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Credit Risk Management in Low Default Portfolios

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Dissertation report presented as partial requirement for obtaining the Master's degree in Statistics and Information Management

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CREDIT RISK MANAGEMENT IN LOW DEFAULT PORTFOLIOS

by

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ABSTRACT

BACKGROUND
This paper considers the problem of the risk parameters quantification, namely the probability of default, for the portfolios where the occurrence of such events is very low.

OBJECTIVE
The objective is to achieve the most realistic PD estimates for the Low Default Portfolios, considering the statistical based approach and not being based purely on judgmental consideration.

METHOD
The presented method is an aggregation of the different approaches. The Cumulative Accuracy Profile is used for the PD estimation and adjusted through the Bayesian inference where the prior distribution uses the Confidence Confidence-Based approach as input.

RESULTS
The PD estimates were derived from a portfolio of sovereign bonds and then compared with the observed defaults based on the portfolio's historical experience. The results show an alignment between expected (PDs) and realized (DRs).

CONCLUSIONS
The proposed method allows for the aggregation of different requirements, such as the conservatism or the monotonicity principle, while preserving the historical experience. In addition, the possibility to use several assumptions brings the sufficiently flexible statistical approach for the PDs estimation considering the nature of Low Default Portfolios.

KEYWORDS

Low Default Portfolios; IRB Requirements; Risk Management; Calibration

INDEX

LIST OF FIGURES

LIST OF TABLES

LIST OF ABBREVIATIONS AND ACRONYMS

AIGV Accord Implementation Group's Validation Subgroup

BCBS Basel Committee on Banking Supervision

BCR Benjamin Cathcart and Ryan Approach

CAP Cumulative Accuracy Profile Approach

CBA Confidence-Based Approach

DR Default Rate

LDP Low Default Portfolio

MOC Margin of Conservatism

PD Probability of Default

1. INTRODUCTION

Nowadays, the predictive ability of financial institutions about the behavior of clients who are unlikely to pay is one of the most important and indispensable tools. But what about those who historically have experienced a low number of defaults, if none at all? This phenomenon is characteristic of the so-called Low Default Portfolio (LDP). There is no exact definition for the LDPs, since they are subjective and vary across different institutions. In this way, the concern about those portfolios has been increasing over the years not only across the banks, but for the regulators as well, and many methods were proposed to comply with the IRB requirements.

The use of internal measures required the incorporation of all relevant and available data derived from historical experience, with a certain level of conservatism that should not be based entirely on subjective evaluation. For this purpose, the use of statistical models and other mechanical methods is considered. However, it could not be easy to satisfy those requirements when it comes to portfolios with few defaults. In this case, the quantifications of the risk parameters from historical experience do not allow for solid conclusions and realized default averages may not be an adequate reflection of the real risk.

To solve the issues related to those portfolios, different techniques were proposed by different authors. Despite the many discussions around this topic, there is still no exact answer to which method is more likely to be applied since all of them show some strengths and limitations. On one hand, the results of those methods sometimes could not be equivalent. On the other hand, methods may be combined to produce more accurate results. Against this background, the method proposed in this paper is a combination of the different approaches and consequently a combination of the different assumptions. The aim is to add some flexibility for risk managers through its assumptions and avoid some limitations, where the restriction of one approach could be complement with information from another.

In this way, the method applies the Cumulative Accuracy Profile (CAP) approach, proposed by Burgt (2008), as a primary technique. Consequently, the estimates are adjusted through the scaling procedure towards the Portfolio Reference PD. The Portfolio Reference PD is derived from the Bayesian inference, where the prior belief of the default distribution uses the estimates from the Confidence-Based approach (CBA), proposed by Pluto and Tasche (2011), as an input.

Through this combination, the following could be achieved: for one side, the primary technique (CAP approach) will allow for the monotonicity of the estimates; for another, the use of the results from the Confidence-Based approach, as an input for the prior belief, avoids the problem of risk managers to estimate the prior parameters, especially when the institutions do not have internal default experience. Additionally, estimating the Portfolio Reference PD allows not only for the adjustment of the primary results but also to manage the direction by which they will be adjusted through the choice of different assumptions and parameters.

To this extent, the remainder of this paper is structured as follows. In Section 2 we briefly review the relevant literature on the subject of LDPs: the definition and the main concerns, as well as the proposed solutions, will be addressed; Section 3 describes the proposed methodology, including the description of the method, the theoretical background about each of techniques used and consequently, the establishment of the respective assumptions; additionally, the data applied for the practical demonstration will be analysed. Section 4 presents and discusses the main results derived from the proposed methodology. The final section concludes and sets some directions for further research.

2. LITERATURE REVIEW

The estimation of Default Probability is one of the main components of the risk management tool that becomes a challenge when banks are confronted with portfolios with the LDP nature.

This section is intended for an introduction to the Low Default Portfolios through the following topics: a general definition of the Probability of Default and how it is seen from a regulatory perspective; the different views of LDPs across the institutions, as well as the main concerns of both market participants and regulators; and finally, some of the techniques designed to address the specificities of LDPs. As previously mentioned, although there is no answer about which of those methods is the most accurate or which should be adopted to meet all requirements, the aim is to compare different views, assumptions, and approaches to face the challenge in the LDP development process.

2.1. PROBABILITY OF DEFAULT

Under the scope of the Internal Risk-Based (IRB) models, banks are allowed to use their estimates for risk quantification. However, several requirements and guidance should be considered.

For purpose of default probability estimation, financial institutions are required to use “all relevant data, information, and methods (…) historical experience and empirical evidence, and not based purely on judgemental considerations. (…) be plausible and intuitive and shall be based on the material drivers of the respective risk parameters.” in accordance with Article 179 of the Regulation (EU) No 575/2013. These requirements are general for all types of portfolios, whether they are low or not.

Furthermore, regarding the data and the observation period used for the PD estimation, is specified that: “irrespective of whether an institution is using external, internal, or pooled data sources, or a combination of the three (..) the length of the underlying historical observation period used shall be at least five years for at least one source” and that “if the available observation period spans a longer period for any source, and this data is relevant, this longer period shall be used” according to the Article 180.

Nevertheless, it is also referred that given the possibility of unpredictable errors over the PD estimation, the margin of conservatism should be added to cover the possible range of errors and “where methods and data are considered to be less satisfactory (…) the margin of conservatism shall be larger.” In addition, the article (Article 180 of the Regulation (EU) No 575/2013) highlights “the importance of judgmental considerations in combining results of techniques and in making adjustments for limitations of techniques and information”. Those points are particularly important given the characteristics of the Low Default Portfolios.

2.2. LOW DEFAULT PORTFOLIOS

There is no straight line between low or not-low default portfolios, neither the IRB Framework in Basel II nor the Capital Requirements Regulation under Basel III define LDPs as a separate asset class.

According to the Basel Committee Accord Implementation Group's Validation Subgroup (AIGV), the specific portfolio is "closer to the LDP end (…) when Bank's internal data systems include fewer loss events" (BCBS, 2005). This subjective definition questions whether the portfolio is considered as LDP or not. As an example, given a certain portfolio of the same nature in two different institutions, in one it might be called as LDP, but not considered as such in another for having data that is richer in default.

The low number of defaults is certainly a common characteristic to recognize the portfolio as LDP. However, in what refers to the risk type, they can differ among themselves. For example, the portfolio that is relatively small in size or with a low number of obligors in the market is quite different from the one that over years has experienced a few occurrences of default events (BCBS, 2005).

It should be noted that low default portfolios are not always low data portfolios, i.e., the ones where the data is scarce. As an example, the mortgage portfolio with highly rated obligors could be considered under the LDP nature despite being relatively big. In this context, "low number of defaults needs to be seen in the light of the size of the portfolio which produce them" (Benjamin et al., 2006). On the other side, the common examples of small size LDPs are usually portfolios with exposures to banks or insurance companies, sovereigns, large corporates, or project finance. This allows us to conclude that the general problem with the LDPs, is that their definition might vary from one institution to another.

2.3. THE MAIN CONCERN FOR THE LDP ESTIMATION

Before taking a closer look at the regulator's or market participant's main concerns, it is important to mention the demonstration of Venter (2016) about why naïve estimators of PD will fail under LDPs. The naïve estimator of PD is a simple quotient between the number of defaults in a specific rating grade, divided by the number of obligors of this rating. Therefore, what will happen with this estimator if only one default will occur by 1000 obligors, or no defaults will be observed in a particular rating grade? Probably, this estimator will significantly underestimate the PD. The opposite could be verified if considering a grade with only 10 obligors.

Following the newsletter issued by Basel Committee in 2005, the concern of market participants regarding the scarcity of statistical data and problems related to the backtesting of the risk parameters will exclude the LDPs from the scope of capital requirements computation. However, even with this concern in mind, those portfolios are still under the consideration of IRB treatment. Furthermore, the data period used for estimation purposes should be representative of the experience through the cycle, and this, in turn, must include the economic downturns. As such, it could not be possible to consider the cases of the portfolios that only cover the relatively benign economic conditions or the institutions that are recent entrants on a certain portfolio type.

Once again, the quantification of the LDPs risk parameters based on historical experience statistically does not make it possible to draw solid conclusions. In addition, another issue appears with the requirements set in Article 180 of the Regulation (EU) No 575/2013), where is explicit that that "institutions shall estimate PDs by obligor grade". According to Benjamin et al. (2006), "it may be simple (…) to produce a historical average, but a distinguishing feature of LDPs is that such estimates may be inadequate reflections of the true risk" and "unreliable or poor in some statistical sense".

Kruger (2015) mentions that to cope with the problem related to the lack of default data, various assumptions should be made, and a certain level of conservatism should be applied to avoid the possibility of model errors. According to the author, the concerns of the regulators and institutions might be different in the way that "the supervisor is mainly concerned about the credit risk being underestimated", while "the bank is concerned about PDs that are too pessimistic since this has an impact on their pricing and economic capital".

2.4. DIFFERENT APPROACHES USED FOR THE LDP'S ESTIMATION

Several approaches were suggested by different authors to give the response to the LDPs issues. This section is dedicated to describing some of those methods to understand the different treatments that were proposed.

Confidence-Based Approach (CBA)

Pluto and Tasche suggest a "*most prudent estimation principle*" by estimating the PD as upper confidence bounds and assuming that the "ordinal borrower ranking is correct". This approach is mostly known as the Confidence Based Approach (CBA) and gives monotone PD estimates (see Pluto and Tasche, 2011, for more details). Under the assumption of this approach, the probability of default of a higher rating grade cannot be greater than the PD of a lower rating grade. This method gives the confidence intervals for PDs of each rating grade which is adjusted by the determination of the confidence level. However, the choice of confidence level is an open question, for which the authors do not give a concrete answer, meaning that it rests on the institution to decide which level to choose. The model can assume both – correlated (with assumptions that are consistent with Basel risk-weighted model) and uncorrelated default events and also be applied to the samples with no defaults. In addition, the authors propose the extension of the methodology in two ways: scaling the estimating PDs to overall portfolios central tendencies or portfolio upper confidence bound for the better fitting of the estimates; or by introduction of the multi-period case assumption.

Benjamin Cathcart and Ryan Approach (BCR)

The next method presented by Benjamin, Cathcart and Ryan – called as BCR approach –, brings some modification for the Pluto and Tasche approach by introducing more conservative PD estimates (see Benjamin et al., 2006 for more details). Under this approach the regulators play an important role: the starting point is the definition of criteria by which the portfolio is considered as being the LDP; the second step generates the look-up table with portfolio-wide PDs, applying a specified and public available methodology, as a function of (1) numbers of company years in the historical sample and (2) numbers of defaults observed in the historical sample. Those PDs are determined based on the regulatory choice of the confidence level as well as the choice of asset and year-to-year correlation. Subsequently, the PD from the look-up table is then compared with the internal weighted average PD. In the case that PD of the institution portfolio is below the look-up PD, the firm should adjust its PD until it will be equal to or greater than the look-up PD. Additionally, for the grade-level PDs adjustments, the simple scaling factor method is used. It should be mentioned, that “PDs can be adjusted upwards but not downward in effect making a firm´s internal PDs act as a floor to the PDs resulting from the process”. According to the authors, this approach should be adopted by regulators to “give firms some assurance that their approaches would embody sufficient conservatism to qualify for IRB”.

Cumulative Accuracy Profile Approach (CAP)

Another perspective for Low Default Portfolios calibration was introduced by Van der Burgt (2008) by using a cumulative accuracy profile (CAP) which is usually used for the measurement of the discriminatory power of the model, i.e., how good the model is in discriminating good obligors from bad obligors. The author suggests modelling the cumulative accuracy profile curve, known as the power curve of Lorentz, from observed data and then driving the calibration from this curve. The concavity, which represents the shape of a curve, is used as the main parameter in conjunction with the functional form of the power curve. This parameter could be also considered as an indicator of the soundness of the rating system under analysis: high and positive values indicate a good discriminatory power of the model while a parameter equal to 0, represents no discriminatory power. Additionally, when the concavity represents negative values, the rating system is subject to adverse selection when the defaults are allocated to the best rating grades.

The calibration is calculated by “taking the derivative of the closed-form equation for the CAP”. In this context, Burgt (2008) mentioned that “the method is based in the fact that assessing the discriminatory power of a credit rating model is easier than calibrating a credit rating model, but calibration can be derived from the discriminatory power (Falkstein et al., 2003)”.

Bayesian Approach

Finally, the Bayesian approach for modelling the default probability is one of the most discussed methods among different authors starting with Dwyer (2006). Under the Bayesian approach the prior distribution, which represents a prior belief of the possible values of the unknown parameters (in this particular case the PDs), is defined before the outcome of the data is observed. Subsequently, the prior beliefs are updated based on the already observed outcome and from the given posterior density function, a Bayesian estimator of the parameter is determined. Those estimators could be derived from three different loss functions, namely square error loss, absolute loss, or 0-1 loss, and correspond respectively to the mean, median, and mode of the marginal posterior distribution. The square error loss, which is the mean, is “the most common estimator occurring in literature” according to Castor and Gerhardsson (2017) used as a point estimate of the unknown parameter.

It is relevant to mention, that the choice of the prior distribution has been varying among different authors. As an example, Beta prior distribution was suggested by Kiefer (2009) while previously Dwyer (2006) proposed the uniform prior (where nothing is known about the value of parameters before seeing the data) that was lately considered by Tasche as conservative prior in 2013; in the same year, Clifford et al. proposed the expert prior, and then the Pareto prior distribution was proposed by Venter

(2016) (see Venter (2016) for more details). Furthermore, those prior distributions can be grouped into two main categories: subjective prior and objective prior (Berning, 2010, as cited in Venter, 2016).

The first category, mostly considered to support the calibration process of the LDPs, is based on expert information, namely on the risk manager's experience on the default distribution. Alternatively, objective prior is an opposite where no expert information is considered, and the prior distribution results from the assumed probability function of the data.

2.5. APPROACHES' MAIN DIFFERENCES AND CONCERNS

As mentioned in the previous section, different approaches were proposed considering the Low Default Portfolios nature, and although there is no answer which one is the most appropriate, all of them offer contributes in some way, as well as present some limitations.

Historical data

In general, most of the models use historical data for parameter estimation. However, this might be one of the criticisms for all approaches since what is desired are forward-looking PDs. Regarding the LDPs, this is one of the main limitations given the fact that such estimations require a large amount of data, that is scarce in most of the cases. Another feature of the LDPs is the data where there are no occurrences of default which is not uncommon across the banks. The Confidence-Based Approach (CBA) provided by Pluto and Tasche (2011) allows for the estimation of PD parameters when there are no defaults, which cannot be possible under the Cumulative Accuracy Profile (CAP) Approach.

Some authors appeal for the significance of the period that is chosen for the estimation. For example, Castor and Gerhardsson (2017) refer that the fact that the BCR approach does not consider "wherein the economic cycle the data is from" should be seen as a possible improvement of the method. A similar idea was highlighted by Kiefer (2009) in respect to the Bayesian approach, outlining that is better "to interrogate experts on what they would expect to see in data, rather than what they would expect of parameter values". Kiefer argues that is critical to specify the data period since the "true" default probability might happen to change over time.

Scaling Factors

According to the authors of the Confidence-Based Approach, "one of the drawbacks (…) is that in few defaults case, for all grades the upper confidence bound PD estimates are higher than the average default rate of the overall portfolio". This phenomenon however is true regarding the method without multi-period case assumption and non-zero default cases and happens due to the defaults of the portfolio being all included in the upper confidence bound estimation even for those grades where the rating is high. As a potential solution, Pluto and Tasche (2011) suggest both: the calibration by scaling factors for all grades towards central tendency, namely the average portfolio default rate; or toward portfolio upper bound PD to avoid the limitation of non-zero defaults. The first option allows the PD estimates to fit the overall portfolio default rate, the second option is used to manage the degree of conservatism.

Under the CBA approach, the scaling procedure is recommended by authors to prevent the idea of market participants that final estimates are being too conservative. While the BCR approach considers the scaling for the opposite purpose. Benjamin et al. argue that PD adjustments should be done only to make the new PDs more conservative. However, it should be mentioned that the reference point under BCR approach is the look-up PD that serves as a benchmark for the institution to make conservative results and its completely different procedure. Despite that, in both approaches, whatever the adjustments are being upward or downward, the scaler is seen as fitting of the PDs estimates for the better reflection of the real default occurrence.

Choice of the parameters

Considering all the methods, the main difference related to the choice of the parameters is that under Bayesian Approach they are regarded as stochastic while in others they are fixed. In this particular case, the use of the Bayesian Approach for the PD estimation could be a solution for those institutions that are not comfortable with the choice of the confidence level under CBA method. Although Pluto and Tasche (2011) remain the issue of the choice of confidence level open, they "argue for moderate confidence levels". On the other side, the flexibility that such choice brings could be useful in the way that it allows for controlling the spread of PDs estimates, i.e., the higher the confidence level, the higher will be the spread, and vice versa. In this context, the use of scaling procedure toward portfolio upper bound PD is suggested by authors instead of the central tendency, due to the possibility to adjust the degree of conservatism, as already mentioned. This is also valid under the BCR approach.

However, not only the choice of confidence level received the attention of market participants. The choice of asset and year-to-year correlation as input parameters, in the extended to the multi-period case Confidence-Based Approach, is being under expert consideration and should be properly justified based on empirical evidence and specific characteristics of the portfolio in analysis. According to Article 153 of the Regulation (EU) No 575/2013, the asset correlation should be standing between 12% and 24%. Although simply the fact that three parameters should be predefined increases the level of complexity and subjectivity of this multi-period version of the approach.

Grade level allocation

In terms of PD's grade level allocation, the Confidence-Based Approach could be used, as it provides the estimation for each of the grades. However, some concerns have a place: "if the relative number of defaults in one of the better rating grades is significantly higher than those in lower rating grades (…), then (…) PD estimates can turn out to be non-monotone". That being the case, the authors question if it should be taken as "an indicator for the non-correctness of the original ranking". On the other side, the Cumulative Accuracy Profile (CAP) approach could be considered as a possible solution for the previously mentioned issue. It does not just estimate the PDs for each rating grade by considering the information for this particular grade, but implicitly takes into consideration all the available data (see Burgt, 2008, for more details) and “as such, the method is based on the economic use of the data, which is available in a limited way”.

The choice of the prior distribution

Under the Bayesian approach, the specification of the prior distribution is not only the main requirement, but it is also considered as a main assumption of the method. According to Castor and Gerhardsson (2017), “is an evident element of uncertainty present in the Bayesian approaches compared with the other methods”. In addition, it should be highlighted the subjectivity which is entailed in case that the institution chooses the subjective prior distribution that “may vary from one expert to another” (Venter, 2016). On the other side, the use of prior distribution allows for the incorporation of the previous experience or expert view as a prior belief of the parameter’s behaviour. This gives institutions some flexibility since almost any distribution could be selected as a prior, depending once again on the belief of the parameter’s true value (Castor and Gerhardsson, 2017).

In conclusion, the risk managers should consider that the final estimates from different methods could not represent the same and consequently could not be compared. As an example, Castor and Gerhardsson (2017) in their paper underline that under the Bayesian approach the resulting PD is the mean of the sample derived from the posterior distribution and therefore is “a point estimate of the sought probability”. On the other side, the BCR approach and accordingly the Confidence-Based approach, produce the estimates of PD through upper bounds from the confidence level, which could be viewed as a conservative method, however, cannot be considered as equivalent or compared with the estimated PD from Bayesian approach.

3. METHODOLOGY

To satisfy the IRB requirements, the PD estimates should be derived from historical default rates, not be based purely on judgmental consideration, and incorporate a certain level of conservatism. This regards all types of portfolios. In this context, the concern about Low Default Portfolios has been the subject of too much discussion and consequently, many proposals were suggested to solve this issue. Some of them were previously mentioned and it is possible to conclude that there is no answer which is more likely to be applied. All of them presents some limitations as well as some advantages while compared to each other.

Sometimes final estimates of different approaches could not be equivalent, however, they could be combined. For this purpose, the method presented in this paper is a combination of those approaches to avoid some limitations or to add some flexibility for the estimation procedure. The approach is based on the rationale that no specific model alone can consistently achieve the best predictive results, but a combination of models can best approximate the actual data generation process. When compared to a single model, ensemble learning or model averaging proved to improve traditional and machine learning methods and is being increasingly used in many scientific areas (see, e.g., Bravo et al., 2021; Ayuso et al., 2021; Stell (2020; Bravo & Ayuso, 2020, 2021; Ashofteh & Bravo, 2021a,b and references therein).

3.1. DESCRIPTION OF THE METHOD

The method proposed in this study is a sort of estimation based on quantitative information that is considered as a start point from which further adjustments such as judgmental consideration or application of the margin of conservatism (MoC) could be made. In this way, the method applies the Cumulative Accuracy Profile (CAP) approach as a primary technique that hence is adjusted by so-called *portfolio reference PD*. The *portfolio reference PD* is an estimated portfolio PD, that is obtained through the Bayesian approach in which the prior distribution is selected based on the output from Confidence-Based method (CBA). Subsequently, for the grade-level allocation, the scaling procedure is applied.
Accordingly, this method is performed by few steps.

1. ***First step***: grade-level PDs are estimated applying the Confidence-Based Approach proposed by Pluto & Tasche (2011);
2. ***Second step***: is based on Bayesian inference where the estimations from the Confidence-Based Approach (step 1) are used as an input to determine the prior belief assuming beta distribution. Afterward, the data is added, the *portfolio reference PD* is estimated.
3. ***Third step***: modelling the power curve and deriving the calibrations from this curve to create the grade-level PDs according to the method proposed by Burgt (2008)
4. ***Fourth step***: the PDs grade-level estimates from the Cumulative Accuracy Profile approach (step 3) would be scaled towards Bayes estimator – the *portfolio reference PD* (step 2), which is considered as a reference point.

The figure below illustrates the method proposed by this research, summarising the aforementioned steps:

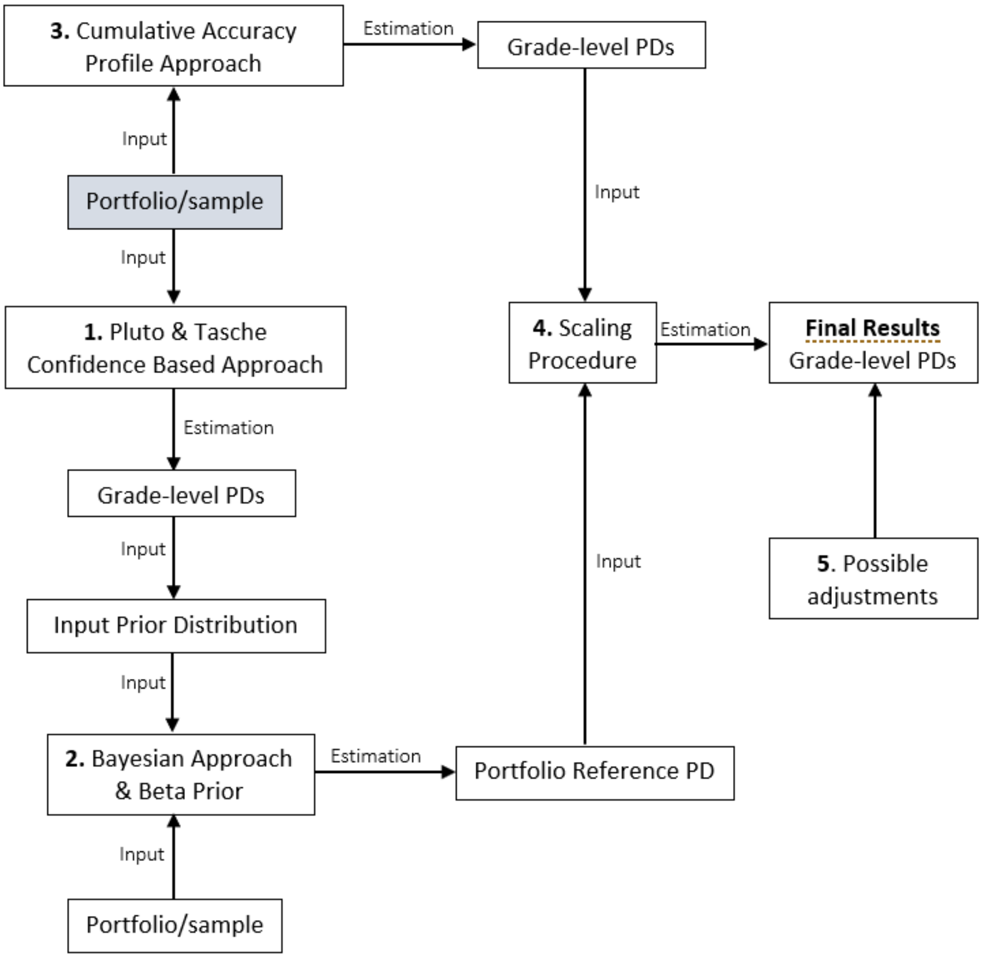

Figure 3.1 – Figure of the proposed Method

3.2. THEORETICAL BACKGROUND

To provide a more comprehensive view of the aforementioned approach, this section contains concepts, theory, and assumptions provided by different authors which methods formed the basis for the methodology proposed by this paper.

3.2.1. Calibration through the Cumulative Accuracy Profile

The Cumulative Accuracy Profile curve, knowing as a CAP curve, is the most used method by institutions for discriminatory power measuring, namely how good the model is in discriminating good obligors from bad obligors. However, a CAP curve could also be applied from the calibration perspective.

To derive a CAP curve, the risk grades are sorted in descending order by level of riskiness, from bad grades to good grades, considering the total number of obligors and the total number of defaults. After that, a CAP curve is modelled through the cumulative percentage of defaults as a function of the cumulative percentage of obligors. The figure below illustrates the CAP curve for the "user model" compared with the curve of the perfect model and random model. Assuming that the perfect model represents the perfect discrimination of bad and good clients and random model has no discriminatory power.

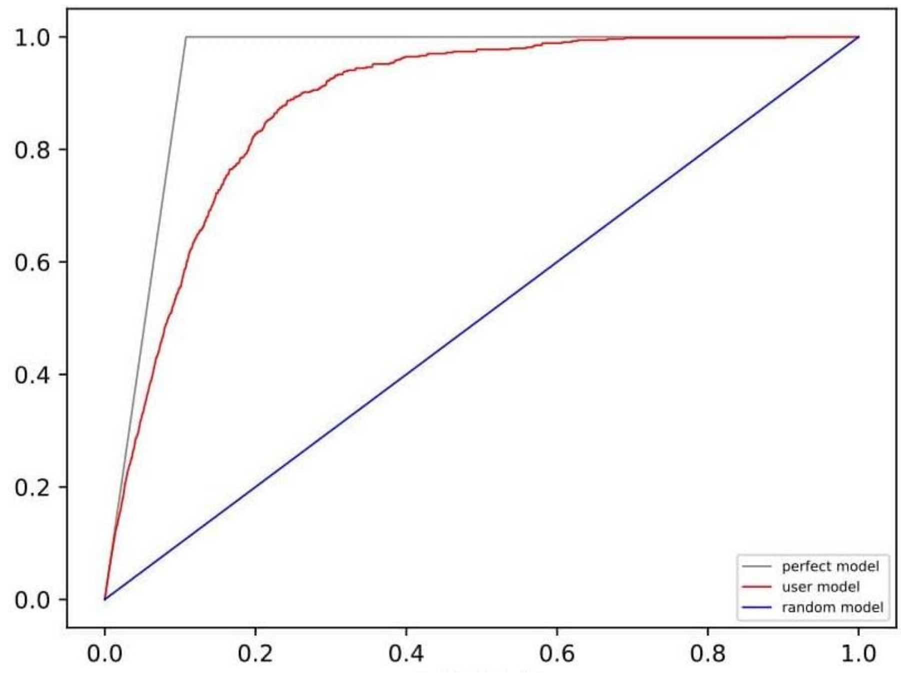

Figure 3.2 - CAP Curve (Source: Burgt, 2008)

In 2008, Burgt introduces the mathematical function for modelling the concave shape of CAP curve, where the main assumption is based on the exponential increase of the PDs with the rating class. For more information about the validity of the assumption see Burgt (2008). The function is presented below:

$$\mathrm{y(x)} = \frac{1 - \mathrm{e}^{-\mathrm{kx}}}{1 - \mathrm{e}^{-\mathrm{k}}} \tag{1}$$

The parameter ***k*** represents the concavity of the curve and when related with the Figure 3.2 - CAP Curve can be interpreted as follows:

- When ***k*** → ꝏ, ***y*** ≈ 1 and the CAP curve corresponds to the perfect model with perfect discriminatory power and the area under the curve is closed to 1;
- When ***k*** → 0, ***y*** ≈ ***x,*** and the CAP curve corresponds to the random model with no discriminatory power and the area under the curve is equal to 0.5.

Another indicator related to the model performance, mentioned above, is the relation between the area under the curve (**A**) and the accuracy ratio (**AR**):

$$\boldsymbol{AR} = \frac{\boldsymbol{A} - \boldsymbol{A_{random}}}{\boldsymbol{A_{perfect}} - \boldsymbol{A_{random}}} \approx \boldsymbol{2A - 1} \tag{2}$$

Where:
$\boldsymbol{A_{random}}$ = 0.5
$\boldsymbol{A_{perfect}}$ = 1

The method of calibrating the PDs is then performed by fitting the constructed CAP curve to the function presented in equation (1). First, the value of the concavity parameter ***k*** is computed by minimizing the root mean square error (**RMSE**) ***E***:

$$\boldsymbol{E} = \sqrt{\frac{\mathbf{1}}{\boldsymbol{N}}\sum_{\boldsymbol{i=1}}^{\boldsymbol{N}}\left\{\boldsymbol{y_i} - \frac{\mathbf{1 - exp}(-\boldsymbol{kx_i})}{\mathbf{1 - exp}(-\boldsymbol{k})}\right\}^{\mathbf{2}}} \tag{3}$$

Where:
$\boldsymbol{y_i}$ – observed cumulative percentage of obligors in each of rating class *i*
$\boldsymbol{x_i}$ – observed cumulative percentage of defaults in each of rating class *i*
N – number of rating classes *i*

Equation (3) implicitly assumes that residuals are normally distributed (see Burgt, 2008, for more details). However, a simpler method could be applied to get the value of ***k*** related to the area under the curve (**A**).

The **A** approaches 1 when k→∞. This is the case when the rating system has perfect discriminatory power. When k→0 the **A** approaches 0.5, which corresponds to a rating system with no discriminatory power. As $\mathbf{e}^{-\mathbf{k}}$ tends to go to zero very fast, equation (4) can be approximated by the equation (5):

$$\mathbf{A} = \frac{\mathbf{1}}{\mathbf{1 - e^{-k}}} - \frac{\mathbf{1}}{\boldsymbol{k}} \tag{4}$$

if assuming that **A** is higher than 0.8:

$$\boldsymbol{k} \approx \frac{\mathbf{1}}{\mathbf{1} - \boldsymbol{A}} \tag{5}$$

Finally, given the value of the parameter $\boldsymbol{k}$, the PDs are derived from the CAP curve applying the following equation:

$$\boldsymbol{PD}(\boldsymbol{R}) = \langle \boldsymbol{D} \rangle \frac{\boldsymbol{dy}}{\boldsymbol{dx}} \tag{6}$$

Where:

$\langle \boldsymbol{D} \rangle$ –observed average default rate (total number of defaults by the total number of obligors of the portfolio)

Combining equation (1) with the equation above, the derivative of the CAP function in (1) is given by:

$$\boldsymbol{PD}(\boldsymbol{R}) = \frac{\boldsymbol{k}\langle \boldsymbol{D} \rangle}{\mathbf{1} - \boldsymbol{e}^{-\boldsymbol{k}}} \boldsymbol{exp}\{-\boldsymbol{k}\boldsymbol{x}_{\boldsymbol{R}}\} \tag{7}$$

with $\boldsymbol{x}_{\boldsymbol{R}}$ denoting the cumulative percentage of obligors in rating class R, computed as

$$\boldsymbol{x}_{\boldsymbol{R}} = \frac{\boldsymbol{Z}_{\boldsymbol{N}} + \boldsymbol{Z}_{\boldsymbol{N}-\mathbf{1}} + \cdots + \boldsymbol{Z}_{\boldsymbol{R}-\mathbf{1}} + \left(\frac{\boldsymbol{Z}_{\boldsymbol{R}}}{\mathbf{2}}\right)}{\boldsymbol{Z}} \tag{8}$$

Where:

$\boldsymbol{Z}$ – total number of obligors

$\boldsymbol{Z}_{\boldsymbol{i}}$ – number of obligors in rating class *i*

$\boldsymbol{N}$ – most risky rating class

3.2.2. Pluto & Tasche Confidence Based Approach

The next point of this section is the more detailed view of Pluto and Tasche approach. As was already mentioned, this method could be used considering the following assumptions:

- No defaults were observed with assumptions of independence of the events;
- Few defaults were observed with an assumption of independence of the events;
- Correlated defaults events with few or no defaults observed;
- A potential extension of the calibration by scaling factors;
- A potential extension to the multi-period case.

Assumption: defaults were observed with an assumption of independence of the events

Although the methodology of this paper applies the method considering only the case of a few defaults and assumption of independence, further information could be consulted in Pluto and Tasche (2011) for more details. Also, the potential calibration by scaling factors will take place in this section.

The crucial assumption of this model is that the ordinal borrower ranking is correct, namely that PDs of each rating grade reflects the decreasing credit-worthiness of the grades:

$$\boldsymbol{p_n \leq \ldots \leq p_N} \tag{9}$$

Where:
$\boldsymbol{p_n}$ – probability of default of the higher rating grade
$\boldsymbol{p_N}$ – probability of default of the lowest rating grade

Subsequently, is also assumed from equation (9) that for the *most prudent estimate* of the higher rating grade (***p****n*), the probability of this rating grade is equal to the probability of the lowest rating grade (***p****N*):

$$\boldsymbol{p_n = \ldots = p_N} \tag{10}$$

From the last assumption is clear that, if the rating grades do not differ in their riskiness, the size of the portfolio is homogeneous:

$$\boldsymbol{s_n = \ldots = s_N} \tag{11}$$

Where:
$\boldsymbol{s_n}$ – total number of obligors in the higher rating grade
$\boldsymbol{s_N}$ – total number of obligors in the lowest rating grade

Following the fact that the number of defaults is binomially distributed as long as the assumption of independence of default events are considered, the probability of not observing more that $\mathbf{d_j}^*$ defaults are given by the expression:

$$\boldsymbol{\sum_{k=0}^{d_j^*} \binom{s_j^*}{k} p_j^k (1 - p_j)^{s_j^* - k}} \tag{12}$$

Where:
$\mathbf{s_j^*} = \sum_{i=j}^{c} s_i$, represents the sample size used in determining the PD for rating grade **j**

$\mathbf{d_j^*} = \sum_{i=j}^{c} d_i$, represents the sum of defaults considered for estimating the PD for rating grade **j**

Consequently, the general definition (Venter, 2016) of most prudent confidence region at predetermined level **γ**, for the probability of default of rating grade j ($\mathbf{p_j}$) is given by the inequality:

$$\boldsymbol{1 - \gamma \leq \sum_{k=0}^{d_j^*} \binom{s_j^*}{k} p_j^k (1 - p_j)^{s_j^* - k}} \tag{13}$$

It is worth to mention, that for determining the confidence region at level ***γ*** for the lowest rating grade (***p****n*), this rating grade should be considered as a standalone portfolio, i.e., it is only taking into account the observation of this rating grade, as well as the respective number of defaults:

$$1-\gamma \leq \sum_{k=0}^{d_N} \binom{s_N}{k} p_N^k (1-p_N)^{s_N-k} \tag{14}$$

Summarizing all aforementioned above, the "confidence region can be described as the set of all admissible values of p_j with the property that the probability of not observing any default during the observation period is not less than 1 – γ", as referred by Pluto and Tasche (2011). The possible maximum value of p_j - as upper confidence bound, is obtained by solving analytically the equation (13) for a certain confidence level γ.

3.2.3. Scaling Factors Procedure

The scaling method is often used for the grade-level estimation by adjusting grade-level PDs through the application of the scaling factor. Lets scaling factor be represented by **K**:

$$Y_{Portfolio} = K\frac{p_n \times s_n + \ldots + p_N \times s_N}{s_n + \cdots + s_N} \tag{15}$$

and consequently:

$$p_{j,scaled} = Kp_j \qquad\qquad j = n, \ldots, N \tag{16}$$

where $\mathbf{Y_{Portfolio}}$ is a parameter towards which the estimates are scaled. Pluto and Tasche (2011) proposed the average portfolio default rate, or the upper confidence bound of the portfolio as a parameter and Benjamin et al. (2006) suggest the PD at the portfolio level.

3.2.4. Bayesian Approach

For the theoretical demonstration of the Bayesian Approach this paper uses the definitions, concepts, and proofs from the Casella and Berger (2002) Statistical Inference:

> *In the classical approach the parameter, ϑ, is thought to be an unknown, but fixed, quantity. A random sample $X_1, \ldots, X_n$ is drawn from a population indexed by ϑ and, based on the observed values in the sample, knowledge about the value of ϑ is obtained. In the Bayesian approach, ϑ is considered to be a quantity whose variation can be described by a probability distribution (called the prior distribution). This is a subjective distribution, based on the experimenter's belief, and is formulated before the data are seen (hence the name prior distribution). A sample is then taken from a population indexed by ϑ and the prior distribution is updated with this sample information. The updated prior is called the posterior distribution. This updating is done with the Bayes' Rule, hence the name Bayesian statistic. (p.324)*

Given this, the posterior distribution could be defined with the following equation:

$$\boldsymbol{\pi(\theta \mid x) = f(x|\theta)\pi(\theta)/m(x)} \tag{17}$$

and

$$\boldsymbol{(f(x \mid \theta)\pi(\theta) = f(x, \theta))} \tag{18}$$

Where:
$\boldsymbol{\pi(\theta)}$ – represents the prior distribution
$\boldsymbol{f(x|\theta)}$ – represents sampling distribution
$\boldsymbol{m(x)}$ **–** represents marginal distribution of X

The marginal distribution of **X** is given by:

$$\boldsymbol{m(x) = \int f(x|\theta)\pi(\theta)d\theta} \tag{19}$$

In this way, the posterior distribution is a conditional distribution of ϑ given the outcome of sample **x**, where ϑ is considered as a random variable. Accordingly, the mean of the posterior distribution could be used as an estimator of this variable.

Assumption: Binomial Bayes Estimator

Under this methodology, in accordance with Casella and Berger (2002), is assumed that $\mathbf{X_1, \ldots, X_n}$ is independent and identically distributed random variables, all Bernoulli trials with success probability ***p***. Then their sum $Y = \sum X_i$ is a Binomial distribution with parameters **n** and ***p***. It is also assumed, that the prior distribution on ***p*** is a Beta distribution, parameterized by two parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

To summarize, the joint distribution of **Y** and **p** follows:

$$f(y,p) = \left[\binom{n}{y}\ p^y(1-p)^{n-y}\right]\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\ p^{\alpha-1}(1-p)^{\beta-1}\right]\binom{Conditional \times Marginal}{f(y|p) \times \pi(p)}$$

$$\boldsymbol{= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\ p^{y+\alpha-1}(1-p)^{n-y+\beta-1}} \tag{20}$$

and the marginal probability density function of **Y**, a distribution is known as the beta-binomial, is given by:

$$\boldsymbol{f(y) = \int_0^1 f(y,p)dp = \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(y+\alpha)\Gamma(n-y+\beta)}{\Gamma(n+\alpha+\beta)}} \tag{21}$$

Finally, combining all aforementioned above, the posterior distribution of **p** given **y** is expressed in the next equation by beta distribution with parameters $\mathbf{y + \alpha}$ and $\mathbf{n - 1 + \beta}$:

$$f(\boldsymbol{y}) = \frac{\boldsymbol{f(y,p)}}{\boldsymbol{f(y)}} = \frac{\boldsymbol{\Gamma(n+\alpha+\beta)}}{\boldsymbol{\Gamma(y+\alpha)\Gamma(n-y+\beta)}}\,\boldsymbol{p^{y+\alpha-1}(1-p)^{n-y+\beta-1}} \tag{22}$$

Bayes estimator

As previously mentioned, the mean of the posterior distribution is used to obtain the natural estimate for the Bayes estimator of the parameter ***p***:

$$\boldsymbol{\hat{p}_B} = \frac{\boldsymbol{y+\alpha}}{\boldsymbol{\alpha+\beta+n}} \tag{23}$$

Originally, ignoring the effect of the data, the best estimate of ***p*** of the prior beta distribution, which is the mean, would be $\alpha/(\alpha+\beta)$. However, without the prior distribution, the naïve estimator of ***p*** of the chosen sample is considered as y/n. In this way, combining all this information, the equation (23) presented above, could be expressed by following:

$$\boldsymbol{\hat{p}_B} = \left(\frac{\boldsymbol{n}}{\boldsymbol{\alpha+\beta+n}}\right)\left(\frac{\boldsymbol{n}}{\boldsymbol{y}}\right) + \left(\frac{\boldsymbol{\alpha+\beta}}{\boldsymbol{\alpha+\beta+n}}\right)\left(\frac{\boldsymbol{\alpha}}{\boldsymbol{\alpha+\beta}}\right) \tag{24}$$

Where $\boldsymbol{\hat{p}_B}$ is a linear combination of the prior mean and the sample mean, with the weights being determined by α, β, and n.

3.3. ASSUMPTIONS OF THE METHOD

The method proposed in this research applies the Cumulative Accuracy Profile approach as a start point from where grade-level PDs are estimated. The main reason why this approach is considered as a better choice despite the other is that implicitly all the available data is considered for default rate calculation and not only the data from specific rating classes. This adds value when the number of obligors in a specific rating grade might is low, which is not uncommon under Low Default Portfolios. And the second reason is based on the assumption that PD varies exponentially with the rating class, which guarantees the monotonicity of the estimating PDs.

On the other side, the Confidence-Based Approach is used as an input for the prior distribution, considering the assumption of independence of default events. This is particularly important due to the following reasons:

- This assumption allows for the uniqueness of the solution for PDs estimates;

- This approach assumes that the number of defaults is binomially distributed (as long as they are independent), which could be expressed in terms of the beta distribution function, assuming that ***p*** varies according to parameters α and β;

- The results are significantly more conservative when compared with the multi-period approach and less conservative under the correlated approach, however, in those cases, the previous assumption will not be applicable (see Pluto and Tasche, 2011, for more details;

Even though the independence of the defaults is often unrealistic and considering all aforementioned, this approach by presenting the desired distribution and delivering conservative results is then chosen as a prior of Bayesian inference to incorporate some conservativism to the estimates.

However, the following should be noted regarding the choice of the parameters under the Bayesian Approach: the parameters of the prior distribution should be selected before considering the data since is considered as a prior belief. Otherwise, the data influence the selection, and the prior belief loses its significance.

Finally, the use of scaling procedure is seen as the fitting of the estimates (derived from the power curve) through the *portfolio reference PD* (from Bayes inference) or rather as a combination of the information that allows for an adjustment considering additional information. On the other side, the method allows for two ways of scaling procedure, namely upward or downward, depending on the value of the *reference PD*.

3.4. DATA DESCRIPTION

For demonstrating the proposed method, the data was collected from Moody's Investors Service. More in detail, the data contains the history of sovereign bond ratings, namely the sovereign's foreign-currency bond rating as well as the default history, considering only sovereign defaults on bond obligations of foreign currency. Consequently, the table below summarises the obligors with sovereign bonds rating and respective defaults for each year within the 1985-2019 period. Also, the annual default rates for each year of the observed period are provided.

Year	# Obligors	# Defaults	DR_{Annual}
1985	13	0	0.0%
1986	21	0	0.0%
1987	23	0	0.0%
1988	28	0	0.0%
1989	33	1	3.0%
1990	36	0	0.0%
1991	36	0	0.0%
1992	37	0	0.0%
1993	42	0	0.0%
1994	50	0	0.0%
1995	55	0	0.0%
1996	71	0	0.0%
1997	88	0	0.0%
1998	95	1	1.1%
1999	100	3	3.0%
2000	100	1	1.0%
2001	101	1	1.0%
2002	100	1	1.0%
2003	99	1	1.0%
2004	101	0	0.0%
2005	102	1	1.0%
2006	104	1	1.0%
2007	109	0	0.0%
2008	110	1	0.9%
2009	109	0	0.0%
2010	113	1	0.9%
2011	115	0	0.0%
2012	120	2	1.7%
2013	125	1	0.8%
2014	129	1	0.8%
2015	131	1	0.8%
2016	134	1	0.7%
2017	137	4	2.9%
2018	138	1	0.7%
2019	143	0	0.0%

Table 3.1 – Sovereign Rating History

From the table above is possible to conclude that it is a sample with reduced number of defaults, with a maximum of 4 defaults per year. The same is observed regarding the number of obligors, that do not exceed the total of 143 per year. Therefore, this could be used as a trial sample, given the LDP nature of the underlying portfolio. The figure above summarises the information presented in the table and shows the volatility of the observed default rates in parallel with the slow but constant grow of the obligors per year since 1985:

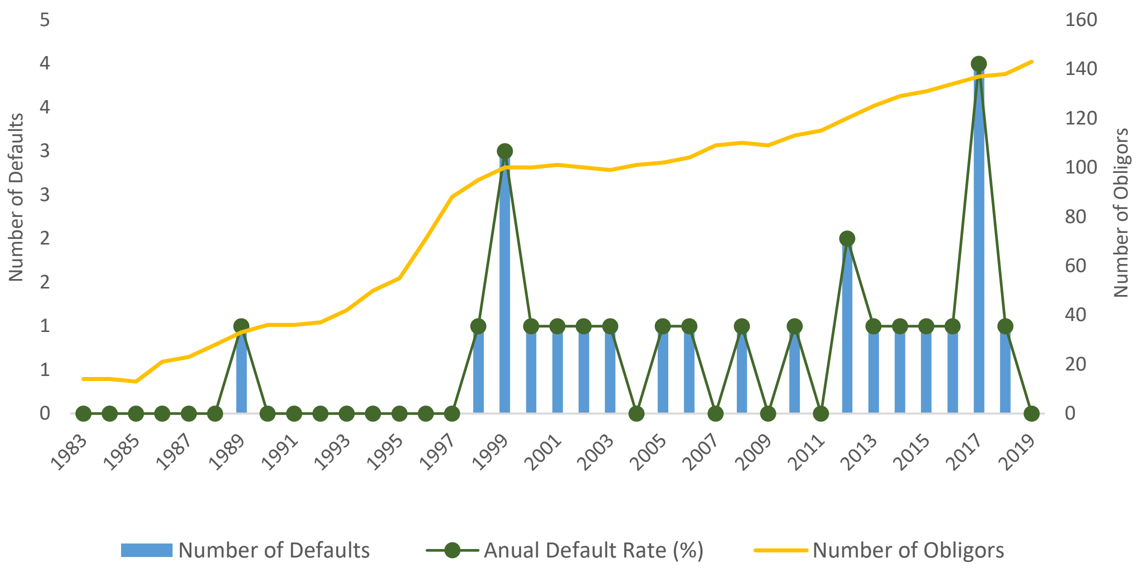

Figure 3.3 – Sovereign Bond Defaults since 1985 (Source: Moody's, 2020)

Nevertheless, the information about each year is not enough neither from the analytical perspective nor from the regulatory view. In the way to provide more deepest view about each rating, obligors have been sorted according to Moody's long-term rating scale, taking into consideration the following assumptions:

- The rating of each obligor at the end of each year was considered;
- For defaulted obligors, the rating at the moment of default event was considered;

In this paper, the observations within the periods are treated as independent and aggregated to a one-time window. Subsequently, the data of all years have been pooled as in the one-year case for each of ratings for the periods from 1985-2019, 2010-2019, and 2015-2019, as well as the observed default rates (DRs) were calculated respectively for each of those periods, as presented in the next tables:

1985-2019			
Rating	**# Obligors**	**# Defaults**	**DR_{Obs}**
Aaa	485	0	0.0%
Aa1	128	0	0.0%
Aa2	156	0	0.0%
Aa3	147	0	0.0%
A1	138	0	0.0%

1985-2019			
A2	152	0	0.0%
A3	139	0	0.0%
Baa1	141	0	0.0%
Baa2	153	0	0.0%
Baa3	215	0	0.0%
Ba1	186	0	0.0%
Ba2	125	0	0.0%
Ba3	142	0	0.0%
B1	230	0	0.0%
B2	189	0	0.0%
B3	158	8	5.1%
Caa1	78	5	6.4%
Caa2	26	3	11.5%
Caa3	21	5	23.8%
Ca	6	2	33.3%
C	3	1	33.3%
Total:	3018	24	0.795%

Table 3.2 - Sovereign Rating History (1985-2019)

2010-1019			
Rating	**# Obligors**	**# Defaults**	**DR_{Obs}**
Aaa	140	0	0.0%
Aa1	28	0	0.0%
Aa2	64	0	0.0%
Aa3	70	0	0.0%
A1	60	0	0.0%
A2	39	0	0.0%
A3	54	0	0.0%
Baa1	52	0	0.0%
Baa2	78	0	0.0%
Baa3	96	0	0.0%
Ba1	69	0	0.0%
Ba2	36	0	0.0%
Ba3	83	0	0.0%
B1	126	0	0.0%
B2	90	0	0.0%
B3	110	2	1.8%
Caa1	33	1	3.0%
Caa2	20	3	15.0%
Caa3	18	3	16.7%
Ca	2	2	100.0%
C	3	1	33.3%
Total:	1271	12	0.944%

Table 3.3 - Sovereign Rating History (2010-2019)

2015-2019			
Rating	**# Obligors**	**# Defaults**	**DR_{Obs}**
Aaa	62	0	0.0%
Aa1	14	0	0.0%
Aa2	37	0	0.0%
Aa3	29	0	0.0%
A1	30	0	0.0%
A2	24	0	0.0%
A3	38	0	0.0%
Baa1	18	0	0.0%
Baa2	42	0	0.0%
Baa3	40	0	0.0%
Ba1	35	0	0.0%
Ba2	20	0	0.0%
Ba3	43	0	0.0%
B1	58	0	0.0%
B2	63	0	0.0%
B3	73	1	1.4%
Caa1	16	0	0.0%
Caa2	17	2	11.8%
Caa3	12	3	25.0%
Ca	0	1	0.0%
C	2	0	0.0%
Total:	673	7	0.892%

Table 3.4 – Sovereign Rating History (2015-2019)

The data was split into three different periods since the time horizon that is chosen for estimation purposes might contain different sources of information. In this way, the period from 1985-2019 provides the historical information from the perspective of a long-time horizon while the period from 2010-2019 could be considered as “recent data”. Finally, the period from 2015-2019 is chosen as an example considering the minimum period prescribed by regulators, where is highlighted that “the length of the underlying historical observation period used shall be at least five years”. However, in the last example, the rating Ca does not have any obligor but presents one default. This occurs due to the assumption that the rating that is assigned to each obligor, is the rating at the end of the year, while the default rating is the rating at the moment of default. Sometimes it might happen that the rating of the defaulted obligor is not equal to the rating that this obligor has at the end of the year. Although, it is important to consider the rating that occurs at the time of the default event for the correct reflection of the rating system.

Given the distribution of obligors and defaults presented above and considering the different time horizons, these samples own general characteristics of Low Default Portfolios and will be considered as a proxy sample for further analysis.

3.5. APPLICATION OF THE METHOD

The results will be presented step by step using the sample of sovereign's foreign-currency bonds; however, different historical observation periods were considered for each of the methods presented by the methodology.

3.5.1. Pluto & Tasche Confidence-Based Approach (based on the data from 1985-2019)

The estimates from the Confidence-Based Approach will be used as an input for the prior distribution of Bayesian inference. The following is assumed:

- Independence of default events assuming the binomial distribution;
- One period case: the sum of one-year frequencies for each of grades over the time horizon from 1985-2019 for both the number of obligors and the number of defaulted obligors.
- Confidence level of 75%

Rating	# Obligors (1985-2019)	# Defaults (1985-2019)	P&T Indep (1985-2019)
Aaa	485	0	0.936%
Aa1	128	0	1.110%
Aa2	156	0	1.172%
Aa3	147	0	1.251%
A1	138	0	1.337%
A2	152	0	1.432%
A3	139	0	1.553%
Baa1	141	0	1.680%
Baa2	153	0	1.837%
Baa3	215	0	2.042%
Ba1	186	0	2.417%
Ba2	125	0	2.873%
Ba3	142	0	3.294%
B1	230	0	3.951%
B2	189	0	5.829%
B3	158	8	9.579%
Caa1	78	5	14.409%
Caa2	26	3	24.496%
Caa3	21	5	34.286%
Ca	6	2	50.199%
C	3	1	67.365%

Table 3.5 –Results from Pluto & Tasche Confidence-Based Approach

The data from the 1985-2019 period was applied to use all available data and incorporate as much information as possible. These results will be used as a prior belief of the defaults behaviour and therefore must also reflect the level of conservatism; given the results from the table presented above, it can be concluded the estimates are quite conservative.

Although there is a possibility of scaling estimated PDs towards the average portfolio default rate or portfolio upper confidence bound, the significant part of the conservatism could be lost and therefore the scaler will not be applied at this step.

The 75% confidence level was chosen since higher confidence levels induce a significant increase in the resulting estimates.

3.5.2. Bayesian Approach & Beta Prior

For the Bayesian inference, the results from the previous step are used to determine the prior belief, namely the prior that is assumed to be Beta distributed. In this way, the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ were estimated from the Table 3.5:

α	β
0.235	1.884

Table 3.6 – Parameters α and β of the Prior Distribution

In this particular example, the estimation of those parameters was based on the data from 1985-2019. Meaning that the same sample (used for the prior) cannot be considered in the future for the posterior distribution estimation. To avoid the data influence on parameters estimation by considering the same sample and incorporate not only historical experience, but also the recent and relevant information, the data from 2010-2019 was applied in conjunction with the prior belief.

Consequently, given the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ from Table 3.6 and the information about the data (**n**=1271 and **y**=12, where **n** is considered to be the number of obligors and **y** the number of defaults) from the Table 3.3, the equation (23) could be applied to obtain the *Portfolio Reference PD*:

Portfolio Reference PD
0.961%

Table 3.7- Portfolio Reference PD

3.5.3. Cumulative Accuracy Profile Approach (based on the data from 2010-2019)

At this point, given the *Portfolio Reference PD*, the new method is applied for the grade-level PDs estimation. The results from the Cumulative Accuracy Profile Approach are based on the data from 2010-2019 period. Since this method is not sensitive to the low number of obligors in the rating grade, it was possible to apply the data with a relatively reduce size. However, the main reason why the data from 2010-2019 was chosen instead of the period from 1985-2019, was to give the light and weight for the more recent and relevant information (from the historical point of view), even if at first place it might be seen as being a scarce information choice. That fact must not be seen as a devaluation of historical experience, but only as a choice or judgemental assumption that could be different among the institutions. For the choice of the temporal period, the lending standards of the underlying portfolio should also be considered.

The table below contains a summary of the results of the CAP method:

Rating	# Obligors (2010-2019)	# Defaults (2010-2019)	CAP (2010-2019)
Aaa	140	0	0.000%
Aa1	28	0	0.000%
Aa2	64	0	0.000%
Aa3	70	0	0.000%
A1	60	0	0.000%
A2	39	0	0.000%
A3	54	0	0.000%
Baa1	52	0	0.000%
Baa2	78	0	0.000%
Baa3	96	0	0.000%
Ba1	69	0	0.000%
Ba2	36	0	0.000%
Ba3	83	0	0.000%
B1	126	0	0.001%
B2	90	0	0.029%
B3	110	2	0.648%
Caa1	33	1	5.898%
Caa2	20	3	13.375%
Caa3	18	3	24.057%
Ca	2	2	32.767%
C	3	1	35.398%

Table 3.8 –Results from Cumulative Accuracy Profile Approach

These results are less conservative while comparing with the results from the Confidence-Based Approach (Table 3.5). However, if comparing with the historical information, this approach seems closer in a way that in the higher rating grades there is no occurrence of defaults since 1985, while the Confidence-Based Approach the rating scale starts immediately with a PD of 0.94%.

3.5.4. Scaling Procedure

The scaling procedure is applied for adjustment of the results from *Table 3.8* with the *Portfolio Reference PD.* The following table illustrates the results from this adjustment:

Rating	Grade PDs $_{CAP}$	# Obligors (2010-2019)	Weighted Grade PD	Grade-scaled PDs
Aaa	0.000%	140	0.000%	0.000%
Aa1	0.000%	28	0.000%	0.000%
Aa2	0.000%	64	0.000%	0.000%
Aa3	0.000%	70	0.000%	0.000%
A1	0.000%	60	0.000%	0.000%

Rating	Grade PDs $_{CAP}$	# Obligors (2010-2019)	Weighted Grade PD	Grade-scaled PDs
A2	0.000%	39	0.000%	0.000%
A3	0.000%	54	0.000%	0.000%
Baa1	0.000%	52	0.000%	0.000%
Baa2	0.000%	78	0.000%	0.000%
Baa3	0.000%	96	0.000%	0.000%
Ba1	0.000%	69	0.000%	0.000%
Ba2	0.000%	36	0.000%	0.000%
Ba3	0.000%	83	0.000%	0.000%
B1	0.001%	126	0.000%	0.001%
B2	0.029%	90	0.002%	0.032%
B3	0.648%	110	0.056%	0.693%
Caa1	5.898%	33	0.153%	6.314%
Caa2	13.375%	20	0.210%	14.320%
Caa3	24.057%	18	0.341%	25.756%
Ca	32.767%	2	0.052%	35.081%
C	35.398%	3	0.084%	37.898%
Total:		1271	**0.898%**	

Table 3.9 - Scaling Procedure

Once again, the sample of the 2010-2019 period was considered instead of the last 5 years or the longer period from 1985 due to the reflection of likely range of variability of default rates for one side and to put more significance of relatively recent information for other.

According to the results presented by the table above, it is possible to conclude that the average PD of the portfolio based on the distribution of obligors per grade from the historical period between 2010-2019, so-called weighted portfolio PD, is **0.898%** and the *Portfolio Reference PD* is **0.961%**. Meaning that the upward adjustments to the grade-level PDs were made by multiplying each of those grades by scaling factor. In this case, the scaling factor is equal to **1.072** and the final results are presented in the last column of Table 3.9, as Grade-scaled PDs.

3.5.5. Possible Adjustments

The ECB Guide of Internal Models in paragraph 49 points that "the results of the statistical model must be complemented by human judgment, especially by taking into account all information not included in the model".

One of the possible adjustments of the final results provided by *Table 3.9* could be set through Article 160, paragraph 1 of the Regulation (EU) No 575/2013. The article defines that "the PD of an exposure to a corporate or an institution shall be at least 0.03%". A similar idea was proposed by Castor and Gerhardsson (2017); however, the authors incorporate the "floor value" before the computation of the weighted portfolio PD. This methodology, instead, proposes the possible addition of the floor values or any other adjustments after the scaling procedure.

Although the underlying portfolio in this paper is composed of sovereign bonds, the abovementioned example will be used for the demonstration. These overrides are purely exemplary and could vary from one institution to another. In this way, the final results were adjusted and summarise in the following table:

Rating	Final Results
Aaa	0.03%
Aa1	0.03%
Aa2	0.03%
Aa3	0.03%
A1	0.03%
A2	0.03%
A3	0.03%
Baa1	0.03%
Baa2	0.03%
Baa3	0.03%
Ba1	0.03%
Ba2	0.03%
Ba3	0.03%
B1	0.03%
B2	0.03%
B3	0.69%
Caa1	6.31%
Caa2	14.32%
Caa3	25.76%
Ca	35.08%
C	37.90%

Table 3.10 – Final Results after Adjustments

No more adjustments were made in this research, trying to provide the methodology with as little expert judgment and supported to the most extent as possible by statistical analysis. However, there will always be a place for some additional corrections, as an example the recent requirement of appropriate adjustments and margin of conservatism (MoC) that should be incorporated in risk parameters.

4. RESULTS

The final estimates were examined through the comparison with the results from different steps of the methodology, i.e., from each of the approaches individually, and with the observed default rates. In addition, the forecast defaults were compared with the observed defaults applying the Binomial test.

4.1. COMPARISON OF THE RESULTS

The results of the Confidence-Based Approach, Cumulative Accuracy Profile, and Grade-scaled PD, from steps 1, 3, and 4 respectively, as well as the observed default rates over the period from 1985-2019 (Table 3.2 - Sovereign Rating History (1985-2019) are summarized in the next figure:

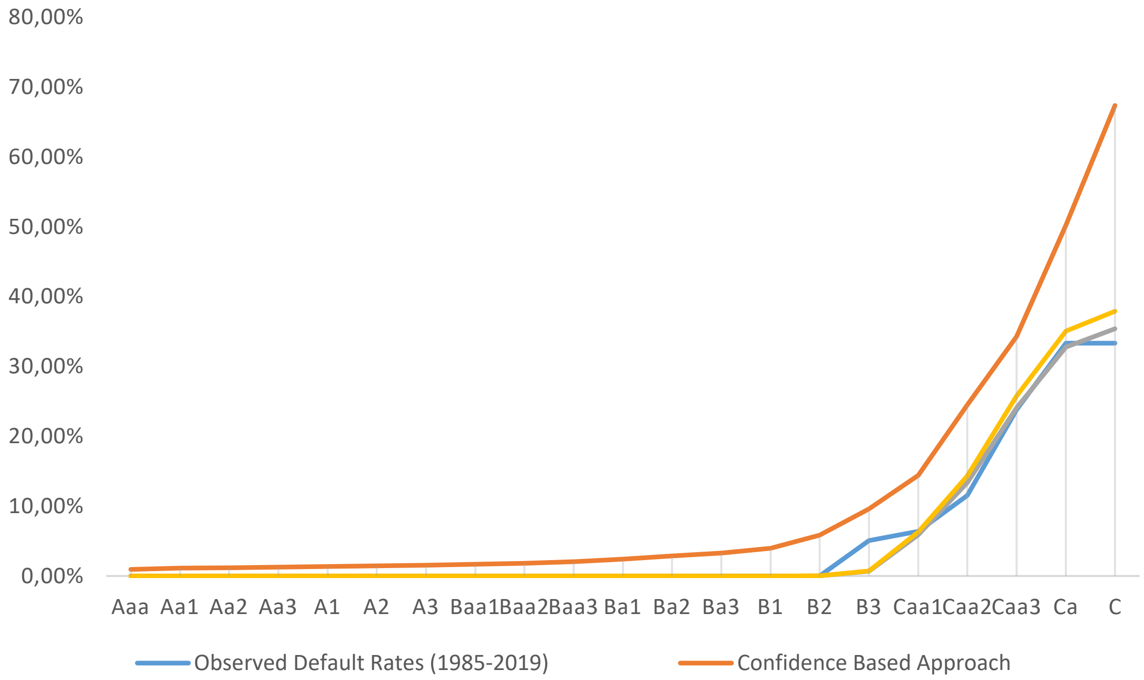

Figure 4.1 - Comparison of different results

From the figure above, the results from the Confidence-Based Approach are significantly higher than estimates from alternative approaches and real defaults. In this way, the main objective of incorporation of the conservatism through the prior belief on Bayesian Inference was achieved. On the other side, the estimates from the Cumulative Accuracy Profile are very closed to the observed default rates and therefore this approach seems to be the most appropriate for a starting point of the proposed method. Final, the Grade-scales PDs are slightly more conservative while compared with the real defaults.

4.2. VALIDATION OF THE RESULTS

To validate the results of the proposed methodology, the adjusted PDs from Table 3.10 were compared with the observed defaults rates used in the previous demonstration. For this purpose, the binomial test was applied. The test examines whether the observed DRs fall within a 75% confidence level around the estimated PDs.

The derivation of the confidence bounds was based on the following equation:

$$\left[PD_{Est} + N^{-1}[(1-\alpha)/2] * \sqrt{\frac{PD_{Est} * (1 - PD_{Est})}{N}} \right.$$

$$\left. PD_{Est} - N^{-1}[(1-\alpha)/2] * \sqrt{\frac{PD_{Est} * (1 - PD_{Est})}{N}} \right] \quad (25)$$

Where the first part refers to the upper bound and the second to the lower bound. The N^{-1} is the inverse of the cumulative normal distribution and α is the confidence bound, which was set at 75%. For N, the number of obligors for each rating grade from period 2010-2019 was assigned.

Rating	DR_{obs}	$Upper_{Bound}$	PD_{est}	$Lower_{Bound}$	Binomial Test
Aaa	0.00%	0.00%	0.03%	0.20%	Within
Aa1	0.00%	0.00%	0.03%	0.41%	Within
Aa2	0.00%	0.00%	0.03%	0.28%	Within
Aa3	0.00%	0.00%	0.03%	0.27%	Within
A1	0.00%	0.00%	0.03%	0.29%	Within
A2	0.00%	0.00%	0.03%	0.35%	Within
A3	0.00%	0.00%	0.03%	0.30%	Within
Baa1	0.00%	0.00%	0.03%	0.31%	Within
Baa2	0.00%	0.00%	0.03%	0.26%	Within
Baa3	0.00%	0.00%	0.03%	0.23%	Within
Ba1	0.00%	0.00%	0.03%	0.27%	Within
Ba2	0.00%	0.00%	0.03%	0.36%	Within
Ba3	0.00%	0.00%	0.03%	0.25%	Within
B1	0.00%	0.00%	0.03%	0.21%	Within
B2	0.00%	0.00%	0.03%	0.25%	Within
B3	5.06%	0.00%	0.69%	1.60%	Out
Caa1	6.41%	1.44%	6.31%	11.18%	Within
Caa2	11.54%	5.31%	14.32%	23.33%	Within
Caa3	23.81%	13.90%	25.76%	37.61%	Within
Ca	33.33%	0.00%	35.08%	73.90%	Within
C	33.33%	5.68%	37.90%	70.12%	Within

Table 4.1 - Upper and Lower Bounds of Estimated PDs

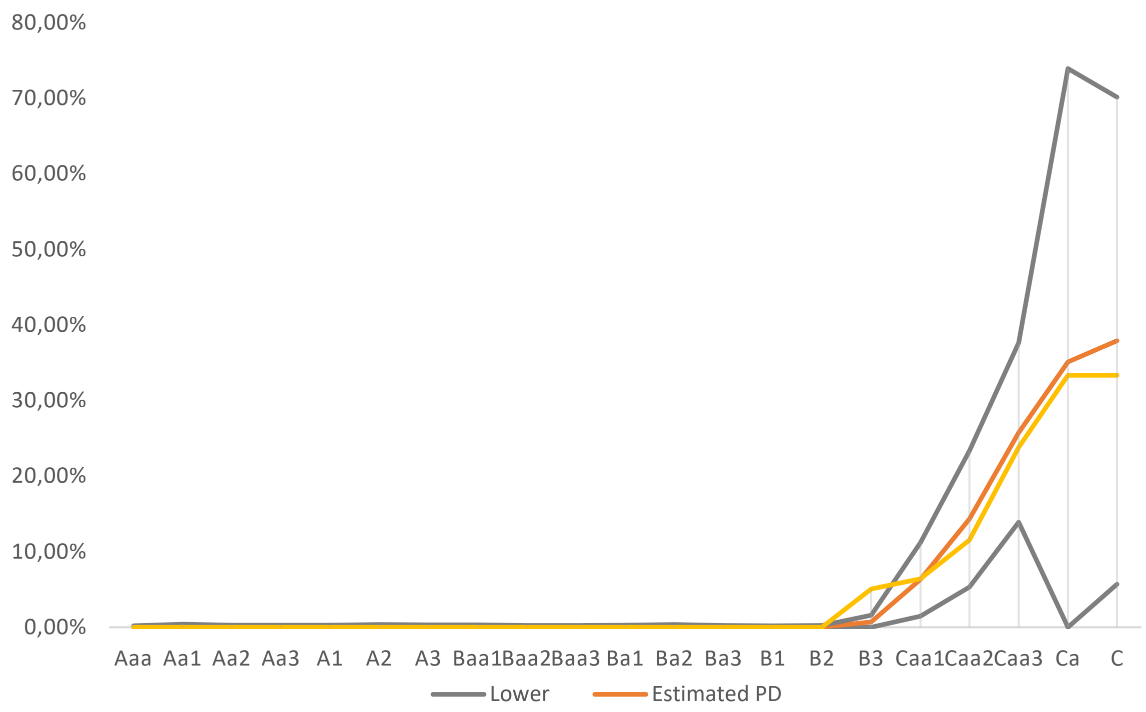

Figure 4.2 -Binomial Results

According to the analyses, the observed DRs are in line and slightly below compared to the estimated PDs and within the confidence interval, except the rating grade B3 for which the observed DR is above both the estimated PD and the upper boundary of the confidence interval. In this way, the estimated PD for rating grade B3 underestimates the true risk and further adjustments should be done to produce a more conservative result.

However, other parameters also should be considered for the application of the binomial test, such as the choice confidence level and the data used for computation of the confidence bounds. For example, increasing the confidence level will increase the confidence interval, as well the use of other data will adjust the bounds in another way.

4.3. SENSITIVITY ANALYSIS

This section aims to compare the results for each of the steps of the methodology, with assumptions different than those already presented. This demonstration allows comprehending how sensitive the outcomes could be by chasing some parameters separately and how to vary the final results by aggregating them.

Additionally, the resulting outcomes could be more or less conservative, and this allows the risk managers to examine which of them are more in line whit the institution's needs and probably adjust some of the assumptions in a different way. This is one of the advantages of the presented methodology as it introduces the possibility of being flexible with the assumption's choice.

4.3.1. Pluto & Tasche Confidence-Based Approach with different confidence levels

First, the results from Pluto & Tasche Approach, applying different confidence levels, assuming the independence of default events are presented in the table below. The period considering is from 1985 to 2019.

Rating	# Obligors (1985-2019)	# Defaults (1985-2019)	Confidence Levels				
			50%	75%	90%	95%	99%
Aaa	485	0	0.82%	0.936%	1.05%	1.12%	1.26%
Aa1	128	0	0.97%	1.110%	1.24%	1.33%	1.50%
Aa2	156	0	1.03%	1.172%	1.31%	1.40%	1.58%
Aa3	147	0	1.10%	1.251%	1.40%	1.50%	1.69%
A1	138	0	1.17%	1.337%	1.50%	1.60%	1.80%
A2	152	0	1.26%	1.432%	1.61%	1.71%	1.93%
A3	139	0	1.36%	1.553%	1.74%	1.86%	2.09%
Baa1	141	0	1.48%	1.680%	1.88%	2.01%	2.26%
Baa2	153	0	1.61%	1.837%	2.06%	2.19%	2.48%
Baa3	215	0	1.79%	2.042%	2.29%	2.44%	2.75%
Ba1	186	0	2.12%	2.417%	2.70%	2.89%	3.25%
Ba2	125	0	2.52%	2.873%	3.22%	3.43%	3.87%
Ba3	142	0	2.89%	3.294%	3.69%	3.94%	4.43%
B1	230	0	3.47%	3.951%	4.42%	4.72%	5.30%
B2	189	0	5.13%	5.829%	6.52%	6.95%	7.80%
B3	158	8	8.44%	9.579%	10.67%	11.36%	12.72%
Caa1	78	5	12.41%	14.409%	16.35%	17.57%	19.98%
Caa2	26	3	20.71%	24.496%	28.14%	30.41%	34.83%
Caa3	21	5	28.58%	34.286%	39.68%	42.99%	49.27%
Ca	6	2	39.31%	50.199%	59.94%	65.51%	75.00%
C	3	1	50.00%	67.365%	80.42%	86.47%	94.11%

Table 4.2 - Pluto & Tasche Confidence Based Approach with different confidence levels

Considering the results presented above, it is possible to conclude that choice of the confidence level allows for the adjustment of the level of conservatism, i.e., increasing the confidence level, the estimated PDs will also increase. However, this choice should be always properly justified by institutions.

4.3.2. Different scenarios for demonstration of the concavity accuracy

Inspiring on the Burgt (2008) paper, to verify the accuracy of the Cumulative Accuracy Profile method, the concavity (***k***) is calculated based on the different possible scenarios. In this way, the defaults are shifted from the original rating grade to the adjacent or opposite grades. The scenarios as well as the respective concavities are summarized in the following table.

	Original Scenario:	Concavity
1.	**1**D. in C, **2**D. in Ca, **3**D. in Caa3, **3**D. in Caa2, **1**D. in Caa1, **2**D in B3	**39.271**
	Scenarios:	
2.	**1**D. in C, **2**D. in Ca, **3**D. in Caa3, **3**D. in Caa2, **1**D. in Caa1, **1**D in B3, **1**D in B2	39.061
3.	**1**D. in C, **2**D. in Ca, **3**D. in Caa3, **3**D. in Caa2, **2**D. in Caa1, **1**D in B3	41.706
4.	**1**D. in C, **2**D. in Ca, **3**D. in Caa3, **3**D. in Caa2, **0**D. in Caa1, **3**D in B3	36.660
5.	**1**D. in C, **2**D. in Ca, **3**D. in Caa3, **4**D. in Caa2, **0**D. in Caa1, **2**D in B3	43.406
6.	**1**D. in C, **2**D. in Ca, **3**D. in Caa3, **2**D. in Caa2, **2**D. in Caa1, **2**D in B3	35.507
7.	**1**D. in C, **2**D. in Ca, **4**D. in Caa3, **2**D. in Caa2, **1**D. in Caa1, **2**D in B3	43.680
8.	**1**D. in C, **2**D. in Ca, **2**D. in Caa3, **4**D. in Caa2, **1**D. in Caa1, **2**D in B3	35.727
9.	**1**D. in C, **3**D. in Ca, **2**D. in Caa3, **3**D. in Caa2, **1**D. in Caa1, **2**D in B3	40.862
10.	**1**D. in C, **1**D. in Ca, **4**D. in Caa3, **3**D. in Caa2, **1**D. in Caa1, **2**D in B3	37.868
11.	**2**D. in C, **1**D. in Ca, **3**D. in Caa3, **3**D. in Caa2, **1**D. in Caa1, **2**D in B3	40.264
12.	**0**D. in C, **3**D. in Ca, **3**D. in Caa3, **3**D. in Caa2, **1**D. in Caa1, **2**D in B3	38.358
13.	**2**D. in C, **2**D. in Ca, **3**D. in Caa3, **3**D. in Caa2, **1**D. in Caa1, **1**D in B3	**55.733**
14.	**0**D. in C, **2**D. in Ca, **3**D. in Caa3, **3**D. in Caa2, **1**D. in Caa1, **3**D in B3	**28.828**
	Average Standard deviation 99% Confidence level	**39.781** 5.941 13.820

Table 4.3 – Different scenarios to access the concavity accuracy

The table above shows that the average concavity is equal to 39.781, which is very close to the concavity from the original scenario that is equal to 39.271. Additionally, this original value is within the confidence interval.

The estimated PD were computed for the maximum and minimum values of ***k***, scenarios 13 and 14 respectively and were compared with the original scenario, as shown below:

PD Original (2010-2019)	**PD Min** (2010-2019)	**PD Max** (2010-2019)
0.000%	0.000%	0.000%
0.000%	0.000%	0.000%
0.000%	0.000%	0.000%
0.000%	0.000%	0.000%
0.000%	0.000%	0.000%
0.000%	0.000%	0.000%
0.000%	0.000%	0.000%
0.000%	0.000%	0.000%
0.000%	0.000%	0.000%

PD Original (2010-2019)	PD Min (2010-2019)	PD Max (2010-2019)
0.000%	0.000%	0.000%
0.000%	0.000%	0.000%
0.000%	0.000%	0.000%
0.000%	0.001%	0.000%
0.001%	0.012%	0.000%
0.029%	0.144%	0.002%
0.648%	1.395%	0.168%
5.898%	7.059%	3.873%
13.375%	12.876%	12.380%
24.057%	19.813%	28.480%
32.767%	24.857%	44.154%
35.398%	26.307%	49.270%

Table 4.4 – Estimated PDs for different scenarios

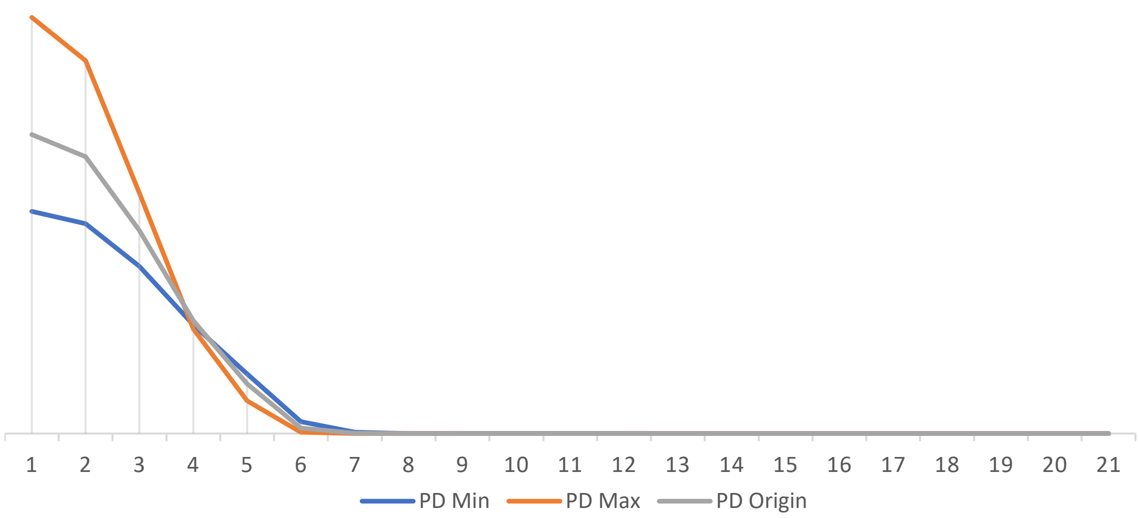

Figure 4.3 - Estimated PDs for different scenarios

In general, the observed results are quite similar, and the main differences appear on the riskiest rating grades. In this way, controlling the spread of the PDs for those grades through the choice of ***k*** could be useful for some institutions.

4.3.3. Scaling Procedure using different Approaches

The scaling procedure could also be applied before the Bayesian inference to both Pluto & Tasche Confidence-Based Approach and Cumulative Accuracy Profile Approach. This could be useful in situations that estimated PDs seem to be very conservative. As was previously mentioned, the estimates could be scaled in two ways: towards the average portfolio default rate or upper bound PD. In this way, using an average as a scaling factor is possible for both methods. However, only under the Pluto & Tasche approach, the upper confidence bound will be applied for the scaling proposal.

The table below summarize the aforementioned results:

Rating	P&T Original Indep.(1985-2019)	P&T Scal by Upp. Bound (1985-2019)	P&T Scal by AvR DR (1985-2019)	CAP Original (2010-2019)	CAP Scal by AvR DR (2010-2019)
Aaa	0.936%	0.256%	0.218%	0.000%	0.000%
Aa1	1.110%	0.304%	0.258%	0.000%	0.000%
Aa2	1.172%	0.321%	0.272%	0.000%	0.000%
Aa3	1.251%	0.342%	0.291%	0.000%	0.000%
A1	1.337%	0.366%	0.311%	0.000%	0.000%
A2	1.432%	0.392%	0.333%	0.000%	0.000%
A3	1.553%	0.425%	0.361%	0.000%	0.000%
Baa1	1.680%	0.460%	0.391%	0.000%	0.000%
Baa2	1.837%	0.502%	0.427%	0.000%	0.000%
Baa3	2.042%	0.559%	0.475%	0.000%	0.000%
Ba1	2.417%	0.661%	0.562%	0.000%	0.000%
Ba2	2.873%	0.786%	0.668%	0.000%	0.000%
Ba3	3.294%	0.901%	0.766%	0.000%	0.000%
B1	3.951%	1.081%	0.919%	0.001%	0.001%
B2	5.829%	1.595%	1.355%	0.029%	0.031%
B3	9.579%	2.620%	2.227%	0.648%	0.681%
Caa1	14.409%	3.942%	3.350%	5.898%	6.203%
Caa2	24.496%	6.701%	5.696%	13.375%	14.068%
Caa3	34.286%	9.379%	7.972%	24.057%	25.304%
Ca	50.199%	13.732%	11.672%	32.767%	34.464%
C	67.365%	18.428%	15.663%	35.398%	37.232%

Table 4.5 – Estimated PDs applying scaling procedure

In the cases when the portfolio average default rate is used as a scaling factor, the new estimated average PD fits exactly the portfolio central tendency. The estimated PDs from the Pluto and Tasche approach decrease significantly although are slightly higher when scaled towards the upper confidence bound. On the other side, the scaled PDs from CAP approach are very similar to the final results presented in the Table 3.9**.**

4.3.4. Portfolio Reference PD using different prior distributions

In the previous sections, for the calculation of *Portfolio Reference PD* under the Bayesian approach, the estimations from the Pluto and Tasche method were used as an input to determine the prior belief. However, there are other possible solutions for the inputs of the prior. The following examples demonstrate how the estimates of the *Portfolio Reference PD* could vary according to different inputs and different time horizons. For this purpose, the results from CAP approach were considering, as well as the results from Table 4.5 considering the scaling procedure.

	1985-2019	2010-2019
P&T Indep Original	0.802%	**0.961%**
P&T Scal by AvR DR	0.803%	0.961%
P&T Scal Upp Bound	0.803%	0.962%
CAP Original	0.802%	0.958%
CAP Scal by AvR DR	0.800%	0.954%

Table 4.6 – Different outcomes for the Portfolio Reference PD

The aforementioned results show that the outcomes are almost equal between different approaches. However, the main difference is observed between two periods of time. For other side, is also clear that even applying the scaling procedure, the results still very close to each other.

5. POTENTIAL APPLICATIONS AND ADVANTAGES

This paper aims to provide a method that is positioned as a starting point and can combine different approaches to infer as much as possible from each of them. For example, it could be useful in cases when there is no knowledge about the portfolio in analysis or when an institution does not have internal default experience. In this case, for the inference of Portfolio Reference PD, which serves as the reference point for the adjustments of the primary PD estimates, the Confidence-Based Approach could be used as an input, since this approach could perfectly work even for the portfolios with zero defaults.

On the other side, the method adds value for being flexible regarding the assumptions and data used, as was demonstrated in previous sections. For example, adopting the Confidence-Based Approach (CBA) as an input for the prior belief, the risk managers can choose between independence or correlation of default events or opt for the multi-period perspective. Furthermore, under CBA, the choice of confidence level could be used to control the level of the conservatism that an institution wants to apply – the higher the confidence level, the higher the estimated results will be. However, if the banks will decide to apply the correlated or multi-period case, the assumption of the numbers of defaults being binomially distributed for the Bayesian inference will have to be adjusted and other parameters, such as asset and year-to-year correlation, should be predefined as well.

What regards to the data, the method allows for considering different periods and therefore there is a possibility to manage where in the economic cycle the data is from. For instance, the company could use different periods of the sample (as it was applied in the example of this paper): from one side collect all available data for the prior belief to generate PDs from the observed default history and consequently derived the Portfolio Reference PD; and for other, incorporate the most recent data for the estimation of grade-level PDs, not forgetting, however, the representativeness of the data and its range of variability.

In addition, the Portfolio Reference PD enables the risk managers to override the estimates in both ways – upward or downward. Additionally, there are also multiple ways to adjust this reference PD: through the assumptions/choices made on Confidence-Based Approach (i.e., the consideration of correlated, independent, or multi-period default events) and consequently on the Bayesian prior (for instance, if the multi-period case will be adopted for the input of prior belief, the estimates are going to be less conservative (see Pluto and Tasche, 2011, for more details)); or through the choice of the sample – using different periods or even different data. As an illustration, in the example of this paper, the reference PD was estimated from the conservative point of view and consequently, the assumptions have been adopted for this purpose, the opposite could also occur.

In this context, the proposed method is based on the assumptions/parameters choices that can be easily adjusted by different needs and views and consequently offering some flexibility in the estimation process.

6. POSSIBLE CRITICISM

One of the possible criticisms of this method could be related to the availability of the data possessed by an institution. In this case, the impact of the data scarcity could be summarized by the following:

- If the data does not include any default event, which is not rare under Low Default Portfolios, the estimation of grade-level PDs is possible under the Confidence-Based Approach but does not work if applying the Cumulative Accuracy Profile method. The portfolios with no defaults are very common, especially if the time window is short or when the portfolio is relatively recent in the market;

- If the institutions do not have enough data to split the portfolio into different-period samples (or using similar in nature portfolio), it could not be considered as a good practice the use of the same sample for the prior distribution and, at the same time, for the posterior distribution while estimating the Portfolio Reference PD.

On the other side, if there is some flexibility about assumptions in the Bayesian inference and Confidence-Based approach, the Cumulative Accuracy Profile approach is based on pre-defined functions. This is particularly important since of these three approaches the Cumulative Accuracy Profile is chosen as a starting point from which estimates are adjusted.

Finally, as was mentioned by Benjamin et all., (2006) “trying to infer too much from too little” can be considered as the main criticism for all methods proposed for solving the issues related to the LDPs, and this method is not an exception.

7. CONCLUSIONS

The concerns around the robustness of the risk parameters estimates will continue to be actual till a portfolio represent the LDP characteristics and all "surprises" in the data will have to be explained. This statement is still being relevant even for the method described in this paper and applies to any other method used as well. In this way, statistical models are combined with an expert judgment trying to avoid some of these specific "surprises" or simply stating some of the model errors. However, using a limited set of data on credit scoring models can be also considered as a source of error.

To give another possible solution to what was already proposed, the aggregation of various views was testing. The method under consideration combines different approaches, which could adjust the results in different ways, based and depending on the choice of different assumptions. If taking separately each of those approaches, the following conclusions were made, based on the practical illustration presented in previous sections:

- The Confidence-Based Approach (CBA) while considering the upper confidence bound for the PDs estimates and under the assumption that defaults events are independent, provides the results that are quite conservative. Additionally, in the cases where the observed default rate was zero for all historical data, the method assigns the PDs that are above 0%.

- On the other side, the Cumulative Accuracy Profile (CAP) method results in less conservative estimates and assigns zero PDs for almost all rating grades where no defaults were observed. However, with the assumption that PDs increase exponentially with the rating class, the principle of monotonicity is not violated.

According to the results described above, the idea was to consider the Cumulative Accuracy Profile approach as a primary technique and to estimate the Portfolio Reference PD, through the conservative estimates from the Confidence-Based Approach, for the adjustments/fitting purposes. In this way, the final estimates are slightly more conservative than those that originally were derived from CAP curve, due to the adjustments of the reference PD.

Nevertheless, the assumptions of the proposed method could be adjusted, and this introduces some flexibility for the estimation procedure. Also, it could be useful for the institutions without internal default experience for the portfolio under analysis. On the other side, the characteristics of the available data play an important role in this approach. For instance, the portfolios without any defaults could not be considered and the data that could be split into different periods is advisable.

Also, some overrides were made given the particularities of the Cumulative Accuracy Profile approach. For high rating grades where the estimated PDs were equal to zero, the addition floor values were added. These adjustments are particular to this practical demonstration and are purely exemplary. However, as prescribed in the Regulation (EU) No 575/2013, "institutions shall recognize the importance of judgmental considerations in combining results of techniques and in making adjustments for limitations of techniques and information". As such, the final estimates derived from this method are not an exemption and further judgmental considerations are always required.

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