, leverage will be bounded and no further constraints will be imposed on financial positions.

2.2.3 Bankruptcy and feasible market plans

Given market prices and repo rates, (p, q, r), agents decide on a plan (x^i, ϕ^i, z^i), consisting of consumption and portfolios in the securities and repo markets. Let us define the budget constraints that these plans must satisfy. At date 0, the repo market opens and agents have initial endowments of goods and securities. Agent i's budget constraint at this date is:

$$p_0(x_0^i - \omega_0^i) + \sum_f q_{f0}(\phi_{f0}^i - o_f^i) + \sum_f \sum_{j \neq i} q_{f0} h_f^{ij} z_f^{ij} \leq 0 \tag{2.3}$$

Repo markets do not open at the second date but agents can still trade in securities, using real payments from their previous securities positions. We allow for the possibility of agents not fulfilling their obligations in the second date. This only happens if agents become insolvent. For every agent, we need to see how do assets set against liabilities, in each state of the second date. On the assets' side we have a first component which is the new market value plus returns associated with the actual amount of each security that the agent has in his possession when he enters that node. This (non-negative) amount is what the agent had in his date 0 box for the corresponding security. By adding up the current market value and returns, across all securities, we get assets' component pertaining to *the value of the past box positions*:

$$\Xi_s^i = \sum_f \left(\phi_{f0}^i + \sum_{j \neq i} z_f^{ij} \right) (q_{fs} + p_s R_{fs})$$

To evaluate assets or liabilities resulting from repo positions taken at the previous node, we need to remember that repo positions consist of securities that the agent has received (borrowed) or pledged (lent) as collateral backing a cash loan. If agent i borrowed security f from agent j at the initial node ($z_f^{ij} > 0$), he must return the security together with the respective returns to its original owner, that is, pass on the non-negative value $z_f^{ij}(q_{fs} + p_s R_{fs})$ to agent j but at the same time agent j must repay (at the gross rate r_f) to i the cash loan he obtained before. Conversely, a repo short agent repays the cash loan and gets back the value of the security he pledged as collateral. The settling of all of agent i's repo transactions with agent j as counterparty is captured by the following term:

$$I_s^{ij} = \sum_f z_f^{ij} \left[q_{f0} h_f^{ij} r_f - (q_{fs} + p_s R_{fs}) \right]$$

If all repo transactions are settled as was agreed upon at the initial date, that is, if every agent is solvent in state s, agent i's corresponding budget constraint is:

$$p_s(x_s^i - \omega_s^i) + \sum_f q_{fs}\phi_{fs}^i \leq \ \Xi_s^i + \sum_{j\neq i} I_s^{ij} = \sum_f \left(\phi_{f0}^i \left(q_{fs} + p_s R_{fs}\right) + \sum_{j\neq i} z_f^{ij} q_{f0} h_f^{ij} r_f \right)$$

Since repos are recourse loans, insolvency will occur in state s only if the agent's assets are insufficient to cover his liabilities. The assets include the value of the past box positions, plus the positive settlements of his repo transactions, plus the garnishable portion (according to some coefficient β) of his commodity endowment in that state. Liabilities consist in the negative settlements of his repo trades. That is, insolvency will occur if, and only if, the following condition is satisfied:

$$\beta p_s \omega_s^i + \Xi_s^i + \sum_{j\neq i} \eta_s^j I_s^{ij+} \geq \sum_{j\neq i} I_s^{ij-}$$

Here, I_s^{ij+} and I_s^{ij-} denote the positive and the negative parts[5], respectively, of I_s^{ij}, while η_s^j denotes the portion of all of agent j's financial obligations that he *effectively* repays given his income in that state. That is, $\eta_s^j = 1$ if j is solvent in state s and $\eta_s^j < 1$ when he declares bankruptcy. In words, agent i declares bankruptcy in state s if, and only if, the garnishable portion of his commodity endowment ($\beta p_s \omega_s^i$), plus the value of the securities in his possession at the beginning of the second date (Ξ_s^i), plus the positive repo repayments the agent gets from all his counterparties ($\sum_{j\neq i} \eta_s^j I_s^{ij+}$) is not sufficient to repay all of agent i's repo obligations (I_s^{ij-}).

Once we allow bankruptcy, agent i's budget constraint in state s of the second date is:

$$p_s(x_s^i - \omega_s^i) + \sum_f q_{fs}\phi_{fs}^i \leq \max\left\{ -\beta p_s \omega_s^i \ , \ \Xi_s^i + \sum_{j\neq i} I_s^{ij+}\eta_s^j - I_s^{ij-} \right\}, \tag{2.4}$$

[5] $I_s^{ij+} = \max\{0, I_s^{ij}\}$ and $I_s^{ij-} = -\min\{0, I_s^{ij}\}$.

where

$$\eta_s^i = \begin{cases} 1\ , \ \beta p_s \omega_s^i + \Xi_s^i + \sum_{j \neq i} \eta_s^j I_s^{ij+} \geq \sum_{j \neq i} I_s^{ij-} \\ \frac{\beta p_s \omega_s^i + \Xi_s^i + \sum_{j \neq i} \eta_s^j I_s^{ij+}}{\sum_{j \neq i} I_s^{ij-}}\ , \quad \text{otherwise} \end{cases} \tag{2.5}$$

At the last date there is no trading of securities. Agents consume from their commodity endowments and security payments according to their security positions constituted at the previous nodes

$$p_{s^+}(x_{s^+}^i - \omega_{s^+}^i) \leq \sum_f \phi_{fs}^i p_{s^+} R_{fs^+} \tag{2.6}$$

Given parameters (p, q, r), an agent's plan of consumption, of securities, and of repo positions (x^i, ϕ^i, z^i), will be called *feasible* if $x^i \geq 0$ and conditions (2.1), (2.2), (2.3), (2.4) and (2.6) hold. We denote by $B^i(p, q, r)$ the set of all feasible plans for agent i, and by $B_\star^i(p, q, r)$ the subset of utility maximizing plans in $B^i(p, q, r)$.

2.2.4 Equilibrium

For this economy, equilibrium is defined as follows:

Definition 1 (Equilibrium). *An equilibrium is an allocation of bundles, securities and repo positions (x, ϕ, z) together with prices (p, q, r) and haircuts implied by $h = (h_f^{ij})$, such that:*

(a) for each agent i, $(x^i, \phi^i, z^i) \in B_\star^i(p, q, r, h)$,

(b) commodity markets clear: $\sum_i \left(x_0^i - \omega_0^i\right) = 0$, $\sum_i \left(x_s^i - \omega_s^i\right) = \sum_f R_{fs}$ for $s = U, D$, and $\sum_i \left(x_{s^+}^i - \omega_{s^+}^i\right) = \sum_f R_{fs^+}$ for $s^+ = U^+, D^+$,

(c) security markets clear: $\sum_i \phi_{fe}^i = 1$ at each event $e = 0, U, D, U^+, D^+$,

(d) repo markets clear: $z_f^{ij} + z_f^{ji} = 0$ for all i, j and f.

2.3 A one-security and two-agent model

We can now take our base model and consider the simplest of economies. Suppose there are only two dates, only one security ($F = 1$ and we dispense with the index f for the security) and only two agents indexed i and j with $\omega_0^i = \omega_0^j = 0$ and $o^i = o^j = 1$. Agents utilities are simply the expected consumption at the second date: $U^i(x_U^i, x_D^i) = a^i x_U^i + (1 - a^i) x_D^i$ and $U^j(x_U^j, x_D^j) = a^j x_U^j + (1 - a^j) x_D^j$. To simplify we have dispensed with the third date and assumed repo to maturity[6], that is, both repo trades are settled at the maturity date of the security (even though the security does not have a price at the second date it can still serve as collateral, since its second date value consists in its returns given by R_s).

As there is only one security and one pair of agents, we simplify notation by letting $h = h^{ij} = h^{ji}$, $z^i = z^{ij}$ and $z^j = z^{ji}$. Suppose $R_U > R_D$ and that agents' subjective probabilities are different enough as to guarantee trade in the repo market: $E^i R > E^j R$, where $E^i R \equiv a^i R_U + (1 - a^i) R_D$. This is the same as assuming that $a^i > a^j$.

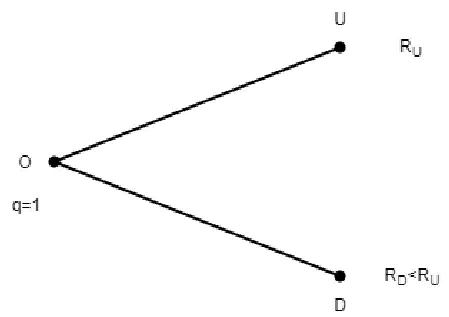


Figure 2.2: Economy with 2 dates. Security's price and payments.

Normalizing prices so that the security price is one, we can write agent i's constraints as:

$$\phi^i + hz^i = o^i, \qquad \text{(First date)}$$

$$\phi^i + z^i \geq 0 \Leftrightarrow o^i + (1 - h)z^i \geq 0, \qquad \text{(Box constraint)}$$

$$\begin{aligned} x_s^i = \omega_s^i + \max\big\{ & -\beta\omega_s^i, (o^i + (1-h)z^i)R_s \\ & + \eta_s^j [(hr - R_s)z^i]^+ - [(hr - R_s)z^i]^- \big\}, \qquad \text{(Second date, state } s\text{)} \end{aligned}$$

[6]This type of repo trades occurs in reality, although some agents (in particular, central banks) are not willing to engage in it.

The box constraint determines what that largest short repo position an agent can take is:

$$z^i = -\frac{o^i}{1-h} \tag{2.7}$$

Note that the magnitude of the position in equation (2.7) can be many times higher than the total initial supply of the security in the economy $(o^i + o^j)$. Building such a large short repo position is possible because of the re-usability of collateral in repo markets. At the same time, the position is not unbounded because of the haircut[7]: how much an agent can leverage his initial endowment of the security is related inversely to the haircut applied in the repo market. This is one good reason to define the *asset specific leverage* as the inverse of the haircut applied to the security when used as collateral in the repo market: $\frac{1}{1-h}$.

At the first date, we will require that long repo positions are also limited by the asset specific leverage:

$$z^i \leq \frac{o^i}{1-h}$$

If we substitute x^i_U and x^i_D into agent i's utility function, we can write his problem as:

Maximize

$$E^i\omega^i + a^i \max\big\{-\beta\omega^i_U, (o^i + (1-h)z^i)R_U + \eta^j_U[(hr - R_U)z^i]^+ - [(hr - R_U)z^i]^-\big\}$$
$$+ (1-a^i)\max\big\{-\beta\omega^i_D, (o^i + (1-h)z^i)R_D + \eta^j_D[(hr - R_D)z^i]^+ - [(hr - R_D)z^i]^-\big\}$$

s. t.

$$o^i + (1-h)z^i \geq 0$$
$$z^i \leq \frac{o^i}{1-h}$$

The only decision variable in the problem is z^i and the agent only needs to decide whether to be long $(z^i > 0)$ or short $(z^i < 0)$ in repo. Given our assumption on security payments and utilities, it is reasonable[8] to search for equilibria in which $hr \in (E^jR, E^iR)$. Given the relative

[7] See Bottazzi, Luque and Páscoa (2012) for a complete example of how these positions are built in repo markets with one security.

[8] In fact, it is easy to see that if $hr > E^iR$ or $hr < E^jR$ both agents will want to take either long or short repo positions, and there cannot be market clearing.

weights of each state in agents i and j utility function it is also reasonable to start for equilibria by assuming agent i to be repo short ($z^i < 0$), and j to be repo long ($z^j > 0$).

Being short in repo, agent i can potentially transfer consumption from state D to state U, which gives him comparatively more utility. In other words, agent i is an optimist with regard to this security (as he puts more weight in the state where the security pays more) and this suggests that he should be long in the security and leverage his long position by being short in repo. However, taking a short repo position is not guaranteed to increase his consumption in state U, or to yield an increase in overall utility, since this depends on the agent's counterparty effective repayment rate in state U (η^j_U).

We cannot completely rule out agent i taking a long repo position even though this would transfer consumption from a high utility state to a state with low utility. The reason for this is that bankruptcy limits the utility loss in state U and, depending on how large his long repo position is allowed to be, the increase of utility experienced in state D could more than compensate this loss.

Ultimately, whether agent i takes a long or short repo position will depend on his endowments on each state and in how much he can leverage his position. If leverage is low enough as to rule out bankruptcy, then i necessarily takes a short position ($z^i < 0$).

Starting with the assumption that i is short in repo, his position will be determined by the box constraint. In fact, as the utility function is linear, agent i will pick the largest possible short repo position. Since we know that $x^i_s \geq (1-\beta)\omega^i_s > 0$ no matter what the portfolio might be, agent i is not constrained in his choice by non negativity of x^i_s in any state. It is just the box that determines what that largest short repo position is. This is $z^i = -\frac{o^i}{1-h} = -\frac{1}{1-h}$.

Market clearing requires that if agent i is short in repo, agent j must be long. Again, the linearity of utilities requires that j takes the maximum position that he can in the repo market. We know this position to be $z^j = \frac{o^j}{1-h} = \frac{1}{1-h}$.

With these repo positions, agents i and j are solvent in states U and D, respectively. In fact, in state U agent i has a non-negative financial income: $(o^i + (1-h)z^i)R_U + \eta^j_U[(hr - R_U)z^i]^+ -$

$[(hr - R_U)z^i]^- = (o^i + (1-h)z^i)R_U + \eta_U^j(R_U - hr)|z^i| \geq 0 > -\beta\omega_U^i$. Therefore, agent i does not become insolvent, actually makes $x_U^i \geq \omega_U^i$ (and analogously for agent j in state D).

However, agent i is decreasing consumption in state D and we cannot be sure of his solvency in that state. The same applies for agent j in state U. If we let $\alpha_s^i = 1$ iff agent i is solvent in state s and $\alpha_s^i = 0$ when he declares bankruptcy, we have four possible cases to consider:

Case	α_U^i	α_D^i	α_U^j	α_D^j
1	1	0	0	1
2	1	0	1	1
3	1	1	0	1
4	1	1	1	1

It will be useful to denote by z^s the position of the short repo agent and z^l the position of the long repo agent. We have argued that the short agent (whoever he is) will be solvent in state U while the long agent will be solvent in state D.

Next we note that whether an agent goes bankrupt or not in a certain state depends entirely on how the agent's obligation in that state compares with the garnishable portion of his income. For a given β and h we can compute the (gross) repo rate that equalizes the two and that we denote r^s for the short agent and r^l for the long agent. In the case of the short agent we have:

$$r^s = \frac{R_D}{h} + \frac{1-h}{h} \cdot \frac{\beta\omega_D^s}{o^s}$$

If $r < r^s$ we have $\alpha_D^s = 1$ and, if $r > r^s$, $\alpha_D^s = 0$.

For the long agent we have that:

$$r^l = \frac{R_U}{h} - \frac{1-h}{h} \cdot \frac{\beta\omega_U^l + 2o^l R_U}{o^l}$$

If $r < r^l$ we have $\alpha_U^l = 0$ and, if $r > r^l$, $\alpha_U^l = 1$.

Suppose that the parameters of the model are such that $R_D/h < r^s < r^l < R_U/h$. If we consider a given haircut and for a fixed β we have that depending on the repo rate, bankruptcy coefficients are necessarily as follow:

Case	α_U^s	α_D^s	α_U^l	α_D^l
$r < r^s$	1	1	0	1
$r^s < r < r^l$	1	0	0	1
$r^l < r$	1	0	1	1

Now, for each r, we can compute the consumption of both short and long agents. For the short agent we have:

$$x_U^s = \omega_U^s + \alpha_U^l(hr - R_U)z^s + (1 - \alpha_U^l)[\beta\omega_U^l + 2o^l R_U] \tag{2.8}$$

$$x_D^s = \omega_D^l + \max\{-\beta\omega_D^s, (hr - R_D)z^s\} \tag{2.9}$$

In (2.8) we have written $\alpha_U^l(hr - R_U)z^s + (1 - \alpha_U^l)[\beta\omega_U^l + 2o^l R_U]$ instead of $\eta_U^l[(hr - R_U)z^s]^+$. The two terms coincide because $\eta_U^l = \alpha_U^l$ when the long agent is solvent in state U and, when the long agent is insolvent, we have that $\beta\omega_U^l + 2o^l R_U = -\eta_U^l(hr - R_U)z_U^l$. From market clearing we have that $z^l = -z^s$, so that:

$$\begin{aligned}
\eta_U^l[(hr - R_U)z^s]^+ = \eta_U^l(hr - R_U)z^s &= -\frac{\beta\omega_U^l + 2o^l R_U}{(hr - R_U)z_U^l}(hr - R_U)z^s \\
&= \frac{[\beta\omega_U^l + 2o^l R_U]z^s}{z^s} = \beta\omega_U^l + 2o^l R_U
\end{aligned}$$

Analogously, we can write the consumption of the long agent as:

$$x_U^l = \omega_U^l + \max\{-\beta\omega_U^l, 2o^l R^U + (hr - R_U)z^l\} \tag{2.10}$$

$$x_D^l = \omega_D^l + 2o^l R_D + \alpha_D^s(hr - R_D)z^l + (1 - \alpha_D^s)\beta\omega_D^s \tag{2.11}$$

From this consumption for the short and long agent, we can compute their respective utilities for a given value of r. The final step to confirm that consumption plans and portfolios correspond to an equilibrium is to check for optimality. This is done by comparing agents' utilities with the levels of utility they would attain by taking the opposite action (e.g. a short agent deciding to take a long position instead) *while considering the choice of the other agent as given.* That is, the short agent must compare his utility with the utility he would get if he chose the portfolio

$z^{sl} > 0$ instead. The consumption implied by this portfolio would be given by:

$$x_U^{sl} = \omega_U^s + \max\{-\beta\omega_U^s, 2o^s R_U + (hr - R_U)z^{sl}\} \tag{2.12}$$

$$x_D^{sl} = \omega_D^l + 2o^s R_D + (hr - R_D)z^{sl} \tag{2.13}$$

Note that in (2.12), even though the (long) counterparty might be insolvent in state U, this does not affect consumption of the short agent because now, when he is also taking a long position, the term $(hr - R_U)z^{sl}$ constitutes an obligation for the agent and the repayment rate of his counterparty is irrelevant.

The long agent must also compare his utility with what he would get if he chose the short position $z^{ls} < 0$ instead. In this case his consumption would be given by:

$$x_U^{ls} = \omega_U^l + (hr - R_U)z^{ls} \tag{2.14}$$

$$x_D^{ls} = \omega_D^l + \max\{-\beta\omega_D^l, (hr - R_D)z^{ls}\} \tag{2.15}$$

We have that the original consumption plans are optimal (and we have an equilibrium) if it is true that $U^s(x_U^s, x_D^s) \geq U^s(x_U^{sl}, x_D^{sl})$ **and** $U^l(x_U^l, x_D^l) \geq U^l(x_U^{ls}, x_D^{ls})$.

We can for example, study an economy with initial parameters:

$\beta = 0.35$	$R_U = 1.4$	$\omega_U^i = 4$	$\omega_U^j = 6$	$a^i = 0.9$
$h = 0.9$	$R_D = 0.1$	$\omega_D^i = 2$	$\omega_D^j = 4$	$a^j = 0.2$

The following are equilibria in which i is repo short and j is repo long for this economy:

r	α^i_D	α^j_U	x^i_U	x^i_D	x^j_U	x^j_D	U^i	U^j
0.7755	0	0	8.9	1.3	3.9	4.9	8.14	4.7
0.8044	0	0	8.9	1.3	3.9	4.9	8.14	4.7
0.8333	0	0	8.9	1.3	3.9	4.9	8.14	4.7
0.8622	0	0	8.9	1.3	3.9	4.9	8.14	4.7
0.8911	0	0	8.9	1.3	3.9	4.9	8.14	4.7
0.9200	0	0	8.9	1.3	3.9	4.9	8.14	4.7
0.9488	0	0	8.9	1.3	3.9	4.9	8.14	4.7
0.9777	0	0	8.9	1.3	3.9	4.9	8.14	4.7
1.0066	0	0	8.9	1.3	3.9	4.9	8.14	4.7
1.0355	0	1	8.68	1.3	4.12	4.9	7.942	4.744
1.0644	0	1	8.42	1.3	4.38	4.9	7.708	4.796
1.0933	0	1	8.16	1.3	4.64	4.9	7.474	4.848
1.1222	0	1	7.9	1.3	4.9	4.9	7.24	4.9
1.1511	0	1	7.64	1.3	5.16	4.9	7.006	4.952
1.1800	0	1	7.38	1.3	5.42	4.9	6.772	5.004
1.2088	0	1	7.12	1.3	5.68	4.9	6.538	5.056
1.2377	0	1	6.86	1.3	5.94	4.9	6.304	5.108
1.2666	0	1	6.6	1.3	6.2	4.9	6.07	5.16
1.2955	0	1	6.34	1.3	6.46	4.9	5.836	5.212
1.3244	0	1	6.08	1.3	6.72	4.9	5.602	5.264

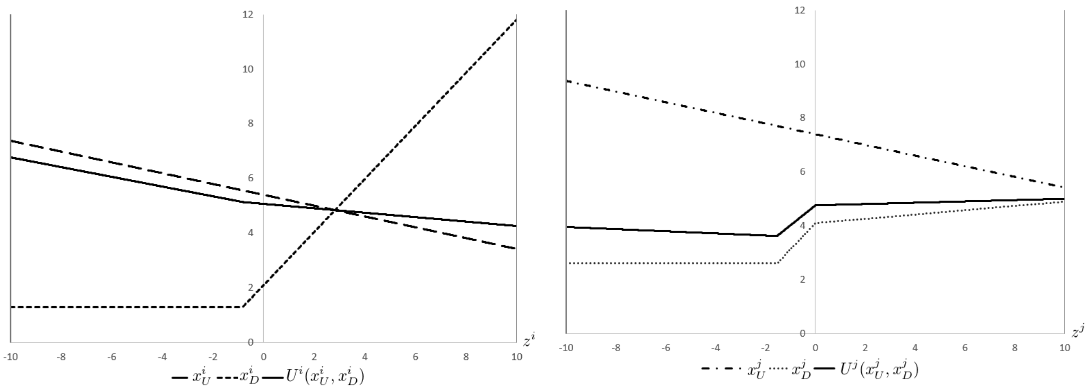
Notably, there are no equilibria corresponding to cases 3 or 4 in this economy. We can compute the exact values of r^s and r^l:

$$r^s = \frac{R_D}{h} + \frac{1-h}{h} \cdot \frac{\beta\omega^s_D}{o^s} = \frac{0.1}{0.9} + \frac{0.1}{0.9} \cdot \frac{0.35 \cdot 2}{1} = 0.1888$$

$$r^l = \frac{R_U}{h} - \frac{1-h}{h} \cdot \frac{\beta\omega^l_U + 2o^l R_U}{o^l} = \frac{1.4}{0.9} - \frac{0.1}{0.9} \cdot \frac{0.35 \cdot 6 + 2 \cdot 1.4}{1} = 1.0111$$

Figure 2.3 shows agents i and j's problems when the (gross) repo rate is 1.18 and clearly show that it is optimal for i to be short in repo (as much as the box constraint allows him) and for agent j it is optimal to take the highest long position that he can. The Figure shows consumption in states U and D for each agent. Sometimes, as with x^i_D, a kink occurs in the agents consumption at the point where z^i is such that the agents obligations equal his garnishable income and the agent is indifferent between being solvent or declaring bankruptcy. In other cases, as for x^i_U, no kink is observed. This is because the repo position that equalizes obligations and garnishable income occurs outside the interval that constraints repo positions. In this case, z^i at which the kink would occur is $z^i = 14.14$, which is the portfolio that satisfies the condition

$-\beta\omega_U^i = (o^i + (1-h)z^i)R_U + (hr - R_U)z^i$. Similarly, for x_U^j the kink where j is marginally solvent occurs for $z^j = 15.9$, also beyond the upper bound on z^j. In the case of x_D^j, two kinks are observed. The one to the left corresponds to the repo position that makes the agent indifferent between being solvent or not. The one at $z^j = 0$ occurs because when $z^j < 0$, the agent is a debtor in state D (meaning that $(hr - R_D)z^j < 0$) and so he is not affected by i's repayment rate $\eta_D^i < 1$. When $z^j > 0$, he is a net creditor, is affected by i's repayment rate (meaning that his income is $\eta_D^i(hr - R_D)z^j$ instead of $(hr - R_D)z^j$) and this reduces the slope of x_D^j as a funtion of z^j.



(a) Agent i's problem.

(b) Agent j's problem.

Figure 2.3: How consumption and utility at the second date relate to repo positions, when $r = 1.18$.

As is clear from the previous discussion, for a given set of parameters, there are multiple equilibria. Regardless of this indeterminacy, we have two conditions that must be satisfied in any equilibrium:

$$\eta_D^i = \frac{\beta\omega_D^i(1-h)}{(rh - R_D)}, \quad \eta_U^j = -\frac{[\beta\omega_U^j + 2R_U](1-h)}{(rh - R_U)} \tag{2.16}$$

These equations suggest that $\frac{\partial\eta_D^i}{\partial h} = \frac{\beta\omega_D^i(R_D - r)}{(rh - R_D)^2}$. For the equilibria we have presented here we have $\frac{\partial h}{\partial\eta_D^i} < 0$, and $\frac{\partial h}{\partial\eta_U^j} < 0$.

Figure 2.4 shows how h is related to the equilibrium values of η_D^i and η_U^j, for h ranging from

0.85 to 0.99.

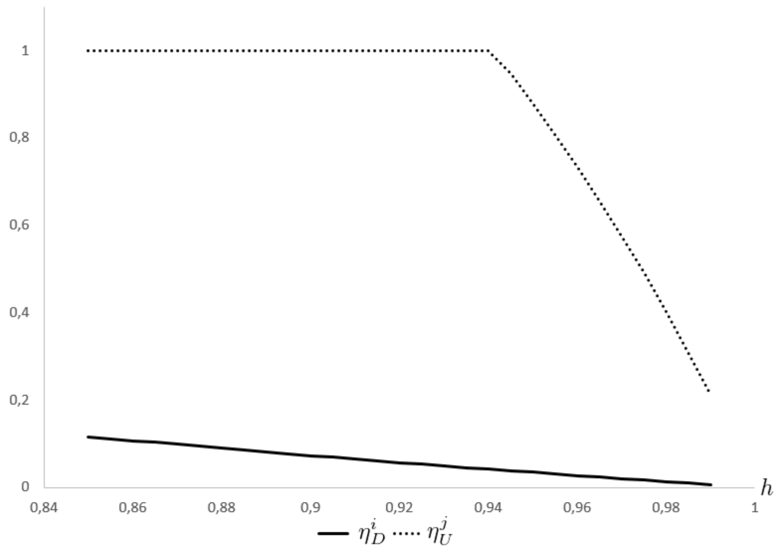


Figure 2.4: How η_D^i and η_U^j relate to h when $r = 1.18$.

Observe that haicuts move together with the counterparty's repayment rate (this must always happen in this 2-agent and 1-security economy. As we will see in section 2.4, in a competitive setting, where many agents trade many securities, the impact of the haircut in one security on the insolvency of an agent becomes less noticeable. It is the other direction that becomes more relevant: haircuts rise in response to lower repayment rates of the counterparty, for securities that involve a risk from the creditor's point of view (have a collateral liquidation value below the promised repo loan settlement). That is, in a competitive setting, creditors tend to focus on how to protect their individual credits rather than trying to influence the solvency of the counterparty.

2.4 Haircuts

Haircuts in pairwise repo trades are endogenously determined in the equilibrium that we defined. Existence was established and characterized for the 2-agent and 1-security case for a set of given parameters. We discuss now what may govern haircuts, that is, how should we expect haircuts to be set in equilibrium, depending on what are the parameters and other equilibrium variables for the relevant pair of repo traders.

Suppose agent i has a possession value for security f at the initial node, that is, a binding box constraint for security f at the initial node - more precisely, the shadow value μ^i_{f0} of this constraint is positive. Denoting by λ^i_e agent i's multiplier for the budget constraint at each node e and ν^{ij}_f the multiplier for the lower bound on repo positions value, from the first order conditions for agent i's problem, we get the following expression for h^{ij}_f

$$h^{ij}_f = \frac{\tau^{ij}_f(p,q) + \frac{\nu^{ij}_f}{\lambda^i_0}}{1 - r_f \sum_s \frac{\lambda^i_s}{\lambda^i_0}\alpha^i_s \kappa^{ij}_s} \tag{2.17}$$

where

$$\tau^{ij}_f(p,q) = \begin{cases} \sum_s \frac{\lambda^i_s}{\lambda^i_0}\alpha^i_s(1-\kappa^{ij}_s)\frac{(q_{fs}+p_s R_{fs})}{q_{f0}}, & \text{if } \mu^i_{f0} = 0 \\ 1 - \sum_s \frac{\lambda^i_s}{\lambda^i_0}\alpha^i_s \kappa^{ij}_s \frac{(q_{fs}+p_s R_{fs})}{q_{f0}}, & \text{if } \mu^i_{f0} > 0 \end{cases}$$

and $\kappa^{ij}_s = \gamma^{ij}_s \eta^j_s + (1-\gamma^{ij}_s)$, $\gamma^{ij}_s = 1$ if $I^{ij}_s > 0$, $\gamma^{ij}_s = 0$ if $I^{ij}_s < 0$, $\alpha^i_s = 1$ if agent i is solvent in state s, $\alpha^i_s = 0$ if i goes bankrupt in state s, and η^i_s satisfies (2.5).

It is worth recalling that η^j_s is the effective percentage of his debt that agent j pays to all of his counterparties, so it can be used as a measure of counterparty risk: the lower η^j_s is, the riskier (or less solvent) agent j is in state s, and this must be taking into account by agents deciding having j and counterparty and, in particular, in setting the terms of repo contracts (h^{ij}_f).

Equation (2.17) is true in any equilibrium and can be used to study *the incentives* that counterparties i and j have to either increase or decrease the haircut $(1-h^{ij}_f)$ associated to their repo transactions in response to an increase in the risk of one of the counterparties. Let's look at the derivative of h^{ij}_f with respect to η^j_D, under the assumption that agents' marginal rates of income substitution remain unchanged. To be more precise,

Assumption (Λ): agent i's marginal rates of substitution of income across the first two dates, λ^i_s/λ^i_0, are not affected by a change in the counterparty j's effective repayment rate η^j_D.

Although we might not want to take this assumption literally, it is useful to get a sense of how haircuts move with counterparty risk in a context where agent i is trading in many securities and has many counterparties, so that a small variation in the default rate of one of them in some state won't affect the optimal inter-nodes deflators of agent i.

This assumption holds for linear utilities (recall that the bankruptcy structure ensures the positivity of consumption in each state, which implies that $DU_s^i(x^i) = \lambda_s^i p_s$) in the case of repo to maturity (dispensing with the third date and allowing for $p_s = 1$) and provided that agent i is consuming at the initial date (so that $DU_0^i(x^i) = \lambda_0^i p_0$) and that the equilibrium commodity price p_0 is not affected by a small change in the effective repayment rate η_D^j of counterparty j in state D.

$$\frac{\partial h_f^{ij}}{\partial \eta_D^j} = \frac{-\alpha_D^i \gamma_D^{ij} \cdot \frac{1}{q_{f0}} \cdot \frac{\lambda_D^i}{\lambda_0^i} \cdot [(q_{fD} + p_D R_{fD}) - h_f^{ij} q_{f0} r_f] + h_f^{ij} \frac{\partial r_f}{\partial \eta_D^j} \sum_s \frac{\lambda_s^i}{\lambda_0^i} \alpha_s^i \kappa_s^{ij}}{1 - r_f \sum_s \frac{\lambda_s^i}{\lambda_0^i} \alpha_s^i \kappa_s^{ij}} \tag{2.18}$$

When most of the response to a variation in counterparty risk is channeled into a change in haircuts, rather than a change in the repo rate, we can be more specific about the direction of change. We say that *repo rates are competitive* if actions by a pair of agents i and j, in particular actions that change their solvency rates (η_s^i and η_s^j) do not affect equilibrium repo rates. This is a reasonable assumption if there are many agents (and therefore, many pairs of counterparties) in the economy, but not to be expected in an economy with only two (or very few) agents, as in the example of section 2.3. We have,

Proposition 1. *Suppose repo rates are competitive. Let us evaluate the impact of η_D^j on h_f^{ij}), under a scenario where agents' marginal rates of substitution are not affected. Say $I_D^{ij} > 0$ (i is a net creditor in the repo market with respect to j) and i is solvent in state D ($\alpha_D^i = \eta_2^i = 1$). If agent j's expected repayment rate η_D^j decreases, agent i will want to:*

- *Increase the haircut $(1 - h_f^{ij})$ he charges (pays) in his repo long (short) positions with agent j, of securities f such that $h_f^{ij} q_{f0} r_f > q_{fD} + p_D R_{fD}$.*
- *Decrease haircuts paid to (charged to) agent j for his short (long) repo positions in securities g such that $h_g^{ij} q_{g0} r_g < q_{gD} + p_D R_{gD}$.*

If $I_D^{ij} < 0$ (i is a net debtor in the repo market with respect to j), or if i is insolvent in state D, *he has no incentives to increase or decrease the haircut* $(1-h_f^{ij})$ *in response to expected changes in* η_D^j.

Remark 1. The last part of the proposition reflects the fact that if i were a net debtor to agent j instead, he would not be entitled to any share in the liquidation of agent j's estate in the event of agent j's bankruptcy. Note that as long as agents i and j trade in the repo market, one of them must be a net creditor and proposition 1 applies to either i or j, as long as the agent is solvent in state D.

Proof. See the appendix. □

Suppose that i is a net creditor with counterparty j, and that $z_f^{ij} > 0$, and that $h_f^{ij} q_{f0} r_f > q_{fD} + p_D R_{fD}$. If agent i anticipates a decrease in j's expected repayment rate, η_D^j, then i would like to charge j a higher haircut (by lowering h_f^{ij}). To understand why this is so, note that the magnitude of j's net debt to i, is given by $z_f^{ij}[q_{f0} r_f h_f^{ij} - (q_{fD} + p_D R_{fD})]$ and lowering h_f^{ij} would reduce this debt and, therefore, the loss resulting from agent j's bankruptcy.

Now suppose $h_f^{ij} q_{f0} r_f < q_{fD} + p_D R_{fD}$. If everything else is as in the previous paragraph, a decrease in η_D^j will be an incentive for i to collect a lower haircut from j (by raising h_f^{ij}). Even though agent i is a net creditor to agent j when adding up all of his repo transactions with j, he has now a debt to j associate to his position on security f with absolute value $z_f^{ij}[(q_{fD} + p_D R_{fD}) - q_{f0} r_f h_f^{ij}]$. That is, the collateral kept by i when lending cash to j has now a higher market value than what j owes to i. If h_f^{ij} increases, he gets to keep more of the collateral in the event of j's bankruptcy.

In both cases, i has incentives to respond in a way that counteracts the loss in income when j becomes more insolvent in state D. The appropriate response depends on the relationship of the value of j's debt $(q_{f0} r_f h_f^{ij})$ with the market value of the collateral $(q_{fD} + p_D R_{fD})$. This relationship is what is usually understood as *asset risk*, the risk is precisely that the value of the collateral might decrease in some future date and become insufficient to cover the value of the debt that it is backing. In practice, haircuts are set so that it is expected that[9] $h_f^{ij} q_{f0} r_f <$

[9] Or if the value of the collateral decreases and doesn't cover the value of the debt, a margin call is issued by

$(q_{fD} + p_D R_{fD})$. This should be considered the most relevant case in the context the proposition.

Proposition 1 is useful to understand agents' incentives when a perceived increase in counterparty risk occurs. It should not be interpreted as a comparative statics analysis. One should expect to observe different haircuts but also different repo positions in equilibria with different repayment rates. One could expect that even the roles of net creditor and net debtor might get reversed when repayment rates change.

2.5 Concluding remarks

Repo markets have attracted a lot of attention in the applied macro and finance literatures, particularly since the 2008 financial crisis. These literatures have focused on how leverage in these markets has impacted on the whole economy, how this leverage depends on repo haircuts and what determines these margins. However, at the theoretical level, these important issues had not been addressed yet, possibly due to the complexity and intrinsic non-convexities involved in a bankruptcy model where counterparty risk could be understood and margins could be explained as endogenous variables.

We contribute in that direction, building a general model, exhibiting concrete examples of bankruptcy equilibrium for a simple economy, and characterizing how counterparty and asset risks interact to determine repo haircuts. Full recognition of the recourse nature of repo loans is crucial. This implies also that default must be modelled in terms of insolvency and according to bankruptcy rules. The model can be made richer, incorporating more detailed institutional aspects, or allowing also for centrally cleared repo, but the main drivers of margins seem to identifiable already in a simple model. Competitiveness versus small numbers of traders or asset risk in a creditor's perspective (suggesting default tensions) versus a debtor's risk perspective (concerned with repo fails instead) are some of the key issues in assessing how counterparty risk impacts on repo haircuts and the resulting leverage.

the lender of funds.

Appendix

Proof of Proposition 1. From (2.18), if r_f is competitive, the derivative of h_f^{ij} with respect to η_s^j reduces to:

$$\frac{\partial h_f^{ij}}{\partial \eta_D^j} = -\alpha_D^i \gamma_D^{ij} \cdot \frac{1}{q_{f0}} \cdot \frac{\lambda_D^i}{\lambda_0^i} \cdot \frac{(q_{fD} + p_D R_{fD}) - h_f^{ij} q_{f0} r_f}{1 - r_f \left[\frac{\lambda_U^i}{\lambda_0^i} \alpha_U^i \kappa_U^{ij} + \frac{\lambda_D^i}{\lambda_0^i} \alpha_D^i \kappa_D^{ij}\right]}$$

Moreover, we have that $\frac{dh_f^{ij}}{d\eta_s^j}$ has the same sign as $[h_f^{ij} q_{f0} r_f - (q_{fs} + p_s R_{fs})]$ and $\frac{d(1-h_f^{ij})}{d\eta_s^j}$ has the same sign as $[(q_{fs} + p_s R_{fs}) - h_f^{ij} q_{f0} r_f]$. To see that this is the case, look at the first order condition of agent i's problem with respect to z_f^{ij}:

$$r_f = \frac{\lambda_0^i}{\sum_s \lambda_s^i \alpha_s^i \kappa_s^{ij}} - \frac{1}{q_{f0} h_f^{ij}} \cdot \frac{\sum_s \lambda_s^i \alpha_s^i [(1 - \kappa_s^{ij})(q_{fs} + p_s R_{fs})]}{\sum_s \lambda_s^i \alpha_s^i \kappa_s^{ij}} - \frac{1}{h_f^{ij}} \cdot \frac{(\mu_{f0}^i / q_{f0}) + \nu_f^{ij}}{\sum_s \lambda_s^i \alpha_s^i \kappa_s^{ij}} \tag{2.19}$$

Lets focus on the term in the middle of the right hand side of (2.19):

$$\frac{\lambda_U^i \alpha_U^i [(1 - \kappa_U^{ij})(q_{fU} + p_U R_{fU})] + \lambda_D^i \alpha_D^i [(1 - \kappa_D^{ij})(q_{fD} + p_D R_{fD})]}{\lambda_U^i \alpha_U^i \kappa_U^{ij} + \lambda_D^i \alpha_D^i \kappa_D^{ij}} \tag{2.20}$$

In an equilibrium as the one which existence we have proven, we have $\alpha_U^i = 1$. The theorem assumes that i is a net creditor so we have $\gamma_D^{ij} = 1$. We have that agent j is solvent in state U. We must have $\kappa_U^{ij} = 1$. We have assumed that i is solvent in state D, so that $\alpha_D^i = 1$, we can suppose that $\eta_D^j \in (0, 1)$ (so that j was risky to begin with and so that it makes sense to take the derivative with respect to η_D^j which belongs in the set $[0, 1]$). From all these considerations, we have that (2.20) is actually positive:

$$\frac{\lambda_D^i [(1 - \eta_D^j)(q_{fD} + p_D R_{fD})]}{\lambda_U^i + \lambda_D^i \eta_D^j} > 0 \tag{2.21}$$

We can conclude from equation (2.19) that $r_f < \lambda_0^i / \sum_s \lambda_s^i \alpha_s^i \kappa_s^{ij}$ as long as at least one of the following conditions holds:

(a) Agent i values the possession of security f ($\mu_{f0}^i > 0$).

(b) Agent i's position z_f^{ij} has a market value at the lower bound.

(c) Counterparty j is *somewhat* risky in state D ($\eta_D^j \in (0, 1)$).

When at least one of the conditions is satisfied we have $r_f \sum_s \frac{\lambda_s^i}{\lambda_0^i} \alpha_s^i \kappa_s^{ij} < 1$. $\square$

Chapter 3

Short Sales and Shareholders' Unanimity

3.1 Introduction

Introducing firms into a model with uncertainty and incomplete markets has been a challenge to the general equilibrium theory at least since the early days when the work by Dreze (1974) and Grossman and Hart (1979) came out. Market incompleteness prevents the firm from having a well defined objective function and shareholders, with different marginal rates of intertemporal and interstate substitution, tend to disagree on what the firm should do. Several solutions have been attempted, and a promising approach relies on assumptions on the competitiveness of the firm in markets where its shares are traded, as in Hart (1979) and Makowski (1983). However, in these papers short sales were ruled out as they seemed to be incompatible with competitiveness. In Makowski's work, short sales are clearly referred to as an externality that cannot be priced in the stock market.

In the current paper we show that, with a more precise modeling of securities trading markets,

[0]Joint work with Mário Páscoa and Jean-Marc Bottazzi.

there is no longer an incompatibility between short sales and the competitiveness that ensures shareholders' unanimity. We abandon the naive modelling of short positions as naked short sales and, instead require agents to borrow the shares before they can be shorted, as actually occurs in reality. Makowski's externality can now be priced in the market where securities are borrowed. Shares that are on high demand for short positions have higher lending fees.

A recent example of the renewed interest in the shareholders' unanimity problem is the paper by Carceles-Poveda and Coen-Pirani (2009), where it is shown that unanimity holds when the technology exhibits constant returns to scale and the bounds on (naked) short sales are non-binding. Our work differs from this interesting paper in that we allow for general technologies and we do not require non-binding portfolio constraints.

Our approach depends crucially on the correct modeling of short sales. In order to short a security, one needs to obtain it through a Securities Financing Transaction (SFT), either by borrowing it against cash collateral (or another security as collateral), as done in the Securities Lending Market, or by accepting the security as collateral in exchange for a cash loan, as is done in repo markets. In the case of bonds, repo markets are dominant, but for equity the securities lending markets are the preferred channel, at least between the beneficial owners and the hedge fund managers that wish to short-sell[1].

Unanimity of shareholders is a convenient property since it simplifies both the definition of the firm's objective and also its agency problem: there is no need for majority voting rules and the firm can simply implement the production plan favored by all its shareholders, the plan that maximizes the firm's present value.

Although very convenient, unanimity does not often occur when markets are incomplete. There are two reasons for disagreement among shareholders. First, ownership of the firm's shares confers access to a stream of dividends that can be used to transfer income across time and across states of nature, that is, the share provides some *hedging* or *spanning* services to the shareholder. Dividends are contingent on the production plan the firm adopts and different

[1] Among dealers or between dealers and the real money agents (money market funds or commercial banks), repo trades using shares tend to be more common (see FSB (2012)).

shareholders, having different inter-temporal and inter-state marginal rates of substitution, will generally disagree on the production plan that provides the best hedging services.

Apart from these different preferences over the *hedging effect*, there is a second reason for disagreement as even the *income effect* may be perceived differently. Under incomplete markets, there are multiple non-arbitrage deflators and, therefore, shareholders might not be coordinating on how to anticipate the impact of production changes on the share price and on their current incomes. Good surveys describing the problem and the literature can be found in Kreps (1979) and in Magill and Quinzii (1996).

An assumption that guarantees shareholders' unanimity when markets are incomplete is *partial spanning* but it has somehow an *ad-hoc* flavour. The firm's shareholders will be unanimous in their production choices if the production set of the firm is spanned by the equilibrium production plans of all firms. Changes in hedging services are always accompanied by a change of the share's price that exactly compensates the loss in utility suffered by the agents and there is no confusion about what are the correct price of the firm's share and the firm's present value. This assumption appears in Diamond (1967), Ekern and Wilson (1974), and Radner (1974). It is also implied by the stronger (but possibly more tangible) assumption made in Carceles-Poveda and Coen-Pirani (2009) that firms' production functions exhibit constant returns to scale. There are some important drawbacks with partial spanning. First, the firm cannot *innovate* in the sense that it cannot introduce into the market anything which is not already marketed. Second, partial spanning is stronger than it might seem. If technologies are such that there is free disposal, partial spanning implies that markets are complete[2].

Another approach, due to Grossman and Hart (1979), uses the assumption of *competitive price perceptions*: each shareholder expects the change in hedging services that he himself experiences, when the production plan is modified, to be exactly compensated by the variation in shares' price. This is not a rational conjecture since the hedging service each agent receives depends on his own preferences and will not coincide with that of other agents. Once this is assumed,

[2]This is not an issue in Carceles-Poveda and Coen-Pirani (2009), where firms are characterized using production functions instead of production sets.

the hedging services become irrelevant and all agents agree on selecting a plan that maximizes the present value of the firm. However, since each agent uses his own personal valuation of the stream of dividends, shareholders still cannot agree on a production plan: the second source of disagreement remains.

Given the important shortcomings of partial spanning and of competitive price perceptions, several authors have tried to provide better assumptions that guarantee shareholders' unanimity when markets are incomplete. These assumptions are related to the competitiveness of the firm's shares. An example is the model by Hart (1979), where the number of firms is small, the number of agents is large and for each shareholder the hedging effect is much less important than the income effect. Another competitiveness approach explores the idea that the firm shares may not be irreplaceable in the optimal portfolios of the traders. Makowski (1983)'s approach is precisely to take this inessentiality as the defining feature of competitiveness itself: perfect competition in markets where shares are traded just captures the property that other assets can replace the firm's shares in reaching utility levels of trade outcomes[3]. In contrast with the competitive price perceptions assumption, Makowski's approach does not suffer from irrational price conjectures. Moreover, competitiveness assumptions do not share the unpleasant features of partial spanning: innovation is not precluded and complete markets are not implied when free disposal is allowed.

Sadly, whatever their advantages are, models that rely on competitiveness assumptions have, until now, shared a very important flaw: short sales were not allowed. This was never by choice as short sales are prevalent in reality and increase efficiency[4] by allowing agents to transfer income from the future into the present and by enlarging their hedging possibilities and, therefore, should not be excluded. But something was missing to make things work with short sales.

Previous authors have argued that short sales are fundamentally incompatible with the com-

[3]For such adequate substitutability the large number (oceanic) case might look like a natural example but is not necessary (as we illustrate in Example 1). Oceanity itself can help but may not be sufficient, as we argue in section 3.3. If equilibrium is not symmetric having plenty of firms with identical technologies may not be enough for competitiveness if these firms choose different production plans and therefore differentiate their dividends. At a symmetric equilibrium however competitiveness holds

[4]See, for example, Chen and Rhee (2010), Bai et al. (2006).

petitiveness of the firm's shares and with shareholders' unanimity. In Makowski (1983)'s model, for example, the source of this incompatibility is the positive externality consisting of the hedging that can only be done by short selling. It seems that this externality could not be priced in the stock market since the stock price reflected just the willingness of agents to take long positions.

In this article we solve the problem: the firm's competitiveness in the secondary market for its stock becomes compatible with the way that short sales are actually done in financial markets. Naked short sales (those that were commonly contemplated in the general equilibrium literature) are usually forbidden in reality[5]. Shares can only be short-sold by borrowing them first in a securities lending market. We extend the competitiveness assumption to this market.

Once we have introduced markets where the shares can be borrowed, we provide, in the spirit of Makowski (1983), a definition of what a perfectly competitive firm is, with respect to both the trading and the lending of its stock. This is a firm whose equity will be traded at prices taken as given by holders of the shares and lent at lending fees taken as given by lenders of the shares. At a slightly higher price, no one would purchase the firm's shares and, at a slightly higher lending fee, no one would borrow the firm's shares. Short sales do not constitute an externality anymore. The hedging services that agents get when short selling the firm's shares are priced in the securities lending market through the lending fee. Shares that provide more valuable hedging services through a short sale, will be on higher demand by borrowers of shares and, therefore, will have higher lending fees. Cash collateral pledged when borrowing a security earns an interest rate that is called the *rebate rate*, which is the money market interest rate minus the lending fee. When the rebate rate on a certain security is below the highest available rebate rate[6], the security is said to be *on special*. We prove that specialness is incompatible with perfect competition.

By shareholders we mean, as it is customary, the agents that legally own shares of the firm rather than those that may be in temporary possession of the shares as a result of having borrowed

[5]See e.g., S.E.C Exchange Act Rule 10b-21. http://www.sec.gov/divisions/marketreg/tmcompliance/rule10b21-secg.htm.

[6]That is, the *general collateral rate*.

them. Even though it may not be immediate to distinguish the former from the latter, we can assume that the former are those that tend to be in possession of the share when voting takes place. It is true that the latter are entitled to voting rights even if they don't have ownership, but it is commonly observed that shares tend to be returned to the owners before a voting decision takes place (see Aggarwal et al. (2015) for evidence in the securities lending market).

We prove that if a firm is perfectly competitive, with respect to both stock trades and stock loans, its shareholders will unanimously agree on choosing the production plan that maximizes the firm's present value. Makowski had provided a counterexample showing why short sales, in the naive naked form, were incompatible with shareholders unanimity. We reformulate this example (see Example 3), replacing naked short sales by short sales (preceded by borrowing), and show that, in equilibria allowing for short sales, unanimity still depends on whether its shareholders and sharelenders face perfectly elastic demands.

The assumptions that we use to achieve shareholders' unanimity are less restrictive than others used in the literature. Perfect competition is more general than partial spanning in that it does not place strong restrictions on the firms' technologies or on the number of firms. Perfect competition *can* be a result of the firms' characteristics but can also occur as a consequence of the agents' preferences. To make this point, we give an example where there is only one firm (Example 1) and consumers have indifference curves with linear sections, which facilitates the substitutability between the firm's shares and a one-period real bond. This example is intended to show that even if there were just one firm, we could still regarded it as being perfectly competitive, provided that consumers manage to find substitutes for *trading the* firm's shares, more precisely, by finding utility equivalent plans using a bond instead of relying on long positions in the firms' shares or on the borrowing of these shares[7].

We have a second example, where there are several firms and the condition that ensures competitiveness is closer to the partial spanning assumption (Example 2), since the production

[7]Linear indifference curves were already contemplated by Makowski (1983) in his examples and also by Ostroy (1980), as a devise that allows for perfectly competitive outcomes in spite of the number of competitors being small.

set of one firm is generated by the equilibrium production plans of the other firms, but differently from partial spanning we also need to have no possession value for the firm's shares, which in turn restricts the rebate rate to be exactly equal to the general collateral rate (see Theorem 4).

We also address the problem of security pricing when the firm is a perfect competitor in the stock and securities lending markets. The way an agent's operations in these two markets are linked is captured by the *box constraint*. This constraint says that the physical title amount of the share in the agent's possession must be non-negative. If the agent is long in the share he can lend all or part of it. On the other hand, if an agent intends to short-sell a certain amount of shares, he must have borrowed at least that amount. Since the share is in positive net supply, agents' box constraints cannot be all binding. For an agent whose box constraint is not binding, either some part of his long security positions is unencumbered or some part of the shares that he borrowed has not been reused. Such agent has null shadow price for the box constraint and, therefore, the present value of dividends, deflated using his personal marginal rates of substitution, will be exactly equal to the equilibrium price of the share. It does not matter whether the agent is a shareholder or a shareborrower. However, that agent can only be used for pricing throughout the event-tree as long as he remains with null box shadow prices and for this reason, like in Makowski (1983), we may need to use different reference agents at different moments in time. Moreover, if the firm is perfectly competitive and such *reference agents* are not initial shareholders, then their marginal rates of substitution can be used to evaluate the price of the share *out of the financial equilibrium*. This result is in the spirit of Makowski but it is important to note that in Makowski (1983) such reference agent was always a buyer of the share whereas in our model the agent may be a borrower of the share and, therefore, even a short-seller.

The valuation, at date t of date $t+1$ dividends, that is obtained when using reference shareholders' deflators, corresponds to the maximum personal valuation of such stream of dividends among all agents. A pricing function that also corresponds to the maximum valuation among agents has appeared recently in some models[8] (e.g., Bisin et al. (2014) and Acharyaa and Bisin

[8]This is the case in Bisin et al. (2014) only when short sales are not allowed. In the case with short sales,

(2014)). As we prove in section 3.4, this function correctly computes the price of the share as long as the firm is a perfect competitor in the economy and there is at least one reference agent, but these two conditions are sometimes ignored in the literature. In an economy with a finite number of firms, if the firm is not perfectly competitive, or if there are no reference agents, the criterion is correct for the plan that maximizes the firm's present value but will be generally incorrect when evaluating other production plans. We illustrate this point in Example 4.

In the next section we present the model, the equilibrium definitions, and make sure that existence of equilibrium is not an issue. In section 3.3 we define perfectly elastic demands by shareholders and share borrowers and provide our unanimity results (including unanimity through time). In section 3.4 we address the pricing of securities and relate no-specialness to perfect competition. Section 3.5 concludes.

3.2 The model

The economy is represented by an event tree D with 3 dates $t \in \{1, 2, 3\}$[9]. There is a unique node at date 1 while $S < \infty$ states of nature occur at $t = 2$. Between dates 2 and 3 there is no uncertainty. Except for the initial node (where the event variable e takes the value $e = 1$), each node is defined by a date and a state, $e = ts$ and each node $2s$ has only one successor, $3s$. Given two nodes $e, u \in D$ we write $u \geq e$ if $u = t's'$ belongs to the sub-tree with root $e = ts$, and we write $u \in e + 1$ if $u \geq e$ and $t' = t + 1$.

There are C commodities. Production is carried out by F firms, each one being characterized by a production possibilities set $Y^f \subset \mathbb{R}^{C(2S+1)}$ $(f = 1, \cdots, F)$. There are $I > 2$ agents trading commodities and firms' shares. A bundle of commodities consumed by agent i is denoted by $x^i = (x^i_e)_e \in X \equiv \mathbb{R}^{C(2S+1)}_+$ and p_{ec} stands for good c price at node e. Agents are described by their commodity endowments ($\omega^i \in X$), nonnegative ownership shares at date 1 (ϕ^{if}_0) and a

which are modeled by introducing a derivative and not through securities lending, the pricing function is different from the one in our model.

[9] We want securities to be traded and have a price at the intermediate date, when the second leg of the securities lending transaction takes place.

non-satiated utility function $U^i : X \to \mathbb{R}$. For each firm f, we normalize the total supply of the firm's shares to one: $\sum_i \phi_0^{if} = 1$.

3.2.1 Security markets

Securities in this model are the ownership shares of the firms. Owning a firm (or a share of the firm) means owning a stream of dividends across dates and states and the right to lend it. The latter tends to be overlooked in a *cash-flows only* view of finance, but will play a major role in this article. Dividends are contingent on the production plan adopted by the firm and can be used to transfer consumption among states.

Shares are traded in stock markets at dates 1 and 2. Each agent decides on an ownership plan $\phi^i \equiv \phi_e^{if} \in \mathbb{R}^{F(S+1)}$, where ϕ_e^{if} represents the position of agent i in firm f's shares at event e. Share prices are denoted by $q \equiv (q_e^f) \in \mathbb{R}^{F(S+1)}$.

Agents can have negative positions of the ownership shares of firms, short sales are permitted. Short selling, however, is not the same as issuing shares. Agents can only trade (perhaps several times over the same settlement date) the finite given amount of shares existing in the economy. Short selling and issuing are different. To understand this difference we need to introduce markets where agents can borrow shares.

3.2.2 Markets for borrowing and lending shares

Our approach to securities lending has some traits in common with the GEI repo model in Bottazzi, Luque and Páscoa (2012). If an agent wishes to short firm f's shares, he must first go to the securities lending market and *borrow* the desired amount of shares before selling them short. This is the way short-selling is done in reality. Naked short sales have been a useful abstraction in general equilibrium models in the past but are simply not allowed in reality. To better focus on short sales we fix issuance at the initial date, and do not allow for naked shorts.

Securities lending works as follows: The lender of the security transfers the security to the borrower and receives collateral in exchange. The exchange is reversed at a future date which can be left open. In the U.S., up to 98% of security loans are made against cash collateral. In Europe, it is most common to use other securities as collateral. In this article, we will focus on transactions done against cash collateral and for a fixed term. A description of the U.S. market can be found in D'Avolio (2002).

The market practice is that the borrower of the security posts initial margin to cover the risks that the lender incurs in the transaction. Here it is assumed that the value of the collateral posted is H times the market value of the securities being lent, where $H > 1$. The borrower of the security must pay a fee to the lender, which is usually embedded in the *rebate rate*, the interest that the lender of the security pays to the borrower in the second leg of the transaction for the use of the cash collateral. The rebate rate is usually below the market rate at which the lender of the security can invest the cash during the term of the transaction, so the fee associated with the loan is this difference between the market rate and the rebate rate. Securities with high fees and low interest rates are said to be "special" and securities with the lowest fees and an interest rate very near the market rate are said to be "general collateral".

The borrower of a security acquires possession rights associated with the security[10]. However, any coupon or dividend paid to the borrower during the term of the transaction is passed through to the original owner; this is called a *manufactured payment* or a *manufactured dividend*[11].

For simplicity, security lending takes place only at date 1. Agent i's position on firm f's shares in the securities lending market is represented by the variable z^{if}. If $z^{if} > 0$ the agent is borrowing the security and if $z^{if} < 0$ he is lending it. Given the margin that is applied, the amount of funds that must be pledged as collateral when borrowing one unit of firm f's shares in the securities lending market is given by $H^f q_1^f$, where $H > 1$. The rebate rate that the lender of firm f's shares pays to the borrower for the use of the cash collateral is denoted by ρ^f. As will become clear (see section 3.3.3 below) by inspection of first order conditions on shares trades and lending, higher lending fees (that is, lower rebate rates) are driven by a significant value of

[10]Together with voting rights although these tend not to be exercised since shares are usually returned to the original owners for the voting date.

[11]See (Maxime Bianconi et al., 2010, section I)

being in possession of the shares in order to short sell them.

An important feature of a borrowing transaction is that the borrower has the right to re-lend the security or to sell it in the stock market. This process may occur many times over for the same settlement period. The reusability of the securities implies that, although issuance is fixed at the initial date, this is not sufficient to guarantee that short positions on the securities are bounded. This is a new instance of Hart (1975)'s problem: when short sales of real assets are not bounded, equilibrium may fail to exist. To circumvent this problem we assume that there is an exogenous bound on the positions agents hold in the securities lending market, $|z^{if}| \leq M$, for all i and all f. This, together with the box constraint is enough to bound short positions. We adopt this exogenous bounds approach since the focus of this article is not on existence but on shareholder's unanimity and perfect competition in the presence of short sales. However, it is important to mention that in reality there are several legal and institutional constraints that bound agents' short positions.

The present model of securities lending is formally quite close to the repo model by Bottazzi, Luque and Páscoa (2012). The key difference is that here cash is used as collateral and it is the securities borrower that pledges the collateral to the securities lender, while in repo there is a cash loan against a security that is pledged as collateral by the cash borrower (security lender). In both cases the value of the collateral usually exceeds that of the loan: actually in securities lending an initial margin is always posted ($H > 1$) while in repo a haircut is many times collected by the cash lender ($H \leq 1$). See for example Bottazzi, Luque and Páscoa (2012) for detailed models of repo where the bounds on short sales, rather than being exogenous, are driven by different institutional arrangements (segregated haircuts or a combination of constrained dealers with non-dealers that do not collect haircut).

3.2.3 Budget and box constraints

Given a set of production decisions $y \equiv (y^f)$, market prices and rebate rates, (p, q, ρ), agents decide on a plan (x^i, ϕ^i, z^i), consisting of consumption and portfolios in the stock and securities lending markets. Let us define the budget constraints that these plans must satisfy.

At date 1, the securities lending market opens. Borrowing and lending positions are bounded exogenously as follows:

$$-M \leq z^{if} \leq M \tag{3.1}$$

Agent i faces the following budget constraint at date 1:

$$p_1(x_1^i - \omega_1^i) + \sum_f q_1^f(\phi_1^{if} - \phi_0^{if}) + \sum_f q_1^f H^f z^{if} \leq \sum_f \phi_0^{if} p_1 y_1^f \tag{3.2}$$

Date 1 budget constraints include initial endowments of goods and securities and dividends paid to initial shareholders.

The fact that, to short-sell a security, an agent must first borrow it, is reflected in a restriction we will refer to as the *box constraint*. This restriction states that the agent must hold a non-negative amount of the security in his possession. That is, his current position plus the amount borrowed (or minus the amount lent) of the security must be nonnegative. At date 1 the box constraint is:

$$\phi_1^{if} + z^{if} \geq 0 \ , \ \forall f \tag{3.3}$$

Securities lending markets do not open at date 2: agents trade securities only in stock markets, and receive dividends from the firms according to their shares positions at the end of date 1. Borrowing and lending transactions that occurred at date 1 are settled at date 2: for transactions where shares of firm f were borrowed, lenders repay the cash they have received as collateral at the rate ρ^f. The budget constraint at node $e = 2s$ for agent i is given by:

$$p_{2s}(x_{2s}^i - \omega_{2s}^i) + \sum_f q_{2s}^f(\phi_{2s}^{if} - \phi_1^{if}) \leq \sum_f \phi_1^{if} p_{2s} y_{2s}^f + \sum_f (1+\rho^f) q_1^f H^f z^{if} \tag{3.4}$$

Observe that the repayment of cash at date 2, given by $(1+\rho^f)q_1^f H^f$, is locked-in at the initial date, it depends on q_1^f (and is therefore related to future dividends) and ρ^f which will both be the same no matter what the state of nature is when the repayment takes place. Since securities lending markets do not open at date 2, the box constraint at node $e = 2s$ reduces to a no short sales constraint:

$$\phi_{2s}^{if} \geq 0 \ , \ \forall f \tag{3.5}$$

At date 3 there is no trading of securities[12], agents consume from their endowments and the dividends associated with their security positions at date 2:

$$p_{3s}(x^i_{3s} - \omega^i_{3s}) \leq \sum_f \phi^{if}_{2s} p_{3s} y^f_{3s} \tag{3.6}$$

Finally, the following non-negativity restrictions must hold: $x^i_e \geq 0$, $\forall e$.

Given production plans and prices (y, p, q, ρ), an agent's plan of consumption, of securities, and of borrowing positions (x^i, ϕ^i, z^i), will be called *feasible* if it satisfies equations (3.1), (3.2), (3.3), (3.4), (3.5) and (3.6). We will represent the set of all feasible plans for agent i by $B^i(y, p, q, \rho)$, and $B^i_\star(y, p, q, \rho)$ will represent the subset of utility maximizing plans in $B^i(y, p, q, \rho)$.

3.2.4 Equilibrium

In this section we present various concepts of equilibrium that will be needed to study the production decisions taken by the firms. We start by defining an *exchange equilibrium*, where production decisions are taken as given.

Definition 2 (Exchange equilibrium). *Given firms' production decisions $y = (y^1, \cdots, y^F)$, $(x, \phi, z, p, q, \rho)(y)$ is an exchange equilibrium parametrized by y, if*

(a) for each agent i, $(x^i, \phi^i, z^i) \in B^i_\star(y, p, q, \rho)$,

(b) commodity markets clear: $\sum_i \left(x^i - \omega^i\right) = \sum_f y^f$,

(c) stock markets clear: $\sum_i \phi^{if}_e = 1$ at each event e,

(d) securities lending markets: $\sum_i z^i = 0$.

$EE(y)$ will denote the set of all exchange equilibria parametrized by y.

We look at another equilibrium concept, where production decisions are not exogenous anymore and firms anticipate the impact of their production choices on dividends and equity value.

[12]The third date serves for guaranteeing that the shares of the firm retain value at the second date, when borrowing and lending transactions are settled.

To do this, we suppose that each firm has conjectures $\upsilon^f = (\upsilon_e^f) : Y^f \to \mathbb{R}^{S+1}$ about how the price of its shares will vary with its production plan.

Production decisions will also affect the rebate rate paid by shares lenders on the cash collateral. Low rebate rates indicate that share borrowers have a possession value driven by their desire to short the shares. Such possession value will ultimately be reflected in the price of the stock (see sections 3.3.3 and 3.4) and this suggests that the firm may also want to have conjectures about the rebate rate paid by lenders of its stock, again conditional on the production plan the firm selects, and given by $\tau^f : Y^f \to \mathbb{R}$.

The net market value of the firm equals dividends paid at node e plus the current equity value[13]. Firm f's conjectured net market value at node e, given production decisions y^f, is given by $\pi_e^f(y^f) \equiv p_e y_e^f + \upsilon_e^f(y^f)$. Current stock price (estimated by $\upsilon_e^f(y^f)$) does not entitle to current dividends but entitles to immediate possession, as the shares can be immediately lent. The link between possession and pricing is addressed in section 3.4 (see in particular Remark 3).

Next we define a *financial equilibrium*, which is an exchange equilibrium with the added requirement that firms choose the production plans that maximize their respective net market values at date 1, for conjectures on equity value that must be fulfilled in equilibrium.

Definition 3 (Financial equilibrium). *$(x, \phi, z, y, p, q, \rho)$ is a financial equilibrium if*

(a) $(x, \phi, z, p, q, \rho)(y) \in EE(y)$,

(b) for each firm f, $y^f \in \operatorname{argmax}_{\bar{y}^f \in Y^f} \pi_1^f(\bar{y}^f)$, and, in each event e, $\upsilon_e^f(y^f) = q_e^f$ and $\tau^f(y^f) = \rho^f$.

FE will denote the set of financial equilibria in the economy.

Finally, we present the concept of *quasi-equilibrium*, which is an exchange equilibrium in which all firms, with the possible exception of one, choose the production plans that maximize

[13] Note that in some nodes, where $p_e y_e^f < 0$, this corresponds to investment in the firm which shareholders must bear. At other nodes, where $p_e y_e^f > 0$, it constitutes positive dividends that the firm distributes among its shareholders.

their respective net market values at date 1. This concept will help us study whether or not present value maximization is a good objective for the firm. By "good" we mean a course of action that maximizes shareholders' utilities.

Definition 4 (Quasi-equilibrium). *We say that* $(\bar{x}, \bar{\phi}, \bar{z}, \bar{y}, p, \bar{q}, \bar{\rho})$ *is a quasi-equilibrium given firm f's production plan* $\bar{y}^f$ *if*

(a) $(\bar{x}, \bar{\phi}, \bar{z}, p, \bar{q}, \bar{\rho})(\bar{y}) \in EE(\bar{y})$,

(b) for each firm $g \neq f$, $\bar{y}^g \in \operatorname{argmax}_{y^g \in Y^g} \pi_1^g(y^g)$,

(c) for each firm g, in each event e, $\upsilon_e^g(y^g) = \bar{q}_e^g$ *and* $\tau^g(y^g) = \bar{\rho}^g$.

The set of quasi-equilibria when firm f's production plan is y^f *will be denoted by* $QE(\bar{y}^f)$.

Observe that conjectures on equity price must also be fulfilled in any quasi-equilibrium.

3.2.5 Existence of exchange equilibrium

It is well understood that the main difficulty when proving the existence of equilibrium arises when short sales are unbounded, as shown by Hart (1975). This is not a problem in our model, where the exogenous upper bound on borrowing of securities, together with the box constraint, bound agents' short positions. We now introduce some assumptions that will be used in some of the results that follow.

Assumption 1. *For every agent i:*

1. *commodity endowments are strictly positive in every node e,*
2. *stock endowments are such that* $\sum_f \phi_0^{if} > 0$,
3. *the agent's preferences satisfy:*
   - *(a)* U^i *is concave and twice continuously differentiable,*
   - *(b)* $DU^i(x) \in \mathbb{R}_{++}^{(1+2S)C}$, $\forall x \in \mathbb{R}_{++}^{(1+2S)C}$,

(c) $\forall c \in \mathbb{R}$*, the set* $[U^i]^{-1}(c)$ *is closed in* $\mathbb{R}^{(1+2S)C}_{++}$*,*

(d) for every (y, p, q, ρ)*:* $(x^i, \phi^i, z^i) \in B^i_\star(y, p, q, \rho)$ *implies that for each event* e*, there is some commodity* c *such that* $x^i_{ec} > 0$ *and* $\partial U^i(x^i)/\partial x^i_{ec} > 0$*.*

Part (3d) of assumption 1 ensures that marginal utilities of income (the Lagrange multipliers of the budget constraints) are positive. The next assumption will be considered in some of our results:

Assumption 2. $y = (y^1, \cdots, y^F)$ *is such that:*

1. *for each good, there is at least one firm producing that good at dates* 2 *and* 3*,*
2. *for every agent* i*, the initial endowment of goods exceeds the agent's initial investment in the several firms he has ownership shares*[14]*:* $\omega^i_1 + \sum_f \phi^{if}_0 y^f_1 \gg 0$*.*

Given that short sales are bounded, the existence of exchange equilibria follows as a natural extension of Bottazzi, Luque and Páscoa (2012) in our framework:

Proposition 2. *If assumption 1 holds and the vector of firms production plans* $y = (y^1, \cdots, y^F)$ *satisfies assumption 2, then there exists an exchange equilibrium with production plans* y*.*

3.3 Unanimity

Before presenting the main unanimity results, we introduce two important definitions that will help us characterize equilibria and the firms for which the *shareholders' unanimity* property holds.

3.3.1 Perfectly elastic demand by shareholders and by shareborrowers

To start with, a perfectly competitive firm is such that both its shareholders and the lenders of its shares will be facing perfectly elastic demands for the shares, that is, the agents that hold the

[14]This is equivalent to assuming that the value of initial endowments ($p_1\omega^i_1$) dominates the cost of the initial investment ($p_1 \sum_f \phi^{if}_0 y^f_1$) regardless of what the relative commodity prices (p_1) might be.

firm's shares have no market power to raise the share's price or lower the rebate rate. Moreover, as we will see next, even though a perfectly competitive firm does influence the shares price and rebate rate through its production decisions, it does so in a very objective way, limited to be exactly as predicted by the conjectures which will be characterized in section 3.4.

Definition 5 (**Perfectly elastic demand**). *Let $(\bar{x}, \bar{\phi}, \bar{z}, \bar{y}, p, \bar{q}, \bar{\rho}) \in QE(\bar{y}^f)$ be a quasi-equilibrium, and let $E_T \equiv \{1, 21, \cdots, 2S\}$ be the set of nodes where there is trading of securities. The demand for firm f's shares is perfectly elastic both in the stock and the securities lending markets, in the quasi-equilibrium, when for every agent i, it is true that:*

If the agent's plan in the quasi-equilibrium $(\bar{x}^i, \bar{\phi}^i, \bar{z}^i)$ is such that $\bar{\phi}_e^{if} > 0$ in some event $e \in E_T$ (agent is a buyer of the share in the stock market), or $\bar{z}^{if} > 0$ (agent is a borrower of the share), then there is another (optimal) plan $(\bar{x}_*^i, \bar{\phi}_*^i, \bar{z}_*^i,) \in B_\star^i(\bar{y}, p, \bar{q}, \bar{\rho})$, in which $\bar{\phi}_{*e}^{if} \leq 0$ in all events $e \in E_T$, and $\bar{z}_*^{if} \leq 0$ (agent is neither a buyer nor a borrower).

If the demand for firm f's shares is perfectly elastic according to definition 5, whoever is trying to sell the share in the stock market is a *price taker* in the sense that if he tried to sell the firm's shares at a higher price, he would loose all of his buyers. Every potential buyer could do just as well (in utility terms) with plans that did not include long positions on the shares of the firm. For the same reason, if the shares of the firm were to be lent at a slightly lower rebate rate, no one would borrow the share any more.

The general collateral rate is the highest rebate rate that lenders of securities pay to borrowers. This rate is associated with the lowest lending fee paid by borrowers, which in turn corresponds to the borrowing of securities for which there is no special demand (hence the term *general collateral*). In other words, a security classified as general collateral has no particular *possession value*. Of the sample of securities studied in D'Avolio (2002), 91% were general collateral, lent for an average fee of 17 basis points. Transactions done with general collateral are more likely to be driven by the supply and demand for cash than the supply and demand for individual securities. The lending fee associated with the firm's shares is the price that a borrower of shares is willing to pay to be in possession of these particular shares in order to reuse them, either by short-selling or by re-lending them. It is the presence of this price that makes the short sales no longer an externality as they were in Makowski (1983)'s model with naked short sales.

If borrowers of the firm's shares are not willing to accept being paid a rebate rate below the general collateral rate, and if this rate coincides with money market interest rate, then the lending fee is zero, and one can infer that there is no *possession value* for the firm's shares and, therefore, the firm faces a perfectly elastic demand in the securities lending market. Examples 1 and 2 will illustrate such absence of possession value. A precise relation between perfectly elastic demands and no-specialness will be established in Theorem 4, in section 3.4.1.

Definition 6 (Perfectly competitive firm). *Let* $(x, \phi, z, y, p, q, \rho) \in FE$ *be a Financial Equilibrium. We say that firm* f *is a* perfect competitor *in the equilibrium if for any production plan* $\bar{y}^f \in Y^f$ *that firm* f *chooses to implement, and for every quasi-equilibrium* $(\bar{x}, \bar{\phi}, \bar{z}, \bar{y}, p, \bar{q}, \bar{\rho}) \in QE(\bar{y}^f)$*, it is true that:*

1. *Firm* f*'s production decision does not affect prices or dividends of other firms:* $p_e \bar{y}_e^g = p_e y_e^g$*,* $\bar{q}_e^g = q_e^g$*,* $\bar{\rho}^g = \rho^g$*, at all events* e*, and for all firms* $g \neq f$*.*
2. *There is a perfectly elastic demand for firm* f*'s shares in the quasi-equilibrium.*

The first part of definition 6 states that firm f's choice of production plan does not affect other firms' prices. The second part says that f is limited in how it can affect its own prices: it can choose a production plan $\bar{y}^f$ which induces a quasi-equilibrium in $QE(\bar{y}^f)$, but the way in which the choice of the production plan $\bar{y}^f$ induces quasi-equilibrium price and rebate rates for the firm's shares is constrained by the fact that there is a perfectly elastic demand for its shares.

That is, the quasi-equilibrium price of the share can not be influenced by either shareholders or by share-borrowers (as these are the two types of agents that sell or short-sell the shares to any buyer). Any attempt to raise the price above the quasi-equilibrium level, would fail since the buyers can always find good substitutes and maintain their utility levels. Similarly, an attempt by lenders of the shares to charge a lending fee (pay a rebate rate) above (below) the quasi-equilibrium level would also fail due to the adequate substitutability that borrowers of shares enjoy.

The above notion of a firm being perfectly competitive in the markets for its shares is not related in a straightforward manner to having many firms. Actually, the large numbers case is not necessary, as we illustrate in the following examples, it may help (as we discuss in example 3) but is not sufficient either. In fact, if firms do not have the same technology or even when a same technology set allows them to pick different production plans, firms end up offering, through the dividends, hedging services that are not perfect substitutes, and therefore, demands would still be non-perfectly elastic in spite of the large numbers involved.

In the literature some other notions of competitiveness are used to obtain unanimity. One important example, the *spanning* condition, implies that firms cannot innovate. Essentially, shareholders of a firm will be unanimous in their preferred production plan if the firm is competitive in the sense that it cannot introduce new goods into the market or produce the same goods in a way that differs much from that of the other firms'. Whatever production plan the firm decides to implement, it is a linear combination of the equilibrium production plans in the economy.

The notion of competition in definition 6 is less restrictive than the spanning condition because firms are allowed to innovate by selecting a production plan that is not spanned by the equilibrium plans of the firms in the economy. However, in any quasi-equilibrium, investors manage to find an alternative optimal portfolio where positions on the shares of this firm are absent. In Makowski (1983) a firm is competitive when investors manage to do without long positions in the firm's shares, but as the hedging done through short selling could not be priced, short selling was an externality that had to be ruled out. In the present article, a firm is competitive when investors manage to find good substitutes for both long and short positions in the firm's shares. In the spirit of Ostroy (1980), trading the shares of a competitive firm, being either long or short, contributes no surplus to the economy.

Let us now give two examples of economies in which a firm is a perfect competitor according to our definition. These examples will show that, in our definition of perfect competition, what matters is not just what the production plans of the firms are, but also the way in which the technology of a firm affects the utility of the agents. Competitiveness also depends on the agents preferences. To illustrate this last point, we present an example of an economy with only one firm. There are no restrictions on the firm's technology but we assume linear utilities and this allows for an easy substitution of the firm's shares by positions in a riskless one-period real bond.

Example 1. Suppose there is only one firm. There is only one commodity produced by the firm and traded at each node. This commodity will be the numeraire (that is, $p_e = 1$, $\forall e$). We assume that apart from hedging in the firm's shares, agents have at their disposal a one-period real bond whose real returns (in the single consumption good) are $1 + \rho_b$. We denote positions in such bond by b^i (measured in the equivalent first-date consumption units). Agents' budget constraints for nodes 1 and $2s$ become:

$$(x_1^i - \omega_1^i) + q_1^f(\phi_1^{if} - \phi_0^{if}) + b^i \leq \phi_0^{if} y_1^f - q_1^f H^f z^{if}$$
$$(x_{2s}^i - \omega_{2s}^i) + q_{2s}^f(\phi_{2s}^{if} - \phi_1^{if}) - (1 + \rho_b)b^i \leq \phi_1^{if} y_{2s}^f + (1 + \rho^f)q_1^f H^f z^{if}$$

Utilities of all the agents are linear: $U^i(x^i) = \sum_{ts} \beta^{t-1}\sigma_{ts}^i x_{ts}^i$, where $\beta \in (0, 1]$ is the discount factor and σ_{ts}^i is the probability agent i assigns to the occurrence of date-state ts. Since the unique state of the world is known at date 1, and there is no uncertainty from date 2 to date 3, we must have $\sigma_1^i = 1$ $\forall i$, and $\sigma_{2s}^i = \sigma_{3s}^i$ $\forall i, \forall s$. Also, $\sum_s \sigma_{2s}^i = 1$ $\forall i$. We will show that demand for the firm's shares is perfectly elastic at the quasi-equilibria where the condition $\rho = \rho_b$ holds.

Consider a quasi-equilibrium in which the solutions to agents' problems are interior, that is, there is positive consumption at all date-states. Let λ_e^i denote the multipliers for the budget constraints at node e. From the agents first order conditions we have that, for interior solutions, $\partial U(x^i)/\partial x_e^i = \lambda_e^i$. Computing $\partial U(x^i)/\partial x_e^i$, we have $\lambda_{ts}^i = \beta^{t-1}\sigma_{ts}^i$. In particular, $\lambda_1^i = 1$, and for every agent i, it must be true that $(1 + \rho_b) = 1/\sum_s(\beta\sigma_{2s}^i) = 1/\beta$ (see appendix).

Suppose $\rho = \rho_b$. To prove that the demand for the firm's shares is perfectly elastic, we must show that whenever an agent has positive long positions, at any date, in the stock or in the securities lending market, it is possible to substitute the agent's plan with some other optimal plan that does not include the firm's shares. We will start by showing that this is true for long positions purchased at date 1.

The first order conditions of the agent i's problem (see appendix) imply that in an equilibrium with $z^{if} < M$ the following equation holds:

$$\sum_s \beta\sigma^i_{2s}\left(\left(1+\rho^f\right)q^f_1 - \left(y^f_{2s} + q^f_{2s}\right)\right) = 0 \tag{3.7}$$

Suppose the agent's portfolio was initially (ϕ^i, z^i, b^i). There is a portfolio $(\tilde{\phi}^i, \tilde{z}^i, \tilde{b}^i)$ which doesn't include shares of the firm, and doesn't change the level of utility attained by the agent. Simply set $\tilde{\phi}^{if}_1 = 0$, $\tilde{z}^{if} = 0$ and $\tilde{b}^i = b^i + q^f_1(\phi^{if} + H^f z^{if})$.

With the new portfolio, consumption at date 1 will remain unaltered. At date 2, state s, the change in consumption (and in income) will be $\phi^{if}_1\left((1+\rho^f)q^f_1 - (q^f_{2s} + p_{2s}y^f_{2s})\right)$. After the change, the agent will consume more in some states and less in others. Suppose that consumption at every state remains positive after the change[15]. Equation (3.7) implies that the overall change in utility after the change in portfolio will be zero. An agent can always do as well by not having long positions on firm f's shares on the securities lending and stock markets. The linearity of the preferences guarantees that the same substitution works in a quasi-equilibrium where $z^{if} = M$.

This proves that the demand for the firm's shares is perfectly elastic at date 1. The same argument could be done for the portfolio at the second date if a one period real bond was traded at that date too. Alternatively, since there is no uncertainty between dates 2 and date 3, the firm would also face a perfectly elastic demand for its shares if there were no inter-temporal discounting, that is, if $\beta = 1$. In this case an agent which initially had a long position $\phi^{if}_{2s} > 0$, can simply choose another portfolio in which $\tilde{\phi}^{if}_{2s} = 0$ instead. The change in portfolio will increase the agent's income at node $2s$ by $q^f_{2s}\phi^{if}_{2s}$ and decrease his income at node $3s$ in the amount $\phi^{if}_{2s}y_{3s}$. In terms of utility, the agent will gain $\beta q^f_{2s}\phi^{if}_{2s}$ and loose $\beta^2\phi^{if}_{2s}y_{3s}$. From the first order conditions the share's price at date 2 satisfies $q^f_{2s} = (\lambda^i_{3s}/\lambda^i_{2s})y^f_{3s}$ and $\lambda^i_{ts} = \beta^{t-1}\sigma^i_{ts}$. Since $\sigma^i_{2s} = \sigma^i_{3s}$, when $\beta = 1$ the increment in expected utility from the change in income at node $2s$ is exactly compensated by the decrease in expected utility from the change at node $3s$.

[15]Consumption after the change of portfolio will remain positive if the cost of the original portfolio of firm f's shares was a small enough fraction of the agent's total income. This in turn will be more likely when there are more firms and shares of firm f constitute only a small portion of the agent's portfolio.

We have seen that, in this example, if the condition $\rho^f = \rho_b$ holds at all quasi-equilibria, the firm will be a perfect competitor in the economy. A stronger result which is valid in general and not only in this example is given in Theorem 4, which states that if firm f is a perfect competitor in the financial equilibrium, then $\rho^f = \rho_b$ in all quasi-equilibria.

This example imposes restrictions on agents' preferences rather than on firms' technologies. What is required from the agents is that they all have linear utilities, so that the null first order effect given by equation (3.7) coincides with the actual utility variation experienced by the agents when they substitute their long positions on the shares of the firm by positions in the one-period bond.

We made no assumptions on spanning. The payments of the risk-free loan will, in general, not be equal to the dividends paid by the firm, but when the interest rates coincide ($\rho^f = \rho_b$) the equality holds in terms of the average across states, weighted by marginal utilities of income. Hence, for linear utilities, the proposed change in portfolio has no effect on agents' expected utility.

The example also highlights another important feature of equilibria and economies in which firms are perfectly competitive in the sense of Definition 6: some linearity of preferences is needed when there are few firms or few agents. To be more precise, plans $(\bar{x}^i, \bar{\phi}^i, \bar{z}^i)$ and $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*,)$ in Definition 5 must be such that $U^i(\alpha\bar{x}^i + (1-\alpha)\bar{x}^i_*) = U^i(\bar{x}^i) = U^i(\bar{x}^i_*)$ for all $\alpha \in [0, 1]$.

In contrast with example 1, the following example depends strongly on the structure of the firms' technologies but has no particular assumptions on preferences.

Example 2. Suppose there are three firms and only one commodity in the economy. The margin applied in borrowing transactions is the same for the three firms $H = H^1 = H^2 = H^3$. Firm 1 will be a perfect competitor in any financial equilibrium if its shares can be substituted by some portfolio consisting of shares of firms 2 and 3 in such a way that income at every node stays the same. In that case, agents optimal consumption remains attainable and agents utilities do not decrease as result of the substitution.

Suppose $(x, \phi, z, \bar{y}, p, q, \rho) \in FE$ is a financial equilibrium, that $\bar{y}^2$ and $\bar{y}^3$ are the production plans that maximize the present value of firms 2 and 3, respectively, and that firm 1's production set satisfies the condition

$$Y^1 \subset \left\{ y \in \mathbb{R}^{1+2S} | y = \alpha_2 \bar{y}^2 + \alpha_3 \bar{y}^3, (\alpha_2, \alpha_3) \in \mathbb{R}^2_+ \right\} \tag{3.8}$$

Then, if $y^1 \in Y^1$ and $y^1 = \alpha_2^{y^1} \bar{y}^2 + \alpha_3^{y^1} \bar{y}^3$, a stock portfolio consisting of one unit of firm 1's shares $(\phi^{i1}, \phi^{i2}, \phi^{i3}) = (1, 0, 0)$, pays the same dividends, in all nodes, as a portfolio consisting of $\alpha_2^{y^1}$ units of firm 2's shares plus $\alpha_3^{y^1}$ units of firm 3's shares $(\phi^{i1}, \phi^{i2}, \phi^{i3}) = (0, \alpha_2^{y^1}, \alpha_3^{y^1})$. More generally, the stock portfolios $(\phi^{i1}, \phi^{i2}, \phi^{i3})$ and $(0, \phi^{i2} + \alpha_2^{y^1} \phi^{i1}, \phi^{i3} + \alpha_3^{y^1} \phi^{i1})$, pay exactly the same dividends in all nodes of the event tree. This substitution leaves the returns from the whole portfolio unchanged but this still does not guarantee that agents will be able to consume exactly the same as before the substitution, due to the change in portfolio costs. Previous consumption is still attainable as long as the condition $y^1 = \alpha_2^{y^1} \bar{y}^2 + \alpha_3^{y^1} \bar{y}^3$, which guaranteed unchanged returns implies also that security and borrowing costs remain the same, given, respectively, by:

$$q^1 = \alpha_2^{y^1} q^2 + \alpha_3^{y^1} q^3 \tag{3.9}$$

$$\rho^1 q_1^1 = \alpha_2^{y^1} \rho^2 q_1^2 + \alpha_3^{y^1} \rho^3 q_1^3 \tag{3.10}$$

When conditions (3.9) and (3.10) hold, an agent that substitutes the whole securities and borrowing portfolio $\big((\phi^{i1}, z^{i1}), (\phi^{i2}, z^{i2}), (\phi^{i3}, z^{i3})\big)$ by the portfolio $\big((0, 0), (\phi^{i2}, z^{i2}) + \alpha_2^{y^1} (\phi^{i1}, z^{i1}), (\phi^{i3}, z^{i3}) + \alpha_3^{y^1} (\phi^{i1}, z^{i1})\big)$ will experience no change in income at every date-state and, therefore, no loss in utility.

For $y^1 = \alpha_2^{y^1} \bar{y}^2 + \alpha_3^{y^1} \bar{y}^3$ to imply (3.9) and (3.10) it is enough that shadow prices of the agents' budget constraints are zero, and in particular, that the shares of the three firms are not on special. Null shadow prices imply that there is no possession value for any of the securities in the economy.

3.3.2 Unanimity results

Let us now present our main unanimity result. The following theorem shows that perfect competition is enough to guarantee that initial shareholders agree on the best production plan for the firm.

Theorem 1 (Shareholders' Unanimity). *Let $(x, \phi, z, y, p, q, \rho) \in FE$ be a financial equilibrium and suppose firm f is a perfect competitor in this equilibrium. Given any production plan $\bar{y} \in Y^f$ let $(\bar{x}, \bar{\phi}, \bar{z}, p, \bar{q}, \bar{\rho})(\bar{y}) \in QE(\bar{y}^f)$ be a quasi-equilibrium. Then:*

(a) Each initial shareholder (i such that $\phi_0^{if} > 0$) is not worse off in the financial equilibrium than in the quasi-equilibrium: $U^i(x^i) \geq U^i(\bar{x}^i)$.
Also, if $\pi_1^f(y^f) > \pi_1^f(\bar{y}^f)$, the inequality is strict: $U^i(x^i) > U^i(\bar{x}^i)$.

(b) An agent with $\phi_0^{if} = 0$ is as well off in the financial equilibrium as in the quasi-equilibrium: $U^j(x^j) = U^j(\bar{x}^j)$.

Remark 2. Remember that, by definition, the difference between y^f and $\bar{y}^f$ is that y^f maximizes the present value of the firm at date 1: $\pi_1^f(y^f) \geq \pi_1^f(\bar{y}^f)$. Note also that both our definition of a perfectly competitive firm and the results in Theorem 1 apply to economies with a finite number of firms. If we had a continuum of firms instead, agents would always be indifferent to the production plan choices of any single firm.

The intuition behind the proof is that if the firm is a perfect competitor in the financial equilibrium, we can find new consumption plans for the agents that are utility-equivalent to the equilibrium and quasi-equilibrium production plans, respectively (the resulting allocations might not be market clearing). These new plans will be utility-comparable, since they are also utility-maximizers in the respective (portfolio constrained) budget sets and one of them (or both) belongs to the other (portfolio constrained) budget set .

Proof. First, note that given any quasi-equilibrium $(\bar{x}, \bar{\phi}, \bar{z}, \bar{y}, p, \bar{q}, \bar{\rho})$, and for every agent i, there exists some alternative plan $(\bar{x}_*^i, \bar{\phi}_*^i, \bar{z}_*^i) \in B_\star^i(\bar{y}, p, \bar{q}, \bar{\rho})$, with $(\bar{\phi}_{*1}^{if}, \bar{z}_*^{if}, (\bar{\phi}_{*2s}^{if})_s) = (0, 0, (0, \cdots, 0))$.

To see this, note that the firm is a perfect competitor in the economy, so we know that agent i could choose a feasible plan $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*) \in B^i_\star(\bar{y}, p, \bar{q}, \bar{\rho})$, such that $(\bar{\phi}^{if}_*, z^{if}_*) \leq 0$.

This alternative plan is feasible, so it satisfies the box constraint, $\bar{\phi}^{if}_{*1} + \bar{z}^{if}_* \geq 0$. This, together with $\bar{\phi}^{if}_{*1} \leq 0$ and $\bar{z}^{if}_* \leq 0$, imply $\bar{\phi}^{if}_{*1} = \bar{z}^{if}_* = 0$. The same for the second date when, again by feasibility, $\bar{\phi}^{if}_{*2s}$ satisfies the box constraint (3.5) and satisfies $\bar{\phi}^{if}_{*2s} \leq 0$, so that we actually have $\bar{\phi}^{if}_{*2s} = 0$.

That is, the box constraints imply that if a firm is a perfect competitor in the financial equilibrium, then every agent can dispense with not only long positions in the shares of the firm, but with all positions altogether. In other words, every agent can ignore the presence of the firm in the economy and not loose any utility by doing that.

Next, given the equilibrium production plan $y^f \in Y^f$, and an arbitrary production plan $\bar{y}^f \in Y^f$, from what we have just proved, for every agent there are plans (x^i_*, ϕ^i_*, z^i_*) and $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*)$ such that $U^i(x^i_*) = U^i(x^i)$ and $U^i(\bar{x}^i_*) = U^i(\bar{x}^i)$.

If $\phi^{if}_0 = 0$ we have that $(x^i_*, \phi^i_*, z^i_*) \in B^i(\bar{y}, p, \bar{q}, \bar{\rho})$ and that $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*) \in B^i(y, p, q, \rho)$, that is, both plans are feasible at the prices corresponding to the other (quasi)equilibrium. This is true because prices and dividends associated with other firms do not change in the quasi-equilibrium, and because both plans exclude firm f from the agent's portfolio. This implies $U^i(\bar{x}^i_*) \geq U^i(x^i_*)$ and $U^i(\bar{x}^i_*) \leq U^i(x^i_*)$, that is, $U^i(\bar{x}^i) = U^i(\bar{x}^i_*) = U^i(x^i_*) = U^i(x^i)$. This proves part (b) of the theorem.

If an agent is an initial shareholder, we have that, from the definitions of financial equilibrium and quasi-equilibrium we know that $\pi^f_1(y^f) = p_1 y^f_1 + q^f_1 \geq p_1 \bar{y}^f_1 + \bar{q}^f_1 = \pi_1(\bar{y}^f)$. This implies that the agent's plan in the quasi-equilibrium is feasible at equilibrium prices: $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*) \in B^i(y, p, q, \rho)$. This in turn implies $U^i(\bar{x}^i) = U^i(\bar{x}^i_*) \leq U^i(x^i_*) = U^i(x^i)$. If $\pi^f_1(y^f) < \pi_1(\bar{y}^f)$ the strict preference of x^i over $\bar{x}^i$ comes from the monotonicity of preferences that we have assumed.

□

Shareholders are the agents that are long in the firm's stock. As mentioned before, an agent that has borrowed the share has, while the share is in his possession, all voting rights associated with it. However, the usual practice is that the share will be returned to its original owner before a vote. See for example Aggarwal et al. (2015)[16].

From the proof of Theorem 1, it is clear that, if an agent held an initial short position ($\phi_0^{if} < 0$) he would, in fact, prefer if the firm implemented the plan that minimizes its present value. By construction, this is not possible in our model with three dates because all initial positions are nonnegative, but it will be possible when we extend the model to accommodate more dates and consider the firms' decisions in the future, when agents' initial positions at date $t+1$ will be the positions they have chosen at the end of t.

An interesting question in an economy with $T > 3$ dates, is whether, at dates $t > 1$, *new shareholders would agree with initial shareholders or would prefer that the firm implemented a different production plan*. To answer this question we need to introduce the following assumption:

Assumption 3. *If* $(\bar{x}, \bar{\phi}, \bar{z}, \bar{y}, p, \bar{q}, \bar{\rho})$ *is a quasi-equilibrium given firm* f*'s production plan* $\bar{y}^f$*, then at every node* e *in the event tree there is some agent* i *who:*

1. *is not an initial shareholder:* $\bar{\phi}_0^{if} = 0$*,*
2. *has a non-binding box constraint at node* e*.*

We will refer to agents that satisfy assumption 3 as *reference agents*. Note that there are always agents satisfying the second part of the assumption. This follows from aggregation of the box constraints and the market clearing conditions[17]. The first part of the assumption is also likely to be satisfied in reality because, although this is not imposed in our model, the securities lending market's supply and demand sides tend to be segmented. On the supply side

[16] Notice that if not all shares borrowed were being returned on time to the original owners before a vote was taken, it would be possible to have more security long positions at that time than the fixed net supply (the 100% ownership) but the voters would be those in possession of the latter (possibly borrowers doing a short sale and, therefore, more interested in minimizing the firm's present value instead, which shows why it is important and a prevailing practice to return shares before voting takes place). Notice that our results address the unanimity of initial shareholders, assumed to have positive endowments of the shares.

[17] Take for example node $e = 1$, we have $\sum_i(\phi_1^{if} + \theta^{if} - \psi^{if}) = \sum_i \phi_1^{if} + \left(\sum_i \theta^{if} - \sum_i \psi^{if}\right) = 1 + 0 > 0$, so we must have $\phi_1^{if} + \theta^{if} - \psi^{if} > 0$ for at least one agent i.

we have *beneficial owners*. These include pension funds, insurance companies, mutual funds, endowments and foundations and sovereign wealth funds. On the other hand, the demand side is mainly comprised of leveraged agents with hedging or shorting needs. Agents on the demand side are unlikely to be initial shareholders.

Assumption 3 only imposes the extra requirement that at least one of the agents with non-binding box constraint is also not an initial shareholder.

As we will see in Theorem 3, reference agents play a fundamental role in our model. These agents' marginal rates of substitution can be used to compute the price of the firm's shares both in and out of equilibrium (that is, in any other quasi-equilibrium).

Consider an economy with $T > 3$ dates in which securities lending markets open in the first $T-2$ dates. There is uncertainty between dates t and $t+1$ for $t = 1, 2, \cdots, T-2$ and there is no uncertainty between dates $T-1$ and T. All definitions and assumptions are to be extended in the natural way.

Let $(x, \phi, z, y, p, q, \rho) \in FE$ be a financial equilibrium and let $D(e) = \{u \geq e\}$ be the sub-tree of events with root e. Define the economy $\mathcal{E}(x, \phi, z, y)(e)$ as an economy in which the consumption sets of the agents are $X^i(e) = \{\bar{x}^i \in X^i | \bar{x}^i_u = x^i_u, \forall u \notin D(e)\}$, preferences of the agents remain the same as in the original economy, production sets of the firms satisfy $Y^f(e) = \{\bar{y}^f \in Y^f | \bar{y}^f_u = y^f_u, \forall u \notin D(e)\}$ for all firms, and the available portfolios for each agent i are in the set $\{(\bar{\phi}^i, \bar{z}^i) | \bar{\phi}^i_u = \phi^i_u, \bar{z}^i_u = z^i_u, \forall u \notin D(e)\}$.

Theorem 2 (Unanimity through time). *Suppose that preferences satisfy assumption 1. Suppose also that $(x, \phi, z, y, p, q, \rho)$ is a Financial Equilibrium in an economy with $T > 3$ dates, and that firm f satisfies assumption 3 and is a perfect competitor in the truncated economy $\mathcal{E}(x, \phi, z, y)(e)$. Then*

1. *agents that immediately prior to node e had positive positions ($\phi^{if}_{e-1} > 0$), will unanimously agree on the production plan $y^f(e)$ that the firm should implement,*
2. *and $y^f(e)$ coincides with $y^f(1)$, the plan chosen by initial shareholders (those with $\phi^{if}_0 > 0$).*

Proof. The proof is given in the Appendix, after the proof of Theorem 3. □

3.3.3 Is competition compatible with Short Sales?

One important shortcoming in other models where shareholders' unanimity is obtained via some assumption regarding the competitiveness of firms, is that in those models there is a fundamental incompatibility between perfect competition and the short selling of the firm's shares. We have proven that this need not be the case when the services agents get by short selling the security are priced correctly in the securities lending market.

A short seller must be willing to pay a lending fee, in other words, end up being receiving a rebate rate on the posted cash collateral that may be lower than what his personal marginal rates of intertemporal substitution would predict. This follows from his first order condition on security borrowing:

$$(1+\rho^f) = \frac{\lambda_1^i}{\sum_s \lambda_{2s}^i} - \frac{\mu_1^{if} - \nu_+^{if} + \nu_-^{if}}{H^f q_1^f \sum_s \lambda_{2s}^i},$$

where λ_e^i and μ_e^{if} are the multipliers of node e's budget and box constraint, respectively and ν_+^{if}, ν_-^{if} are the multipliers of the upper and lower bounds on z^{if}, respectively. The first term on the right hand side is what the agent would normally expect to receive when lending cash. The second term, measures the possession value of the security. Possession has value because it enables the agent to short-sell the security. In order to short sell, he may have to accept an interest rate lower than his valuation of the cash flow corresponding to the loan and the difference is the cost the agent pays for the benefits of short-selling (see Section 3.4.1).

To illustrate that there is no incompatibility between shareholders' unanimity and short sales, we adapt an example from Makowski (1983). While in Makowski (1983) the example illustrated why naked short sales prevented unanimity, once we adapt the example by allowing for borrowing of shares, we see that unanimity may prevail in spite of short selling. It all depends on whether firms are competitive (in both the stock and securities lending market).

Example 3 (**On Makowski's counterexample**).

3.1 A counterexample with naked short sales

This example is adapted from Makowski (1983). Consider an economy in which there are only two dates ($T = 2$) and no uncertainty ($S = 1$). There are 2 agents. One of the agents, denoted by j, has initial endowments $\phi_0^j = 0$, $\omega^j = (9,2)$, the other agent, denoted by k, has endowments $\phi_0^k = 1$, $\omega^k = (3,2)$. The production set is given by $Y = \{(0,0),(-2,1)\}$. Agents preferences are $U^j(x_1^j, x_2^j) = x_1^j + x_2^j$ and $U^k(x_1^k, x_2^k) = x_1^k + \xi x_2^k$, where $0 < \xi < 1/2$. Utilities are linear for convenience. As discussed in example 1, some linearity in agents' utilities is necessary if firms are to be perfect competitors[18]. Given the restriction on ξ, agent k will prefer to transfer as much consumption as possible from date 2 to date 1.

For the moment we will assume that naked short sales are allowed, by which we mean that agents do not need to borrow in the securities lending market before short selling, that is, there are no securities lending markets and there is no box constraint.

In this economy, a quasi-equilibrium in which each agent consumes his initial endowment and there is no production, is supported by prices $p = (1,1)$ and $q_1 = 0$. In this quasi-equilibrium, the present value of the firm is 0.

When we have production and the plan adopted by the firm is $y = (-2,1)$, a quasi-equilibrium emerges where $x^k = (4,0)$, $x^j = (6,5)$, $\phi_1^k = -2$, $\phi^j = 3$. This quasi-equilibrium is supported by prices $p = (1,1)$, $q_1 = 1$. Note that in this case, the present value of the firm is $\pi(y) = y_2 + q_1 = -2 + 1 = -1$.

Now that we know the present value of the firm in every quasi-equilibrium we can label the one with no production as the unique financial equilibrium for this economy because it is in this quasi-equilibrium that the firm's present value is maximized. Note that agent k, the initial shareholder, is strictly better off in the quasi-equilibrium with production while agent j attains the same level of utility in both quasi-equilibria. Also note that the firm is a perfect competitor in this economy[19]. This shows that the result in Theorem 1 cannot be extended to apply to

[18]There are two differences between this example and Example 2 , in Makowski (1983). First, Makowski does not specify a utility function, just shows some indifference curves. Second, in Makowski (1983) there are J agents of type j while in ours, $J = 1$. In Makowski (1983) the section of indifference curves that must be linear for the firm to be a perfect competitor, shrinks as J increases. In the limit, linearity is reduced to differentiability.

[19]To obtain the correct definition of perfectly competitive firm in an economy without securities lending markets, the definition for perfectly elastic demand should be modified to refer only to long positions in the stock market, as there are no lending markets.

naked short sales: The initial shareholder of a perfectly competitive firm is better off in the quasi-equilibrium where the present value of the firm is the lowest.

3.2 The example redone with securities lending markets.

We now consider the same economy with the addition of securities lending markets. When the production plan is $(0,0)$ the unique equilibrium is, as before, one in which each agent consumes his initial endowment and $q_1 = 0$ so the value of the firm is 0.

More interesting, when the production plan is $(-2,1)$ it is possible to transfer consumption from one date to another, something that benefits agent k because he prefers to transfer as much consumption as possible from date 2 to date 1.

Although there may be multiple quasi-equilibria when $y = (-2,1)$, we will restrict ourselves to studying only those in which agent j has a perfectly elastic demand for the firm's shares both in the stock and the securities lending market. That is, only to quasi-equilibria in which, if the agent holds a long position in the securities lending market, he does no worse and no better than when he consumes his initial endowment. This is the condition for perfectly elastic demand in this special case in which there is only one firm and there are no other securities with which the agent can substitute the firm's shares. We restrict our attention to this case because, otherwise, we already know that the hypotheses of Theorem 1 are not satisfied.

If agent k transfers as much consumption as possible from the second to the first date, and agent j has a perfectly elastic demand for the firm's shares both in the stock and securities lending markets, portfolios must be of the form $(\phi_1^k, z^k) = (\eta, -\eta)$ and $(\phi_1^j, \theta^j, \psi^j) = (1-\eta, \eta)$, where $\eta \geq 0$ satisfies the condition:

$$0 < \eta = \min\left\{\frac{2}{H-1}, M\right\}. \tag{3.11}$$

This quasi-equilibrium is supported by prices $q_1 = 1 + \rho = 1$ and consumption is $(x_1^k, x_2^k) = (2 + \eta(H-1), 2 - \eta(H-1))$ and $(x_1^j, x_2^j) = (8 - \eta(H-1), 3 + \eta(H-1))$.

Table 3.1 summarizes some variables of interest in the two quasi-equilibria we have found: Suppose $M > 2/(H-1)$. Here are the important facts:

– From (3.11) we get $\eta = 2/(H-1)$.

Table 3.1: Quasi-equilibria with securities lending markets

QE	y	$U^k(x^k)$	$U^j(x^j)$	$\pi(y)$
1	$(0,0)$	$3+2\xi$	11	0
2	$(-2,1)$	$2(1+\xi)+\eta(H-1)(1-\xi)$	11	-1

– Initial shareholder k utilities are $3+2\xi$ and 4 in quasi-equilibria 1 and 2, respectively. Since $\xi < 1/2$, he is better off in quasi-equilibrium 2.

– Firm's present value is greater in quasi-equilibrium 1 than in quasi-equilibrium 2.

According to Theorem 1, if the firm is a perfect competitor, initial shareholders should be better off in the quasi-equilibrium where the value of the firm is maximized. This is not the case in this economy and the reason why this happens is that the firm is not a perfect competitor.

If the firm were a perfect competitor, every agent, and in particular agent k should be able to do just as well with a plan that does not include borrowing of the firm's shares ($z^k = 0$) as with plans that do include borrowing ($z^k > 0$). Since in this economy there are no other firms, whose shares could serve as substitutes, the firm is a perfect competitor only if every agent is indifferent between having the firm's shares in his portfolio or not.

When we add the securities lending market to Makowski's economy we end up with a counterexample but of a different kind. While Makowski's example showed the incompatibility of unanimity and short sales, ours shows how crucial the competitiveness assumption is to get the unanimity result of Theorem 1, which allows for short sales.

3.4 Pricing of securities

Our definitions of equilibrium and quasi-equilibrium require that agents have conjectures on how the price of the stock of a firm varies in response to a change in its production plan. Pricing securities is straightforward when markets are complete. In that case, marginal rates of substitution between future and present consumption are equalized across agents, and these rates can be used by any agent as deflators when computing the price of a security.

In the economy we are considering, where markets may be incomplete, marginal rates of substitution will generally differ across agents and so, their personal valuations of a given security may disagree. We will show that, under certain assumptions, some agents (the reference agents) will have the correct valuation of the firm's stock price in equilibrium, and the price computed using these agents deflators in any quasi-equilibrium will also be correct.

The following theorem characterizes the price of the stock of the firm, contingent on the firm's production decision. A remarkable feature is that the equilibrium deflators can also be used to compute the price of the firm's stock in any other quasi-equilibrium, for production plans that differ from the financial equilibrium ones.

Theorem 3 (Pricing of stock and of the rebate rates). *Let $(x, \phi, z, y, p, q, \rho)$ be a financial equilibrium. Suppose assumption 1 holds and that firm f satisfies assumption 3 and is a perfect competitor. Then, the following functions for date 1 and date-state $2s$, correctly compute the price of firm f's stock for any given production plan $\bar{y}^f \in Y^f$ that the firm decides to implement:*

$$\upsilon_1^f(\bar{y}^f) = \max_{i \in I} \sum_s \frac{\lambda_{2s}^i}{\lambda_1^i} \left(p_{2s}\bar{y}_{2s}^f + \upsilon_{2s}^f(\bar{y}^f) \right) \tag{3.12}$$

$$\upsilon_{2s}^f(\bar{y}^f) = \max_{i \in I} \frac{\lambda_{3s}^i}{\lambda_{2s}^i} \left(p_{3s}\bar{y}_{3s}^f \right) \tag{3.13}$$

Moreover, function $\tau^f(\bar{y}^f)$ correctly computes the rebate rate:

$$\tau^f(\bar{y}^f) = \max_{i \in I} \sum_s \frac{\lambda_{2s}^i}{\lambda_1^i} - 1 \tag{3.14}$$

These functions use agent i's marginal utilities of income (λ_e^i), <u>in the financial equilibrium</u>.

Remark 3. Formula (3.12) follows from the following first order conditions (FOC)

$$q_1^f = \sum_s \frac{\lambda_{2s}^i}{\lambda_1^i} \left(p_{2s}y_{2s}^f + q_{2s}^f \right) + \frac{\mu_1^{if}}{\lambda_1^i}$$

The maximal valuation in (3.12) uses the marginal rates of substitution (MRS) of reference agents, as these agents have null shadow values μ_1^i for the box constraint. This does not mean that (3.12) ignores the frictions experienced by non-reference agents, in particular their desire to go short and how the box binds them in doing so. For example, in a two-agent economy, if the agent with $\mu^i > 0$ is a short-seller and the reference agent is long in the security (as in Example 3), then the pricing formula implies that the MRS of the latter must be high enough. The latter is the agent transfering income into today by accepting a cash collateral above the value of the security that he lends. His high MRS reflects the opportunity created in equilibrium by the desire of the counterparty to borrow the security in order to go short[20].

Remark 4. Notice, however, that the correct valuation of the share might also happen to be done using as deflators the MRS of agents that are short in the shares. The reason for this is that reference agents can be both shareholders or shareborrowers (possibly shorting the share). In other price functions appearing in the literature, the correct price of the shares is computed using a weighted average of *shareholders'* valuations but never considering short-sellers' valuations. In Grossman and Hart (1979), the correct price of the stock is given by a weighted average of initial shareholders' personal valuations in which the weights, are given by the proportion of total shares each agent holds. Diamond (1967) and Dreze (1974) propose a weighted average of new shareholders instead.

A version of the pricing function given in Theorem 3 in which only shareholders can act as reference agents appears in Makowski (1983) and, more recently, when short sales are not allowed, in Bisin et al. (2014) and in Acharyaa and Bisin (2014).

We should point out that there is another important difference between our work and that in Bisin et al. (2014), where, when short sales are not allowed, the pricing function in Theorem 3 is assumed to be correct[21] and, from that, unanimity of shareholders follows. However, unanimity

[20] If we had modeled short sales in the naive traditional way, as just negative positions, unrelated to the borrowing of shares, such shadow prices would not be present at all: the pricing equation would just say that *for every agent* the equity price is a discounted expectation of future prices and dividends.

[21] In Bisin et al. (2014), short sales are not modeled using securities lending markets. Instead, financial intermediaries who can issue claims corresponding to short positions on the firm's stock are introduced and, a pricing function different from the one in Theorem 3 is used by firms in this environment.

is not the main focus of their paper. We prove the converse implication: we characterize perfect competition and, *from that*, both unanimity and the pricing function in Theorem 3 follow.

To see that this implication is not at all vacuous we give a counterexample. In general, it is not *rational* to expect the pricing function defined in Theorem 3 to be correct (not even in the simpler case when short sales are not allowed). Given assumption 3, this pricing function will always be correct for the financial equilibrium production plans but, if the firm is not a perfect competitor or, if there are no reference shareholders in the economy, it may be incorrect for other production plans, as the following example shows.

Example 4. Consider an economy with one good ($C = 1$), one firm ($F = 1$) and three agents ($I = 3$) which are denoted j, k, l. There are two dates and two states at the second date. The firm chooses one of two possible production plans: $Y = \{y^1, y^2\} = \{(-1, 1, 1/6), (-1, 1/6, 1)\}$. Agent l is the only initial owner of the firm at date one $\phi_0^l = 1$, $\phi_0^j = \phi_0^k = 0$. Initial endowments of the commodity are as follow: $(\omega_1^l, \omega_{21}^l, \omega_{22}^l) = (0, 1, 1)$, $(\omega_1^j, \omega_{21}^j, \omega_{22}^j) = (3, 2, 1)$, $(\omega_1^k, \omega_{21}^k, \omega_{22}^k) = (3, 1, 2)$. For simplicity, suppose there are no markets for borrowing shares[22].

The utility functions of the agents are $U^l(x_1^l, x_{21}^l, x_{22}^l) = 10{x_1^l}^{\frac{9}{10}} + {x_{21}^l}^{\frac{9}{10}} + {x_{22}^l}^{\frac{9}{10}}$, $U^j(x_1^j, x_{21}^j, x_{22}^j) = {x_1^j}^{\frac{9}{10}} + 2{x_{21}^j}^{\frac{9}{10}} + \varepsilon^j {x_{22}^j}^{\frac{9}{10}}$ and $U^k(x_1^k, x_{21}^k, x_{22}^k) = {x_1^k}^{\frac{9}{10}} + \varepsilon^k {x_{21}^k}^{\frac{9}{10}} + 2{x_{22}^k}^{\frac{9}{10}}$, where $\varepsilon^j = 12(3^{\frac{1}{10}} - 1)(7/18)^{\frac{1}{10}}$ and $\varepsilon^k = 6\left[(399/200)/(201/200)^{\frac{1}{10}} - 2/3^{\frac{1}{10}}\right](7/6)^{\frac{1}{10}}$.

All utilities are increasing in consumption in every date-state so agents will consume whatever they have available for consumption in that date-state. The only way to transfer consumption from date one to date two is by buying the share of the firm. Since agent l values consumption at date 1 much more than consumption at date 2, we will suppose (correctly) that he will sell his share and use the proceeding for consumption. Considering this, agent l's consumption will be $(x_1^l, x_{21}^l, x_{22}^l) = ((q-1), 1, 1)$.

Agents $i \in \{j, k\}$ will consume according to how much of the share they buy: $(x_1^i, x_{21}^i, x_{22}^i) = (\omega_1^i - q\phi_1^i, \omega_{21}^i + \phi_1^i y_{21}, \omega_{22}^i + \phi_1^i y_{22})$.

[22]The definition for perfectly elastic demand should be modified to refer only to long positions in the security market and exclude long positions in the securities lending market. In this example, a firm will be a perfect competitor only if it is so in the sense of Makowski (1983).

A quasi-equilibrium for this economy, when $y = y^1 = (-1, 1, 1/6)$ is supported by the price $q = 2$. Agents' positions on the firm's share are $\phi_1^l = \phi_1^k = 0$, $\phi_1^j = 1$. The corresponding allocations are: $(x_1^l, x_{21}^l, x_{22}^l) = (1, 1, 1)$, $(x_1^j, x_{21}^j, x_{22}^j) = (1, 3, 7/6)$ and $(x_1^k, x_{21}^k, x_{22}^k) = (3, 1, 2)$. With these allocations and prices, markets clear and each agent maximizes his utility. The present value of the firm at $t = 1$ is $y_1 + q = 1$.

When the production plan adopted by the firm is $y = y^2 = (-1, 1/6, 1)$, a quasi-equilibrium is supported by the price $q = 399/200$. Agents' positions on the firm's share are $\phi_1^l = \phi_1^j = 0$, $\phi_1^k = 1$. Agents' consumption is $(x_1^l, x_{21}^l, x_{22}^l) = (1, 1, 1)$, $(x_1^j, x_{21}^j, x_{22}^j) = (3, 2, 1)$ and $(x_1^k, x_{21}^k, x_{22}^k) = (201/200, 7/6, 3)$. With these allocations, markets clear and each agent maximizes his utility. In this case, the present value of the firm at $t = 1$ is $y_1 + q = 199/200 < 1$.

Now that we know that the quasi-equilibrium with production plan $y = y^1$ maximizes present value of the firm, we can label it as the (unique) financial equilibrium. It is easy to see that pricing function $v(\cdot)$, defined in Theorem 3, correctly computes the price of the share in equilibrium. First, we must compute the deflators derived from the multipliers of the agents' problem *in the financial equilibrium*: $\lambda_{21}^l/\lambda_1^l = (1/10)$, $\lambda_{22}^l/\lambda_1^l = (1/10)$, $\lambda_{21}^j/\lambda_1^j \approx 1.7919$, $\lambda_{22}^j/\lambda_1^j \approx 1.2484$, $\lambda_{21}^k/\lambda_1^k \approx 1.3743$, $\lambda_{22}^k/\lambda_1^k \approx 2.0827$.

With these deflators in hand, we can compute the share's price from the Theorem 3's function: $v(y^1) = \max_{i \in I} \left\{(\lambda_{21}^i/\lambda_1^i)y_{21}^1 + (\lambda_{22}^i/\lambda_1^i)y_{22}^1\right\} = \max\{(1/10)(1) + (1/10)(1/6), (1.7919)(1)$ $+ (1.2484)(1/6), (1.3743)(1) + (2.0827)(1/6)\} = \max\{0.11\overline{66}, 2, 1.7214\} = 2$.

This is not surprising at all since the pricing function is derived from the first order conditions of the agents' problem *in equilibrium*. Note that the correct price corresponds to the valuation of the reference agent[23], which in this case is agent j: $v(y^1) = (\lambda_{21}^j/\lambda_1^j)y_{21}^1 + (\lambda_{22}^j/\lambda_1^j)y_{22}^1 = 2$.

However, if we were to use $v(\cdot)$ to compute the price of the share in the quasi-equilibrium we would get: $v(y^2) = \max_{i \in I} \left\{(\lambda_{21}^i/\lambda_1^i)y_{21}^2 + (\lambda_{22}^i/\lambda_1^i)y_{22}^2\right\} = \max\{(1/10)(1/6)+(1/10)(1), (1.7919)$ $(1/6)+(1.2484)(1), (1.3743)(1/6)+(2.0827)(1) = \max\{0.11\overline{66}, 1.5471, 2.3118\} \approx 2.3118 \neq 399/200$.

So the function $v(\cdot)$ is incorrect in the quasi-equilibrium. Note that the assumptions of Theorem 3 are not satisfied. In particular, the firm is not a perfect competitor in the financial

[23] In a model without short sales, a reference agent is simply a buyer of the share who is not an initial shareholder.

equilibrium.

3.4.1 Perfect competition, possession value and specialness

In our discussion of examples 1 and 2 we pointed out that, in both examples, null shadow prices for the agents' constraints involving firm f's shares, are enough to guarantee that the firm is a perfect competitor in the economy. Null shadow prices imply, in particular, an absence of *possession value* for the firm's shares. Let us be more precise.

Suppose first that there were no bounds on securities borrowing and lending. A positive shadow price μ_e^{if} for the box constraint, of firm f's shares at node e, can be interpreted either as the utility a shares borrower would gain if he were allowed to short-sell more than the amount he has borrowed, or as the utility that a shares lender would gain if he were allowed to lend more than what he has in his possession. In both of these interpretations, the agent is limited in his short-selling or in his lending by his (limited) possession of the share. This makes the possession of the share valuable for the agent.

If in addition there is a bound on shares borrowing, the looser is this bound, the lower will be the value attached to the possession of the borrowed shares. On the other hand, in the presence on a bound on shares lending, the looser is the bound, the higher will be the value attached to the possession of shares that can be lent. That is, the possession value for firm f's shares is defined as $\mu_1^{if} - \nu_+^{if} + \nu_-^{if}$, where ν_+^{if} and ν_-^{if} are the multipliers for the restrictions $z^{if} \leq M$ and $z^{if} \geq -M$, respectively.

Another context where null shadow prices are related to unanimity of shareholders can be found in Carceles-Poveda and Coen-Pirani (2009). In that article, short sales are naked (there are no securities lending markets) and bounded exogenously. Null shadow prices (but for different constraints) together with a specific technology are assumed by these authors in order to derive their unanimity result.

In example 1 the *no possession value* follows from the fact that the rebate rate is equal to the yield rate of a one-period real bond $\rho = \rho_b$. This makes the shares of the firm substitutable by trading the price-indexed bond. In example 2, when all shadow prices are null, the shares of firm 1 can be replaced by a portfolio consisting of shares of firms 2 and 3.

While in the above examples absence of possession value implied that the firm is a perfect competitor, the converse will be established in Theorem 4: possession value for the firm's shares must be null when the firm is perfectly competitive.

When a share is *General Collateral* (GC) there is no particular demand to borrow it or special desire to lend it more than any other share. General collateral securities are substitutable among them and are traded at the common rate ρ^{GC}, the *General Collateral Rate.* General collateral securities are high quality and liquid. The GC rate can be thought of as being driven by the supply and demand for cash and not by the supply and demand for individual assets. It is therefore reasonable to assume that, at least for some agents, the GC has no possession value, as follows:

Assumption 4. *For any quasi-equilibrium $(\bar{x}, \bar{\phi}, \bar{z}, \bar{y}, p, \bar{q}, \bar{\rho})$ induced by production plan $\bar{y}^f$ of firm f, there exists at least one non-initial shareholder $\tilde{i}$ ($\bar{\phi}_0^{\tilde{i}f} = 0$) for whom*

$$(1 + \bar{\rho}^{\mathrm{GC}}) = \frac{\bar{\lambda}_1^{\tilde{i}}}{\sum_s \bar{\lambda}_{2s}^{\tilde{i}}},$$

where $\bar{\lambda}_e^{\tilde{i}}$ are the multipliers corresponding to node e's budget constraint in agent $\tilde{i}$'s problem.

A security is said to be trading *on special* on the securities lending market, whenever the rebate rate associated with the security is below the corresponding GC rate. In other words, a security trades on special if agents pay a higher lending fee when borrowing it than when they borrow general collateral.

We will first note that a perfectly competitive firm can never influence the general collateral rate, that is, ρ^{GC} will be the same regardless of the production plan implemented by the firm (see Lemma 3 in the Appendix).

Second, we note that the actions by a perfectly competitive firm can never determine whether its shares trade on special or not.

Lemma 1 (**A competitive firm cannot induce specialness**). *Suppose assumptions 1 and 4 hold, and that firm f is a perfect competitor. Then the firm's production plans cannot affect whether the firm's shares are on special or not.*

The next result takes a step forward. It says that it is enough to have one agent not interested in reusing the share in some quasi-equilibrium, to infer that the share is not on special (in any quasi-equilibrium and in particular in the financial equilibrium).

Lemma 2. *Under assumption 1, given a quasi-equilibrium $(\bar{x}, \bar{\phi}, \bar{z}, \bar{y}, p, \bar{q}, \bar{\rho})$ for some production plan $\bar{y}^f$, and agent $\tilde{i}$ as in assumption 4, we can say that* ***for any quasi-equilibrium*** *$(\tilde{x}, \tilde{\phi}, \tilde{z}, p, \tilde{q}, \tilde{\rho})(\tilde{y}) \in QE(\tilde{y}^f)$ we have: If $0 < \bar{z}^{\tilde{i}f} < M$ and $\bar{\phi}_1^{\tilde{i}f} = 0$, then $\tilde{\rho}^f = \rho^{\mathrm{GC}}$.*

We are now ready to state an important necessity result. Assuming that the null vector can always be chosen as a production plan we can construct the corresponding quasi-equilibrium (with $\tilde{q}_1^f = 0$) as in Lemma 2 and infer that the shares of a perfectly competitive firm are never on special.

Theorem 4 (**Absence of specialness is necessary for perfect competition**). *Under assumptions 1 and 4, if $0 \in Y^f$, and firm f is perfectly competitive, then in equilibrium (and in any quasi-equilibrium) the firm's shares are not on special ($\tilde{\rho}^f = \rho^{\mathrm{GC}}$).*

(The proof follows from Lemma 2, as already explained).

3.5 Concluding remarks

This paper contributes to two strands of the literature on general equilibrium with incomplete markets. The first is an old strand concerned with the relation between shareholders' unanimity and perfect competition in the markets where the shares of the firm are traded. The second is a recent attempt to capture in a GEI framework the increasing role of Securities Financing Transactions (SFT), which are transactions where securities are used to borrow cash (as in repo markets), or cash is used to borrow securities (as in securities lending markets).

We have proven that, if a firm is a perfect competitor in both the stock and the securities lending markets, its shareholders will be unanimous in their selection of the production plan. Previously, it was believed that short sales and shareholders' unanimity where incompatible, in models where unanimity was driven by perfect competition. We have shown that this is not the case once short sales are modeled the way they are actually done in reality (by borrowing first the shares), and not in the naive way of previous models. Under such reformulation of short sales, what was a counterexample in Makowski (1983) no longer illustrates that unanimity driven by perfect competition is incompatible with allowing for short sales (see Example 3).

The concept of a perfectly competitive firm may appear to be abstract but we relate it to the simpler concept of *no specialness*, which refers to observable variables. The shares of the firm are said to be *on special* if the rebate rate at which they are borrowed is below the general collateral rate. Our results on the relation between possession value, specialness and perfect competition provide important insights that were absent in the previous literature. First, a perfectly competitive firm can never influence whether or not the firm's shares trade on special. Second, assuming that inaction is a possible production choice for the firm, *no specialness* becomes a necessary condition for the firm to be a perfect competitor.

Conversely, we illustrate in Examples 1 and 2 that when the shadow values of all constraints involving the firm's shares are null (which in particular implies no possession value) the firm becomes a perfect competitor.

Finally, Example 4 serves as an important reminder that Makowski's criterion for pricing securities should not be used if the firm is not a perfect competitor. Otherwise, the pricing function shown in Theorem 3 will be correct when the firm selects the plan that maximizes its present value but will otherwise be incorrect.

Appendix

This appendix contains proofs and computations for some of the results and examples.

Proof of Proposition 2. Agents' borrowing positions are bounded by (3.1). This, together with the box constraint implies $\phi_1^{if} \geq -M$ for every i and f. Also, market clearing allocations must satisfy $\phi_1^{if} \leq 1 + IM$. Since, at date 2 there is no borrowing of securities, attainable allocations satisfy $\phi_{2s}^{if} \leq 1$. Finally, agents' consumption is bounded from bellow by 0 and from above by $x_e^i \leq \omega_e^i + \sum_f y_e^f$, for all i and all e. This proof follows that of Proposition 1 in Bottazzi, Luque and Páscoa (2012) with some minor adjustments. We consider a sequence of auxiliary economies in which the box constraint and the lower bound on securities lending positions are relaxed. Step 1 of the proof must be modified as follows.

Rebate rates are decided at date 1, when borrowing contracts are negotiated. Let $R^f \equiv 1/(1+\rho^f)$. Let $\tilde{z}^{if} \equiv (1+\rho^f)q_1^f z^{if}$. The agent's (relaxed) budget constraints can be rewritten as:

$$p_1 x_1^i + \sum_f q_1^f \phi_1^{if} + \sum_f R^f H^f z^{if} \leq p_1 \omega_1^i + \sum_f \phi_0^{if}(q_1^f + p_1 y_1^f) \tag{3.15}$$

$$q_1^f \phi_1^{if} + R^f \tilde{z}^{if} \geq -1/n \ , \forall f \tag{3.16}$$

$$R^f \tilde{z}^{if} \leq q_1^f M + 1/n \ , \forall f \tag{3.17}$$

$$R^f \tilde{z}^{if} \geq -(q_1^f M + 1/n) \ , \forall f \tag{3.18}$$

$$p_{2s} x_{2s}^i + \sum_f q_{2s}^f \phi_{2s}^{if} \leq p_{2s}\omega_{2s}^i + \sum_f \phi_1^{if}(q_{2s}^f + p_{2s} y_{2s}^f) + \sum_f H^f \tilde{z}^{if} \tag{3.19}$$

$$\phi_{2s}^{if} \geq -1/n \ , \forall f \tag{3.20}$$

$$p_{3s} x_{3s}^i \leq p_{3s}\omega_{3s}^i + \sum_f \phi_{2s}^{if} p_{3s} y_{3s}^f \tag{3.21}$$

Equations (3.15), (3.19) and (3.21) are the budget constraints with modified variables at nodes 1, $2s$ and $3s$, respectively. Equations (3.16) and (3.20) are the new box constraints, after the change of variables, and being relaxed. With the change of variables, the relaxed bounds on borrowing positions are given by (3.17) and (3.18). Finally, we must have $x_e^i \geq 0$, $\forall e$.

Auctioneer at date 1 selects $(p_1, q_1, R) \in \Delta^{L+2F}$ to maximize:

$$p_1 \left(\sum_i (x_1^i - \omega_1^i) - \sum_f y_1^f \right) + q_1 \sum_i (\phi_1^i - \phi_0^i) + \sum_f R^f H^f \sum_i \tilde{z}^{if}$$

Auctioneer at node $2s$ selects $(p_{2s}, q_{2s}) \in \Delta^{L+F}$ to maximize:

$$p_{2s} \left(\sum_i (x_{2s}^i - \omega_{2s}^i) - \sum_f y_{2s}^f \right) + q_{2s} \sum_i (\phi_{2s}^i - \phi_1^i)$$

Finally, auctioneer at node $3s$ selects $(p_{3s}) \in \Delta^L$ to maximize

$$p_{3s} \left(\sum_i (x_{3s}^i - \omega_{3s}^i) - \sum_f y_{3s}^f \right) + q_{3s} \sum_i (\phi_{3s}^i - \phi_{2s}^i)$$

In step 2 of the proof we need to modify the argument showing the lower semicontinuity of the consumers' constraint correspondences. This is achieved by finding an interior point in the correspondence for any given set of prices. Let

$$d_{ts} \equiv \begin{cases} 1, \text{ if } p_{ts} y_{ts}^f \geq 0 \ \forall f \\ -\min_f \{p_{ts} y_{ts}^f\}, \text{ otherwise} \end{cases} \quad \text{and} \quad g_{ts} \equiv \begin{cases} 1, \text{ if } p_{ts} y_{ts}^f \leq 0 \ \forall f \\ \max_f \left\{ p_{ts} y_{ts}^f \right\}, \text{ otherwise} \end{cases}$$

If $p_1 \neq 0$. Let $x^i = 0$, $\tilde{z}^{if} = 0$, $\phi_{2s}^{if} = 0$, and

$$\phi_1^{if} = \frac{1}{2F} \min \left\{ p_1 \left(\omega_1^i + \sum_f \phi_0^{if} y_1^f \right), \left(\frac{p_{2s} \omega_{2s}^i}{d_{2s}} \right)_s \right\}$$

If $p_1 = 0$, $q_1 \neq 0$, then there is some firm g such that $q_1^g \neq 0$. Let $x^i = 0$; $\tilde{z}^{if} = 0$, $\forall f$; and let

$$\phi_{2s}^{if} = -\frac{1}{2F} \min \left\{ \frac{p_{3s} \omega_{3s}^i}{g_{3s}}, \frac{F}{n} \right\} \quad \text{and} \quad \phi_1^{if} = -\frac{1}{2F} \min \left\{ \left(\frac{p_{2s} \omega_{2s}^i}{g_{2s}} \right)_s, \left(-F \phi_{2s}^{if} \right)_s \right\}$$

If $p_1 = q_1 = 0$, $R^f \neq 0$. Let $x^i = 0$, $\phi_{2s}^i = 0$. Also, let

$$\phi_1^{if} = \frac{1}{2F} \min \left\{ \left(\frac{p_{2s} \omega_{2s}^i}{d_{2s}} \right)_s \right\} \text{ and } H^f \tilde{z}^{if} = -\frac{1}{2F} \min \left\{ \left(p_{2s} \omega_{2s}^i + \sum_f \phi_1^{if} (p_{2s} y_{2s}^f + q_{2s}^f) \right)_s, \frac{F}{n} \right\}$$

The last modification that we must do to the proof of Proposition 1 in Bottazzi, Luque and Páscoa (2012) is the argument that bounds from below ${}^n\hat{R}^f$, the renormalized rebate rates corresponding to the n-th economy in the sequence (the one in which the budget constraints have been relaxed by subtracting or adding $1/n$). The agent's first order conditions, with respect to $\tilde{z}^{if}$, in the n-th economy in the sequence give us:

$$ {}^{n}\hat{R}^{f} = \sum_{s} \frac{{}^{n}\lambda^{i}_{2s}}{{}^{n}\lambda^{i}_{1}} + \frac{({}^{n}\mu^{if}_{1} - {}^{n}\nu^{if}_{+} + {}^{n}\nu^{if}_{-}){}^{n}\hat{R}^{f}}{{}^{n}\lambda^{i}_{1}H^{f}} \tag{3.22} $$

Where ${}^{n}\lambda^{i}_{e}$ are the Lagrange multipliers corresponding to node e's budget constraint, ${}^{n}\mu^{if}_{1}$ the multipliers corresponding to security f's box constraint at date 1, ${}^{n}\nu^{if}_{+}$ and ${}^{n}\nu^{if}_{-}$ the multipliers corresponding to the upper and lower bounds on borrowing positions. In equilibrium there must always be an agent i such that ${}^{n}\nu^{if}_{+} = 0$, so (3.22) implies that ${}^{n}\hat{R}^{f} \geq \sum_{s} \frac{{}^{n}\lambda^{i}_{2s}}{{}^{n}\lambda^{i}_{1}}$.

□

Computational details for Example 1. Let $\lambda^{i}_{e} \geq 0$ denote the multiplier for the budget constraints at node e, $\mu^{if}_{e} \geq 0$ the multipliers for the corresponding box constraint, $\underline{\nu}^{if} \geq 0$ and $\overline{\nu}^{if} \geq 0$ the multipliers corresponding to the lower and upper bound on z^{if}, respectively. In equilibrium, the following equations, (obtained from the first order conditions with respect to the variables ϕ^{if}_{2s}, ϕ^{if}_{1}, z^{if}, and b^{i}), must hold:

$$ q^{f}_{2s} = \frac{\lambda^{i}_{3s}}{\lambda^{i}_{2s}}\left(y^{f}_{3s}\right) + \frac{\mu^{if}_{2s}}{\lambda^{i}_{2s}}\ , \forall i \tag{3.23} $$

$$ q^{f}_{1} = \sum_{s} \frac{\lambda^{i}_{2s}}{\lambda^{i}_{1}}\left(y^{f}_{2s} + q^{f}_{2s}\right) + \frac{\mu^{if}_{1}}{\lambda^{i}_{1}}\ , \forall i \tag{3.24} $$

$$ (1+\rho^{f}) = (1+\rho_{b}) - \frac{\mu^{if}_{1}}{q^{f}_{1}H^{f}\sum_{s}\lambda^{i}_{2s}} + \frac{(\overline{\nu}^{if} - \underline{\nu}^{if})}{q^{f}_{1}H^{f}\sum_{s}\lambda^{i}_{2s}}\ , \forall i \tag{3.25} $$

$$ (1+\rho_{b}) = \frac{\lambda^{i}_{1}}{\sum_{s}\lambda^{i}_{2s}}\ , \forall i \tag{3.26} $$

If $\rho^{f} = \rho_{b}$, we see that equation (3.25) implies $\mu^{if}_{1} + \underline{\nu}^{if} = \overline{\nu}^{if}$. If $z^{if} < M$ then $\overline{\nu}^{if} = 0$, which implies $\mu^{if}_{1} = 0$ and $\underline{\nu}^{if} = 0$. In this case, equations (3.24) and (3.25) imply:

$$ \sum_{s} \frac{\lambda^{i}_{2s}}{\lambda^{i}_{1}}\left((1+\rho^{f})q^{f}_{1} - \left(y^{f}_{2s} + q^{f}_{2s}\right)\right) = 0 \tag{3.27} $$

Substituting $\lambda^{i}_{ts} = \beta^{t-1}\sigma^{i}_{ts}$ in equation (3.27), we have:

$$ \sum_{s} \beta\sigma^{i}_{2s}\left(\left(1+\rho^{f}\right)q^{f}_{1} - \left(y^{f}_{2s} + q^{f}_{2s}\right)\right) = 0 \tag{3.7} $$

□

Proof of Theorem 3. Let $(\bar{x}, \bar{\phi}, \bar{z}, \bar{y}, p, \bar{q}, \bar{\rho}) \in QE(\bar{y}^f)$ and let $(\bar{\lambda}^i, \bar{\mu}^{if})$ be the multipliers corresponding to the budget and box constraints, when the optimal consumption plan chosen by the agent is $(\bar{x}^i, \bar{\phi}^i, \bar{z}^i)$. Since firm f is a perfect competitor in the financial equilibrium, there is some plan $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*) \in B^i_\star(\bar{y}, p, \bar{q}, \bar{\rho})$ with $\bar{\phi}^{if}_{*e} = 0 \ \forall e$ and $\bar{z}^{if}_* = 0$ (as we saw in the proof of Theorem 1). Since $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*)$ is utility maximizing, and given the concavity assumptions that we have made for the agents' utilities, $(\bar{\lambda}^i, \bar{\mu}^{if})$ also satisfy KT first order conditions at the point $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*)$. From assumption 1 and the first order conditions we have $\bar{\lambda}^i_e = (\partial U^i(\bar{x}^i_*)/\partial x^i_{ec})(1/p_{ec})$. Also, **in equilibrium**, the following relations between share prices and dividends are satisfied:

$$q^f_{2s} = \frac{\lambda^i_{3s}}{\lambda^i_{2s}}\left(p_{3s}y^f_{3s}\right) + \frac{\mu^{if}_{2s}}{\lambda^i_{2s}} \ , \forall f \tag{p.1}$$

$$q^f_1 = \sum_s \frac{\lambda^i_{2s}}{\lambda^i_1}\left(p_{2s}y^f_{2s} + q^f_{2s}\right) + \frac{\mu^{if}_1}{\lambda^i_1} \ , \forall f \tag{p.2}$$

Here (λ^i, μ^{if}) are the **equilibrium** multipliers corresponding to agent i's budget and box constraints in the financial equilibrium $(x, \phi, z, y, p, q, \rho)$. Note that the maximum valuation of the dividends, $\sum_s \frac{\lambda^i_{2s}}{\lambda^i_1}\left(p_{2s}y^f_{2s} + q^f_{2s}\right)$ and $\frac{\lambda^i_{3s}}{\lambda^i_{2s}}\left(p_{3s}y^f_{3s}\right)$, is realized for agents whose box constraint multipliers satisfy $\mu^{if} = 0$. As we saw when we proved Theorem 1, if $\phi^{if}_0 = 0$ then $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*) \in B^i_\star(y, p, q, \rho)$ and so, if $\phi^{if}_0 = 0$, (λ^i, μ^{if}) are also multipliers for $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*)$ and $\lambda^i_e = (\partial U^i(\bar{x}^i_*)/\partial x^i_{ec})(1/p_{ec}) = \bar{\lambda}^i_e$.

From assumption 3 there is some agent i such that $\bar{\phi}^{if}_0 = 0$ and such that $\bar{\phi}^{if}_{2s} > 0$. For this agent, $\bar{\mu}^{if}_{2s} = 0$ and

$$\bar{q}^f_{2s} = \frac{\bar{\lambda}^i_{3s}}{\bar{\lambda}^i_{2s}}\left(p_{3s}\bar{y}^f_{3s}\right)$$

But by our previous discussion, $\bar{\lambda}^i_{3s} = \lambda^i_{3s}$ and $\bar{\lambda}^i_{2s} = \lambda^i_{2s}$, so actually

$$\bar{q}^f_{2s} = \frac{\bar{\lambda}^i_{3s}}{\bar{\lambda}^i_{2s}}\left(p_{3s}y^f_{3s}\right) = \frac{\lambda^i_{3s}}{\lambda^i_{2s}}\left(p_{3s}y^f_{3s}\right) = \upsilon_{2s}(\bar{y})$$

Again, by assumption 3 there is some agent j such that $\bar{\phi}^{jf}_0 = 0$ and such that $\bar{\phi}^{jf}_1 + \bar{z}^{jf} > 0$. For this agent, $\bar{\mu}^{jf}_1 = 0$ and

$$\bar{q}^f_1 = \sum_s \frac{\bar{\lambda}^j_{2s}}{\bar{\lambda}^j_1}\left(p_{2s}\bar{y}^f_{2s} + \bar{q}^f_{2s}\right)$$

But by our previous discussion, $\bar{\lambda}^j_{2s} = \lambda^j_{2s}$ and $\bar{\lambda}^j_1 = \lambda^j_1$, so actually

$$\bar{q}^f_1 = \sum_s \frac{\bar{\lambda}^j_{2s}}{\bar{\lambda}^j_1}\left(p_{2s}\bar{y}^f_{2s} + \bar{q}^f_{2s}\right) = \sum_s \frac{\lambda^j_{2s}}{\lambda^j_1}\left(p_{2s}\bar{y}^f_{2s} + \bar{q}^f_{2s}\right) = \upsilon_1(\bar{y})$$

$\square$

Proof of Theorem 2. If an agent i is an initial shareholder in node e, that is, if $\phi^{if}_{e-1} > 0$, and if firm f is a perfect competitor in the truncated economy $\mathcal{E}(x, \phi, z, y)(e)$, then the same arguments as in the proof of Theorem 1 apply. In that case, if $\pi^f_e(y^f) \equiv p_e y^f_e + q^f_e \geq p_e\bar{y}^f_e + \bar{q}^f_e = \pi^f_e(\bar{y}^f)$, for all $\bar{y}^f \in Y^f(e)$, then $U^i(x^i) \geq U^i(\bar{x}^i)$.

What needs to be proven then, is that $y^f \in \operatorname{argmax}_{\tilde{y}^f \in Y^f(e)} \pi^f_e(\tilde{y}^f)$, where y^f is the plan chosen by initial shareholders at date 1. The only difficulty in proving this lies in the way in which share prices might change in the truncated economies. It will be enough to prove that if e is a node at time t and e' is a successor of e at time $t+1$, then $\pi^f_e(y^f) \geq \pi^f_e(\bar{y}^f)$ implies $\pi^f_{e'}(y^f) \geq \pi^f_{e'}(\bar{y}^f)$, for all $\bar{y}^f \in Y^f(e')$. This result will imply $y^f \in \operatorname{argmax}_{\tilde{y}^f \in Y^f(e)} \pi^f_e(\tilde{y}^f)$, just start at $e = 0$ and apply the result in successive nodes of the relevant branches of the events tree until you reach the root of the sub-tree that defines the truncated economy. The next argument is inspired by Makowski (1983).

Now, let $\lambda^{\hat{i}}$ be the multipliers of a node e reference agent in the quasi-equilibrium where production is y^f, and let $\lambda^{\tilde{i}}$ be the multipliers of a node e reference agent in the quasi-equilibrium when production is $\bar{y}^f$. In particular, we have that (see the proof of Theorem 3):

$$\sum_{u \in e+1} \frac{\lambda^{\tilde{i}}_u}{\lambda^{\tilde{i}}_e}(p_u\bar{y}^f_u + \upsilon^f_u(\bar{y}^f)) \geq \sum_{u \in e+1} \frac{\lambda^j_u}{\lambda^j_e}(p_u\bar{y}^f_u + \upsilon^f_u(\bar{y}^f))\ ,\ \forall j \in I, \tag{3.28}$$

where $e+1$ is the set of successor nodes of e. Then we have that:

$$\begin{aligned}
p_e y_e^f + \upsilon_e^f(y^f) &= p_e y_e^f + \sum_{u \in e+1} \frac{\lambda_u^{\hat{i}}}{\lambda_e^{\hat{i}}}(p_u y_u^f + \upsilon_u^f(y^f)) \\
&\overset{(a)}{\geq} p_e \bar{y}_e^f + \sum_{u \in e+1} \frac{\lambda_u^{\tilde{i}}}{\lambda_e^{\tilde{i}}}(p_u \bar{y}_u^f + \upsilon_u^f(\bar{y}^f)) \\
&\overset{(b)}{\geq} p_e \bar{y}_e^f + \sum_{u \in e+1} \frac{\lambda_u^{\hat{i}}}{\lambda_e^{\hat{i}}}(p_u \bar{y}_u^f + \upsilon_u^f(\bar{y}^f)) \\
&\overset{(c)}{=} p_e y_e^f + \sum_{\substack{u \in e+1 \\ u \neq e'}} \frac{\lambda_u^{\hat{i}}}{\lambda_e^{\hat{i}}}(p_u y_u^f + \upsilon_u^f(y^f)) + \frac{\lambda_{e'}^{\hat{i}}}{\lambda_e^{\hat{i}}}(p_{e'} \bar{y}_{e'}^f + \upsilon_{e'}^f(\bar{y}^f))
\end{aligned}$$

A brief explanation of these inequalities:

(a) follows from our initial hypothesis: $\pi_e^f(y^f) = p_e y_e^f + \upsilon_e^f(y^f) \geq p_e \bar{y}_e^f + \upsilon_e^f(\bar{y}^f) = \pi_e^f(\bar{y}^f)$.

(b) follows from (3.28), as we saw in the proof of Theorem 3, the valuation of reference agents is always greater or equal to the valuation of all other agents.

(c) follows from the fact that both y^f and $\bar{y}^f$ belong in $Y^f(e')$ so we actually have that $y_e^f = \bar{y}_e^f$ and $y_u^f = \bar{y}_u^f$ for $u \in e+1$, $u \neq e'$. These inequalities imply that $\pi_{e'}^f(y) = p_{e'} y_{e'}^f + \upsilon_{e'}^f(y^f) \geq p_{e'} \bar{y}_{e'}^f + \upsilon_{e'}^f(\bar{y}^f) = \pi_{e'}^f(\bar{y})$, as we wanted to prove.

$\square$

Lemma 3 (**The GC rate is unique across quasi-equilibria**). *Suppose assumptions 1 and 3 hold, and that firm f is a perfect competitor. Let $(x, \phi, z, y, p, q, \rho)$ be a financial equilibrium. Given any production plan $\bar{y} \in Y^f$ let $(\bar{x}, \bar{\phi}, \bar{z}, p, \bar{q}, \bar{\rho})(\bar{y}) \in QE(\bar{y}^f)$ be a quasi-equilibrium. Then $\bar{\rho}^{\mathrm{GC}} = \rho^{\mathrm{GC}}$.*

Proof of Lemmas 1 and 3. Let $(\tilde{x}, \tilde{\phi}, \tilde{z}, p, \tilde{q}, \tilde{\rho})(\tilde{y}) \in QE(\tilde{y}^f)$ and $(\bar{x}, \bar{\phi}, \bar{z}, p, \bar{q}, \bar{\rho})(\bar{y}) \in QE(\bar{y}^f)$ be two different quasi-equilibria, each of which corresponds to a different selection of production plan by firm f. We will prove that $\bar{\rho}^{\mathrm{GC}} = \tilde{\rho}^{\mathrm{GC}}$, and that $\tilde{\rho}^f < \tilde{\rho}^{\mathrm{GC}} \Leftrightarrow \bar{\rho}^f < \bar{\rho}^{\mathrm{GC}}$ and $\tilde{\rho}^f = \tilde{\rho}^{\mathrm{GC}} \Leftrightarrow \bar{\rho}^f = \bar{\rho}^{\mathrm{GC}}$.

Suppose agent i is agent $\tilde{i}$ in assumption 4. In particular, $1 + \tilde{\rho}^{\mathrm{GC}} = \tilde{\lambda}_1^i / \sum_s \tilde{\lambda}_{2s}^i$. Let $(\bar{\lambda}^i, \bar{\mu}^{if})$ be the multipliers corresponding to the budget and box constraints, and $(\bar{\nu}_+^{if}, \bar{\nu}_-^{if})$ be the multipliers corresponding to upper and lower bound restrictions on borrowing positions, respectively, when the optimal consumption plan chosen by the agent is $(\bar{x}^i, \bar{\phi}^i, \bar{z}^i)$. Since firm f is a perfect competitor in the financial equilibrium, there is some plan $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*) \in B^i_\star(\bar{y}, p, \bar{q}, \bar{\rho})$ with $\bar{\phi}^{if}_{*e} = 0\ \forall e$ and $\bar{z}^{if}_* = 0$ (as we saw in the proof of Theorem 1). The same applies to quasi-equilibrium $(\tilde{x}, \tilde{\phi}, \tilde{z}, p, \tilde{q}, \tilde{\rho})(\tilde{y})$, where $(\tilde{\lambda}^i, \tilde{\mu}^{if}, \tilde{\nu}^{if}_+, \tilde{\nu}^{if}_-)$ are the multipliers for the agent's problem when he chooses consumption plan $(\tilde{x}^i, \tilde{\phi}^i, \tilde{z}^i)$. There is also an optimal plan $(\tilde{x}^i_*, \tilde{\phi}^i_*, \tilde{z}^i_*) \in B^i_\star(\tilde{y}, p, \tilde{q}, \tilde{\rho})$ with $\tilde{\phi}^{if}_{*e} = 0\ \forall e$ and $\tilde{z}^{if}_* = 0$.

This implies that if a firm f is perfectly competitive, then $\bar{\rho}^f \leq \bar{\rho}^{\mathrm{GC}}$. This is because the multipliers $(\bar{\lambda}^i_*, \bar{\mu}^{if}_*, \bar{\nu}^{if}_{*+}, \bar{\nu}^{if}_{*-})$ associated with the plan $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*)$ are such that $\bar{\nu}^{if}_{*+} = \bar{\nu}^{if}_{*-} = 0$ because $\bar{z}^i_* = 0$ and this implies that:

$$\begin{aligned} 1 + \bar{\rho}^f &= \frac{\bar{\lambda}^i_*}{\sum_s \bar{\lambda}^i_{*2s}} + \frac{-\bar{\mu}^{if}_{*1} + \bar{\nu}^{if}_{*+} - \bar{\nu}^{if}_{*-}}{H^f q^f_1 \sum_s \bar{\lambda}^i_{*2s}} = \frac{\bar{\lambda}^i_*}{\sum_s \bar{\lambda}^i_{*2s}} - \frac{\bar{\mu}^{if}_{*1}}{H^f q^f_1 \sum_s \bar{\lambda}^i_{*2s}} \\ &\leq \frac{\bar{\lambda}^i_*}{\sum_s \bar{\lambda}^i_{*2s}} = 1 + \bar{\rho}^{\mathrm{GC}} \end{aligned}$$

Given the concavity assumptions that we have made for the agents' utilities, $(\bar{\lambda}^i, \bar{\mu}^{if}, \bar{\nu}^{if}_+, \bar{\nu}^{if}_-)$ and consumption plan $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*)$ satisfy the KT first order conditions, as do multipliers $(\tilde{\lambda}^i, \tilde{\mu}^{if}, \tilde{\nu}^{if}_+, \tilde{\nu}^{if}_-)$ and consumption plan $(\tilde{x}^i_*, \tilde{\phi}^i_*, \tilde{z}^i_*)$. As we saw in the proof of Theorem 3, when $\phi^{if}_0 = 0$, as is the case for agent i given assumption 4, we have that $(\tilde{x}^i_*, \tilde{\phi}^i_*, \tilde{z}^i_*) \in B^i_\star(\bar{y}, p, \bar{q}, \bar{\rho})$, and that $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*) \in B^i_\star(\tilde{y}, p, \tilde{q}, \tilde{\rho})$. Again, we will have that $(\bar{\lambda}^i, \bar{\mu}^{if}, \bar{\nu}^{if}_+, \bar{\nu}^{if}_-)$ and consumption plan $(\tilde{x}^i_*, \tilde{\phi}^i_*, \tilde{z}^i_*)$ satisfy the KT first order conditions, as do multipliers $(\tilde{\lambda}^i, \tilde{\mu}^{if}, \tilde{\nu}^{if}_+, \tilde{\nu}^{if}_-)$ and consumption plan $(\bar{x}^i_*, \bar{\phi}^i_*, \bar{z}^i_*)$, and, as in the proof of Theorem 3, $\bar{\lambda}^i = \tilde{\lambda}^i$. This implies, from the first order conditions, $\tilde{\rho}^{\mathrm{GC}} = \bar{\rho}^{\mathrm{GC}} = \rho^{\mathrm{GC}}$. The first order conditions also imply that

$$\begin{aligned} 1 + \tilde{\rho}^f &= \frac{\tilde{\lambda}^i_1}{\sum_s \tilde{\lambda}^i_{2s}} + \frac{1}{H^f \tilde{q}^f_1}\left(\frac{-\tilde{\mu}^{if} + \tilde{\nu}^{if}_+ - \tilde{\nu}^{if}_-}{\sum_s \tilde{\lambda}^i_{2s}}\right) = 1 + \rho^{\mathrm{GC}} + \frac{1}{H^f \tilde{q}^f_1}\left(\frac{-\bar{\mu}^{if} + \bar{\nu}^{if}_+ - \bar{\nu}^{if}_-}{\sum_s \bar{\lambda}^i_{2s}}\right) \\ &= 1 + \rho^{\mathrm{GC}} + \frac{1}{H^f \tilde{q}^f_1}\left(\frac{-\bar{\mu}^{if} + \bar{\nu}^{if}_+ - \bar{\nu}^{if}_-}{\sum_s \tilde{\lambda}^i_{2s}}\right) \end{aligned}$$

and

$$\begin{aligned} 1 + \bar{\rho}^f &= \frac{\bar{\lambda}^i_1}{\sum_s \bar{\lambda}^i_{2s}} + \frac{1}{H^f \bar{q}^f_1}\left(\frac{-\bar{\mu}^{if} + \bar{\nu}^{if}_+ - \bar{\nu}^{if}_-}{\sum_s \bar{\lambda}^i_{2s}}\right) = 1 + \rho^{\mathrm{GC}} + \frac{1}{H^f \bar{q}^f_1}\left(\frac{-\tilde{\mu}^{if} + \tilde{\nu}^{if}_+ - \tilde{\nu}^{if}_-}{\sum_s \tilde{\lambda}^i_{2s}}\right) \\ &= 1 + \rho^{\mathrm{GC}} + \frac{1}{H^f \bar{q}^f_1}\left(\frac{-\tilde{\mu}^{if} + \tilde{\nu}^{if}_+ - \tilde{\nu}^{if}_-}{\sum_s \bar{\lambda}^i_{2s}}\right) \end{aligned}$$

So we have $\bar{\rho}^{f} \left(\substack{=\\<}\right) \rho^{\mathrm{GC}} \Leftrightarrow -\tilde{\mu}^{if} + \tilde{\nu}_{+}^{if} - \tilde{\nu}_{-}^{if} \left(\substack{=\\<}\right) 0 \Leftrightarrow -\bar{\mu}^{if} + \bar{\nu}_{+}^{if} - \bar{\nu}_{-}^{if} \left(\substack{=\\<}\right) 0 \Leftrightarrow \tilde{\rho}^{f} \left(\substack{=\\<}\right) \rho^{\mathrm{GC}}$.

$\square$

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