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Principles of fractional signal processing

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A B S T R A C T

A general framework for fractional signal processing is described and used to derive several interesting formulations. This scheme is based on the Liouville approach that gave rise to the classic Riemann-Liouville and Liouville-Caputo derivatives, here dismissed. Liouville's idea consisted of fractionalizing the transfer function of the basic definition of derivative. Various coherent formulations are introduced from suitable derivative definitions and the corresponding ARMA-type linear systems are obtained. In particular, the Euler discrete-time, classic continuous-time, bilinear (Tustin), and scale-invariant systems are introduced and studied as applications of the proposed scheme. The two-sided derivatives and systems are also considered.

1. Introduction

Born in the troubled times of the Second World War, Signal Processing (SP) has since been linked to major scientific and technical achievements that have changed the world. In an interesting recent article, A. Petropulu et al. [1] present the history of Signal Processing over the last 75 years and its influence on the most important technological advances. The main areas of influence and the underlying mathematical tools are mentioned. People who are really involved in Signal Processing know that it has recovered, reinterpreted, reformulated and made sense of many theories and mathematical tools and, in parallel, introduced new ones, which are now active areas of research. However, strangely, SP has been almost absent in the world of fractional calculus (FC) and the reverse is also true, although fractional behavior is present in many of its traditional areas. An Internet search reveals more than a million references to the term "fractional derivative". Among these, we find around 250 with the expression "fractional signal processing". We wonder why this discrepancy occurs, given that fractional calculus is being increasingly adopted in the 21st century to model many natural and artificial phenomena, since traditional tools cannot give fair descriptions of their behavior [2–7]. This happens, for example, in Physics [8–15], Mechanics [16–19], Viscoelasticity [20–22], Biomedical Engineering [23–28], Civil Engineering [29–31], Electrochemistry (constant phase elements) [32–35] and Electromagnetics [36–40]. Other interesting applications include the long range processes, $1/f$ noise, fractional chaos, fractional Gaussian noise, and fractional Brownian motion (fBm) [41–47]. Since the early 1990's, some scientists and engineers have been working, having in mind the perspective of practical applications [48–51]. It is important to refer the pioneering works of A. Oustaloup's group in mechanical damping/suspension [52] and control [53]. They started with system identification [54,55,43,56–61]. The fractional electrical circuits began being investigated [62–66,33,40,67]. In the last 10 years a number of papers were published describing realizations of fractional systems [68–73]. Other applications include neural networks [74,75], convection [76], fluid kinetic [77], diffusion [78], wind turbines [79], robotics [3], irrigation [80], and image processing [81–84,18,85].

In a sequence of articles published since 2000, M. Ortigueira sought to formalize Fractional Systems Theory by generalizing the classical tools, namely the impulse response (IR), the transfer function (TF), and the frequency response (FR), while maintaining backward compatibility [86,87], proposing that FC would be the tool for modeling systems in the 21st century. A more general overview of fractional systems and their applications was presented in [88,89]. In the last decade, many applications have emerged and important topics such as analysis, modeling, and synthesis have been considered [90]. However, a closer look reveals that there are other integer-order tools that need to be extended to the fractional framework [91]. Not all proposed formulations for fractional operators are suitable for accomplishing this task.

This movement has essentially focused on the case of systems that are continuous-time and shift–invariant, almost forgetting the scale–invariant and discrete–time systems [92,93]. In two recent papers, M. Ortigueira and J. Machado described the state-of-the-art of the compatible fractional system theory, considering both the continuous– [94] and discrete–time [93] formulations, but only for systems described by the usual convolution (shift-invariant systems). The equivalence of the two ARMA type systems was also studied by M. Ortigueira and R. Magin

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[95]. The framework we are proposing makes up for this shortcoming, allowing for a unified formulation.

In this context, we wonder why the SP has a presence in this movement that can be considered negligible. We will try to explain the reasons. Subsequently, we will describe a way of formulating a fractional signal processing (FSP) environment in order to facilitate the work of interested new researchers. Probably the main difficulty experienced by a beginner is the huge number of fractional derivatives and other operators that are described in the main textbooks and articles published in recent years [96,97]. However, most of them are incompatible with the theory and practice of SP. It is therefore important to develop a "guide" to help newcomers enter a somewhat difficult and mathematically complicated world. We will try to address this task and present a framework, using the language of SP, that will allow any researcher to enter the world of fractional processing without feeling that they have left their traditional environment.

We will start with a brief history of fractional calculus, highlighting the role played by J. Liouville (Section 2). The nomenclature and some mathematical tools are described in Section 4. In this paper we will consider linear systems of the ARMA type. In particular, we will work with cases of commensurable order (21). The proposed general structure will be introduced in subsection 5. We will give examples of the use of the structure with discrete-time Euler systems 6.1, continuous-time systems 6.2, discrete-time bilinear systems 6.3, and scale invariant systems 6.4. The application to stochastic processes leads us to introduce a corresponding formulation that suggests the use of bilateral derivatives. Finally, some conclusions are presented in Section 8.

Remark 1. The word "fractional" came into vogue more than 30 years ago and we can find it associated with various mathematical or scientific tools. Perhaps the most interesting for us is the fractional Fourier transform, of which there are several versions, in discrete and continuous time. The most used in applications has been dealt with at length in [98,99]. However, it has no direct relation to the topics we propose to deal with in this article. It is based in a rotation in the time-frequency plane. It is not clear why it is called "fractional". It has already been called "angular Fourier transform" [100], but the most common name is fractional Fourier transform (FrFT).

2. A short history of a cloudy world

Fractional calculus is a generalization of conventional calculus that arose from a Leibniz idea expressed in 1695, in an exchange of letters between him and J. Bernoulli [101]. The idea was taken up by Euler and Lacroix who proposed a formula for the non-integer derivative of the power function. In his work, "Théorie Analytique du Chaleur", Fourier introduced an integral formula which implicitly had the derivative of non-integer order, but he didn't take the idea any further. Abel solved the essentially fractional tautochrone problem, but did not explore it from the point of view of the fractional derivative. The real birth of what is now called "fractional calculus" occurred in 1832. In his first paper [102], Liouville began by generalizing the derivative of an exponential to non-integer orders and proposing a fractional integral (anti-derivative), $D^{-\alpha}$, $\alpha > 0$, defined by [102,103]

$$D^{-\alpha} f(t) = \frac{1}{(-1)^{\alpha}\Gamma(\alpha)} \int_{0}^{\infty} f(t+\tau)\tau^{\alpha-1} d\tau \tag{1}$$

from where he introduced two ways of expressing the α–order derivative

$$D^{\alpha} f(t) = D^{n} D^{\alpha-n} f(t) \tag{2}$$

and

$$D^{\alpha} f(t) = D^{\alpha-n} D^{n} f(t) \tag{3}$$

with $n > \alpha$, $n \in \mathbb{N}$.

It should be noted that, at the time, the bilateral Laplace transform was unknown (only the unilateral one, from the works of Euler and Laplace) and, more importantly, its inverse was also unknown. This prevented Liouville from expressing a function in terms of exponentials in a simple way and forced him to resort to various tricks, not accepted by most mathematicians. On the other hand, Liouville didn't realize that his proposal was a right derivative (in fact, an anti-causal operator). It wasn't until about 20 years later that Serret introduced the causal version immediately adopted by Liouville [104]. In the meantime, a controversy arose involving many mathematicians due to the strange results obtained with Liouville's approach when dealing with power functions. The appearance of Riemann's proposal (1847) [105] allowed Sonine (1872) to formulate a synthesis called the *Riemann-Liouville* (RL) derivative [106,103,107], based on both formulations. This theory remained almost a "standard" until the end of the 20th century, when it was replaced by the (Dzherbashian-)Caputo (C) derivative, a particular case of the second Liouville formula [108,107]. The "solution" was to modify Liouville's formulas to accept functions defined on subsets of the real line.

In the second paper [109], Liouville introduced definitions of derivatives based on fractional versions of the incremental ratio, using the binomial theorem. However, these did not attract much attention. They were recovered by Grünwald (1867) and Letnikov (1868) and applied to causal functions. E. Post (1930) continued the study. M. Ortigueira (2006) generalized these derivatives to the complex domain and introduced a two-sided derivative which gave rise to a unified formulation [110,111].

Other interesting derivative versions were introduced by Hadamard (1892), Weyl (1917), Marchaud (1927), Riesz (1949), and Feller (1952). Gerasimov (1948), Caputo (1967), and Dzhrbashyan and Nersesyan (1968) rediscovered the Liouville's second formula and applied it to causal signals. It became very famous in the last 20 years, under the name "Caputo derivative" (some people use "Dzhrbashyan-Caputo" and others "Liouville-Caputo") [108,107,14]. This was due to the initial conditions problem [112]. The Hilfer derivative is a mixture of RL and C [96]. The Hadamard approach was explored by Samko et al. [103], but only recently has it been used to set up scale-invariant linear systems [92,137,113,114].

Over the last 20 years, many modifications and/or combinations of these derivatives have been introduced, along with other operators claiming to be "fractional derivatives". Some are mere high-pass filters and others are order 1 derivatives in disguise. A list of these and other more interesting fractional derivatives and operators is given in [96,97]. Although many have some mathematical interest, they are completely useless in SP and also incompatible with our traditional tools and practices. The difficulties that a newcomer experiences can be found in the interesting article [115], where they are very well described. However, the solutions proposed are not necessarily the right ones from our point of view.

Most derivatives and fractional operators, in particular, RL and C, have several drawbacks. We are going to list them for the nowadays most used C derivative (for the RL, the situation is identical). Let ${}^{C}D^{\alpha} f(t)$ denote the Caputo derivative [108,107,21] of a given function $f(t)$ having Fourier transform $\mathcal{F} f(t) = F(\omega)$. We can show that [116,117]

- The C derivative of a sinusoid is not a sinusoid. This prevent us from defining the frequency response of linear systems. We find frequently that $\mathcal{F}\left[{}^{C}D^{\alpha} f(t)\right] = (i\omega)^{\alpha} F(\omega)$, that is incorrect for this derivative.
- The C derivative makes a confusion between the Heaviside unit step and the constant function.
- The C derivative of the unit step is null; so, the same happens to the derivative of the Dirac impulse. This prevents us to define correctly the impulse response of linear systems.

- The simultaneous use of the C derivative and the unilateral Laplace transform leads to the introduction of incorrect initial conditions in the systems [118,112].

Another disadvantage of these derivatives, which is important for those who really want to implement fractional systems, is the lack of commutativity/associativity of the orders; in general, $D^{\alpha}D^{\beta}f(t) \neq D^{\alpha}D^{\beta}f(t) \neq D^{\alpha+\beta}f(t)$ which is unacceptable in hardware implementations. These combinations of derivatives are time dependent [119] preventing the systems from being shift invariant. We therefore need a suitable framework to avoid these drawbacks. We will start by considering derivatives and shift-invariant systems. Later, we will present a brief approach to scale-invariant systems.

3. A look into some problems

3.1. About the fractionalization

Consider a continuous-time shift-invariant linear system defined by the simple differential equation

$$\frac{dy(t)}{dt} + ay(t) = b_1\frac{dx(t)}{dt} + b_0x(t) \;\; t \in \mathbb{R}, \tag{4}$$

where $a, b_0, b_1 \in \mathbb{R}$. This equation is the mathematical model of a highpass RC filter. As it is well-known, if the input is an exponential $x(t) = e^{st}$, $t \in \mathbb{R}$, the output is the same exponential multiplied by a constant (eigenvalue), $y(t) = H(s)e^{st}$, called transfer function and given by:

$$H(s) = \frac{b_1 s + b_0}{s + a}. \tag{5}$$

It must be stressed an important fact: $H(s)$ expressed by the fraction in (5) represents two different transfer functions, $H_f(s)$ and $H_b(s)$ corresponding to two different regions in the complex plane defined by $Re(s) > -a$ and $Re(s) < -a$. To simplify, we set $b_1 = 1$ and $b_0 = b \neq a$. Considering $H(s)$ as a Laplace transform, it represents really two different time functions that we call impulse responses. The first is causal

$$h_f(t) = \begin{cases} \delta(t) + (b-a)e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{6}$$

while the second is anti-causal or anticipatory given by

$$h_b(t) = \begin{cases} 0 & t > 0 \\ \delta(t) + (b-a)e^{-at} & t \leq 0 \end{cases} \tag{7}$$

Now, we set several questions:

1. How do we obtain fractional version(s) of (4)?
2. Which derivative(s) should we use?
3. Why not do it from (5)?
4. Shouldn't the obtained fractional relations be generalizations of the above ones?

In many papers, the answers to these questions are based on private taste or considerations linked to initial conditions or even fashion. Very frequently we find causal applications using anti-causal derivatives. The most common approach is to convert (4) into

$$\frac{d^{\alpha}y(t)}{dt^{\alpha}} + ay(t) = b_1\frac{d^{\alpha}x(t)}{dt^{\alpha}} + b_0x(t) \;\; t \in \mathbb{R}, \tag{8}$$

using the RL or C derivatives, left or right. It is interesting to make a comparison of the step responses obtained with these derivatives.

3.2. Discretizations

There are two forms of discretization: using discrete-time approximations for the derivatives or sampling the expressions obtained from the analytical solutions of the equation in continuous time [95]. Currently, the first approach is more useful in computer applications. It basically involves obtaining discrete-time systems that are equivalent to continuous-time systems. There are basically two ways of doing this, which consist of using Euler's or Tustin's substitutions (we'll deal with them more carefully later). The Euler's approaches set $t = nh$, $n \in \mathbb{Z}$, $h \in \mathbb{R}^+$ being the sampling interval, to obtain

$$\frac{y(t) - y(t-h)}{h} + ay(t) = b_1\frac{x(t) - x(t-h)}{h} + b_0x(t) \;\; t \in \mathbb{R}, \tag{9}$$

or

$$\frac{y(t+h) - y(t)}{h} + ay(t) = b_1\frac{x(t+h) - x(t)}{h} + b_0x(t) \;\; t \in \mathbb{R}. \tag{10}$$

Obviously, the first is causal and the second anti-causal. This fact is often overlooked, leading to surprising results. It is important to note that discrete-time systems exist on their own, without the need for a continuous-time system to give rise to them.
The Tustin substitution is preferably done in the transfer function by setting

$$H(z) = \frac{b_1 \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} + b_0}{\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} + a}. \tag{11}$$

These possibilities complicate obtaining fractional systems in discrete time.

4. Nomenclature and mathematical tools

The nomenclature used in fractional calculus does not correspond to the terminology and concepts used in SP. Some of them are completely ignored. Notions such as causal and anti-causal are not normally used. The terms "forward" and "backward" are used in the opposite way to ours. As far as the Laplace transform is concerned, the bilateral one is unknown to many people; the unilateral one is used because of its initial conditions which, as mentioned above, are often wrong. Here we present the most important tools for clarifying the meaning of certain terms.

- System [120,121]
  Any operator that transforms signals into signals. We will often use the terms system and operator interchangeably.
- Causal operator or system [120,121]
  A system is *causal* if the output at any instant depends only on the values of the input and/or output at the present and past instants. The nabla derivative is an example of a causal system.
- Forward
  Normal time flow—from past to future.
- Anti-causal [122]
  A system is *anti-causal*, if the output at any instant depends only on values of the input and/or output at the present and future time instants. An anti-causal system is causal under reverse time flow. The delta derivative is an example of an anti-causal system.
- Anti-derivative
  The operator that is simultaneously the left and right inverse of the derivative will be called *anti-derivative*. It must not be confused with "primitive" that is right inverse only.
- Backward
  Reverse time flow—from future to past.
- Fractional
  Fractional will have the meaning of a non-integer real number.
- Scale-invariant system
  A system is scale-invariant if a stretching or shrinking in the input produces the same stretching/shrinking in the output.
- Shift-invariant system
  A system is shift-invariant if a delay or lead in the input produces the same delay/lead in the output.

Some interesting mathematical tools are:

1. Heaviside unit step
If $t \in \mathbb{R}$ or $t \in h\mathbb{Z},\ h \in \mathbb{R}^+$,
$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0. \end{cases}$$
Any derivative of this function will be called *impulse*.
2. Usual (D'Alembert's or Duhamel's) convolution [120]
$$y(t) = x(t) * g(t) = \int_{-\infty}^{+\infty} x(t-\eta)g(\eta)\mathrm{d}\eta,$$
that is suitable for expressing the input/output relation in shift-invariant systems [94]
3. Mellin convolution [123,92]
$$y(\tau) = x(\tau) \star g(\tau) = \int_0^{\infty} x\left(\frac{\tau}{\eta}\right) g(\eta)\frac{d\eta}{\eta},$$
used for expressing the input/output relation in scale-invariant systems [92]
4. The (bilateral) Laplace transform (LT) is given [124]:
$$\mathcal{L}[h(t)] = H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,\mathrm{d}t,\ s \in \mathbb{C}, \tag{12}$$
that is assumed to converge in some non-void region (region of convergence—ROC) which may degenerate into the imaginary axis, giving rise to the Fourier transform (with $s = i\omega$.)
5. We define the inverse LT by the Bromwich integral
$$h(t) = \mathcal{L}^{-1}F(s) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} H(s)e^{st}\,\mathrm{d}s,\ t \in \mathbb{R}, \tag{13}$$
where $a \in \mathbb{R}$ is called abscissa of convergence. Frequently, we denote by γ the integration path.
6. In a similar way, we define the Mellin transform (MT) by [92]
$$\mathcal{M}[h(t)] = H(v) = \int_0^{\infty} h(t)t^{-v-1}\,\mathrm{d}t,\ v \in \mathbb{C}, \tag{14}$$
with an inverse similar to (13)
$$h(t) = \mathcal{M}^{-1}H(v) = \frac{1}{2\pi i}\int_{\gamma} H(v)t^{v}\,\mathrm{d}v,\ t \in \mathbb{R}^+. \tag{15}$$
The MT in (14) has a parameter sign change $-v \to v$ relatively to the usual MT [123]. However, this definition establishes a better parallelism with the LT, concerning the region of convergence.
7. We define the Z transform [125,122] by
$$\mathcal{Z}[f(n)] = F(z) = \sum_{n=-\infty}^{+\infty} f(n)z^{-n}, \quad z \in \mathbb{C}, \tag{16}$$
with the inverse given by the Cauchy integral
$$f(n) = \frac{1}{2\pi\ i}\oint_c F(z)z^{n-1}\mathrm{d}z, \tag{17}$$
where c is the unit circle. With the change of variable $z = e^{i\omega}$, $-\pi < \omega \leq \pi$, we obtain the discrete-time Fourier transform.
8. The convolution associated with this transform reads
$$x(n) * y(n) = h\sum_{k=-\infty}^{\infty} x(kh)y(nh-kh). \tag{18}$$
9. The binomial theorem
Let $\alpha \in \mathbb{R} - \mathbb{Z}^-$. Then,
$$(1-z)^{\alpha} = \sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}z^k, \quad |z| < 1. \tag{19}$$
We can extend it for negative integer values of α through the Pochhammer symbol for the rising factorial
$$(a)_n = \prod_{k=0}^{n-1}(a+k) = \frac{\Gamma(a+n)}{\Gamma(a)},\ \text{ with } (a)_0 = 1,$$
obtaining
$$(1-z)^{\alpha} = \sum_{k=0}^{\infty}\frac{(-\alpha)^k}{k!}z^k, \quad |z| < 1. \tag{20}$$
10. The general binomial coefficients
$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)\Gamma(\beta+1)}$$
enjoy interesting properties [103,122]:
   - $$\left|\binom{\alpha}{\beta}\right| \leq \frac{A}{\beta^{\alpha+1}},\ \text{for } \beta \to \infty,$$
   with $A > 0$;
   - $$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)n!} = (-1)^n\frac{(-\alpha)_n}{n!}, \quad n \in \mathbb{N};$$
   - $$\binom{\alpha}{n} = (-1)^n\binom{n-\alpha-1}{n};$$
   - $$\binom{\alpha+n}{n} = \binom{\alpha+n}{\alpha};$$
   - $$\sum_{m=0}^{\infty}\binom{\alpha}{m}\binom{\beta}{n-m} = \binom{\alpha+\beta}{n}.$$

We will assume to work with general fractional autoregressive-moving average (FARMA) type linear systems.

Definition 4.1. We define fractional linear systems of ARMA-type through the following equation [94,93]
$$\sum_{k=0}^{N} a_k\mathcal{D}^{\alpha_k}y(t) = \sum_{k=0}^{M} b_k\mathcal{D}^{\beta_k}x(t), \tag{21}$$
where a_k and b_k $(k = 0, 1, \cdots)$ with $a_N = 1$ are real numbers. The operator $\mathcal{D}$ is any derivative or difference that we must define previously. The system orders N and M are any positive integers; for simplicity or stability, we will assume that $N > M$. The positive real numbers α_k and β_k with $k = 0, 1, \cdots$, form strictly increasing sequences.

This general case is usually difficult to treat [126,127]. We will assume commensurate order systems
$$\sum_{k=0}^{N} a_k\mathcal{D}^{k\alpha}y(t) = \sum_{k=0}^{M} b_k\mathcal{D}^{k\alpha}x(t). \tag{22}$$
Without loosing generality, we will consider that $0 < \alpha \leq 1$. Of course, we can study systems with varying parameters, both coefficients and derivative orders. We will not do it here [107,128,113].

5. Framework for fractional linear systems

We are going to present the steps required to define and study FARMA systems. In particular, we will compute the transfer function and, from it, the impulse response, $g(t)$. The step response $r_u(t)$, can also be computed. The steps are:

1. Let $\mathbb{T}$ denote any non-empty closed subset of $\mathbb{R}$ that is called time scale [129] or time sequence [130].
2. In $\mathbb{T}$, we define an operator $\mathcal{D}_f$ which we will generically call *derivative* and which is basically an incremental ratio. We can consider three types: nabla (causal), delta (anti-causal) and theta (bilateral). The treatment of the anti-causal is similar to that of the causal. To avoid repetition, we won't do so. Bilateral will have a different approach, which will be described later.
3. Define the impulse distribution by
$$\delta_f(t) = D_f u(t). \tag{23}$$
In the usual domain ($\mathbb{R}$) with the classic derivative, $\delta_f(t)$ coincides with the Dirac impulse.
4. For such a derivative, compute the pair eigenfunction/eigenvalue from
$$\mathcal{D}_f e_f(t,s) = s\, e_f(t,s), \tag{24}$$
where $t \in \mathbb{T}$ and $s \in \mathbb{C}$. The eigenfunction $e_f(t,s)$ will be called *generalized exponential*.
5. Define the corresponding fractional derivative, $\mathcal{D}^\alpha_f$, through
$$\mathcal{D}^\alpha_f e_f(t,s) = s^\alpha\, e_f(t,s), \tag{25}$$
where $\alpha \in \mathbb{R}$ is the derivative order. The elemental system characterized by $H(s) = s^\alpha$ with a suitable region of convergence is called *differintegrator*.
6. With (25),
   (a) Set $x(t) = e_f(t,s)$ and $y(t) = G(s)e_f(t,s)$;
   (b) Insert them in (21) and use (25) to obtain
$$G(s) = \frac{\sum\limits_{k=0}^{M} b_k s^{\beta_k}}{\sum\limits_{k=0}^{N} a_k s^{\alpha_k}},$$
that is the eigenvalue of the FARMA differential equation and is the so-called *transfer function*. This general case is a bit difficult to treat [127]. For simplicity, we will continue with the (commensurate) TF
$$G(s) = \frac{\sum\limits_{k=0}^{M} b_k s^{k\alpha}}{\sum\limits_{k=0}^{N} a_k s^{k\alpha}}, \tag{26}$$
with s in a suitable region. The roots of $\sum\limits_{k=0}^{M} b_k z^k$ and of $\sum\limits_{k=0}^{N} a_k z^k$ are called pseudo-zeroes and pseudo-poles of the system [131]. These allow us to obtain the partial fraction decomposition
$$G(s) = \sum_{k=1}^{N_p} \frac{R_k}{\left(s^\alpha - p_k\right)^{n_k}}, \tag{27}$$
where $R_k, p_k, n_k,\ k = 1, 2, \cdots$ are residue, pseudo-pole, and multiplicity; N_p is the number of pseudo-poles [122]. Any pseudo-pole can be a true pole, in agreement with α and its location in the complex plane [131].

Remark 2. It can easily be shown that, if n is a positive integer and $\frac{d}{dp}$ represents the ordinary derivative
$$\frac{1}{(s^\alpha - p)^n} = \frac{1}{(n-1)!}\frac{d^{n-1}}{dp^{n-1}}\left[\frac{1}{(s^\alpha - p)}\right]. \tag{28}$$
Therefore, we will consider the $n = 1$ case in the following.

   (c) To obtain the inverse of the elemental fraction $\frac{1}{(s^\alpha - p)}$ we can use the expansions in geometric series:
$$\frac{1}{(s^\alpha - p)} = \begin{cases} -\frac{1}{p}\sum_{k=0}^{\infty} p^{-k} s^{k\alpha} & |s^\alpha| < |p| \\ \sum_{k=0}^{\infty} p^k s^{-(k+1)\alpha} & |s^\alpha| > |p|. \end{cases} \tag{29}$$
The choice of one of these relations depends on the analyticity region the TF (27). The limiting case $|s^\alpha| = |p|$ will not be treated here.
7. With the generalized exponential, above introduced, define a (generalized Laplace) transform,
$$X(s) = \mathcal{L}_f\left[x(t)\right],$$
so that
   - Any function, $x(t)$, can be expressed in terms of the eigenfunction (synthesis equation) by
$$x(t) = \mathcal{L}_f^{-1} X(s) = \frac{1}{2\pi j}\int_\gamma X(s) e_f(t,s) \mathrm{d}s, \tag{30}$$
where γ is in general a closed simple integration path in a region of the complex plane where $X(s)$ is analytic. It may happen we have to make an adjustment in (30) so that
$$\mathcal{L}_f\left[\delta_f(t)\right] = 1,$$
in the whole complex plane.
   - From (30) and using (25), the classic relation
$$\mathcal{L}_D\left[\mathcal{D}^\alpha x(t)\right] = s^\alpha X_D(s) \tag{31}$$
emerges and is valid in a suitable ROC.
   - The previous relation implies that the transforms of input and output of a FARMA system are related through
$$Y(s) = G(s)X(s), \tag{32}$$
for s in the intersection of the ROCs of $X(s)$ and $G(s)$. Therefore,
$$y(t) = \mathcal{L}_f^{-1}\left[G(s)X(s)\right]. \tag{33}$$
This relation defines a convolution operation
$$y(t) = g(t) * x(t), \tag{34}$$
relating $g(t) = \mathcal{L}_f^{-1} G(s)$ and $x(t) = \mathcal{L}_f^{-1} X(s)$.
The function $g(t)$ is called *impulse response* of the FARMA system.
   - From (33), we conclude that the convolution is commutative and, as $s^\alpha G(s)X(s) = G(s)s^\alpha X(s)$, then
$$D^\alpha_f\left[g(t) * x(t)\right] = D^\alpha_f g(t) * x(t) = g(t) * D^\alpha_f x(t). \tag{35}$$
   - Attending to the definition of the impulse (23), we can conclude that the step response is given by
$$r_u(t) = g(t) * u(t) = g(t) * D_f^{-1}\delta_f(t) = D_f^{-1} g(t), \tag{36}$$
since, from (33) and (34) the impulse is the neutral element of the convolution.
   - The analysis equation (direct transform) is defined, for each case, in agreement with the properties of the eigenfunction.

Remark 3. The real values w for which $s = \phi(w)$ lead to $|e_f(t,s)| = 1$ are called frequencies and $G(w)$ is the FR of the system. In such case, the generalized Laplace transform degenerates into a Fourier transform.

The above definition of fractional derivative, *an elemental system having TF equal to* $H(s) = s^\alpha$, brings some properties that are fundamental in applications and are easily deduced from (30) [89]:

1. Associativity of the orders
$$\mathcal{D}_f^\alpha \mathcal{D}_f^\beta x(t) = \mathcal{D}_f^{\alpha+\beta} x(t); \tag{37}$$
This relationship ensures that a given system can be implemented using different structures.
2. Identity
$$\mathcal{D}_f^0 x(t) = x(t); \tag{38}$$
3. Inverse (anti-derivative)
$$\mathcal{D}_f^\alpha \mathcal{D}_f^{-\alpha} x(t) = \mathcal{D}_f^{-\alpha} \mathcal{D}_f^\alpha x(t) = x(t). \tag{39}$$
This result establishes a fundamental difference between anti-derivative and primitive.

These properties, together with the generalized Leibniz rule, constitute a criterion for deciding if a given elemental system can be considered as a fractional derivative [119]. However, in applications to SP, such a relation is not very important.

We are going to apply this framework for several useful systems.

Remark 4. It is important to highlight some interesting facts:

1. Although from a purely mathematical point of view, the derivative order can be a complex number, this cannot be the case from a signal processing point of view, since a complex order leads to non-Hermitian derivatives and systems [122,132].
2. For a non integer order, s^α represents a multivalued expression. To define a function we need to introduce a branchcut line, starting at $s = 0$. The complete location is specified in agreement with the causality.
3. In the case of transient responses dependent on initial conditions, we must be careful. In fact, the initial conditions required depend on the structure of the systems and not on a unilateral transform [118,112] (this subject requires more study).

6. Special interesting cases

6.1. Discrete-time Euler systems

We will study the generalized discrete-time Euler fractional causal linear time-invariant systems. Therefore, we will assume that our domain is the discrete-time (DT) sequence $\mathbb{T} = h\mathbb{Z}$. The positive real parameter, h, is the graininess or sampling interval of the time sequence. Set $t = nh$. We define the *nabla derivative* by:
$$D_\nabla f(t) = \frac{f(t) - f(t-h)}{h} \tag{40}$$
and the *delta derivative* by
$$D_\Delta f(t) = \frac{f(t+h) - f(t)}{h}. \tag{41}$$
As it can be seen, the first derivative is causal, while the second is anti-causal. We will consider the causal case, only. As, $t = nh$, $n \in \mathbb{Z}$, we can sometimes simplify the notation by setting $f(t) = f(nh) = f_n$

Therefore, taking this derivative we deduce immediately:

1. Nabla exponential
The nabla exponential
$$e_\nabla(nh, s) = (1 - sh)^{-n}, \quad n \in \mathbb{Z}, s \in \mathbb{C}, \tag{42}$$
is the eigenfunction of the nabla derivative. To show it, assume that $e_\nabla(0, s) = 1$. We begin by doing
$$\frac{f(t) - f(t-h)}{h} = s f(t)$$
that gives
$$f(t) = \frac{1}{(1-sh)} f(t-h) = \frac{1}{(1-sh)^2} f(t-2h)$$
$$= \frac{1}{(1-sh)^3} f(t-3h) \cdots = \frac{1}{(1-sh)^n} f(t-nh)$$
and
$$e_\nabla(nh, s) = \frac{1}{(1-sh)^n}, \quad n \in \mathbb{Z}.$$
The corresponding complex sinusoid is obtained when s is over the circle $[1 - sh] = 1$. This is called right Hilger circle [129,133].
2. Nabla fractional derivative and differintegrator
We define the nabla fractional derivative through
$$D_\nabla^\alpha e_\nabla(nh, s) = s^\alpha e_\nabla(nh, s) \tag{43}$$
For non integer orders we have to consider a branchcut line starting at $s = 0$ and lying in the left complex half-plane. The differintegrator is the elemental system with TF $H(s) = s^\alpha$, having the Hilger disk as ROC.
3. Frequency response
If s is on the Hilger circle, we obtain
$$D_\nabla^\alpha e^{i\omega n} = \left[\frac{1 - e^{-i\omega}}{h}\right]^\alpha e^{i\omega n}, \tag{44}$$
meaning that the fractional derivative of a sinusoid is a sinusoid. The corresponding FR is
$$H(i\omega) = \left[\frac{1 - e^{-i\omega}}{h}\right]^\alpha = \left[2\frac{\sin(\omega/2)}{h}\right]^\alpha e^{-j\alpha\omega/2} e^{j\alpha\frac{\pi}{2}\operatorname{sgn}(\omega)}.$$
4. Nabla Laplace transform
The transform implied by the (42) is called nabla Laplace transform (NLT) [134]. The analysis equation for the NLT is given by
$$\mathcal{N}[f(nh)] = F_\nabla(s) = h \sum_{n=-\infty}^{+\infty} f(nh) e_\nabla(-nh, s). \tag{45}$$
Its inverse transform (synthesis equation) is given by
$$f(nh) = -\frac{1}{2\pi i} \oint_\gamma F_\nabla(s) e_\nabla((n+1)h, s)\, ds, \tag{46}$$
where the integration path, γ, is any simple closed contour in a region of analyticity of the integrand that includes the point $s = \frac{1}{h}$. The simplest path is a circle with centre at $s = \frac{1}{h}$.
With the substitution $s = \frac{1-z^{-1}}{h}$ we recover the usual Z transform (see Fig. 1). Therefore the associated convolution is (18).
(a) Impulse response
If $|z| > 1$, that implies $|1 - hs| < 1$ (s inside the Hilger circle), then
$$s^\alpha = \left[\frac{1 - z^{-1}}{h}\right]^\alpha = h^{-\alpha} \sum_{k=0}^{\infty} \frac{(-\alpha)_k}{k!} z^{-kh}$$
$$= h^{-\alpha} \sum_{k=0}^{\infty} \frac{(-\alpha)_k}{k!} (1 - hs)^k = \mathcal{N}\left[h^{-\alpha} \frac{(-\alpha)_k}{k!} u(k)\right]$$
and, from (43) and (18).
$$D_\nabla^\alpha x(nh) = h^{-\alpha} \sum_{k=0} \frac{(-\alpha)_k}{k!} x(nh - kh), \tag{47}$$

that is the discrete-time nabla fractional derivative valid for any real (or complex) order. Therefore, the impulse response, $g(n)$, is a direct consequence of (47)

$$g(n) = \mathcal{L}^{-1} s^\alpha = h^{-\alpha} \frac{(-\alpha)_n}{n!} u(nh). \tag{48}$$

(b) Step response
This function is obtained immediately:

$$r_u(n) = \mathcal{L}^{-1} s^{\alpha-1} = h^{1-\alpha} \frac{(1-\alpha)_n}{n!} u(nh). \tag{49}$$

5. Partial fraction inversion
To finish this algorithm for dealing with discrete-time FARMA linear systems, we only need compute the inverse NLT of a partial fraction of the type

$$G(s) = \frac{1}{(s^\alpha - p)}. \tag{50}$$

Attending fact that s must be inside the Higer circle, we have two possibilities

- $|p| > \frac{1}{h}$ which forces us to use the first expansion in (29) and the integration path must be at the intersection of the two regions and include the point $s = \frac{1}{h}$;
- $|p| < \frac{1}{h}$ that implies the use of the second expansion.

We do not consider the limiting case $|p| = \frac{1}{h}$ that can be treated with a slight modification of the integration path. For $|p| > \frac{1}{h}$ we obtain

$$G(s) = -\frac{1}{p} \sum_{k=0} p^{-k} s^{k\alpha}.$$

The ROC is defined by all points distancing from $1/h$ less than $\left|p - \frac{1}{h}\right|$. The NLT inverse is obtained using (46). It gives

$$g(nh) = -\frac{1}{p}\delta(nh) - \frac{1}{p}\sum_{k=1}^{\infty} p^{-k} h^{-k\alpha} \frac{(-k\alpha)_n}{n!} u(n). \tag{51}$$

Remark 5. For numerical computation, we must note that we can write

$$\frac{(-k\alpha)_n}{n!} = \prod_{m=0}^{n-1} \frac{-k\alpha + m}{1+m},$$

that is better from numerical point of view.

Example 6.1. The expression (51) may not lead to a simple closed form. However, in the $\alpha = 1$ case, it gives a simpler expression that will be used in the following example. We have

$$g(nh) = -\frac{1}{p}\delta(nh) - \frac{1}{p}\sum_{k=1}^{\infty} (ph)^{-k} \frac{(-k)_n}{n!} u(n).$$

It is easy to observe that

$$g(0) = -\frac{1}{p}\sum_{k=0}^{\infty} (ph)^{-k} = \frac{h}{1-ph}$$

and, if $n > 0$, we can write

$$\frac{(-k)_n}{n!} = (-1)^n k(k-1)\cdots(k-n+1)$$
$$= (-1)^n \frac{k!}{(k-n)!n!} = (-1)^n \binom{k}{n},$$

and

$$g(nh) = -\frac{1}{p}(-1)^n \sum_{k=0}^{\infty} \binom{k}{n} (ph)^{-k},$$

but

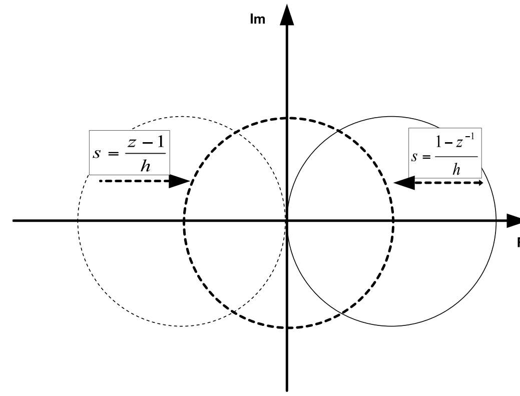

Fig. 1. Transformations from the generalized LT to the ZT.

$$\sum_{k=0}^{\infty} \binom{k}{n} q^k = \frac{q^n}{(1-q)^{n+1}}.$$

Then

$$g(nh) = -\frac{1}{p}(-1)^n \frac{(ph)^{-n}}{(1-(ph)^{-1})^{n+1}}$$
$$= -\frac{1}{p}(-1)^n \frac{ph}{(ph-1)^{n+1}}$$

and, finally

$$g(nh) = \begin{cases} 0 & n < 0 \\ \dfrac{h}{(1-ph)^{n+1}} & n \geq 0. \end{cases} \tag{52}$$

Example 6.2. Consider the simple classic difference equation

$$y(n) + ay(n-1) = x(n),$$

where $a \in \mathbb{R}^+$. Its IR is $g(n) = (-a)^n u(n)$. We want to fractionalize this equation, meaning that we want to find a α–order equation that recovers the above one for $\alpha = 1$. We rewrite the equation as

$$(1+a)y(n) - a\,[y(n) - y(n-1)] = x(n),$$

or

$$-\frac{1+a}{a} y(n) + [y(n) - y(n-1)] = -\frac{1}{a} x(n)$$

that we can write as

$$D_\nabla y(n) - \frac{1+a}{a} y(n) = -\frac{1}{a} x(n)$$

and fractionalized to give

$$D_\nabla^\alpha y(n) - py(n) = -\frac{1}{a} x(n)$$

with

$$p = \frac{1+a}{a}.$$

Using (52) with $h = 1$, we obtain

$$g(nh) = -\frac{1}{a}\frac{1}{(1-p)^{n+1}} u(n),$$

that leads to the expected result.

Remark 6. It is obvious that using the delta derivative (41) we can obtain a slightly different sets of functions and tools, namely a different (delta) Laplace transform [134].

6.2. Continuous-time systems

We consider now the continuous-time fractional causal linear shift-invariant systems. Therefore, we will assume that our domain is the time scale $\mathbb{T}=\mathbb{R}$. The basic elemental system is given by one of the classic derivatives that we represent as

$$\begin{cases} D_f f(t) = \lim_{h\to 0^+} \dfrac{f(t)-f(t-h)}{h} & causal \\ D_b f(t) = \lim_{h\to 0^+} \dfrac{f(t+h)-f(t)}{h} & anti-causal. \end{cases} \tag{53}$$

Remark 7. We can obtain all the required results from the Euler case by letting $h\to 0^+$. To see it, start from $e_\nabla(nh,s)=\dfrac{1}{(1-sh)^n}$ and set $n=\frac{t}{h}$. We have

$$\lim_{h\to 0^+}(1-sh)^{\frac{-t}{h}} = \lim_{h\to 0^+} e^{-t\frac{\ln(1-sh)}{h}} = e^{-st}\frac{\ln(1-sh)}{h}$$

because $\ln(1-sh)\approx sh,\ h\ll 1$.

However, we prefer to deduce everything by using the proposed framework. Therefore, setting $D_f f(t)=\frac{df(t)}{dt}$, the classic derivative, we deduce immediately:

1. Eigenfunction
   The eigenfunction is the usual exponential
   $$e_f(t)=e^{st}, \quad t\in\mathbb{R}, s\in\mathbb{C} \tag{54}$$
2. Fractional derivative (Liouville)
   The fractional derivative was introduced by Liouville (1832) through [102]
   $$\frac{d^\alpha e^{st}}{dt^\alpha}=s^\alpha e^{st}, \tag{55}$$
   with ROC defined by $Re(s)\ge 0,\ s\ne 0$.
3. Associated transform
   The transform implied by the (54) is the usual (bilateral) Laplace transform. Then, by (13)
   $$\frac{d^\alpha x(t)}{dt^\alpha}=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} s^\alpha X(s)e^{st}\,\mathrm{d}s, \tag{56}$$
   with $t\in\mathbb{R}$ and $a>0$
4. Differintegrator
   The differintegrator has TF $H(s)=s^\alpha$, being valid for $Re(s)>0$ or $s=\pm i\omega, \omega\in\mathbb{R}^+$. For non integer orders, we have to consider a branchcut line in the left complex half plane. This gives
   (a) Frequency response
   If $s=i\omega$,
   $$\frac{d^\alpha e^{i\omega t}}{dt^\alpha}=(i\omega)^\alpha e^{i\omega t} \tag{57}$$
   Again, the fractional derivative of a sinusoid is a sinusoid. The corresponding frequency response is
   $$H(i\omega)=(i\omega)^\alpha=|\omega|^\alpha e^{j\alpha\frac{\pi}{2}\,\mathrm{sgn}(\omega)}. \tag{58}$$
   Therefore, the Bode plots are constituted by straight lines. In the amplitude spectrum, the slope is not necessarily multiple of $20dB/dec$
   (b) (Liouville–)Grünwald-Letnikov (GL) derivative
   As
   $$s^\alpha=\left[\frac{1-e^{-sh}}{h}\right]^\alpha = h^{-\alpha}\sum_{k=0}\frac{(-\alpha)_k}{k!}e^{-ksh},$$
   if $Re(s)>0$, then
   $$\frac{d^\alpha x(t)}{dt^\alpha}=\lim_{h\to 0^+} h^{-\alpha}\sum_{k=0}\frac{(-\alpha)_k}{k!}x(t-kh), \tag{59}$$
   that expresses the so-called Grünwald-Letnikov derivative that is valid for any real (or complex) order.
   (c) Impulse response
   The IR reads [89]
   $$\mathcal{L}^{-1}s^\alpha=\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}u(t) \tag{60}$$
   (d) Liouville (anti-)derivatives
   The impulse response and the convolution give another representation for the fractional anti-derivative:
   $$\frac{d^\alpha x(t)}{dt^\alpha}=\int_0^\infty x(t-\tau)\frac{\tau^{-\alpha-1}}{\Gamma(-\alpha)}\mathrm{d}\tau, \tag{61}$$
   that is the causal version of Liouville's first integral formula. He noted that this formula should not be used for positive values of α (the case of the derivative), since the integral can be singular. To avoid the difficulty, Liouville introduced two procedures stated in (2) and (3) ($s^\alpha=s^{\alpha-N}s^N=s^N s^{\alpha-N}, \alpha<N\in\mathbb{N}$) that originated the RL and C derivatives [122]. Alternatively, (61) can be regularized [89,111].
5. Partial fraction inversion
   To finish the algorithm for dealing with continuous-time FARMA linear systems, we only have to compute the inverse LT of the partial fraction (attending to Remark 2)
   $$G_m(s)=\frac{1}{\left(s^\alpha-p_m\right)} \tag{62}$$
   Such function is called α–exponential and reads
   $$g_m(t)=\sum_{k=1}^\infty p_m^{k-1}\frac{t^{k\alpha-1}}{\Gamma(k\alpha)}u(t). \tag{63}$$
   Frequently, it is expressed in terms of the Mittag-Leffler function [108,107]
   $$E_\alpha(p_m t^\alpha)=\sum_{k=0}^\infty p_m^{k-1}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}$$
   The corresponding step response is given by:
   $$r_{u,m}(t)=\sum_{k=1}^\infty p_m^{k-1}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}u(t), \tag{64}$$
   that can be written as
   $$\begin{aligned} r_{u,m}(t) &= \frac{1}{p_m}\left[\sum_{k=0}^\infty p_m^{k-1}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}-1\right]u(t) \\ &= \frac{1}{p_m}\left[E_\alpha(p_m t^\alpha)-1\right]u(t). \end{aligned} \tag{65}$$

   Remark 8. If $\alpha=1$, we obtain the classic impulse and step responses
   $$g_m(t)=\sum_{k=1}^\infty p_m^{k-1}\frac{t^{k-1}}{(k-1)!}u(t)=e^{p_m t}u(t),$$
   and
   $$r_{u,m}(t)=\sum_{k=1}^\infty p_m^{k-1}\frac{t^k}{k!}u(t)=\frac{1}{p_m}\left[e^{p_m t}-1\right]u(t).$$
   These results show the backward coherence of the proposed framework.

Example 6.3. Recently an interesting problem attracted the attention of people interested in fractional circuit theory: the relation involving the charge and voltage in a variable capacitance fractional capacitor. This should generalize the classic relation

$$q(t) = Cv(t), \tag{66}$$

where C is the capacitance expressed in Farad (F). The discussed problem was introduced first by S. Das that proposed the relation

$$q(t) = c(t) * v(t) = \int_{-\infty}^{t} c(\tau)v(t-\tau)\mathrm{d}\tau, \tag{67}$$

where $c(t)$ is a variable capacitance. On the other hand, V. Pandey proposed another way of dealing with the problem as follows

$$q(t) = c(t) * v'(t) = \int_{-\infty}^{t} c(\tau)v'(t-\tau)\mathrm{d}\tau, \tag{68}$$

where $v'(t) = \frac{dv(t)}{dt}$ is the usual derivative. This has given rise to a discussion that seems to be finished with the proposal of the relation [40]

$$\frac{d^{1-\alpha}q(t)}{dt^{1-\alpha}} = c(t)v(t) \tag{69}$$

that really generalizes the classic formulae.

6.3. Discrete-time bilinear systems

The Tustin (bilinear) transformation has long been used to approximate continuous-time systems by discrete-time systems [120,135,125]. However, it is possible to define discrete-time systems based on this transformation without any relation to continuous-time systems [124]. The Tustin transformation establishes a bijection between the left (right) complex half-plane and the inside (outside) of the unit disk. This property allows us to define derivatives in discrete time and corresponding systems that mimic the analogous versions in continuous-time. This feature allows us to adopt the tools and results available in the discrete-time domain for continuous-time fractional systems. In addition, the derivatives and systems required have the important characteristic of being suitable for implementation using the FFT.

Let us assume again that our domain is the time sequence $t \in \mathbb{T} = h\mathbb{Z}$. As above, we will consider the causal case only. We define the order 1 *forward (nabla) bilinear derivative* $\nabla_b x(nh)$ of $x(nh)$ as the solution of the difference equation

$$\nabla_b x(nh) + \nabla_b x(nh-h) = \frac{2}{h}\left[x(nh) - x(nh-h)\right]. \tag{70}$$

With this definition we can deduce the following results.

1. The bilinear exponential
   It is the eigenfunction of the equation (70) [124]

$$e_b(nh,s) = \left(\frac{2+hs}{2-hs}\right)^n, \quad n \in \mathbb{Z},\ s \in \mathbb{C}, \tag{71}$$

   so that

$$\nabla_b e_b(nh,s) = s\, e_b(nh,s).$$

   These relations suggest us to consider the discrete-time exponential function,

$$z^n = \left(\frac{2+hs}{2-hs}\right)^n, \ n \in \mathbb{Z}.$$

   Therefore, we have: work in the s–plane or in the z–plane (see Fig. 2). If we adopt this second case, we define the forward bilinear derivative as an elemental discrete-time system such that

$$\nabla_b z^n = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} z^n. \tag{72}$$

   Remark 9. This change shows that, with this formulation, it is indifferent to work in the plane of the variable s or in the usual "discrete-time" plane of the variable z. We can show that, leaving $h \to 0^+$, the bilinear exponential (71) becomes e^{st} [89,124], giving another justification for Tustin's procedure.

2. Fractional bilinear derivative and differintegrator
   In agreement with our scheme, the bilinear fractional derivative is defined through

$$\nabla_b^\alpha e_b(nh,s) = s^\alpha e_b(nh,s), \tag{73}$$

   with suitable ROC. Therefore, the bilinear differintegrator has TF

$$H_b(z) = \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha, \qquad |z| > 1. \tag{74}$$

   For non-integer orders, we again have to consider a branchcut line. In this case, it is any line that lies inside the unit disk, joining the points $-1, +1$.
3. Frequency response
   We can extend the domain of validity of $H_b(z)$ to include the unit circumference, $z = e^{i\omega n}$, $|\omega| \in (0,\pi)$, with exception of the points $z = \pm 1$. Therefore, with z on the unit circle, we obtain

$$\nabla_b^\alpha e^{i\omega n} = \left[\frac{2}{h}\frac{1-e^{-i\omega}}{1+e^{-i\omega}}\right]^\alpha e^{i\omega n} \tag{75}$$

   meaning that the fractional derivative of a sinusoid is a sinusoid. The corresponding FR is

$$\begin{aligned} H(i\omega) &= \left[\frac{2}{h}\frac{1-e^{-i\omega}}{1+e^{-i\omega}}\right]^\alpha \\ &= \left|\frac{2}{h}\tan(\omega/2)\right|^\alpha e^{j\alpha\frac{\pi}{2}\operatorname{sgn}(\omega)} \quad -\pi < \omega \le \pi. \end{aligned}$$

4. Z transform
   Following the procedure in subsection 6.1, we could use the bilinear exponential to construct a bilinear discrete-time LT. However, formula (71) tells us that $z = \frac{2+hs}{2-hs}$ leads to the ZT, since such transformation sets the unit circle $|z| = 1$ as the image of the imaginary axis in s, independently of the value of h. Therefore, we will use the ZT.
   (a) GL type derivative
   From (73) and (17) we conclude that, if $x(n)$ is a function with ZT $X(z)$, analytic in the ROC defined by $z \in \mathbb{C} : |z| > a,\ a < 1$, then

$$\nabla_b^\alpha x(n) = \frac{1}{2\pi i}\oint_\gamma \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha X(z) z^{n-1}\mathrm{d}z, \tag{76}$$

   with the integration path outside the unit disk. This implies that

$$\mathcal{Z}\left[\nabla_b^\alpha x(n)\right] = \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha X(z), \qquad |z| > 1. \tag{77}$$

   It can be shown [124] that, letting ψ_k^α, $k = 0, 1, \cdots$, be the inverse ZT of $\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha$

$$\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha = \sum_{k=0}^{\infty} \psi_k^\alpha z^{-k},\ |z| > 1,$$

   we define the α-order bilinear nabla derivative as

$$\nabla_b^\alpha x(n) = \left(\frac{2}{h}\right)^\alpha \sum_{k=0}^{\infty} \psi_k^\alpha x(n-k), \tag{78}$$

   analog to the GL derivative.
   (b) Impulse response
   If $\alpha \in \mathbb{R}$ but $\alpha \notin \mathbb{Z}^-$, then [124,93]

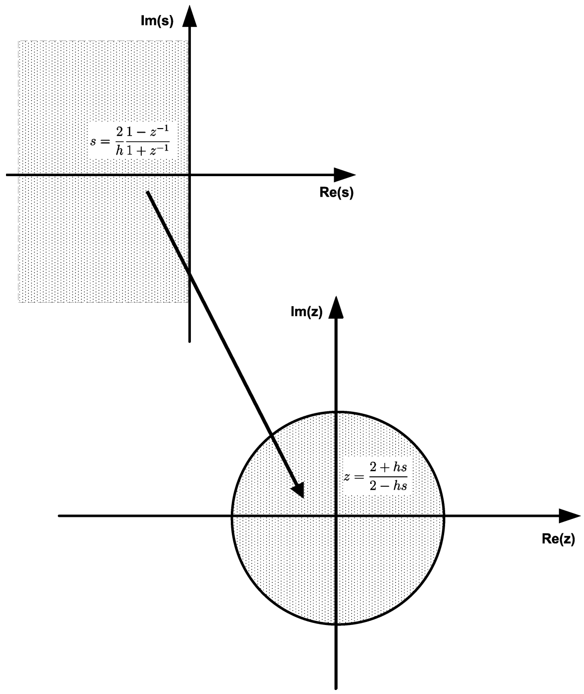

Fig. 2. Transformations from the generalized bilinear LT to the ZT.

$$\psi_k^{\alpha} = (-1)^k \frac{(\alpha)_k}{k!} \sum_{m=0}^{k} \frac{(-\alpha)_m(-k)_m}{(-\alpha-k+1)_m} \frac{(-1)^m}{m!} u(k) \tag{79}$$

and

$$\psi_k^{-N} = \frac{(N)_k}{k!} \sum_{m=0}^{min(k,N)} \frac{(-N)_m(-k)_m}{(-N-k+1)_m} \frac{(-1)^m}{m!} u(k), \tag{80}$$

when $\alpha = -N,\ N \in \mathbb{Z}^+$.
The computation of the step response requires some care [124]. We will not do it here.

5. Partial fraction inversion
To finish this algorithm for dealing with discrete-time FARMA linear systems, we only need compute the inverse transform of the partial fraction

$$G_m(s) = \frac{1}{\left(s^{\alpha} - p_m\right)}, \tag{81}$$

that we can write as

$$G_m(s) = s^{-\alpha} \sum_{k=0} p_m^k s^{-k\alpha} = \frac{1}{p_m} \sum_{k=1} p_m^k s^{-k\alpha},$$

if we assume that $|s| > |p_m|$. It can be written as

$$G_m(z) = \frac{1}{p_m} \sum_{k=1} p_m^k \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^{-k\alpha}. \tag{82}$$

The ZT inverse gives the bilinear version of the α–exponential

$$g_m(nh) = \sum_{k=1} p_m^{k-1} \psi_n^{-k\alpha}\, u(n). \tag{83}$$

Remark 10. We must note that:

1. The ROC is independent on the graininess, h, and we can establish a one to one correspondence between the unit disk, in z, and the left half-plane, in s given by $s = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}$.
2. We can obtain the inverse ZT of $G_m(z)$ through the simple procedure
   (a) Set $z = e^{i\omega}$, $-\pi < \omega \le \pi$, to get a discrete-time Fourier transform $G_m(e^{i\omega})$ of $g_m(nh)$ [122]. Sometimes we must avoid the effect of the branchcut points at $z = \pm 1$ by moving them slightly inside the unit circle.
   (b) Compute $N(e^{i\omega}) = \mathcal{F}([1, -1+\epsilon])$, $\epsilon \ge 0$ and $D(e^{i\omega}) = \mathcal{F}([1, 1-\epsilon])$, where $\mathcal{F}$ represents the discrete Fourier transform that is implemented by the fast Fourier transform algorithm;
   (c) Obtain

$$\begin{aligned} G_m(e^{i\omega}) &= \frac{1}{\left(\left[\frac{2}{T}\frac{N(e^{i\omega})}{D(e^{i\omega})}\right]^{\alpha} - p_m\right)} \\ &= \frac{T^{\alpha} D^{\alpha}(e^{i\omega})}{2^{\alpha} N^{\alpha}(e^{i\omega}) - p_m T^{\alpha} D^{\alpha}(e^{i\omega})} \end{aligned} \tag{84}$$

   that inverted gives $g_m(nh)$.

Example 6.4. To get an idea of the behavior and make a fair comparison, we present illustrations of Bode diagrams and step responses of a second-order system using three models: usual continuous time (black), Euler in discrete time (red) and bilinear (blue) (Fig. 3). The system is described by the equation

$$D^{2\alpha} y(t) + \zeta\omega_n D^{\alpha} y(t) + \omega_n^2 = \omega_n^2 x(t),$$

where $\zeta = 0.1$ and $\omega_n = \sqrt{2}$. The graininess was set to $h = 0.005$. As observed the bilinear model is better than the Euler, although the jump at the right hand extremity of the band, due to the branchcut point at $z = -1$. In Figs. 4 and 5 we illustrate the time behavior through the step responses for two different values of the natural frequency ω_n. Perhapps surprisingly, the results are very similar (see [95] for a deeper study of this subject). It is important to refer that in the continuous-time case, we proceeded as follows:

1. The transfer function was decomposed in two partial fractions, corresponding to a pair of conjugate pseudo-poles;
2. From (65), we can show that the step response corresponding to a given partial fraction $\frac{1}{s^{\alpha}-p}$, aside a constant, is the difference between the Mittag-Leffler function and the unit step.
3. For computing the Mittag-Leffler function we used the algorithm described in [136].

In the discrete-time cases, we used the fast Fourier transform to compute the IR that was accumulated to give the step response, using the corresponding difference equations.

6.4. Scale-invariant systems

Another interesting example of our framework is given by the scale-invariant systems. Let $q > 1$ and $\tau \in \mathbb{R}^+$. Define a *stretching derivative* by

$$\mathfrak{D}_{s+} x(\tau) = \lim_{q \to 1^+} \frac{x(\tau) - x(\tau q^{-1})}{\ln q}. \tag{85}$$

Similarly, we can define a *shrinking derivative* by

$$\mathfrak{D}_{s-} x(\tau) = \lim_{q \to 1^+} \frac{x(\tau q) - x(t)}{\ln q}. \tag{86}$$

We will continue with the first. Our framework gives

1. Eigenfunction
   The eigenfunction is the usual power function [92]

$$e_v(t) = \tau^{v}, \tag{87}$$

   where $v \in \mathbb{C}$

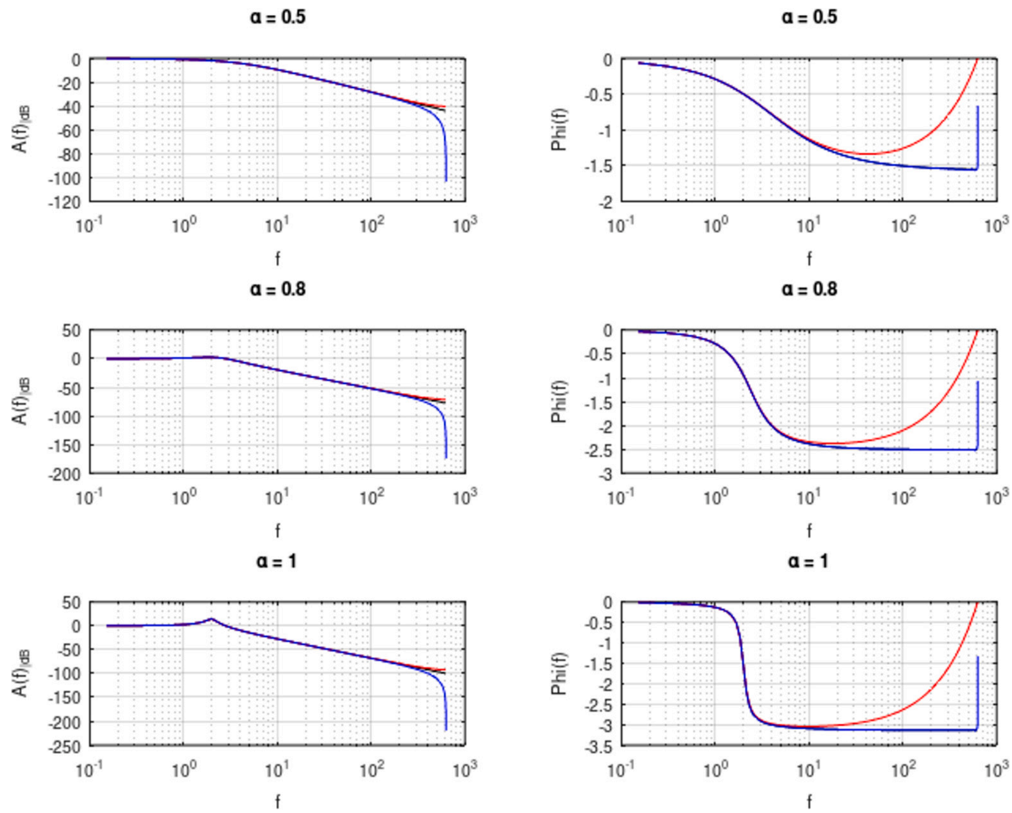

Fig. 3. Bode diagrams for $\alpha = 0.5, 0.8, 1.$, corresponding to continuous-time (black), Euler (red), and bilinear (blue).

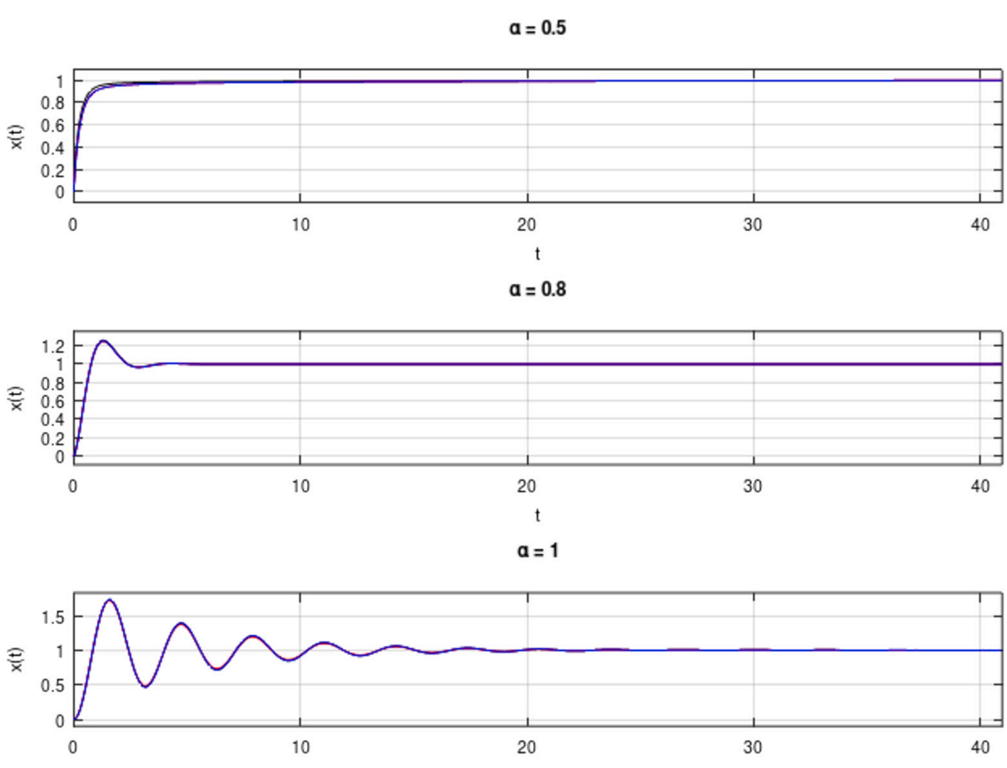

Fig. 4. Step responses for $\alpha = 0.5, 0.8, 1$, with $\zeta = 0.1$ and $\omega_n = \sqrt{2}$, corresponding to continuous-time (black), Euler (red), and bilinear (blue).

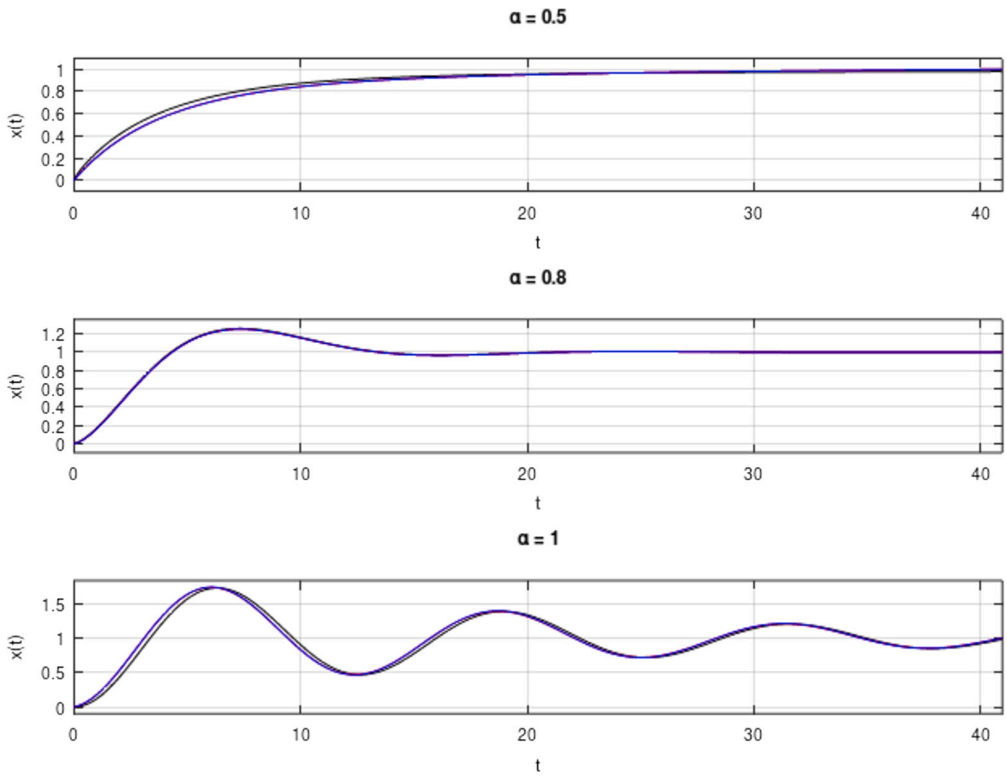

Fig. 5. Step responses for $\alpha = 0.5, 0.8, 1$, with $\zeta = 0.1$ and $\omega_n = 0.5$, corresponding to continuous-time (black), Euler (red), and bilinear (blue).

2. Associated transform
   The transform implied by the (87) is the usual Mellin transform (14). Then, by (15)

$$\mathfrak{D}_{s+}x(\tau) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \upsilon X(\upsilon)\tau^{\upsilon}\,\mathrm{d}\upsilon, \tag{88}$$

   with $\tau \in \mathbb{R}^+$ and $a > 0$.
3. Fractional derivative (Hadamard)
   The scale fractional derivative was introduced by Hadamard (1892) through [103,92]

$$\mathfrak{D}^{\alpha}_{s+}\tau^{\upsilon} = \upsilon^{\alpha}\tau^{\upsilon}, \tag{89}$$

   with suitable ROC, leading to

$$\mathfrak{D}^{\alpha}_{s+}x(\tau) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \upsilon^{\alpha} X(\upsilon)\tau^{\upsilon}\,\mathrm{d}\upsilon, \tag{90}$$

   with $\tau \in \mathbb{R}^+$ and $a > 0$.
4. Differintegrator
   Again, the differintegrator has TF $H(s) = \upsilon^{\alpha}$, being valid for $Re(\upsilon) > 0$ or $\upsilon = \pm i\omega, \omega \in \mathbb{R}^+$, leading to
   (a) Frequency response
   If $\upsilon = i\omega$, obtaining complex scale-periodic sinusoid

$$\mathfrak{D}^{\alpha}_{s+}e^{i\omega\ln(\tau)} = (i\omega)^{\alpha}e^{i\omega\ln(\tau)}. \tag{91}$$

   So, the fractional derivative of a scale-periodic sinusoid is a scale-periodic sinusoid. The corresponding FR is

$$H(i\omega) = (i\omega)^{\alpha} = |\omega|^{\alpha}e^{i\alpha\frac{\pi}{2}\,\mathrm{sgn}(\omega)}.$$

   (b) GL-type scale derivative [92]
   As

$$\begin{aligned}\upsilon^{\alpha} &= \lim_{q\to 1^+}\left[\frac{(1-q^{-\upsilon})}{\ln q}\right]^{\alpha} \\ &= \lim_{q\to 1^+}\sum_{n=0}^{\infty}\frac{(-\alpha)_n}{n!}q^{-n\upsilon}, \quad Re(\upsilon) > 0,\end{aligned}$$

   then

$$\mathfrak{D}^{\alpha}_{s+}x(\tau) = \lim_{h\to 0^+} h^{-\alpha}\sum_{n=0}^{\infty}\frac{(-\alpha)_n}{n!}x(\tau q^{-n}), \tag{92}$$

   that expresses the so-called stretching GL-type derivative.
   (c) Impulse response [92]
   The impulse response is the derivative of an impulse at $\tau = 1$, $\delta(\tau - 1)$. If $Re(\upsilon) > 0$, we have

$$\mathcal{M}^{-1}\upsilon^{\alpha} = \frac{\ln^{-\alpha-1}(\tau)}{\Gamma(-\alpha)}u(\tau - 1). \tag{93}$$

   (d) Hadamard (anti-)derivatives
   The impulse response and the convolution give another representation for the fractional anti-derivative. Let $\alpha < 0$ and $\tau \in \mathbb{R}^+$. The relation stated in (93) used in the Mellin convolution leads to the scale anti-derivative given by

$$\begin{aligned}\mathfrak{D}^{\alpha}_{s+}x(\tau) &= \frac{1}{\Gamma(-\alpha)}\int_{1}^{\infty} x(\tau/\eta)\ln^{-\alpha-1}(\eta)\frac{\mathrm{d}\eta}{\eta} \\ &= \frac{1}{\Gamma(-\alpha)}\int_{0}^{\tau} x(\eta)\ln^{-\alpha-1}(\tau/\eta)\frac{\mathrm{d}\eta}{\eta}.\end{aligned} \tag{94}$$

   These relations express the usually called Hadamard integrals [103,107]. As in the Liouville derivative case, these integrals become singular when the derivative order is positive. To avoid the problem, we can regularize them or adopt Liouville procedures based on the relations ($\upsilon^{\alpha} = \upsilon^{\alpha-N}\upsilon^{N} = \upsilon^{N}\upsilon^{\alpha-N}, \alpha < N \in \mathbb{N}$) [92].

5. Partial fraction inversion
The treatment of scale-invariant FARMA linear systems is similar to those in the previous sub-sections [137]. We reduced the problem to the computation of the inverse MT of the partial fraction

$$G_m(v) = \frac{1}{\left(v^\alpha - p_m\right)}. \tag{95}$$

It gives

$$g_m(\tau) = \sum_{n=0}^{\infty} p^n \frac{\ln^{n\alpha}(\tau)}{\Gamma(n\alpha+1)} u(\tau-1), \tag{96}$$

which by analogy can be called α-log-exponential function [137].

Example 6.5. The classical Black-Scholes equation reads [138]

$$\frac{\partial v(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v(S,t)}{\partial S^2} + rS\frac{\partial v(S,t)}{\partial S} - rv(S,t) = 0, \quad 0 < S < +\infty, \quad 0 < t < T \tag{97}$$

where $v(S,t)$ is the price of the option as a function of stock price S and time $t; r$ is the risk-free interest rate, σ is the volatility of the stock, and T is the expiry date. The equation is subject to the terminal value condition

$$v|_{t=T} = \max(S-K,0) \tag{98}$$

for a call option, or

$$v|_{t=T} = \max(K-S,0) \tag{99}$$

for a put option, where K is the strike price of the option. Looking into the structure of the BSE, we note that

1. The equation must express a causal system;
2. The presence of terms with the form $S^k \frac{\partial^k v}{\partial S^k}$, $k = 1,2$, suggest the introduction of scale derivatives, instead, since such terms are scale independent;
3. Defining a scale variable by $\theta = \frac{S}{K}$, the relations (98) and (99) can be substituted by $\theta > 1$ and $\theta < 1$, expressing stretching and shrinking operations, respectively.

These considerations lead us to rewrite the BSE in the form

$$\frac{\partial v(\theta,t)}{\partial t} + \frac{1}{2}\sigma^2 \left[\mathfrak{D}^2_{s,\theta} v(\theta,t) - \mathfrak{D}_{s,\theta} v(\theta,t)\right] + r\mathfrak{D}_{s,\theta} v(\theta,t) - rv(\theta,t) = 0, \quad t \in \mathbb{R}, \theta \in \mathbb{R}^+,$$

that can be set as

$$\frac{\partial v(\theta,t)}{\partial t} + \frac{1}{2}\sigma^2 \mathfrak{D}^2_{s,\theta} v(\theta,t) + \left[r - \frac{1}{2}\sigma^2\right] \mathfrak{D}_{s,\theta} v(\theta,t) - rv(\theta,t) = 0, \quad t \in \mathbb{R}, \theta \in \mathbb{R}^+. \tag{100}$$

It is important to remark that the initial conditions in the scale must be taken at $\theta = 1$. Obviously, we have two ways of fractionalizing this equation, one relatively to time, other to scale, leading to

$$\frac{\partial^\alpha v(\theta,t)}{\partial t^\alpha} + \frac{1}{2}\sigma^2 \mathfrak{D}^{2\beta}_{s,\theta} v(\theta,t) + \left[r - \frac{1}{2}\sigma^2\right] \mathfrak{D}^{\beta}_{s,\theta} v(\theta,t) - rv(\theta,t) = 0, \quad t \in \mathbb{R}, \theta \in \mathbb{R}^+, \tag{101}$$

where $\alpha, \beta > 0$.

7. Fractional stochastic processes and the two-sided derivatives

In the previous section, we dealt with unilateral derivatives. In particular, we studied the causal and stretch derivatives and the corresponding systems. However, there are other derivatives that appear in some applications. If the independent variable is, for example, space instead of time, we find no reason to use the causal derivative, discarding the anti-causal one. On the other hand, we can have systems that depend simultaneously on values to the left and to the right. Thus, we are led to consider the derivative:

$$D_c f(t) = \lim_{h\to 0^+} \frac{f(t+h/2) - f(t-h/2)}{h} \tag{102}$$

or

$$\mathfrak{D}_{cs} x(\tau) = \lim_{q\to 1^+} \frac{x(\tau q^{1/2} - x(\tau q^{-1/2})}{\ln q} \tag{103}$$

We will continue with the first only. The development from the second is similar.

In [139] we introduced two fractional derivatives resulting from two generalizations of (102) and showed that they could be obtained by combining a causal and an anti-causal derivatives. In terms of the Bode diagrams we were generalizing (58) by introducing an asymmetry parameter

$$H(i\omega) = |\omega|^\alpha e^{j\theta\frac{\pi}{2}\,\mathrm{sgn}(\omega)}. \tag{104}$$

This approach has proved important in the treatment of stochastic processes [140,46,94]. To see how, consider a continuous-time linear system with TF given by $G(s)$, with no poles on the imaginary axis, with $x(t)$ as input into the system a stationary stochastic process with autocorrelation function

$$R_{xx}(t) = E\left[x(\tau+t)x(\tau)\right], \tag{105}$$

so that the output, $y(t) = g(t) * x(t)$, has autocorrelation function

$$R_{yy}(t) = g(t) * g(-t) * R_{xx}(t). \tag{106}$$

Therefore,

$$S_{yy}(s) = G(s)G(-s)S_{xx}(s). \tag{107}$$

In the general fractional case, $S_{yy}(s)$ has a void ROC, since $G(s)$ exists only for $Re(s) > 0$. This leads us to define

$$S_{yy}(i\omega) = \lim_{s\to i\omega} G(s)G(-s)\cdot S_{xx}(i\omega) = |G(i\omega)|^2 S_{xx}(i\omega), \tag{108}$$

that relates the input and output power spectral densities, $S_{xx}(i\omega)$ and $S_{yy}(i\omega)$.

Assume that the input is white noise, $w(t)$, with variance σ^2 and autocorrelation function

$$R_{ww} = \sigma^2 \delta(t). \tag{109}$$

Therefore, the output spectrum of a linear system is

$$S_{yy}(i\omega) = \sigma^2 \left|G(i\omega)\right|^2, \tag{110}$$

stating an important relation suitable for stochastic modeling and identification. Let us go back to equation (107) and substitute there the expression for the TF stated in (26). We have

$$S_{yy}(s) = \frac{\sum_{k=0}^{M} b_k s^{k\alpha}}{\sum_{k=0}^{N} a_k s^{k\alpha}} \cdot \frac{\sum_{k=0}^{M} b_k (-s)^{k\alpha}}{\sum_{k=0}^{N} a_k (-s)^{k\alpha}} S_{xx}(s). \tag{111}$$

Letting $s \to i\omega$, we have

$$\sum_{k=0}^{N}\sum_{m=0}^{N} a_k a_m (i\omega)^{k\alpha}(-i\omega)^{m\alpha} S_{yy}(i\omega) = \sum_{k=0}^{M}\sum_{m=0}^{M} b_k b_m (i\omega)^{m\alpha}(-i\omega)^{k\alpha} S_{xx}(i\omega). \tag{112}$$

This relation suggests we introduce a two-sided derivative D^γ_θ to obtain a differential equation

$$\sum_{k=0}^{N}\sum_{m=0}^{N} a_k a_m D^{k\alpha+m\alpha}_{k\alpha-m\alpha} R_{yy}(t) = \sum_{k=0}^{M}\sum_{m=0}^{M} b_k b_m D^{k\alpha+m\alpha}_{k\alpha-m\alpha} R_{xx}(t), \tag{113}$$

that defines a new linear system relating the autocorrelation functions of input and output signals. The new derivative is defined by

$$D^{\alpha+\beta}_{\alpha-\beta} f(t) = \mathcal{F}^{-1}\left[(i\omega)^{\alpha}(-i\omega)^{\beta} F(i\omega)\right], \tag{114}$$

that has the FR

$$\Psi^{\alpha+\beta}_{\alpha-\beta}(i\omega) = (i\omega)^{\alpha}(-i\omega)^{\beta} = |\omega|^{\alpha+\beta}\, e^{j(\alpha-\beta)\frac{\pi}{2}\operatorname{sgn}(\omega)}, \tag{115}$$

so that the FR of the above system (113) is

$$|G(i\omega)|^2 = \frac{\sum_{k=0}^{M}\sum_{m=0}^{M} b_k b_m\, |\omega|^{(k+m)\alpha}\, e^{j(k-m)\frac{\alpha\pi}{2}\operatorname{sgn}(\omega)}}{\sum_{k=0}^{N}\sum_{m=0}^{N} a_k a_m\, |\omega|^{(k+m)\alpha}\, e^{j(k-m)\frac{\alpha\pi}{2}\operatorname{sgn}(\omega)}}. \tag{116}$$

The two-sided derivatives were formally introduced in [139,89] and were unified in a formulation that included the one-sided (forward/backward) Grünwald-Letnikov and Liouville derivatives [141, 110].

Definition 7.1. Let $f(t)$, $t \in \mathbb{R}$, be a real function and $\gamma, \theta \in \mathbb{R}$ two real parameters. We define a two-sided GL type FD of $f(t)$ by

$$D^{\gamma}_{\theta} f(t) = \lim_{h\to 0^+} h^{-\gamma} \sum_{n=-\infty}^{+\infty} \frac{(-1)^n \Gamma(\gamma+1) f(t-nh)}{\Gamma(\frac{\gamma+\theta}{2}-n+1)\Gamma(\frac{\gamma-\theta}{2}+n+1)}, \tag{117}$$

where γ is the derivative order, and θ is an asymmetry parameter. The situation corresponding to $\gamma = -N$, $N \in \mathbb{N}$ deserves particular attention [110,122].

For absolutely or square integrable functions, the FT of (117) is given by

$$\mathcal{F}\left[D^{\gamma}_{\theta} f(t)\right] = \Psi^{\gamma}_{\theta}(\omega)\mathcal{F}\left[f(t)\right], \tag{118}$$

where $\Psi^{\gamma}_{\theta}(\omega)$ is defined in (115). This derivative enjoys also an integral representation and has some interesting particular cases like [110]: Riesz potential, Feller derivative, and Hilbert transform. Frequently, they are used to model systems defined by partial differential equations [107,14,21,142] and in image processing [82,143].

Example 7.1. Consider the partial differential equation

$$\frac{\partial\psi(x,t)}{\partial t} = \frac{\partial\psi(x,t)}{\partial x}, \quad x \in \mathbb{R}, t \geq 0,$$

which is classic model to the, very important, phenomenon of the diffusion in one dimension. When fractionalizing this equation, it was obvious that we should not use the same derivative in both time and space. While the time derivative must express the causality, the space derivative has to be bilateral. This was recognized a long time ago. After some different attempts, the following version, expressed with our derivative definitions, is generally accepted Let $0 \leq \alpha \leq 2$, $0 < \beta \leq 2$, and $-2 < \theta \leq 2$, with $t, x \in \mathbb{R}$. We define the fractional diffusion equation by [21,142]:

$$\frac{\partial^{\beta}}{\partial t^{\beta}} u(x,t) = D^{\alpha}_{\theta,x} u(x,t), \quad x \in \mathbb{R}, t \geq 0. \tag{119}$$

Example 7.2. The fractional Brownian motion (fBm) was studied first by Mandelbrot and Van Ness [41] that suggested it as a model for non-stationary signals, having stationary increments. These are suitable to model phenomena exhibiting long range or $1/f^{\alpha}$ dependences. Let $H \in (0,1)$ be the so-called Hurst parameter [144] and let $b_0 \in \mathbb{R}$. The fBm, $B_H(t)$, with parameter H was defined by

$$B_H(t) - B_H(0) = \frac{1}{\Gamma(H+1/2)}\left\{\int_{-\infty}^{0}\left[(t-\tau)^{H-1/2} - (-\tau)^{H-1/2}\right] \mathrm{d}B(\tau) + \int_{0}^{t} (t-\tau)^{H-1/2}\, \mathrm{d}B(\tau)\right\}, \tag{120}$$

where $B_H(0) = b_0$, and $B(t)$ is the standard Brownian motion that, although not differentiable, can have assigned to it a generalised derivative, the *white noise*, $w(t)$, $t \in \mathbb{R}$, so that $\mathrm{d}B(t) = w(t)\,\mathrm{d}t$. The definition in (120) is not easy to interpret. However, we can show that the fBm can be defined in a way similar to the classic Brownian motion [140,46]:

$$v_H(t) = B_H(t) - B_H(0) = \int_0^t r_{\alpha}(\tau)\,\mathrm{d}\tau, \tag{121}$$

where

$$r_{\alpha}(t) = \frac{d^{\alpha} w(\tau)}{d\tau^{\alpha}} = \frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{t} w(\tau)(t-\tau)^{-\alpha-1}\,\mathrm{d}\tau, \tag{122}$$

is the Liouville forward derivative of order $-H+1/2$ (92). As $H \in (0,1)$, then $\alpha \in (-1/2, 1/2)$. Expression (121) is similar to the current definition of Brownian motion, provided that $\alpha = 0$ (i.e. that $H = 1/2$). Moreover, the autocorrelation function $R_r(t,\tau) = E\left[r_{\alpha}(t+\tau) r_{\alpha}(\tau)\right]$ depends only on t, not on τ. Therefore, the fractional noise $r_{\alpha}(t)$ is a wide sense stationary stochastic process and its autocorrelation function is

$$R_{\alpha}(t) = \sigma^2 \frac{|t|^{-2\alpha-1}}{2\Gamma(-2\alpha)\cos(\alpha\pi)}, \tag{123}$$

that is the impulse response of a bilateral system.

8. Conclusions

The basic principles underlying fractional signal processing were introduced. The fundamental idea is to formalize a general framework that supports the development of tools and systems, while maintaining complete coherence with classical signal processing methods, in order to create a structure for modeling many fractional systems we find in practice. We have shown that the proposed scheme allows us to obtain different structures for dealing with continuous/discrete shift-invariant or scale-invariant systems. This opens many doors for applications.

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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