Endomorphisms of semigroups of order-preserving partial transformations

Vítor H. Fernandes* and Paulo G. Santos

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Abstract

We characterize the monoids of endomorphisms of the semigroup of all order-preserving partial transformations and of the semigroup of all order-preserving partial permutations of a finite chain.

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1 Introduction

Let n be a natural number. Let Ω_n be a finite set with n elements, say $\Omega_n = \{1, 2, \ldots, n\}$. We denote by $\mathcal{PT}_n$ the monoid (under composition) of all partial transformations of Ω_n. The submonoids of $\mathcal{PT}_n$ of all full transformations and of all partial permutations are denoted by $\mathcal{T}_n$ and $\mathcal{I}_n$, respectively. Also, denote by $\mathcal{S}_n$ the symmetric group on Ω_n, i.e., the subgroup of $\mathcal{PT}_n$ of all permutations of Ω_n. For $s \in \mathcal{PT}_n$, we denote the domain of s by $\mathrm{Dom}(s)$, the image of s by $\mathrm{Im}(s)$, the kernel of s by $\mathrm{Ker}(s)$, and the set of fix points of s by $\mathrm{Fix}(s)$, i.e., $\mathrm{Fix}(s) = \{x \in \mathrm{Dom}(s) \mid xs = x\}$.

Let us consider Ω_n endowed with the usual (linear) order. An element $s \in \mathcal{PT}_n$ is said to be *order-preserving* if $x \leq y$ implies $xs \leq ys$, for all $x, y \in \mathrm{Dom}(s)$. Denote by $\mathcal{PO}_n$ the submonoid of $\mathcal{PT}_n$ of all order-preserving partial transformations. As usual, we denote by $\mathcal{O}_n$ the monoid $\mathcal{PO}_n \cap \mathcal{T}_n$ of all full transformations that preserve the order and by $\mathcal{POI}_n$ its injective counterpart, i.e., the inverse monoid $\mathcal{PO}_n \cap \mathcal{I}_n$ of all order-preserving partial permutations of Ω_n.

Semigroups of order-preserving transformations have been considered in the literature for over than fifty years. Starting in 1962, Aĭzenštat [1, 2] gave a presentation for $\mathcal{O}_n$, from which it can be deduced that, for $n > 1$, $\mathcal{O}_n$ only has one non-trivial automorphism, and characterized the congruences of $\mathcal{O}_n$. Also in 1962, Popova [26] exhibited a presentation for $\mathcal{PO}_n$. In 1971, Howie [20] calculated the cardinal and the number of idempotents of $\mathcal{O}_n$ and later (1992), jointly with Gomes [18], determined the ranks and the idempotent ranks of $\mathcal{O}_n$ and $\mathcal{PO}_n$. More recently, Laradji and Umar [22, 23] presented more combinatorial properties of these two monoids. Certain classes of divisors of the monoid $\mathcal{O}_n$ were determined by Higgins [19] and by Vernitskii and Volkov [30] in 1995, by Fernandes [8] in 1997 and by Fernandes and Volkov [15] in 2010. On the other hand, the monoid $\mathcal{POI}_n$ has been object of study by the first author in several papers [8, 9, 10, 11, 12], by Derech [6], by Garba [17], by Cowan and Reilly [4], by Delgado and Fernandes [5], by Ganyushkin and Mazorchuk [16], by Dimitrova and Koppitz [7], among other authors and papers.

For general background on semigroups, we refer the reader to Howie's book [21].

Let $n \geq 2$.

Let $S \in \{\mathcal{O}_n, \mathcal{POI}_n, \mathcal{PO}_n\}$. We have the following descriptions of the Green relations in the semigroups S:

$s\mathcal{L}t$ if and only if $\mathrm{Im}(s) = \mathrm{Im}(t)$,

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$s\mathcal{R}t$ if and only if $\operatorname{Ker}(s)=\operatorname{Ker}(t)$,

$s\mathcal{J}t$ if and only if $|\operatorname{Im}(s)|=|\operatorname{Im}(t)|$, and

$s\mathcal{H}t$ if and only if $s=t$,

for all $s,t\in S$. If $S=\mathcal{POI}_n$, for the Green relation $\mathcal{R}$, we have, even more simply,

$s\mathcal{R}t$ if and only if $\operatorname{Dom}(s)=\operatorname{Dom}(t)$,

for all $s,t\in S$. Let

$$J_k=J_k^S=\{s\in S\mid |\operatorname{Im}(s)|=k\}\quad\text{and}\quad I_k=I_k^S=\{s\in S\mid |\operatorname{Im}(s)|\le k\},$$

for $0\le k\le n$. If $S\in\{\mathcal{POI}_n,\mathcal{PO}_n\}$ then

$$S/\mathcal{J}=\{J_0<_\mathcal{J} J_1<_\mathcal{J}\cdots<_\mathcal{J} J_n\}$$

and $\{\emptyset\}=I_0\subset I_2\subset\cdots\subset I_n=S$ are all the ideals of S. On the other hand, if $S=\mathcal{O}_n$ then

$$S/\mathcal{J}=\{J_1<_\mathcal{J} J_2<_\mathcal{J}\cdots<_\mathcal{J} J_n\}$$

and $I_1\subset I_2\subset\cdots\subset I_n=S$ are all the ideals of S. See [18, 10].

Recall that a *Rees congruence* ρ of a semigroup S is a congruence associated to an ideal I of S: $s\rho t$ if and only if $s=t$ or $s,t\in I$, for all $s,t\in S$.

Observe that Aĭzenštat [2] proved the congruences of $\mathcal{O}_n$ are exactly the identity and its n Rees congruences. See [24] for another proof. Analogously, the congruences of $\mathcal{POI}_n$ and $\mathcal{PO}_n$ are exactly their $n+1$ Rees congruences. This has been shown, for $\mathcal{POI}_n$, by Derech [6] and, independently, by Fernandes [10] and, for $\mathcal{PO}_n$, by Fernandes et al. [13].

Let S be a monoid such that $S\setminus\{1\}$ is an ideal of S. Let e and f be two idempotents of S such that $ef=fe=f$. Then, clearly, the mapping $\phi:S\longrightarrow S$ defined by $1\phi=e$ and $s\phi=f$, for all $s\in S\setminus\{1\}$, is an endomorphism (of semigroups) of S. This applies to any $S\in\{\mathcal{O}_n,\mathcal{POI}_n,\mathcal{PO}_n\}$.

Let M be a monoid, let S be a subsemigroup of M and let g be a unit of M such that $g^{-1}Sg=S$. Then, it is easy to check that the mapping $\phi^g:S\longrightarrow S$ defined by $s\phi^g=g^{-1}sg$, for all $s\in S$, is an automorphism of S.

Consider the following permutation of Ω_n:

$$\sigma=\begin{pmatrix}1 & 2 & \cdots & n-1 & n\\ n & n-1 & \cdots & 2 & 1\end{pmatrix}.$$

Let $S\in\{\mathcal{O}_n,\mathcal{POI}_n,\mathcal{PO}_n\}$. It is easy to verify that $\sigma^{-1}S\sigma=S$. Therefore, the permutation σ induces a non-trivial automorphism ϕ^σ of S. In fact, this is the unique non-trivial automorphism of S. See [3, Corollary 5.2].

Finding the automorphisms and endomorphisms of transformation semigroups is a classical problem. They have been determined for several transformation semigroups, for instance, $\mathcal{T}_n$ [28], $\mathcal{I}_n$ [27], and the Brauer-type semigroups [25]. Furthermore, regarding semigroups of order-preserving transformations, Fernandes et al. [14] proved the following description of the endomorphisms of $\mathcal{O}_n$:

Theorem 1.1 ([14, Theorem 1.1]). *Let $\phi:\mathcal{O}_n\to\mathcal{O}_n$ be any mapping. Then ϕ is an endomorphism of the semigroup $\mathcal{O}_n$ if and only if one of the following holds:*

(a) *ϕ is an automorphism and so ϕ is the identity or $\phi=\phi^\sigma$;*

(b) *there exist idempotents $e,f\in\mathcal{O}_n$ with $e\neq f$ and $ef=fe=f$ such that $1\phi=e$ and $(\mathcal{O}_n\setminus\{1\})\phi=\{f\}$;*

(c) *ϕ is a constant mapping with idempotent value.*

And, as a corollary:

Theorem 1.2 ([14, Theorem 1.2]). *The semigroup $\mathcal{O}_n$ has $2+\sum_{i=0}^{n-1}\binom{n+i}{2i+1}F_{2i+2}$ endomorphisms, where F_{2i+2} denotes the $(2i+2)$th Fibonacci number.*

In this paper we describe the monoids of the endomorphisms of the semigroups $\mathcal{POI}_n$ and $\mathcal{PO}_n$. As an application of these descriptions, we compute the number of such endomorphisms. This paper is organized as follows. After the current section, we give a miscellaneous of auxiliary results in Section 2. Finally, in Section 3 we present and prove our main results.

2 Preliminary results

In this section we present a series of auxiliary results. We also construct a certain type of endomorphisms of $\mathcal{POI}_n$ and $\mathcal{PO}_n$.

Our first two lemmas have some general nature.

Lemma 2.1. *Let S be a regular semigroup. Let I be an ideal of S and ϕ be an endomorphism of S such that the kernel of ϕ is the Rees congruence associated to I. Let $s,t\in S\setminus I$. Then:*

1. *$s\mathcal{L}t$ if and only if $s\phi\mathcal{L}t\phi$;*
2. *$s\mathcal{R}t$ if and only if $s\phi\mathcal{R}t\phi$.*

Proof. We prove the lemma for the Green relation $\mathcal{L}$. The proof for $\mathcal{R}$ is similar.

First, notice that $I\phi\phi^{-1}=I$ and the restriction of ϕ to $S\setminus I$ is injective.

Let $s,t\in S\setminus I$. If $s\mathcal{L}t$, then $s\phi\mathcal{L}t\phi$, since any homomorphism of semigroups preserve Green relations (i.e., images of related elements are related). Conversely, suppose that $s\phi\mathcal{L}t\phi$. As S is regular, then $S\phi$ is also regular, whence $s\phi$ and $t\phi$ are also $\mathcal{L}$-related in $S\phi$ and so, for some $u,v\in S$, we have $s\phi=(u\phi)(t\phi)$ and $t\phi=(v\phi)(s\phi)$. Thus, $s\phi=(ut)\phi$ and so $ut\in S\setminus I$, since $s\in S\setminus I$ and $I\phi\phi^{-1}=I$. Moreover, as ϕ is injective in $S\setminus I$, it follows that $s=ut$. Analogously, $t=vs$ and so $s\mathcal{L}t$, as required. $\square$

Lemma 2.2. *Let S be any subsemigroup of $\mathcal{PT}_n$ which contains transformations with arbitrary images of size less than n. Let $s\in S$ and $k\in\mathbb{N}$ be such that $1\le\operatorname{rank}(s)\le k\le n-2$. Then, there exists $t\in S$ such that $\operatorname{rank}(t)=k+1$ and $st\neq s$.*

Proof. Let A be any subset of Ω_n such that $\operatorname{Im}(s)\subset A$ and $|A|=k+2$. Let t be any element of S such that $\operatorname{Im}(t)=A\setminus\{\min(\operatorname{Im}(s))\}$ (notice that $\operatorname{Im}(s)\neq\emptyset$). Then, $\operatorname{rank}(t)=k+1$ and $\operatorname{Im}(st)\subseteq\operatorname{Im}(t)$. Since $\min(\operatorname{Im}(s))\notin\operatorname{Im}(t)$, then also $\min(\operatorname{Im}(s))\notin\operatorname{Im}(st)$, whence $\operatorname{Im}(s)\neq\operatorname{Im}(st)$, and so $s\neq st$, as required. $\square$

Observe that the previous lemma applies to any $S\in\{\mathcal{O}_n,\mathcal{POI}_n,\mathcal{PO}_n\}$.

Next, we give two particular properties of our semigroups.

Lemma 2.3. *Let $k\in\{0,1,\dots,n-2\}$. Then, there exist idempotents $h_1,\dots,h_n\in J_{k+1}^{\mathcal{POI}_n}$ such that $h_ih_j\in I_k^{\mathcal{POI}_n}$, for all $1\le i<j\le n$.*

Proof. We have $\binom{n}{k+1}$ distinct subsets of Ω_n with $k+1$ elements. As $0\le k\le n-2$, then $1\le k+1\le n-1$, and so $\binom{n}{k+1}\ge n$. Therefore, we may take (at least) n distinct subsets $Y_1,\dots,Y_n$ of Ω_n with $k+1$ elements. Let h_i be the partial identity on Y_i, for $1\le i\le n$. Then, for $1\le i<j\le n$, the transformation h_ih_j is the partial identity on $Y_i\cap Y_j$ and, since $Y_i\neq Y_j$, we obtain $|Y_i\cap Y_j|<|Y_j|=k+1$, whence $h_ih_j\in I_k^{\mathcal{POI}_n}$, as required. $\square$

For $k \in \{2, 3, \ldots, n\}$, consider the following two transformations of $\mathcal{O}_n$ with rank $n-1$:

$$f_k = \begin{pmatrix} 1 & \cdots & k-1 & k & k+1 & \cdots & n \\ 1 & \cdots & k-1 & k-1 & k+1 & \cdots & n \end{pmatrix} \quad \text{and} \quad g_k = \begin{pmatrix} 1 & \cdots & k-2 & k-1 & k & \cdots & n \\ 1 & \cdots & k-2 & k & k & \cdots & n \end{pmatrix}.$$

We have:

Lemma 2.4. *Each of the $n-1$ $\mathcal{R}$-classes of transformations of rank $n-1$ of $\mathcal{O}_n$ has exactly two idempotents, namely f_k and g_k, for some $k \in \{2, 3, \ldots, n\}$.*

Proof. Let R be an $\mathcal{R}$-class of $\mathcal{O}_n$ contained in $J_{n-1}^{\mathcal{O}_n}$. Then all elements of R have the same kernel, which is associated to a partition of Ω_n of the form $\{\{i\} \mid i \in \{1, 2, \ldots, k-2, k+1, \ldots, n\}\} \cup \{\{k-1, k\}\}$, for some $k \in \{2, 3, \ldots, n\}$. Clearly, f_k and g_k are two distinct idempotents of R. Moreover, let

$$e = \begin{pmatrix} 1 & \cdots & k-1 & k & k+1 & \cdots & n \\ i_1 & \cdots & i_{k-1} & i_{k-1} & i_k & \cdots & i_{n-1} \end{pmatrix},$$

with $1 \le i_1 < \cdots < i_{n-1} \le n$, be an (arbitrary) idempotent of R. Then $\mathrm{Fix}(e) = \mathrm{Im}(e)$ and so $|\mathrm{Fix}(e)| = n-1$, whence there exists a unique $i \in \Omega_n$ such that $(i)e \ne i$. Since $((i)e)e = (i)e$, we have $((i)e, i) \in \mathrm{Ker}(e)$, and so $\{(i)e, i\} = \{k-1, k\}$. If $i = k$ then, clearly, $e = f_k$. Otherwise, $i = k-1$ and then, clearly, $e = g_k$. This proves the lemma. $\square$

For $i \in \{1, 2, \ldots, n\}$, let

$$e_i = \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & \cdots & n \end{pmatrix}.$$

It is clear that each of the n $\mathcal{R}$-classes of transformations of rank $n-1$ of $\mathcal{POI}_n$ has exactly one idempotent (notice that $\mathcal{POI}_n$ is an inverse semigroup), namely e_i, for some $i \in \{1, 2, \ldots, n\}$.

Notice that $J_{n-1}^{\mathcal{PO}_n} = J_{n-1}^{\mathcal{O}_n} \cup J_{n-1}^{\mathcal{POI}_n}$ (a disjoint union). Hence, by the above observations, $J_{n-1}^{\mathcal{PO}_n}$ contains n $\mathcal{R}$-classes exactly with one idempotent and $n-1$ $\mathcal{R}$-classes exactly with two idempotents. Moreover, each $\mathcal{R}$-class contained in $J_{n-1}^{\mathcal{PO}_n}$ has precisely n transformations (one for each possible image with $n-1$ elements). Additionally, it is clear that each $\mathcal{R}$-class contained in $J_1^{\mathcal{PO}_n}$ is determined by the domain of the transformations (as it happens in general in $\mathcal{POI}_n$), and so we have $2^n - 1$ $\mathcal{R}$-classes of transformations of rank 1 of $\mathcal{PO}_n$, each with exactly as many idempotents as elements in the corresponding domain. Such as for $J_{n-1}^{\mathcal{PO}_n}$, each $\mathcal{R}$-class contained in $J_1^{\mathcal{PO}_n}$ has precisely n transformations (one for each possible image with 1 element).

Next, we aim to construct an endomorphism of $\mathcal{PO}_n$.

Let us define a mapping $\phi_1 : \mathcal{PO}_n \longrightarrow \mathcal{PO}_n$ by:

1. $1\phi_1 = 1$;
2. For $s \in J_{n-1}^{\mathcal{POI}_n}$, let $s\phi_1 = \binom{i}{j}$, where $i, j \in \{1, 2, \ldots, n\}$ are the unique indices such that $e_i \mathcal{R} s \mathcal{L} e_j$;
3. For $s \in J_{n-1}^{\mathcal{O}_n}$, let $s\phi_1 = \begin{pmatrix} k-1 & k \\ k_s & k_s \end{pmatrix}$, where $\{k_s\} = \Omega_n \setminus \mathrm{Im}(s)$ and $k \in \{2, 3, \ldots, n\}$ is the unique index such that $s \mathcal{R} f_k$ (and $s \mathcal{R} g_k$);
4. $I_{n-2}^{\mathcal{PO}_n}\phi_1 = \{\emptyset\}$.

Clearly, ϕ_1 is well defined mapping. Moreover, since $\mathcal{PO}_n$ (such as $\mathcal{POI}_n$) is an $\mathcal{H}$-trivial semigroup (and so each element of $\mathcal{PO}_n$ is perfectly defined by its $\mathcal{L}$-class and $\mathcal{R}$-class, i.e., its image and kernel), the restriction of ϕ_1 to $\mathcal{PO}_n \setminus I_{n-2}^{\mathcal{PO}_n}$ is injective. Furthermore, it is a routine matter to prove the following lemma.

Lemma 2.5. *The mapping ϕ_1 is an endomorphism of $\mathcal{PO}_n$ such that $\mathcal{POI}_n\phi_1 \subseteq \mathcal{POI}_n$. Consequently, the restriction of ϕ_1 to $\mathcal{POI}_n$ may also be seen as an endomorphism of $\mathcal{POI}_n$.*

All endomorphisms similar to ϕ_1 have the following property.

Lemma 2.6. *Let $S \in \{\mathcal{POI}_n, \mathcal{PO}_n\}$. Let ϕ be an endomorphism of S such that $1\phi = 1$, $J_{n-1}\phi \subseteq J_1$ and $I_{n-2}\phi = \{\emptyset\}$. Then ϕ is perfectly defined by the images of the idempotents $e_1, \ldots, e_n$. Moreover, $|\operatorname{Dom}(s\phi)| = 1$, for all $s \in J_{n-1}^{\mathcal{POI}_n}$, and $|\operatorname{Dom}(s\phi)| = 2$, for all $s \in J_{n-1}^{\mathcal{O}_n}$.*

Proof. We begin by observing that the kernel of ϕ must be the Rees congruence associated to I_{n-2} and so ϕ is injective in $S \setminus I_{n-2}$. Next, as ϕ applies $\mathcal{R}$-related transformations of J_{n-1} in $\mathcal{R}$-related transformations of J_1, ϕ injective in J_{n-1} and the $\mathcal{R}$-classes contained in J_{n-1} and in J_1 have the same number of elements, namely n, the endomorphism ϕ applies each $\mathcal{R}$-class of J_{n-1} bijectively in a $\mathcal{R}$-class of J_1. It follows that, for each $\mathcal{R}$-class R of J_{n-1}, the number of idempotents of R and of the $\mathcal{R}$-class $R\phi$ of J_1 must be the same. Thus, in particular, $J_{n-1}^{\mathcal{POI}_n}\phi = J_1^{\mathcal{POI}_n}$.

For each $i \in \{1, 2, \ldots, n\}$, let $k_i \in \Omega_n$ be such that $e_i\phi = \binom{k_i}{k_i}$. Notice that $\begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}$ is a permutation of Ω_n.

Let $s \in J_{n-1}^{\mathcal{POI}_n}$. Then, there exists a unique pair $(i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ such that $s\mathcal{R}e_i$ and $s\mathcal{L}e_j$. Then, $s\phi\mathcal{R}e_i\phi$ and $s\phi\mathcal{L}e_j\phi$, and so $s\phi = \binom{k_i}{k_j}$.

If $S = \mathcal{POI}_n$ then the proof is completed. Thus, from now on, we suppose that $S = \mathcal{PO}_n$.

Let $i \in \{2, 3, \ldots, n\}$. Then $f_i\mathcal{L}e_i$ and $g_i\mathcal{L}e_{i-1}$, whence $f_i\phi\mathcal{L}e_i\phi$ and $g_i\phi\mathcal{L}e_{i-1}\phi$, and so $\operatorname{Im}(f_i\phi) = \{k_i\}$ and $\operatorname{Im}(g_i\phi) = \{k_{i-1}\}$. Hence, $k_i \in \operatorname{Im}(f_i\phi) = \operatorname{Fix}(f_i\phi) \subseteq \operatorname{Dom}(f_i\phi)$ and $k_{i-1} \in \operatorname{Im}(g_i\phi) = \operatorname{Fix}(g_i\phi) \subseteq \operatorname{Dom}(g_i\phi)$. Besides, as $f_i\mathcal{R}g_i$, we have $f_i\phi\mathcal{R}g_i\phi$ and so $\operatorname{Dom}(f_i\phi) = \operatorname{Dom}(g_i\phi)$. Since $|\operatorname{Dom}(f_i\phi)| = 2$ and $k_{i-1} \neq k_i$, it follows that $\operatorname{Dom}(f_i\phi) = \{k_{i-1}, k_i\}$.

Now, let $s \in J_{n-1}^{\mathcal{O}_n}$. Then, there exists a unique pair $(i, j) \in \{2, 3, \ldots, n\} \times \{1, 2, \ldots, n\}$ such that $s\mathcal{R}f_i$ and $s\mathcal{L}e_j$. Thus, $s\phi\mathcal{R}f_i\phi$ and $s\phi\mathcal{L}e_j\phi$, and so $s\phi = \begin{pmatrix} k_{i-1} & k_i \\ k_j & k_j \end{pmatrix}$, which finishes the proof of this lemma. □

Observe that we may also conclude that, within the conditions of the previous lemma, we have at most $n!$ endomorphisms.

Now, we recall that a subsemigroup T of $\mathcal{PT}_n$ is said to be $\mathcal{S}_n$-*normal* if $g^{-1}Tg \subseteq T$, for all $g \in \mathcal{S}_n$. In 1975, Sullivan [29, Theorem 2] proved that $\operatorname{Aut}(T) \simeq \mathcal{S}_n$, for any $\mathcal{S}_n$-normal subsemigroup T of $\mathcal{PT}_n$ containing a constant mapping. Moreover, we obtain an isomorphism $\Theta : \mathcal{S}_n \longrightarrow \operatorname{Aut}(T)$ by defining $g\Theta = \theta_g$, where θ_g denotes the inner automorphism of T associated to g (i.e., $t\theta_g = g^{-1}tg$, for all $t \in T$), for all $g \in \mathcal{S}_n$.

Let $S \in \{\mathcal{POI}_n, \mathcal{PO}_n\}$. Let $I_1^1 = I_1 \cup \{1\}$. It is clear that I_1^1 is an $\mathcal{S}_n$-normal subsemigroup of $\mathcal{PT}_n$ containing a constant mapping. Therefore, by Sullivan's Theorem, we have

$$\operatorname{Aut}(I_1^1) = \{\theta_g \mid g \in \mathcal{S}_n\} \simeq \mathcal{S}_n,$$

where θ_g denotes the inner automorphism of I_1^1 associated to g, for all $g \in \mathcal{S}_n$.

Let $\phi_g = \phi_1|_S\theta_g$ (where $\phi_1|_S$ denotes the restriction of ϕ_1 to S), considered as a mapping from S to S, for all $g \in \mathcal{S}_n$. Clearly, $\{\phi_g \mid g \in \mathcal{S}_n\}$ is a set of $n!$ distinct endomorphisms of S. Moreover, it is easy to conclude the following result, with which we finish this section.

Lemma 2.7. *Let $S \in \{\mathcal{POI}_n, \mathcal{PO}_n\}$. Let ϕ be an endomorphism of S such that $1\phi = 1$, $J_{n-1}\phi \subseteq J_1$ and $I_{n-2}\phi = \{\emptyset\}$. Then, $\phi = \phi_g$, for some $g \in \mathcal{S}_n$.*

3 Main results

Let $S \in \{\mathcal{POI}_n, \mathcal{PO}_n\}$. Let ϕ be an endomorphism of the semigroup S. Then, $\operatorname{Ker}(\phi)$ is the Rees congruence of S associated to I_k, for some $k \in \{0, 1, \ldots, n\}$. Observe that the restriction of ϕ to $S \setminus I_k$ is an injective mapping and $|I_k\phi| = 1$. Let $f \in S$ be such that $I_k\phi = \{f\}$. Clearly, f is an idempotent of S. Notice also that $\{f\}\phi^{-1} = I_k$.

Suppose that ϕ is neither a constant mapping nor an automorphism. Then, since $\operatorname{Ker}(\phi)$ is not trivial and not universal, we have $1 \le k \le n-1$.

Let us admit that $k = n-1$ and take $e = 1\phi$. Then e is also an idempotent, $e \neq f$ and, for $s \in I_{n-1}$, we have $ef = (1\phi)(s\phi) = (1\cdot s)\phi = s\phi = f = s\phi = (s\cdot 1)\phi = (s\phi)(1\phi) = fe$.

From now on, suppose that $1 \le k \le n-2$. Since any homomorphism of semigroups preserve Green relations, there exists $\ell \in \{0,1,\dots,n\}$ such that $J_{k+1}\phi \subseteq J_\ell$. Under these conditions, we prove two lemmas.

Lemma 3.1. *Under the above conditions, one has* $\operatorname{rank}(f) < \ell$.

Proof. First, notice that $(\cup_{i=k+1}^n J_i)\phi \subseteq \cup_{i=\ell}^n J_i$, since $J_{k+1}\phi \subseteq J_\ell$ and any homomorphism preserves the quasi-order $\le_{\mathcal{J}}$.

Next, we show that $\operatorname{rank}(f) \le \ell$. Take $s \in J_k$ and $t \in J_{k+1}$. Then, $s <_{\mathcal{J}} t$, and so $s\phi \le_{\mathcal{J}} t\phi$. Since $J_k\phi = \{f\}$ and $J_{k+1}\phi \subseteq J_\ell$, we have $s\phi = f$ and $t\phi \in J_\ell$, whence $\operatorname{rank}(f) \le \ell$.

Now, we prove that $\operatorname{rank}(f) \le k$. By contradiction, suppose that $\operatorname{rank}(f) \ge k+1$. Then, $I_k \subseteq I_{\operatorname{rank}(f)-1}$, and so

$$(S \setminus I_k)\phi = (\cup_{i=k+1}^n J_i)\phi \subseteq \cup_{i=\ell}^n J_i \subseteq \cup_{i=\operatorname{rank}(f)}^n J_i = S \setminus I_{\operatorname{rank}(f)-1} \subseteq S \setminus I_k.$$

It follows that $(S\setminus I_k)\phi = S \setminus I_k$, since the restriction of ϕ to $S \setminus I_k$ is injective. As $\operatorname{rank}(f) \ge k+1$ also implies that $f \in S \setminus I_k$, we deduce that $I_k \cap S \setminus I_k = \{f\}\phi^{-1} \cap S \setminus I_k \neq \emptyset$, which is a contradiction. Thus, we must have $\operatorname{rank}(f) \le k$.

Finally, we prove that $\operatorname{rank}(f) < \ell$. Suppose, by contradiction, that $\operatorname{rank}(f) = \ell$. Take $s \in J_{k+1}$. Since $J_{k+1}\phi \subseteq J_\ell$, we have $s\phi \mathcal{J} f$. Moreover, as $f \in I_k$, we also have $sf, fs \in I_k$, whence $(sf)\phi = (fs)\phi = f\phi = f$. From $f = (fs)\phi = f\phi s\phi = f(s\phi)$, it follows that $\operatorname{Im}(f) = \operatorname{Im}(f(s\phi)) \subseteq \operatorname{Im}(s\phi)$. From $f = (sf)\phi = s\phi f\phi = (s\phi)f$, it follows that $\operatorname{Ker}(f) = \operatorname{Ker}((s\phi)f) \supseteq \operatorname{Ker}(s\phi)$. Since $s\phi\mathcal{J}f$, we have $|\operatorname{Ker}(f)| = |\operatorname{Im}(f)| = |\operatorname{Im}(s\phi)| = |\operatorname{Ker}(s\phi)|$, and so $\operatorname{Im}(f) = \operatorname{Im}(s\phi)$ and $\operatorname{Ker}(f) = \operatorname{Ker}(s\phi)$, whence $f\mathcal{L}s\phi$ and $f\mathcal{R}s\phi$, i.e., $f\mathcal{H}s\phi$. Thus, $f = s\phi$, which is a contradiction (being the case that $s \in J_{k+1}$ and $\{f\}\phi^{-1} = I_k$). Therefore, $\operatorname{rank}(f) < \ell$, as required. □

Lemma 3.2. *Under the above conditions,* $1\phi = 1$, $J_{n-1}\phi \subseteq J_1$, *and* $I_{n-2}\phi = \{\emptyset\}$.

Proof. Since $k \in \{1,2,\dots,n-2\}$, by Lemma 2.3, we may take idempotents $h_1,\dots,h_n \in J_{k+1}^{\mathcal{POI}_n}$ such that $h_ih_j \in I_k^{\mathcal{POI}_n}$, for all $1 \le i < j \le n$.

Recall that $f \in I_k$, and so $f\phi = f$. Let $i \in \{1,2,\dots,n\}$. Then, $f(h_i\phi) = f\phi h_i\phi = (fh_i)\phi = f$, since $fh_i \in I_k$. Hence, $\operatorname{Im}(f) \subseteq \operatorname{Im}(h_i\phi)$.

Next, let $1 \le i < j \le n$. By the previous paragraph, we have $\operatorname{Im}(f) \subseteq \operatorname{Im}(h_i\phi) \cap \operatorname{Im}(h_j\phi)$. Conversely, let $a \in \operatorname{Im}(h_i\phi) \cap \operatorname{Im}(h_j\phi)$. Since $h_i\phi$ and $h_j\phi$ are idempotents and $h_ih_j \in I_k$, we have $\operatorname{Im}(h_i\phi) = \operatorname{Fix}(h_i\phi)$, $\operatorname{Im}(h_j\phi) = \operatorname{Fix}(h_j\phi)$ and $(h_ih_j)\phi = f$, whence $af = a(h_ih_j)\phi = (a(h_i\phi))(h_j\phi) = a(h_j\phi) = a$, and so $a \in \operatorname{Im}(f)$. Then, $\operatorname{Im}(h_i\phi) \cap \operatorname{Im}(h_j\phi) \subseteq \operatorname{Im}(f)$, and thus $\operatorname{Im}(h_i\phi) \cap \operatorname{Im}(h_j\phi) = \operatorname{Im}(f)$.

Let $E_i = \operatorname{Im}(h_i\phi) \setminus \operatorname{Im}(f)$, for $1 \le i \le n$. Clearly, $\operatorname{Im}(f) \cap (\cup_{i=1}^n E_i) = \emptyset$ and, for $1 \le i < j \le n$, the equality $\operatorname{Im}(h_i\phi) \cap \operatorname{Im}(h_j\phi) = \operatorname{Im}(f)$ implies that $E_i \cap E_j = \emptyset$. Moreover, since $h_i \in J_{k+1}$, then $\operatorname{rank}(h_i\phi) = \ell > \operatorname{rank}(f)$, by Lemma 3.1, and so $|E_i| \ge 1$, for $1 \le i \le n$. Thus $|E_1| = |E_2| = \cdots = |E_n| = 1$ and $\cup_{i=1}^n E_i = \Omega_n$, from which follows that $\operatorname{Im}(f) = \emptyset$.

Now, observe that $\ell = \operatorname{rank}(h_1\phi) = |\operatorname{Im}(h_1\phi)| = |E_1 \cup \operatorname{Im}(f)| = |E_1| = 1$. Then, $J_{k+1}\phi \subseteq J_1$, and so we cannot have more than n elements of $J_{k+1}\phi$ in distinct $\mathcal{L}$-classes. Thus, by Lemma 2.1, we cannot have more than n elements of J_{k+1} in distinct $\mathcal{L}$-classes, i.e. $\binom{n}{k+1} \le n$, and so $k+1 \in \{0,1,n-1,n\}$. Since $1 \le k \le n-2$, it follows that $k = n-2$.

It remains to show that $e = 1$. Since $J_{n-1}\phi \subseteq J_1$ and $I_{n-2}\phi = \{\emptyset\}$, the reasoning of the first paragraph of the proof of Lemma 2.6 applies here and we can conclude that $J_{n-1}^{\mathcal{POI}_n}\phi = J_1^{\mathcal{POI}_n}$.

Suppose, by contradiction, that $e \neq 1$. Then, $|\operatorname{Im}(e)| < n$ and so we may take $i \in \Omega_n \setminus \operatorname{Im}(e)$. Let $h \in J_{n-1}^{\mathcal{POI}_n}$ be such that $h\phi = \binom{i}{i}$. Hence, $\emptyset = e\binom{i}{i} = (1\phi)(h\phi) = (1 \cdot h)\phi = h\phi = \binom{i}{i}$, a contradiction. Thus, $1\phi = 1$, as required. $\square$

Now, by Lemmas 3.2 and 2.7, we can deduce that, if $1 \leq k \leq n-2$, then ($k = n-2$ and) $\phi = \phi_g$, for some $g \in \mathcal{S}_n$. This concludes the proof of the following description of the monoid of endomorphisms of S.

Theorem 3.3. *Let $S \in \{\mathcal{POI}_n, \mathcal{PO}_n\}$. Let $\phi : S \longrightarrow S$ be any mapping. Then, ϕ is an endomorphism of the semigroup S if and only if one of the following holds:*

(a) *ϕ is an automorphism and so ϕ is the identity or $\phi = \phi^\sigma$;*

(b) *there exist idempotents $e, f \in S$ with $e \neq f$ and $ef = fe = f$ such that $1\phi = e$ and $(S \setminus \{1\})\phi = \{f\}$;*

(c) *$\phi = \phi_g$, for some $g \in \mathcal{S}_n$;*

(d) *ϕ is a constant mapping with idempotent value.*

As a corollary of the this theorem, we finish this paper by counting the number of endomorphisms of $\mathcal{POI}_n$ and of $\mathcal{PO}_n$.

For $S \in \{\mathcal{POI}_n, \mathcal{PO}_n\}$ and for each idempotent $e \in S$, let

$$E_S(e) = \{f \in S \mid f^2 = f \text{ and } fe = ef = f\}.$$

Then, according to Theorem 3.3, we have $\sum_{e^2=e\in S} |E_S(e)|$ endomorphisms of S of type (b) and (d).

We start by considering $\mathcal{POI}_n$.

Theorem 3.4. *The semigroup $\mathcal{POI}_n$ has $2 + n! + 3^n$ endomorphisms.*

Proof. First, recall that the idempotents of $\mathcal{POI}_n$ are all the partial identities of Ω_n, and so we precisely have $\binom{n}{k}$ idempotents of $\mathcal{POI}_n$ with rank k, for all $0 \leq k \leq n$.

Let e be an idempotent of $\mathcal{POI}_n$. Then, it is easy to show that $f \in E_{\mathcal{POI}_n}(e)$ if and only if $\operatorname{Im}(f) \subseteq \operatorname{Im}(e)$, for any idempotent f of $\mathcal{POI}_n$. Hence, $|E_{\mathcal{POI}_n}(e)| = 2^{|\operatorname{Im}(e)|}$.

It follows that the number of endomorphisms of $\mathcal{POI}_n$ of type (b) and (d) is equal to

$$\sum_{e^2=e\in\mathcal{POI}_n} |E_{\mathcal{POI}_n}(e)| = \sum_{e^2=e\in\mathcal{POI}_n} 2^{|\operatorname{Im}(e)|} = \sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n,$$

and so, since we have $n!$ endomorphisms of type (c), as observed at the end of Section 2, and two automorphisms, we obtain a total of $2 + n! + 3^n$ endomorphisms of $\mathcal{POI}_n$, as required. $\square$

Counting the number of endomorphisms of $\mathcal{PO}_n$ is more elaborated than for $\mathcal{POI}_n$. First, we prove two lemmas.

Lemma 3.5. *Let $1 \leq k \leq n$ and let e be an idempotent of $\mathcal{PO}_n$ with rank k. Then, the set $E_{\mathcal{PO}_n}(e)$ has as many elements as the number of idempotents of $\mathcal{PO}_k$, i.e., $E_{\mathcal{PO}_n}(e)$ has $1 + (\sqrt{5})^{k-1}((\frac{\sqrt{5}+1}{2})^k - (\frac{\sqrt{5}-1}{2})^k)$ elements.*

Proof. Let X be any finite chain and denote by $\mathcal{PO}(X)$ the monoid of all order-preserving partial transformations on X. Observe that $\mathcal{PO}(X)$ and $\mathcal{PO}_{|X|}$ are isomorphic monoids.

Let $1 \leq k \leq n$ and let e be an idempotent of $\mathcal{PO}_n$ with rank k. Given an idempotent $g \in \mathcal{PO}(\operatorname{Im}(e))$, it is a routine matter to check that $eg \in E_{\mathcal{PO}_n}(e)$ and, moreover, that the map $\{f \in \mathcal{PO}(\operatorname{Im}(e)) \mid f^2 = f\} \longrightarrow E_{\mathcal{PO}_n}(e)$, $g \longmapsto eg$, is a bijection. Therefore, $|E_{\mathcal{PO}_n}(e)| = |\{f \in \mathcal{PO}_k \mid f^2 = f\}|$.

Now, it remains to recall that Laradji and Umar proved that the number of idempotents of $\mathcal{PO}_k$ is equal to $1 + (\sqrt{5})^{k-1}((\frac{\sqrt{5}+1}{2})^k - (\frac{\sqrt{5}-1}{2})^k)$ (see [22, Theorem 3.8]). $\square$

Lemma 3.6. *The number of idempotents of $\mathcal{PO}_n$ with rank k is $\sum_{i=k}^{n}\binom{n}{i}\binom{i+k-1}{2k-1}$, for $1 \le k \le n$.*

Proof. Let X be any finite chain and denote by $\mathcal{O}(X)$ the monoid of all order-preserving full transformations on X. As for order-preserving partial transformations, we have that $\mathcal{O}(X)$ and $\mathcal{O}_{|X|}$ are isomorphic monoids.

Let $1 \le k \le n$. Since an idempotent fixes its image, given an idempotent e of $\mathcal{PO}_n$, it is clear that $\mathrm{Im}(e) \subseteq \mathrm{Dom}(e)$, and so $e \in \mathcal{O}(\mathrm{Dom}(e))$. Hence, an element e of $\mathcal{PO}_n$ is an idempotent with rank k if and only if e is an idempotent of $\mathcal{O}(X)$ with rank k, for some subset X of Ω_n such that $|X| \ge k$. Therefore, the number of idempotents of $\mathcal{PO}_n$ with rank k is

$$\sum_{i=k}^{n}\binom{n}{i}|\{e \in J_k^{\mathcal{O}_i} \mid e^2 = e\}|,$$

and so, since $|\{e \in J_k^{\mathcal{O}_i} \mid e^2 = e\}| = \binom{i+k-1}{2k-1}$, by [23, Corollary 4.4], for $i \ge k$, the lemma is proved. □

Observe that, by the previous lemma and [22, Theorem 3.8], we obtain the following equality:

$$\sum_{k=1}^{n}\sum_{i=k}^{n}\binom{n}{i}\binom{i+k-1}{2k-1} = (\sqrt{5})^{n-1}\left(\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{\sqrt{5}-1}{2}\right)^n\right). \tag{1}$$

Now, we can calculate the number of endomorphisms of $\mathcal{PO}_n$.

Theorem 3.7. *The semigroup $\mathcal{PO}_n$ has*

$$3+n!+(\sqrt{5})^{n-1}\left(\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{\sqrt{5}-1}{2}\right)^n\right)+\sum_{k=1}^{n}(\sqrt{5})^{k-1}\left(\left(\frac{\sqrt{5}+1}{2}\right)^k - \left(\frac{\sqrt{5}-1}{2}\right)^k\right)\sum_{i=k}^{n}\binom{n}{i}\binom{i+k-1}{2k-1}$$

endomorphisms.

Proof. The number of endomorphisms of $\mathcal{PO}_n$ of type (b) and (d) is equal to

$$\begin{aligned}
\textstyle\sum_{e^2=e\in\mathcal{PO}_n}|E_{\mathcal{PO}_n}(e)| &= 1+\textstyle\sum_{e^2=e\in\mathcal{PO}_n\setminus\{\emptyset\}}|\{f \in \mathcal{PO}_{|\mathrm{Im}(e)|} \mid f^2=f\}| \\
&= 1+\textstyle\sum_{k=1}^{n}|\{f \in \mathcal{PO}_k \mid f^2=f\}||\{e \in J_k^{\mathcal{PO}_n} \mid e^2=e\}| \\
&= 1+\textstyle\sum_{k=1}^{n}(1+(\sqrt{5})^{k-1}((\frac{\sqrt{5}+1}{2})^k-(\frac{\sqrt{5}-1}{2})^k))\sum_{i=k}^{n}\binom{n}{i}\binom{i+k-1}{2k-1} \\
&= 1+\textstyle\sum_{k=1}^{n}\sum_{i=k}^{n}\binom{n}{i}\binom{i+k-1}{2k-1}+\sum_{k=1}^{n}(\sqrt{5})^{k-1}((\frac{\sqrt{5}+1}{2})^k-(\frac{\sqrt{5}-1}{2})^k)\sum_{i=k}^{n}\binom{n}{i}\binom{i+k-1}{2k-1},
\end{aligned}$$

by Lemmas 3.5 and 3.6, i.e. equal to

$$1+(\sqrt{5})^{n-1}((\frac{\sqrt{5}+1}{2})^n-(\frac{\sqrt{5}-1}{2})^n)+\sum_{k=1}^{n}(\sqrt{5})^{k-1}((\frac{\sqrt{5}+1}{2})^k-(\frac{\sqrt{5}-1}{2})^k)\sum_{i=k}^{n}\binom{n}{i}\binom{i+k-1}{2k-1},$$

by the equality (1). Once again we have $n!$ endomorphisms of type (c) and two automorphisms, so the result follows. □

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Vítor H. Fernandes, CMA, Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade NOVA de Lisboa, Monte da Caparica, 2829-516 Caparica, Portugal; e-mail: vhf@fct.unl.pt.

Paulo G. Santos, Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade NOVA de Lisboa, Monte da Caparica, 2829-516 Caparica, Portugal; e-mail: pgd.santos@campus.fct.unl.pt.