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Greedy Solvable Knapsacks: Identification and Relaxations

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ABSTRACT

The family $\mathcal{K}$ of the feasible solutions for a 0-1 Knapsack with positive coefficients, $\mathcal{K}=\{I\subset\mathcal{N}:\Sigma_{i\in I}a_i\leq b\}$, is an independence system over $\mathcal{N}=\{1,\ldots,n\}$. In some cases, for instance when all the a_i have the same value, this independence system is a matroid over $\mathcal{N}$. We will say then that the knapsack is greedy solvable.

In the first part of this paper we study the conditions for a knapsack to be greedy solvable. We present necessary and sufficient conditions, verifiable in polynomial time, for $\mathcal{K}$ to be a member of a finite family of matroids over $\mathcal{N}$.

When those conditions are not met it seems natural to look for greedy solvable relaxations for the knapsack problem. In the second part of the paper we study a family of matroidal relaxations for the 0-1 knapsack problem. We prove that those relaxations dominate the usual linear programming one for this problem and we present some computational results in order to illustrate that domination.

KEYWORDS: 0-1 Knapsack, relaxations, matroids.

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1. INTRODUCTION

Let $\mathcal{K}$ denote the set of feasible solutions for a *0-1 Knapsack* problem with positive coefficients, i.e

$$\mathcal{K} = \{ x\in\{0,1\}^n : \sum_{i=1}^{n} a_i x_i \le b \} ,$$

where $b\in\mathbb{N}$ and $a_i\in\mathbb{N}$ for all i.

We will use the word *knapsack* either to denote $\mathcal{K}$ or to refer the constraint $\sum_{i=1}^{n} a_i x_i \le b$ but the difference will be clear from the context.

Now let $\mathcal{N} = \{1,\dots,n\}$. To each feasible solution $x^j\in \mathcal{K}$ we can uniquely associate the subset I^j of $\mathcal{N}$ for which x^j is the corresponding incidence vector, i.e

$$I^j = \{ i\in\mathcal{N} : x^j_i = 1 \}.$$

Let $\mathcal{F}$ be the family of all these subsets, i.e $\mathcal{F} = \{I^1,I^2,\dots,I^s\}$, $s=|\mathcal{K}|$. Obviously $\mathcal{K}$ and $\mathcal{F}$ are in a one to one correspondence so we will use $\mathcal{K}$ and $\mathcal{F}$ (also x^j and I^j) interchangeably.

It is well known that

Proposition 1.1: $\mathcal{F}$ is an independence system over $\mathcal{N}$.

It is also well known that the problem of finding a maximum weight set on an independence system can be solved by the greedy algorithm *iff* the independence system is a matroid over $\mathcal{N}$. We recall now two equivalent definitions of a matroid over $\mathcal{N}$, see White (1986).

Definition 1.2: The independence system $(\mathcal{N},\mathcal{F})$ is a *matroid iff* for any $A\subset\mathcal{N}$, all the maximal (for inclusion) independent sets of A have the same cardinality.

Definition 1.3: The independence system $(\mathcal{N},\mathcal{F})$ is a *matroid iff* for any I_p and $I_{p+1}\in \mathcal{F}$ such that $|I_p|=$ p and $|I_{p+1}|=$ p+1, there is an $e \in I_{p+1}\setminus I_p$ for which $I_p\cup\{e\} \in \mathcal{F}$.

In the first part of this paper we shall discuss the problem of finding whether the independence system defined by a knapsack is a matroid over $\mathcal{N}$.

When this happens we shall say that the knapsack is *greedy solvable*.

We present now two examples that illustrate the case where a knapsack is greedy solvable, example 1.4, and where a knapsack is not greedy solvable, example 1.5.

Example 1.4: Let $\mathcal{K} = \{\ x \in \{0,1\}^3 : x_1 + 2x_2 + 3x_3 \leq 4\ \}$. Then
$$\mathcal{F} = \{\ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}\ \}$$
and, of course, $\mathcal{F}$ is a matroid over $\mathcal{N} = \{1,2,3\}$.

Example 1.5: Let $\mathcal{K} = \{\ x \in \{0,1\}^3 : x_1 + 2x_2 + 3x_3 \leq 3\ \}$ Then
$$\mathcal{F} = \{\ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}\ \}$$
and $\mathcal{F}$ is not a matroid over $\mathcal{N} = \{1,2,3\}$ because $\{3\}$ and $\{1,2\}$ are maximal independent subsets of $\mathcal{N}$ with different cardinalities.

In the last part of this paper we study a family of matroidal relaxations for the 0-1 knapsack problem. We prove that those relaxations dominate the usual linear programming one for this problem and we present some computational results in order to illustrate that domination.

2. Cardinality equivalent knapsacks

We start this section with the following remark: the independence system associated with a cardinality knapsack $\mathcal{C}_k=\{x\in\{0,1\}^n:\Sigma_{i=1}^n x_i\le k\}$ exhibits a special structure: the independent sets are the subsets of $\mathcal{N}$ with a cardinality not greater than k. Denote by $\mathcal{F}_{o,k}$ the corresponding family of independent sets.

It is easily proved, using definition 1.2, that $\mathcal{C}_k$ is greedy solvable. In this section we try to find other greedy solvable knapsacks that "look like" $\mathcal{C}_k$. We start by giving a definition:

<u>Definition</u> <u>2.1</u>: Two knapsacks are called *$\mathcal{F}$-equivalent* if they have the same family $\mathcal{F}$ of independent sets. If a knapsack is equivalent to $\mathcal{C}_k$ we will call it $\mathcal{F}_{o,k}$-equivalent.

<u>Example</u> <u>2.2</u>: The two knapsacks $\mathcal{C}_2=\{x\in\{0,1\}^3:x_1+x_2+x_3\le 2\}$ and $\mathcal{K}=\{x\in\{0,1\}^3:100x_1+101x_2+102x_3\le 204\}$ have the same family of independent sets $\mathcal{F}_{o,2}=\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$ so $\mathcal{K}$ is $\mathcal{F}_{o,2}$-equivalent.

Obviously when a knapsack is $\mathcal{F}_{o,k}$-equivalent it is greedy solvable. Our aim now is to characterize the knapsack instances where this happens.

Let $\mathcal{K}=\{x\in\{0,1\}^n:\Sigma_{i=1}^n a_i x_i\le b\}$ be a knapsack instance and assume, with no loss of generality, that $0<a_1\le a_2\le\ldots\le a_n\le b$. We can now state and prove the following

<u>Proposition</u> <u>2.3</u>: The knapsack $\mathcal{K}=\{x\in\{0,1\}^n:\Sigma_{i=1}^n a_i x_i\le b\}$ is $\mathcal{F}_{o,k}$-equivalent, and therefore greedy solvable, iff $b\in[\Sigma_{i=n-k+1}^n a_i,\Sigma_{i=1}^{k+1} a_i[$.

<u>Proof</u>: A knapsack is $\mathcal{F}_{o,k}$-equivalent iff no $(k+1)$-set is independent and all the (k)-sets are independent.

Therefore the $(k+1)$-set with the smallest weight must be dependent so we must have $b<\Sigma_{i=1}^{k+1} a_i$. On the other hand all the (k)-sets must be independent so this must happen for the heaviest of those sets and then $b\ge\Sigma_{i=n-k+1}^n a_i$. □

3. OTHER GREEDY SOLVABLE KNAPSACKS

In the previous section we learned how to polynomially identify greedy solvable knapsacks which are $\mathcal{F}_{o,k}$-equivalent. We now address the question of identifying other classes of greedy solvable knapsacks which are no longer $\mathcal{F}_{o,k}$-equivalent but still closely related to the cardinality ones.

We shall start by analyzing the case of independence systems where the family of independent sets $\mathcal{F}_{1,k}$ consists of all the sets with cardinality not greater than k and all the $(k+1)$-sets containing $1\in N$, i.e. the element of N with the smallest weight a_1.

<u>Proposition 3.1</u>: Let $(N,\mathcal{F}_{1,k})$ be an independence system over N with a family of independent sets $\mathcal{F}_{1,k}$ such that the following properties hold:

(a) $\mathcal{F}_{1,k} \supset \mathcal{F}_{0,k}$

(b) Let $I\subset N$ and $|I|=k+1$. Then $I\in\mathcal{F}_{1,k}$ iff $1\in I$

(c) Let $I\subset N$ and $|I|>k+1$. Then $I\notin\mathcal{F}_{1,k}$.

Then $(N,\mathcal{F}_{1,k})$ is a matroid over N.

<u>Proof</u>:

The proof will be given in the more general context of proposition 3.5.□

The fact that $(N,\mathcal{F}_{1,k})$ is a matroid over N raises the question of identifying, if possible polynomially, all the $\mathcal{F}_{1,k}$-equivalent knapsacks. We shall start by looking at one example:

<u>Example 3.2</u>: Consider $K=\{x\in\{0,1\}^4: 4x_1+100x_2+101x_3+102x_4\leq 110\}$. One can easily see that the family of independent sets of this knapsack is $\mathcal{F}_{1,1}=\{\emptyset,\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\}\}$. Note that the heaviest set in $\mathcal{F}_{1,1}$, $\{1,4\}$, has weight 106 while the lightest set not in $\mathcal{F}_{1,1}$, $\{2,3\}$, has a weight 201 and the right hand side has a value of 110.

This example suggests that the knapsack $K=\{x\in\{0,1\}^n:\Sigma_{i=1}^n a_i x_i\leq b\}$ will be $\mathcal{F}_{1,k}$-equivalent iff the two following conditions hold:

(1) The heaviest set in the knapsack will be such that $x_1=1$, $x_i=0$ for $i=2,\dots,(n-k)$ and $x_i=1$ for $i=(n-k+1),\dots,n$.

(2) The lightest set not in the knapsack will be such that $x_1=0$, $x_i=1$ for $i=2,\dots,(k+2)$ and $x_i=0$ for $i=(k+3),\dots,n$.

Then the following results hold:

<u>Proposition 3.3</u>: The knapsack $K=\{x\in\{0,1\}^n:\Sigma_{i=1}^n a_i x_i \le b\}$ will be $\mathcal{F}_{1,k}$-equivalent, and therefore greedy solvable, iff $b\in[a_1+\Sigma_{i=n-k+1}^n a_i, \Sigma_{i=2}^{k+2} a_i[$.

<u>Proposition 3.4</u>: The condition in proposition 3.3 can be checked in $O(n)$ operations.

We will now build, recursively, a finite chain of matroids over N, with independence families $\mathcal{F}_{0,k}\subset\mathcal{F}_{1,k}\subset\mathcal{F}_{2,k}\subset\dots\subset\mathcal{F}_{i,k}\subset\dots$, such that checking if a knapsack is $\mathcal{F}_{i,k}$-equivalent can be done in polynomial time.

The family $\mathcal{F}_{2,k}$, for instance, will consist of all the subsets of N with cardinality not greater than k and of the $(k+1)$-sets containing at least one of the two lightest elements in N, i.e. containing $1\in N$ or $2\in N$.

The following proposition makes this idea more precise.

<u>Proposition 3.5</u>: Let $(N,\mathcal{F}_{i,k})$, $0<i<n-k+1$, be an independence system over N with a family of independent sets $\mathcal{F}_{i,k}$ such that the following properties hold:

(a) $\mathcal{F}_{i,k}\supset\mathcal{F}_{i-1,k}$

(b) Let $I\subset N$ and $|I|=k+1$. Then $I\in\mathcal{F}_{i,k}$ iff $\{1,2,\dots,i\}\cap I\neq\emptyset$.

(c) Let $I\subset N$ and $|I|>k+1$. Then $I\notin\mathcal{F}_{i,k}$.

Then $(N,\mathcal{F}_{i,k})$ is a matroid over N.

<u>Proof</u>:

We will use the second definition of matroid given in section 1. From the definition of $\mathcal{F}_{i,k}$ it is clear that we shall have to worry only with (k)-sets and $(k+1)$-sets.

Let $I_k, I_{k+1}\in\mathcal{F}_{i,k}$ such that $|I_k|=k$ and $|I_{k+1}|=k+1$.

If $I_k\subset I_{k+1}$ then I_k can be trivially amplified to $I_{k+1}\in\mathcal{F}_{i,k}$.

If I_k is not contained in I_{k+1} but there exists $j\in\{1,2,\ldots,i\}$ such that $j\in I_k$ then, recalling the definition of $\mathcal{F}_{i,k}$, I_k can be amplified using any element of $I_{k+1}\backslash I_k$.

Finally suppose that I_k is not contained in I_{k+1} and there exists no $j\in\{1,2,\ldots,i\}$ such that $j\in I_k$. As $I_{k+1}\in\mathcal{F}_{i,k}$ there exists $e\in\{1,2,\ldots,i\}$ such that $e\in I_{k+1}$. Then $I_k\cup\{e\}\in\mathcal{F}_{i,k}$ and the proof is complete.□

The two following propositions can be easily proved:

Proposition 3.6: The knapsack $\mathcal{K}=\{x\in\{0,1\}^n:\Sigma_{i=1}^n a_i x_i\leq b\}$ will be $\mathcal{F}_{i,k}$-equivalent for $i\in\{1,2,\ldots,n-k\}$, and therefore greedy solvable, iff $b\in[a_i+\Sigma_{j=n-k+1}^n a_j, min\{\Sigma_{j=i+1}^{k+i+1} a_j, \Sigma_{j=1}^{k+2} a_j\}[$.

Proposition 3.7: The condition in proposition 3.6 can be checked in $O(n)$ operations.

Lets now have a closer look at $\mathcal{F}_{n-k,k}$. This family will consist of all the subsets of N with cardinality not greater than k and of the $(k+1)$-sets containing at least one of the elements in $\{1,2,\ldots,n-k\}$. But one can easily see that all the $(k+1)$-sets of N have this last property so the following holds:

Corollary 3.8: $\mathcal{F}_{n-k,k}=\mathcal{F}_{0,k+1}$.

Proposition 3.9: Let $\mathcal{F}$ be the independence family associated with the knapsack $\mathcal{K}=\{x\in\{0,1\}^n:\Sigma_{j=1}^n a_i x_i\leq b\}$ and suppose that $\mathcal{F}_{0,k}\subset\mathcal{F}\subset\mathcal{F}_{0,k+1}$, $\mathcal{F}\neq\mathcal{F}_{0,k}$ and $\mathcal{F}\neq\mathcal{F}_{0,k+1}$. If $\mathcal{K}$ is greedy solvable then $\mathcal{F}=\mathcal{F}_{i,k}$, for some $i\in\{1,\ldots,n-k-1\}$.

Proof:

Note first that $\mathcal{F}\subset\mathcal{F}_{0,k+1}$ and $\mathcal{F}\neq\mathcal{F}_{0,k+1}$ implies $b<min\{\Sigma_{j=1}^{k+2} a_j, \Sigma_{j=n-k}^n a_j\}$.

Note also that, because we have $\mathcal{F}\subset\mathcal{F}_{0,k}$ and $\mathcal{F}\neq\mathcal{F}_{0,k}$, the set $I_{k+1}=\{1,2,\ldots,k+1\}$ is in $\mathcal{F}$. Since $\mathcal{K}$ is greedy solvable and $I_k=\{n-k+1,\ldots,n\}\in\mathcal{F}$ we must have, because of definition 1.3, $\{e,n-k+1,\ldots,n\}\in\mathcal{F}$ for some $e\in I_{k+1}\backslash I_k$

and, as 1 is the lightest element in I_{k+1}, we will have $\{1,n-k+1,\dots,n\}\in\mathcal{F}$ so

$$b\in[a_1+\Sigma^n_{j=n-k+1}a_j, min\{\Sigma^{k+2}_{j=1}a_j,\Sigma^n_{j=n-k}a_j\}[.$$

The proof will proceed by dividing this interval and analyzing what happens to all possible b values along the distinct subintervals.

If $b<\Sigma^{k+2}_{j=2}a_j$ then $b\in[a_1+\Sigma^n_{j=n-k+1}a_j,\Sigma^{k+2}_{j=2}a_j[$ and, by proposition 3.6, $\mathcal{K}$ is $\mathcal{F}_{1,k}$- equivalent.

Otherwise $b\geq\Sigma^{k+2}_{j=2}a_j$ and then $I_{k+1}=\{2,3,\dots,k+2\}\in\mathcal{F}$. Recall that $\mathcal{K}$ is greedy solvable and $I_k=\{n-k+1,\dots,n\}\in\mathcal{F}$ so, again because of the matroidal nature of $\mathcal{F}$, we will have $\{2,n-k+1,\dots,n\}\in\mathcal{F}$. Then $b\geq a_2+\Sigma^n_{j=n-k+1}a_j$.

Now if $b<\Sigma^{k+3}_{j=3}a_j$ then $b\in[a_2+\Sigma^n_{j=n-k+1}a_j, min\{\Sigma^{k+2}_{j=1}a_j,\Sigma^{k+3}_{j=3}a_j\}[$ and $\mathcal{K}$ is $\mathcal{F}_{2,k}$-equivalent. If $b\geq\Sigma^{k+3}_{j=3}a_j$ we continue the reasoning as before.

It's easy to see that this is a finite proof since $b<\Sigma^n_{j=n-k}a_j$ and so we need to divide the first interval in, at most, $n-k-1$ sub-intervals.□

Up to now we have built, for every $k\in\mathcal{N}$, a finite chain of matroids over $\mathcal{N}$ with independence families $\mathcal{F}_{0,k}\subset\mathcal{F}_{1,k}\subset\mathcal{F}_{2,k}\subset\dots\subset\mathcal{F}_{i,k}\subset\dots\subset\mathcal{F}_{n-k,k}=\mathcal{F}_{0,k+1}$ such that checking if a knapsack is $\mathcal{F}_{i,k}$-equivalent, and therefore greedy solvable, can be done in polynomial time.

We also have proved that "between" the cardinality knapsacks, the only ones which are greedy solvable are the $\mathcal{F}_{i,k}$-equivalents.

However there exist greedy solvable knapsacks which are not equivalent to any of the matroids in the aforementioned chain as in the following example:

Example 3.10: Consider $\mathcal{K}=\{x\in\{0,1\}^4:10x_1+11x_2+100x_3+101x_4\leq130\}$. One can easily see that the family of independent sets of this knapsack is $\mathcal{F}=\{\emptyset,\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{1,2,3\},\{1,2,4\}\}$.

The knapsack is greedy solvable but $\mathcal{F}$ is not in the chain.

4. Matroidal relaxations for knapsack problems

When the conditions stated in the previous sections are not met it seems natural to look for greedy solvable relaxations for the knapsack problem.

In this section we study a family of matroidal relaxations for the 0-1 knapsack problem. We prove that those relaxations dominate the usual linear programming bound for this problem and we present some computational results in order to illustrate that domination.

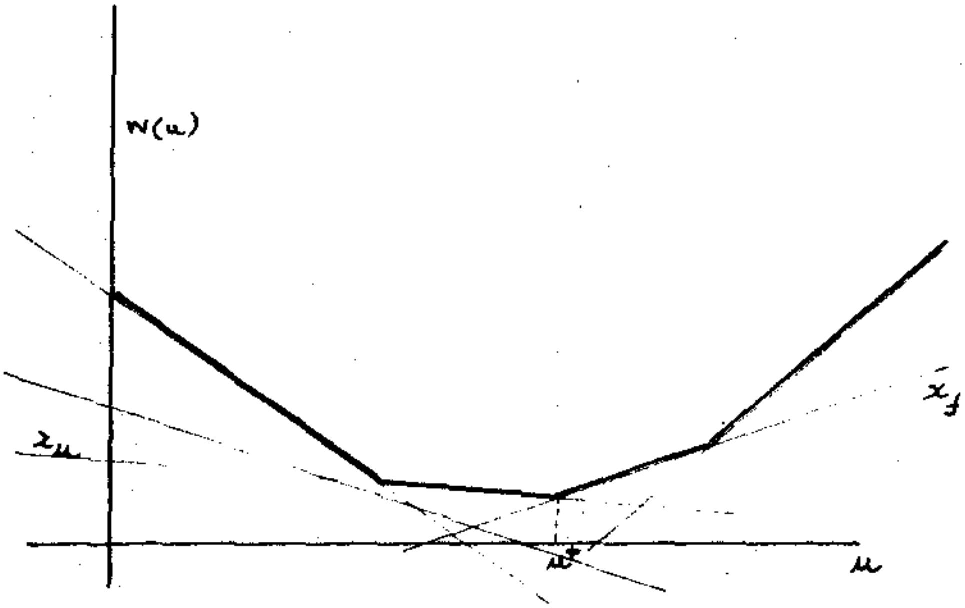
Consider then a n-variable 0-1 knapsack optimization problem $z=\max\{cx:ax\leq b, x\in\{0,1\}^n\}$ with positive coefficients.

It is well known, see Martello & Toth (1990), that the LP bound for this problem $z_{lp}=\max\{cx:ax\leq b, x\in[0,1]^n\}$ can be computed, in $O(n \log n)$ time, taking the variables in the decreasing order of their relative efficiency c_i/a_i, making $x_i=1$ while the sum of the a_i is smaller or equal than b and taking a fraction of the *critical* variable x_k (the variable that would make the sum of the a_i bigger than b) in order to make the constraint $ax\leq b$ binding.

It is also well known that z_{lp} can be obtained, using Lagrangean duality, as $z_{lp}=\min\{w(u):u\geq 0\}$ where the dual function $w(u)$ is $w(u)=\max\{cx+u(b-ax):x\in\{0,1\}^n\}$. In this case it is easy to show that $u^*=c_k/a_k$, where k is the index of the critical variable, is an optimal multiplier, i.e. $z_{lp}=w(u^*)$.

If $z\neq z_{lp}$ and there is no other variable with the same relative efficiency of the critical variable, i.e. $u^*\neq c_i/a_i$ for all $i\neq k$, the problem $w(u^*)=\max\{cx+u^*(b-ax):x\in\{0,1\}^n\}$ will have two, and only two, alternative optimal solutions, x_f and x_u, one feasible $ax_f\leq b$ the other unfeasible $ax_u>b$. From now on we shall assume, as a non-degeneracy hypothesis, that this will always be the case.

To make this more clear consider the following figure. Each point $x\in\{0,1\}^n$ corresponds to one straight line with slope $(b-ax)$ and such that for $u=0$ it crosses the w axis at cx. The (convex) function $w(u)$ will then be the upper envelope of those lines and its minimum will, generally, be attained at the crossing of two lines corresponding to the points x_f (feasible point and line with a positive slope) and x_u (unfeasible point and line with negative slope).



Our aim is to remove x_u, and hopefully quite a few more unfeasible points, using some greedy solvable knapsack as a relaxation for the constraint $K=\{x\in\{0,1\}^n: ax\le b\}$. In fact one can easily prove the following:

<u>Proposition 4.1</u>: Let M be a matroidal relaxation of K removing x_u, i.e. M is a greedy solvable knapsack such that $M\supset K$ and $x_u \notin M$. Now let

$$z_m = \min\{w_m(u): u\ge 0\} \text{ with } w_m(u)=\max\{cx+u(b-ax): x\in M\}$$

Under the non-degeneracy hypothesis sated above we have the following inequalities: $z\le z_m < z_{lp}$.

<u>Proof</u>: The first inequality is implied by the fact that $M\supset K$. The second (strict) inequality is true since $x_u \notin M$, the non-degeneracy hypothesis holds, the line corresponding to x_u has a negative slope and $w(u)$ is convex.□

<u>Proposition 4.2</u>: If checking independence in M can be done in $O(n)$ time the matroidal bound z_m can be computed in polynomial, $O(n^2)$, time.

<u>Proof</u>: z_m is the optimal value of the Lagrangean dual of the problem $z=\max\{cx: ax\le b, x\in M\}$. This problem is a *matroidal knapsack* in Camerini & Vercellis sense, then if checking independence in M can be done in $O(n)$ time its Lagrangean bound can be computed in $O(n^2)$ time, see Camerini & Vercellis (1984).□

Now we deal with an effective way of building $\mathcal{M}$ according to the specifications of proposition 4.1. We start by stating a very simple lemma that will be useful later on:

Lemma 4.3: Consider $\mathcal{G}\subset\mathcal{N}$ and let $\mathcal{F}$ be a matroid on the ground set $\mathcal{G}$. Then $\mathcal{F}\times 2^{\mathcal{N}\setminus\mathcal{G}}$ is a matroid over $\mathcal{N}$.

Proof: The proof is direct from definition 1.2.□

As before consider the variables sorted in decreasing order of their relative efficiency and let k be the index of the critical variable.

Note first that one has $x_{u_i}=1$ for $i=1,\ldots,k$ and $x_{u_i}=0$ for $i=k+1,\ldots,n$. Clearly x_f looks like x_u with the critical variable set to zero, i.e. $x_{f_i}=1$ for $i=1,\ldots,k-1$ and $x_{f_i}=0$ for $i=k,\ldots,n$.

Then an obvious candidate for building the matroidal bound z_m is $\mathcal{M}_0=\{x\in\{0,1\}^k:\Sigma_{i=1}^k x_i\le k-1\}\times\{0,1\}^{(n-k)}$. Of course the first "factor" is a (cardinality) greedy solvable knapsack which is a matroid over $\{1,\ldots,k\}\subset\mathcal{N}$ so, because of lemma 4.3, $\mathcal{M}_0$ is a matroid over $\mathcal{N}$. On the other hand one has $\mathcal{M}_0\supset\mathcal{K}$ and $x_u\notin\mathcal{M}_0$ so $\mathcal{M}_0$ can be used.

However x_u and all its supersets are the only points in $\{0,1\}^n$ excluded by the constraint $\Sigma_{i=1}^k x_i\le k-1$ so the bound obtained may not be very good!

We shall try now to build $\mathcal{M}$, according to the specifications of proposition 4.1, but in such a way that $\mathcal{M}$ preserves more information about the structure of $\mathcal{K}$ than $\mathcal{M}_0$. We shall do that by improving on the ground set of the first "factor" and by using the greedy solvable knapsacks identified in section 3.

We start by defining the ground set $\mathcal{G}\subset\mathcal{N}$ on which to build the first "factor", $\mathcal{F}$, of $\mathcal{M}$.

The ground set $\mathcal{G}$ on which to build $\mathcal{F}$ will be defined as follows:

Let w_k denote the weight of the lightest (k)-set in $\mathcal{G}$ and w_{k-1} denote the weight of lightest $(k-1)$-set in $\mathcal{G}\setminus\{1,\ldots,k-1\}$ (if no such set exists set w_{k-1} equal to $+\infty$).

With this notation build $\mathcal{G}$ as follows:

$\mathcal{G}=\{1,\dots,k\}$

<u>for</u> $i=k+1,\dots,n$ <u>do</u>

$\mathcal{G}=\mathcal{G}\cup\{i\}$

<u>compute</u> w_{k-1}, w_k.

<u>if</u> $w_{k-1}\le b$ <u>or</u> $w_k\le b$ <u>then</u>

$\mathcal{G}=\mathcal{G}\setminus\{i\}$

<u>end if</u>

<u>end do</u>

Obviously we have $\{1,\dots,k\}\subset\mathcal{G}\subset\mathcal{N}$. Now we define $\mathcal{F}$ as follows: all the $(k-2)$-sets of $\mathcal{G}$ are in $\mathcal{F}$, no (k)-set is in $\mathcal{F}$ and the only $(k-1)$-sets in $\mathcal{F}$ must include some element of $\{1,\dots,k-1\}$. In fact one can write $\mathcal{F}$ as a greedy solvable knapsack $\mathcal{F}=\{x\in\{0,1\}^{|\mathcal{G}|}:\Sigma_{i=1}^{|\mathcal{G}|}g_i x_i\le g_0\}$ where $g_0=k^2-2$, $g_i=k$ for $i\in\{1,\dots,k-1\}$ and $g_i=k+1$ for $i\in\mathcal{G}\setminus\{1,\dots,k-1\}$. Therefore $\mathcal{F}$ is an $\mathcal{F}_{(k-1),(k-2)}$ defined in $\mathcal{G}$.

Finally we set $\mathcal{M}=\mathcal{F}\times 2^{\mathcal{N}\setminus\mathcal{G}}$.

We can now state and prove the following:

<u>Proposition</u> <u>4.4</u>: **The following statements hold:**

(a) $\mathcal{M}$ is a matroid over $\mathcal{N}$ and independence can be checked in $O(n)$ time.

(b) $\mathcal{K}\subset\mathcal{M}\subset\mathcal{M}_0$

(c) $x_u\notin\mathcal{M}$

Thus $\mathcal{M}$ can be used to compute the matroidal bound z_m.

<u>Proof</u>: **(a) is obvious because of proposition 3.5 and lemma 4.3; (c) is true because $x_u\cap\mathcal{G}$ is a (k)-set of $\mathcal{G}$. Now lets look at statement (b). The inclusion $\mathcal{M}\subset\mathcal{M}_0$ comes from the fact that the dependent sets of $\mathcal{M}_0$ can be written as $\{\{1,\dots,k\}\}\times 2^{\mathcal{N}\setminus\{1,\dots,k\}}$ so, because $\mathcal{G}\supset\{1,\dots,k\}$, those sets will have (k)-sets of $\mathcal{G}$ as subsets and therefore will also be dependent in $\mathcal{M}$.**

Finally to prove $\mathcal{K}\subset\mathcal{M}$ note that:

- Either the dependent sets of $\mathcal{M}$ will have as subsets (k)-sets of $\mathcal{G}$ and then they will also be dependent (unfeasible) on $\mathcal{K}$ because we always have $w_k>b$.

- Or the dependent sets of $\mathcal{M}$ will have as subsets $(k-1)$-sets of $\mathcal{G}$ with no element in $\{1,\dots,k-1\}$ and then they will also be dependent on $\mathcal{K}$ because $w_{k-1}>b$ is always verified.

So we have the inclusion $\mathcal{K}\subset\mathcal{M}$ and the proof is complete.□

We have just constructed a matroidal relaxation $\mathcal{M}$ which is better than $\mathcal{M}_0$, because $\mathcal{M}\subset\mathcal{M}_0$.

Of course there are many other ways of building matroidal relaxations for $\mathcal{K}$ and one important open question is the following: given a knapsack $\mathcal{K}$ is there a "best" matroidal relaxation for $\mathcal{K}$? If so can one build it in polynomial time? How?

We shall now terminate this section reporting some preliminary computational experience with the matroidal bound z_m built with the matroidal relaxation of $\mathcal{K}$ that we just described, $\mathcal{M}$.

We coded the procedure and used it within a modified version of Martello & Toth (1990) MT1 routine.

We tested it on randomly generated knapsacks with sizes ranging from 50 to 500 variables. In "purely random" problems although the new bound manages to bridge about 20% of the duality gap its impact on the computing time of MT1 is generally meaningless (an average reduction of 10%). However if the problems are generated in order to have some "matroidal features" the result is quite different.

In fact if we add a big enough constant to all the a_i coefficients of a "purely random" problem we get a knapsack that looks like a cardinality one. These problems are very hard for MT1. In fact the routine can seldom solve a problem of more than 100 variables in less than 10 minutes on a VAX 4200. However if the new bound is used not only a large portion of the problems can be solved but this is done in a relatively small amount of time. Next table reports some statistics that illustrate these facts.

# of variables	*50*	*75*	*100*	*150*	*200*	*250*	*300*	*400*	*500*
% of problems solved only with z_m	*0*	*0*	*3*	*11*	*40*	*59*	*18*	*11*	*37*
% of the gap bridged	*12*	*23*	*25*	*13*	*22*	*35*	*21*	*20*	*18*
mean gain in speed(%)	*5*	*75*	*60*	*22*	*60*	*140*	*37*	*95*	*60*

5. CONCLUSIONS

In this paper we learned how to identify some families of greedy solvable knapsacks. However there exist greedy solvable knapsacks which are not equivalent to any of the matroids in the aforementioned families so its an open field the identification of more of these problems.

This is specially important because, as we have seen in section 4, those families can be used to build matroidal relaxations that may help in solving difficult instances of this problem.

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