[10] E. Enochs, Injective and flat covers, envelopes and resolvents. Israel J. Math. 39 (1981), no. 3, 189–209.
[11] E. Enochs, Flat covers and flat cotorsion modules. Proc. Amer. Math. Soc. 92 (1984), 179–184.
[12] E. Enochs and S. Estrada, Relative homological algebra in the category of quasi-coherent sheaves. Adv. Math. 194 (2005), 284-295.
[13] E. Enochs and O.M. Jenda, The flat cover conjecture and its solution. International Symposium on Ring Theory (Kyongju, 1999), 117-122, Trends Math., Birkhuser Boston, Boston, MA, 2001
[14] A. Facchini, *Module Theory*, Birkhäuser Verlag, Vol. 167, 1998.
[15] J.L. García and J. Martínez Hernández, When is the category of flat modules abelian?, Fundamenta Mathematicae 147 (1995), 83-91.
[16] J.L. Gómez Pardo and P.A. Guil Asensio, Big direct sums of copies of a module have well behaved indecomposable decompositions. J. Algebra 232 (2000), 86-93.
[17] P.A. Guil Asensio and I. Herzog, Left cotorsion rings, Bulletin of the London Mathematical Society, **36** (2004), 303-309.
[18] P.A. Guil Asensio and I. Herzog, Sigma-cotorsion rings, Advances in Mathematics 191 (2005), 11-28.
[19] P.A. Guil Asensio and I. Herzog, Sigma-cotorsion modules and divisibility matrix subgroups, preprint, 2004.
[20] P.A. Guil Asensio and I. Herzog, Indecomposable flat cotorsion modules, preprint, 2005.
[21] D.K. Harrison, Infinite abelian groups and homological methods, *Annals of Mathematics*, **69**(2) (1959), 366-391.
[22] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer Verlag, 1977.
[23] I. Herzog, Pure-injective envelopes, Journal of Algebra and its Applications, **2**(4) (2003), 397-402.
[24] M. Makkai and R. Paré, *Accessible Categories: The Foundations of Categorical Model Theory*, Contemporary Mathematics **104**, American Mathematical Society, 1989.
[25] M. Prest, *Model theory and modules*, London Mathematical Society Lecture Note Series, **130**. Cambridge University Press, 1988.
[26] J. Trlifaj, Ext and inverse limits. Special issue in honor of Reinhold Baer (1902–1979). Illinois J. Math. 47 (2003), 529-538.
[27] J.J. Rothman, 'An introduction to homological algebra', Academic Press, New York (1979).
[28] J. Xu, *Flat Covers of Modules*, Lecture Notes in Mathematics **1634** Springer, 1996.
[29] R. Wisbauer, *Foundations of module and ring theory. A handbook for study and research.* Algebra, Logic and Applications, **3**. Gordon and Breach Science Publishers, Philadelphia, PA, 1991.
[30] M. Ziegler, Model Theory of Modules, Annals of Pure and Applied Logic 26(2) (1984), 149-213.
[31] B. Zimmermann-Huisgen and W. Zimmermann, Algebraically compact ring and modules, Math. Z. 161 (1978), 81-93.
[32] W. Zimmermann, Rein injektive direkte Summen von Moduln, Comm. Algebra 5 (1977), 1083-1117.

Departamento de Matemáticas Universidad de Murcia, 30100 Espinardo, Murcia, Spain
E-mail address: paguil@um.es

The Ohio State University at Lima, Lima, OH 45804 USA
E-mail address: herzog.23@osu.edu

Contemporary Mathematics
Volume **419**, 2006

Eigenvalues, Multiplicities and Graphs

Charles R. Johnson, António Leal Duarte, Carlos M. Saiago, and David Sher

Abstract. For a given graph, there is a natural question of the possible lists of multiplicities for the eigenvalues among the spectra of Hermitian matrices with that graph (no constraint is placed upon the diagonal entries of the matrices by the graph). Here, we survey some of what is known about this question and include some new information about it. There is a natural focus upon the case in which the graph is a tree. In this event, there is remarkable structure to the possible lists. Both the general theory and a summary of specific results is given. At the end, this allows to give, in compact tabular form, all lists for trees on fewer than 11 vertices (a potentially valuable tool for further work). There is a brief discussion of non-trees.

1. Introduction

For an n-by-n real symmetric or complex Hermitian matrix $A = (a_{ij})$, the undirected graph of A, denoted by $G(A)$, is the graph on n vertices, labelled $1, 2, \ldots, n$, with an edge between i and j if and only if $a_{ij} \neq 0$. We assume basic terminology from graph theory, but a good reference is [**CL**]. For standard terms or concepts from matrix analysis, see [**HJ**]. As we will be interested in properties of A that are permutation similarity invariant, primarily eigenvalues and their multiplicities, we will generally view a graph as unlabelled, except when referencing by labels is convenient.

For a given undirected graph G on n vertices, let $\mathcal{S}(G)$ (resp. $\mathcal{H}(G)$), denote the set of all n-by-n real symmetric (resp. complex Hermitian) matrices A such that $G(A) = G$. No restriction is placed upon the diagonal entries of A by G, except that they are real.

Our primary interest lay in the following very general question. Given G, what are all the possible lists of multiplicities for the eigenvalues that occur among matrices in $\mathcal{S}(G)$ (resp. $\mathcal{H}(G)$)? It is important to distinguish two possible interpretations of "multiplicity list". Since the eigenvalues of a real symmetric or complex

2000 *Mathematics Subject Classification.* Primary 15A18; Secondary 05C05, 05C50.
Key words and phrases. Eigenvalue, graph, Hermitian matrix, multiplicities, Parter vertex, symmetric matrix, tree.
This work was carried out as part of the research activities of *Centro de Matemática e Aplicações* da Universidade Nova de Lisboa (C. M. Saiago) and of *Centro de Matemática da Universidade de Coimbra* (A. Leal Duarte).

Hermitian matrix are real numbers, they may be placed in numerical order. If the multiplicities are placed in an order corresponding to the numerical order of the underlying eigenvalues, then we refer to such a way of listing the multiplicities as *ordered multiplicities*. If, alternatively, the multiplicities are simply listed in nonincreasing order of the values of the multiplicities themselves, we refer to such a list as *unordered multiplicities*. For example, if A has eigenvalues $-3, 0, 0, 1, 2, 2, 2, 5, 7$, the list of ordered multiplicities is $(1, 2, 1, 3, 1, 1)$ while the list of unordered multiplicities is $(3, 2, 1, 1, 1, 1)$. In either case, such a list means that there are exactly 6 different eigenvalues, of which 4 have multiplicity 1.

If a graph G is not connected, then, to the extent that distinct eigenvalues may be arbitrarily enough chosen for the multiplicity lists of each component, the multiplicity lists for G may be deduced from those of its components via superposition. In any event, the multiplicity lists for G are constrained by such superposition. Also, graphs with many edges admit particularly rich collections of multiplicity lists. For example, the complete graph admits all multiplicity lists with the given number of eigenvalues, except the list in which all eigenvalues are the same. For these reasons, a natural beginning, for the study of multiplicity lists for $\mathcal{S}(G)$ or $\mathcal{H}(G)$, is the case in which $G = T$, a tree. In addition, trees present several attractive features for this problem. Among them is the fact that for trees, every element of $\mathcal{H}(T)$ is diagonally unitarily similar to one in $\mathcal{S}(T)$, so that at least for trees, the possible multiplicity lists, ordered or unordered, (indeed, the possible spectra) in $\mathcal{H}(T)$ are the same as those in $\mathcal{S}(T)$. In fact, most of our discussion will be about trees.

The multiplicity list consisting of all 1's occurs for any graph G, so that it is typically suppressed in discussion. Only for a path is it the only multiplicity list. For a given graph G, we denote the collection of all multiplicity lists by $\mathcal{L}(G)$. If it is not clear from the context we will distinguish the unordered lists as $\mathcal{L}_u(G)$ from the ordered lists $\mathcal{L}_o(G)$. Of course, the latter usually includes a greater number of lists. But, it is usually not the case that every permutation of an unordered list occurs as an ordered list. If a list contains several consecutive repetitions of the same positive integer, we use a natural exponent notation. For example, the shorthand for the unordered list $(7, 5, 2, 2, 2, 1, 1, 1, 1)$ would be $7, 5, 2^3, 1^4$. In the last section we give, in condensed form, all (unordered) multiplicity lists for trees on fewer than 11 vertices.

2. Background

We adopt standard notation for submatrices of an m-by-n matrix A. The submatrix lying in rows $\alpha \subseteq \{1, \ldots, m\}$ and columns $\beta \subseteq \{1, \ldots, n\}$ is denoted by $A[\alpha, \beta]$ and the complementary one resulting from deletion of α, β by $A(\alpha, \beta)$. If A is square, the principal submatrices $A[\alpha, \alpha]$ and $A(\alpha, \alpha)$ are abbreviated to $A[\alpha]$ and $A(\alpha)$, respectively. If α consists of a single index i, we write, for example, $A(i)$. Note that if G is an undirected graph and $A \in \mathcal{H}(G)$, then $A(i) \in \mathcal{H}(G - i)$; we often view induced subgraphs and principal submatrices interchangeably. For example, if G' is a subgraph of G induced by α, we also write $A(G')$ (resp. $A[G']$) instead of $A(\alpha)$ (resp. $A[\alpha]$).

The classical interlacing inequalities (see, e.g., [**HJ**]) for the eigenvalues of an Hermitian matrix A and a principal submatrix $A(i)$ are fundamental to our question. If an n-by-n Hermitian matrix A has eigenvalues

$$a_1 \leq a_2 \leq \cdots \leq a_n$$

and $A(i)$ has eigenvalues

$$a_{i,1} \leq a_{i,2} \leq \cdots \leq a_{i,n-1},$$

then

$$a_1 \leq a_{i,1} \leq a_2 \leq a_{i,2} \leq \cdots \leq a_{i,n-1} \leq a_n,$$

$i = 1, \ldots, n$.

We denote the multiplicity of λ as an eigenvalue of A by $m_A(\lambda)$. A consequence of the interlacing inequalities is that

$$m_A(\lambda) - 1 \leq m_{A(i)}(\lambda) \leq m_A(\lambda) + 1$$

for $i = 1, \ldots, n$. This means that the multiplicity lists for a graph are closely related to those for single-vertex-deleted subgraphs and that the former are tightly constrained.

One might expect that in passing from A to $A(i)$, multiplicities typically decline. A fundamental result is counter to this intuition in the case for trees. A rather complete statement has evolved through a series of papers ([**Pa**], [**Wi**], [**JLS1**]).

THEOREM 2.1. *Let T be a tree and $A \in \mathcal{H}(T)$. Suppose that there exists an index i and a real number λ such that $\lambda \in \sigma(A) \cap \sigma(A(i))$. Then,*

(a) *there is an index j such that $m_{A(j)}(\lambda) = m_A(\lambda) + 1$;*
(b) *if $m_A(\lambda) \geq 2$, then j may be chosen so that $\deg_T(j) \geq 3$ and so that there are at least three components T_1, T_2 and T_3 of $T - j$ such that $m_{A[T_k]}(\lambda) \geq 1$, $k = 1, 2, 3$; and*
(c) *if $m_A(\lambda) = 1$, then j may be chosen so that there are two components T_1 and T_2 of $T - j$ such that $m_{A[T_k]}(\lambda) = 1$, $k = 1, 2$.*

The components of $T - j$ are naturally called *branches* of T at j. If G is a graph, $A \in \mathcal{H}(G)$, $\lambda \in \sigma(A)$ and j is an index such that

$$m_{A(j)}(\lambda) = m_A(\lambda) + 1,$$

then, for historical reasons [**Pa**], j is called a *Parter index* or vertex (for λ, A and G). If the requirements in (b) above are satisfied, j is called a *strong* Parter vertex. In particular, theorem 2.1 says that when T is a tree and $m_A(\lambda) \geq 2$, there must be a strong Parter vertex, because, by interlacing, the hypothesis $\lambda \in \sigma(A) \cap \sigma(A(i))$ must be satisfied for any i. However, i itself need not be a Parter vertex. Even when $m_A(\lambda) \geq 2$, it can happen that (a) be satisfied with $\deg_T(j) = 1$ or $\deg_T(j) = 2$ or λ appears in only one or two components of $T - j$, even if $\deg_T(j) \geq 3$. There may, as well, be several Parter vertices and even several strong Parter vertices. Much information about Parter vertices may be found in [**JLSSW**] and [**JLS1**].

How, then may we recognize Parter vertices? One useful way is via so-called "downer branches". A *downer vertex* i in a graph G (for $\lambda \in \sigma(A)$ and $A \in \mathcal{H}(G)$) is the natural antithesis of a Parter vertex, namely

$$m_{A(i)}(\lambda) = m_A(\lambda) - 1.$$

A *downer branch* of T at j is a branch T_i at j, determined by a neighbor i of j such that i is a downer vertex in T_i (for λ and $A[T_i]$). For completeness a vertex that is neither Parter nor a downer is called *neutral*. A result that identifies Parter vertices is then the following [**JLS1**].

THEOREM 2.2. *For $A \in \mathcal{H}(T)$, T a tree, j is a Parter vertex for λ if and only if there is a downer branch at j for λ.*

We note that theorem 2.2 is valid even if $m_A(\lambda) = 0$ and $m_{A(j)}(\lambda) = 1$; in fact, theorem 2.1 could (trivially) be stated so as to include an increase in multiplicity from 0 to 1.

Though a notion of "Parter vertex" can be defined for nontrees [**Sa**], there is a strong converse to theorem 2.1 that shows that its remarkable conclusions are generally valid only for trees [**JL4**].

THEOREM 2.3. *Suppose that G is graph on n vertices that is not a tree. Then:*

(a) *there is a matrix $A \in \mathcal{S}(G)$ with an eigenvalue λ such that there is an index j so that $m_A(\lambda) = m_{A(j)}(\lambda) = 1$ and $m_{A(i)}(\lambda) \leq 1$ for every $i = 1, \ldots, n$; and*

(b) *there is a matrix $B \in \mathcal{S}(G)$ with an eigenvalue λ such that $m_B(\lambda) \geq 2$ and $m_{B(i)}(\lambda) = m_B(\lambda) - 1$, for every $i = 1, \ldots, n$.*

If the graph G is a path on n vertices, then G is a tree and if the vertices are labelled consecutively, any matrix in $\mathcal{H}(G)$ is irreducible, tridiagonal. Conversely, the graph of an irreducible Hermitian tridiagonal matrix is a path. The very special spectral structure of such matrices has been of interest for some time for a variety of reasons. Two well known classical facts are that all eigenvalues are distinct (i.e., all 1's is the only multiplicity list) and, if a pendant vertex is deleted, the interlacing inequalities are strict. Both statements follow from theorem 2.1, but more can be gotten from theorem 2.1 as well. If A is n-by-n Hermitian and $1 \leq i \leq n$, then as many as $n-1$ of the eigenvalues of $A(i)$ might coincide with some eigenvalue of A. We refer to such an occurance as an "interlacing equality". If a pendant vertex is removed from a path, no interlacing equalities can occur, but if an interior vertex is removed, interlacing equalities can occur. The complete picture in this regard may be also be deduced from theorem 2.1.

COROLLARY 2.4. *Let A be an n-by-n irreducible Hermitian tridiagonal matrix. Then,*

(1) *A has distinct eigenvalues;*

(2) *in $A(i)$, there are at most $\min\{i-1, n-i\}$ interlacing equalities and this number may occur; and*

(3) *for each interlacing equality that does occur, the relevant eigenvalues must be an eigenvalue (of multiplicity 1) of both irreducible principal submatrices of $A(i)$.*

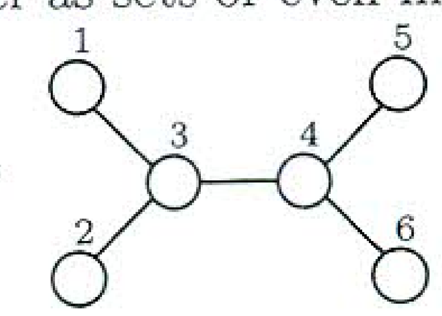
The classical information is a special case of the above general result.

Finally, we recall the notion of "majorization" and "conjugate partition". Let $u = (u_1, \ldots, u_b)$, $u_1 \geq \cdots \geq u_b$, and $v = (v_1, \ldots, v_c)$, $v_1 \geq \cdots \geq v_c$, be two partitions of integers M and N respectively, $M \leq N$, such that $u_1 + \cdots + u_s \leq v_1 + \cdots + v_s$ for all s, interpreting u_s or v_s as 0 when s exceeds b or c, respectively. If $M = N$, we say that u is *majorized* by v and write $u \preceq v$. If $M < N$ we denote by u_e the partition of N obtained from u by appending 1's to the partition u. It is easy to see that $u_e \preceq v$. Note that if $M = N$ then $u_e = u$. We denote by $u^* = (u_1, \ldots, u_b)^* = (u_1^*, \ldots, u_{u_1}^*)$ the *conjugate partition* of u, where u_i^* is the number of j's such that $u_j \geq i$ so that, $u^* = (u_1^*, \ldots, u_{u_1}^*)$ with $u_1^* \geq \cdots \geq u_{u_1}^* \geq 1$.

3. General Constraints upon Multiplicities

Theorem 2.1 and the interlacing inequalities clearly constrain possible multiplicities. They may constitute the only binding constraints for $\mathcal{H}(T)$ when T is a tree, but this is not known. In any event, the way in which they constrain is subtle. Here, we summarize known explicit constraints upon multiplicity lists, primarily for trees. These govern the "shape" of multiplicity lists and include the maximum individual multiplicity, the minimum number of distinct eigenvalues (the minimum length of a list) and the number of 1's that must appear in any list for a given tree. Together, for small numbers of vertices, these constrain lists enough that they "explain" all possible multiplicity lists.

3.1. The Maximum Multiplicity for a Tree. Let T be a tree. The maximum multiplicity $M(T)$ for a single eigenvalue among matrices in $\mathcal{H}(T)$ fortunately has a nice topological graph theoretic description. The *path cover number* $P(T)$ is simply the minimum number of induced paths of T, that do not intersect, but do cover all vertices of T. There may be many minimum path covers [**JL1**] ,[**HoJo**]. If T is a path, $P(T) = 1$; otherwise $P(T) > 1$. Another important, purely graph theoretic notion, that is closely related is $\Delta(T)$, defined by $\Delta(T) = \max[p - q]$ over all ways in which q vertices may be deleted from T, so as to leave p paths. Isolated vertices count as (degenerate) paths. Of course, maximizing sets of removed vertices are also not unique, either as sets or even in number. For example, in

$T =$ 

$P(T) = 2$ (e.g. 1-3-2 and 5-4-6 constitute a path cover of the vertices), and $\Delta(T) = 2$, as removal of vertex 4 leaves the 3 paths 1-3-2, 5 and 6 (and neither can be improved upon). Note that if submatrices $A[\{1,2,3\}]$, $A[\{5\}]$, and $A[\{6\}]$ of $A \in \mathcal{H}(T)$ are constructed so that λ is an eigenvalue of each (this is always possible and no higher multiplicity in any of them is possible), then $m_A(\lambda) \geq 3 - 1 = 2$, which is the maximum possible.

If an n-by-n matrix has rank $k \leq n$, its *rank deficiency* is $n - k$. Because algebraic and geometric multiplicity are the same for Hermitian matrices, and because of translation of a matrix by a multiple of the identity correspondingly translates the eigenvalues, $M(T) = m(T)$, the *maximum rank deficiency* for T,

$$m(T) = n - \min_{A \in \mathcal{H}(T)} \operatorname{rank}(A).$$

The maximum multiplicity may now be characterized [**JL1**].

THEOREM 3.1. *For each tree T,*

$$M(T) = \Delta(T) = P(T) = m(T).$$

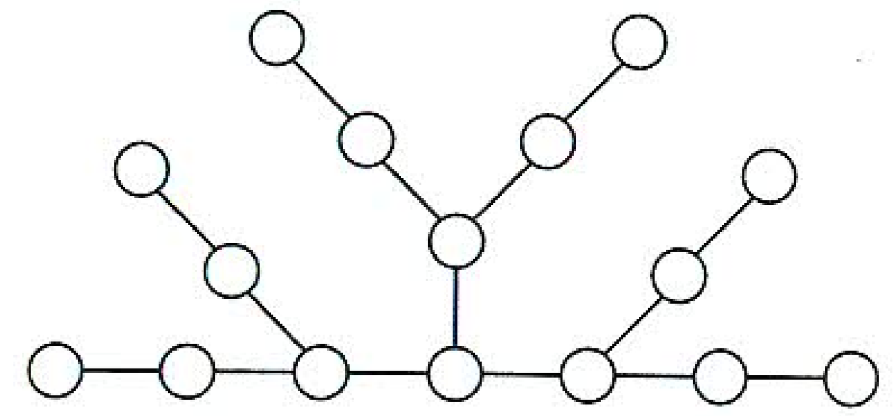
For general graphs G, no corresponding general result is known. The graphs for which $M(G) = 1$ have long been known [**Fi**]; they are just the paths (this is for the real field, but there are a few exceptions for 5-by-5 matrices and the field $\mathbb{Z}_3$, only [**B L-D**]) and the graphs for which $M(G) = 2$ have recently been described [**JoLoSm**].

3.2. The Minimum Number of Distinct Eigenvalues. Since each entry in a multiplicity list represents a distinct eigenvalue, the "length" of a list represents the number of different eigenvalues. This number can be as large as n (the number of vertices), of course, but it cannot be too small. Restrictions upon length limit the possible multiplicity lists. Just as a path has many distinct eigenvalues, a long (chordless) path occurring as an induced subgraph of a tree forces a large number of distinct eigenvalues. We define the *diameter* $d(T)$ of a tree T in terms of vertices: $d(T)$ = the maximum number of vertices in a path occurring as an induced subgraph of T.

The important fact [**JL2**] is

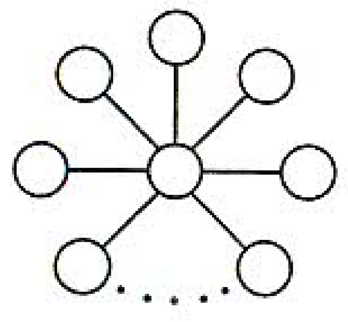
THEOREM 3.2. *Let T be a tree. The minimum, among $A \in \mathcal{H}(T)$, number of distinct eigenvalues of A is at least $d(T)$.*

For many trees T, there exist matrices $A \in \mathcal{H}(T)$ attaining as few distinct eigenvalues as $d(T)$. In [**JoSa3**], it is shown that for any tree T for which $d(T) < 6$, $d(T)$ may be attained. However, for the tree [**BF**]

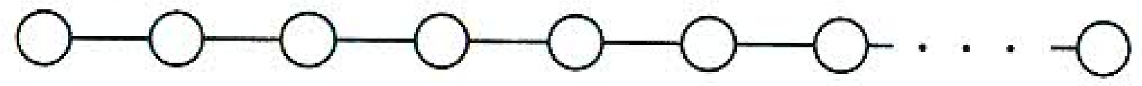
$T =$ 

$d(T) = 7$, and the minimum number of distinct eigenvalues is 8. It is not known how to deduce the minimum number of distinct eigenvalues from the structure of the tree, in general.

3.3. The Number of Multiplicities = 1. As with the length of lists, it is relatively easy to have many 1's in a multiplicity list. The more interesting issue is how few 1's may occur among lists for a given tree T? It certainly depends upon the tree, as the star



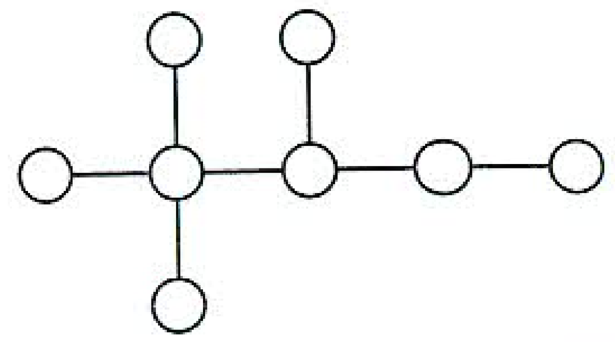
may have just two 1's, while a path



always has as many as the number of vertices. Let $\mathcal{U}(T)$ be the minimum number of 1's among multiplicity lists occurring for T. We then have

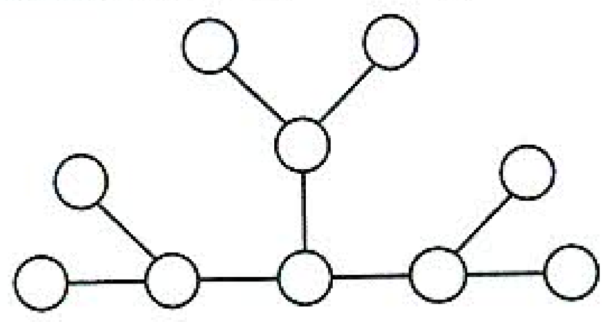
THEOREM 3.3. *For any tree T, $\mathcal{U}(T) \geq 2$ and, for any number of vertices, there exist trees T for which $\mathcal{U}(T) = 2$. Moreover, the largest and smallest eigenvalues of any $A \in \mathcal{H}(T)$ necessarily have multiplicity 1.*

Of course, if T has a diameter that is large relative to its number of vertices, then it may have to have a minimum number of distinct eigenvalues that forces $\mathcal{U}(T)$ to be much greater than 2. In particular [**Sh**], $\mathcal{U}(T) \geq 2d(T) - n$. However, $\mathcal{U}(T)$ may be greater than 2 for other reasons. For example, for

$T =$ 

$d(T) = 5$, $n = 8$, but $\mathcal{U}(T) = 3$. It is not known how $\mathcal{U}(T)$ is determined by T, and it appears to be quite subtle.

3.4. The Inverse Eigenvalue Problem for Trees. More precise than even the ordered multiplicity list problem for trees is the *Inverse Eigenvalue Problem* (IEP): given a tree T what are all possible spectra that occur among matrices in $\mathcal{S}(T)$, or $\mathcal{H}(T)$. For remarkably many trees, often including the classes to be discussed in the next section, the ordered multiplicity lists are equivalent to the IEP, i.e., a spectrum occurs if and only if it is consistent with some ordered list. Again this is not always the case. Extremal multiplicity lists for larger numbers of vertices can force numerical relations upon the eigenvalues, as in the tree [**BF**]



for the list $(4, 2, 2, 1, 1)$.

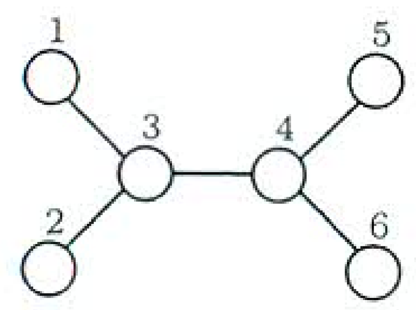
It is not known, for which trees the ordered lists are equivalent to the IEP. Even when the two are not equivalent for some ordered list, they may be for all other ordered lists. In the case of the all 1's list, every spectrum with distinct components is known to exist for every tree, and even every graph.

3.5. Construction of Trees with Given Multiplicities. It is often more difficult (than giving necessary restrictions) to construct matrices $A \in \mathcal{H}(T)$ with a given, especially extremal, multiplicity list, even when that list does occur. There are three basic approaches besides ad hoc methods and computer assisted solution of equations. They are

(a) manipulation of polynomials, viewing the nonzero entries as variables and targeting a desired characteristic polynomial (see [**Le1**] for an initial reference);
(b) division of the tree into understood parts and using the interlacing inequalities to give lower bounds that are forced to be attained by known constraints (this is along the lines of the brief discussion/example in Section 3.1 involving $\Delta(T)$, but for larger trees can lead to complicated simultaneity conditions); and
(c) careful use of the implicit function theorem (initiated in [**JSW**]).

All methods have shortcomings. For example verification that a certain Jacobian is nonsingular makes application of the implicit function theorem difficult, though this has been nicely systematized in some cases. In the case of any distinct eigenvalues (for any graph) the relevant Jacobian is the discriminant, which is necessarily nonsingular.

As an example of method (b) and its subtleties, consider again the tree

$T =$ 

Since $P(T) = 2$, the maximum multiplicity is 2, and because $d(T) = 4$, there must be at least 4 distinct eigenvalues, two of which have multiplicity 1. This leaves the question of whether the list $(2, 2, 1, 1)$ (which would have to be the ordered list $(1, 2, 2, 1)$) can occur. It can, but this is the first nontrivial example. Suppose that the two multiple eigenvalues are λ and μ. We want $A \in \mathcal{H}(T)$ with $m_A(\lambda) = 2$ and $m_A(\mu) = 2$. Each must have a Parter vertex, which must be either vertex 3 or 4. One must be for λ (and not μ) and the other for μ (and not λ), as two consecutive eigenvalues cannot share a Parter vertex [**JLSSW**]. So assume that 3 is Parter for λ and 4 for μ. Then, we must have $A[\{1\}] = \lambda = A[\{2\}]$ and $\lambda \in \sigma(A[\{4,5,6\}])$; and $A[\{5\}] = \mu = A[\{6\}]$ and $\mu \in \sigma(A[\{1,2,3\}])$. A calculation (or other methods) shows this can be achieved simultaneously.

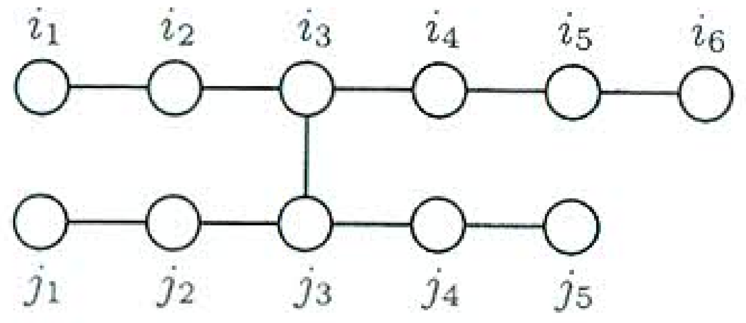
The polynomial method (a), based on some nice formulas for the characteristic polynomial in the case of a tree (see, e.g., [**Pa**], [**MODW**]), can be quite tedious for larger, more complicated trees.

4. Possible Multiplicities for Specific Graphs

4.1. Paths and Double Paths. It is well known that an Hermitian tridiagonal matrix has only simple eigenvalues and so the list of possible multiplicities for a path on n vertices is just an n-uple consisting of 1's. In fact, for a path, much more is known namely the solution of the *Generalized Inverse Eigenvalue Problem* (GIEP): the description of all the list of real numbers that may occur as eigenvalues of A and $A(i)$, for a given i, as A varies over the set of Hermitian tridiagonal matrices. It turns out that any set of real numbers verifying the interlacing inequalities together with the restrictions imposed by corollary 2.4 ([**JL3**, theorem 7]; for the case of strict interlacing see also [**Ch**] and references therein) is a solution of the GIEP for $\mathcal{H}(T)$, T a path. Note that the number of possible equalities in interlacing depends on the vertex i chosen.

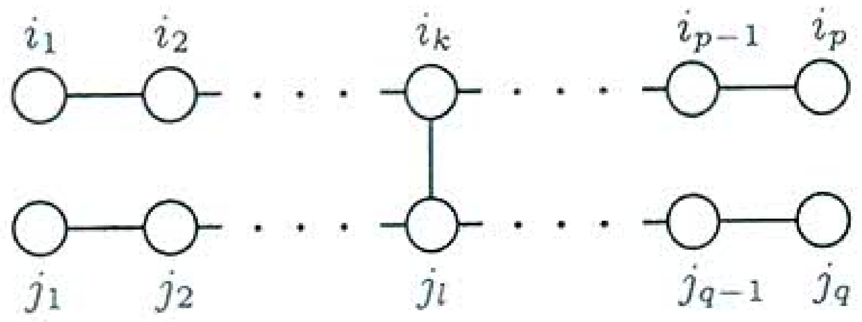
In the construction of multiplicity lists for a tree it is often useful (and, perhaps necessary) to know the solution of the GIEP (or some weak form of it) for some of the subtrees of the tree, as the following example shows.

Consider the following tree on 11 vertices.



Now pick real numbers $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$. Then, call T_1 the path with vertices $i_1, \ldots, i_6$ and T_2 the path with vertices $j_1, \ldots, j_5$ (subgraphs of T induced by the mentioned vertices). Construct A_1 with graph T_1 such that A_1 has eigenvalues $\lambda_1, \mu_1, \lambda_2, \mu_2, \lambda_3, \lambda_4$ and such that the eigenvalues of $A_1[\{i_1, i_2\}]$ and $A_1[\{i_4, i_5, i_6\}]$ are μ_1, μ_2 and μ_1, μ_2, μ_3 respectively; construct A_2 with graph T_2 such that A_2 has eigenvalues $\mu_1, \lambda_2, \mu_2, \lambda_3, \mu_3$ and such that the eigenvalues of both $A_2[\{j_1, j_2\}]$ and $A_2[\{j_4, j_5\}]$ are λ_2, λ_3. According to the GIEP these constructions are possible. Now construct A with graph T and such that $A[T_1] = A_1$ and $A[T_2] = A_2$. Then i_3 is a Parter vertex for μ_1, μ_2, while j_3 is Parter for λ_2, λ_3 and so that $(2, 2, 2, 2, 1, 1, 1) \in \mathcal{L}_u(T)$.

The above tree is an example of what is called a *double path*: a tree whose path cover number is 2. A double path may be represented as



in which the only constraint on the connecting edge $\{i_k, j_l\}$ is that not both $k \in \{1, p\}$ and $l \in \{1, q\}$. The upper (i) path has $k-1$ vertices to the left of the connecting vertex and another $p-k$ vertices to the right; set $s_1 = \min\{k-1, p-k\}$. Similarly, set $s_2 = \min\{l-1, q-l\}$.

If G is a double path and $A \in \mathcal{H}(G)$, the maximum multiplicity of an eigenvalue of A is 2, but how many multiplicity 2 eigenvalues may $A \in \mathcal{H}(G)$ have? Since G has $p+q$ vertices and the length of the longest path in G is $\max\{p, q, p+q-(s_1+s_2)\}$, $A \in \mathcal{H}(G)$ has at least $\max\{p, q, p+q-(s_1+s_2)\}$ distinct eigenvalues and thus at most

$$s = \min\{q, p, s_1 + s_2\}$$

multiplicity 2 eigenvalues. In the above example we have $s = 4$ and we have 4 double eigenvalues. In fact the above constructions can be adapted to general double trees and that numbers can always be achieved, as the next theorem shows [**JL3**, theorem 8].

THEOREM 4.1. *Let G be a double path whose paths have p and q vertices, respectively and define s_1, s_2 and s as above. Then the possible list of unordered multiplicities for G, $\mathcal{L}_u(G)$, consists of all partitions of $p+q$ into parts each one not greater than two and with at most s equal to 2.*

It follows that if G is a double path, any list in $\mathcal{L}_u(G)$ has at least $n-2s$ 1's, in which n is the total number of vertices and s is as above. Of course, this number could be quite large if there is a big disparity in the lengths of the two paths of G or if the connecting edge is far from the center of the two paths. On the other hand there might be only two ones: this happens if and only if $k-1 = p-k$ and $l-1 = q-l$ and $|p-q| \le 2$ (i.e., p and q are odd numbers such that $|p-q| \le 2$ and vertices k and l are in the center of each path).

4.2. Stars and Generalizations. We call a tree T in which there is at most a vertex of degree greater than two, a *generalized star*. In a generalized star, we call a vertex v a *central vertex* if its neighbors are pendant vertices of their branches, and each branch is a path which we call an *arm* of T. Supposing that $\deg_T(v) = k$ we will denote by $T_1, \ldots, T_k$ these arms and by $l_1, \ldots, l_k$ the *lengths* (number of vertices) of $T_1, \ldots, T_k$, respectively. Note that if in a generalized star there is a vertex of degree grater than two, such a vertex must be unique and is the central vertex and the arms are completely determined; we will be primarily interested on those generalized stars, but for technical reasons, our definition of generalized stars also consider as a (degenerate) generalized star a tree with no vertex of degree greater than two: these trees are just the paths in which any vertex is a central vertex. When referring to a path as a generalized star we always suppose that one vertex has been fixed as the central vertex and so we will generally refer to the central vertex of a generalized star.

For generalized stars the solution of the GIEP is known, when the central vertex is deleted [**JLS2**, theorem 11] (for other vertices the solution of the GIEP seems to be quite different and quite complicated).

THEOREM 4.2. *Let T be a generalized star on n vertices with central vertex v of degree k, $l_1, \ldots, l_k$ be the lengths of the arms $T_1, \ldots, T_k$, and $f(x)$, $g_1(x), \ldots, g_k(x)$ be monic polynomials with all their roots real in which* $\deg f = n$, $\deg g_1 = l_1, \ldots,$ $\deg g_k = l_k$.

There exists $A \in \mathcal{H}(T)$ such that A as characteristic polynomial $f(x)$ and $A[T_i]$ has characteristic polynomial $g_i(x)$ if and only if

(1) *Each $g_i(x)$ has only simple roots;*
(2) *If λ is a root of $g_1(x) \cdots g_k(x)$ of multiplicity $m \geq 1$ then λ is a root of $f(x)$ of multiplicity $m-1$; and*
(3) *The roots of $f(x)$ that are not roots of $g_1(x) \cdots g_k(x)$, are simple and strictly interlace the set of roots of $g_1(x) \cdots g_k(x)$ (multiple roots counting only once).*

Note that for a generalized star T whose central vertex v has degree greater than two, the only possible strong Parter vertex is v and so to describe the set $\mathcal{L}_u(T)$ the main question is the (combinatorial) problem of allocating the multiple eigenvalues among the arms of T (note that in each principal submatrix corresponding to an arm of T, an eigenvalue occurs with multiplicity one so that, if it occurs with multiplicity k in $A \in \mathcal{H}(T)$ it must occur in $k+1$ principal submatrices of $A(v)$, each one corresponding to an arm of T). The Gale-Ryser theorem (see, e.g., [**Ry**]) provides a tool for dealing with that combinatorial problem and we have the following result [**JL3**, theorem 9].

THEOREM 4.3. *Let T be a generalized star on n vertices with central vertex of degree s and arm lengths $l_1 \geq \cdots \geq l_s$. Then $(p_1, \ldots, p_r) \in \mathcal{L}_u(T)$ if and only if*

(a) $\sum_{i=1}^{r} p_i = n$;
(b) $r \geq l_1 + l_2 + 1$;
(c) $p_h = p_{h+1} = \cdots = p_r = 1$, *in which* $h = \lceil \frac{r+1}{2} \rceil$;
(d) $(p_1, p_2, \ldots, p_{r-l_1-1}) \preceq (l_1^* - 1, \ldots, l_{l_1}^* - 1)$.

From theorem 4.2 we can also obtain a description of $\mathcal{L}_o(T)$ and the solution of the IEP for generalized stars [**JLS2**, theorems 14 and 15].

THEOREM 4.4. *Let T be a generalized star on n vertices with central vertex of degree s and arm lengths $l_1 \geq \cdots \geq l_s$. Then $(q_1, \ldots, q_r) \in \mathcal{L}_o(T)$ if and only if*

(a) $\sum_{i=1}^{r} q_i = n$;
(b) *if* $q_i > 1$ *then* $1 < i < r$ *and* $q_{i-1} = 1 = q_{i+1}$; *and*
(c) $(q_{i_1} + 1, \ldots, q_{i_h} + 1)_e \preceq (l_1, \ldots, l_s)^*$, *in which* $q_{i_1} \geq \cdots \geq q_{i_h}$ *are the entries of the r-tuple $(q_1, \ldots, q_r)$ greater than 1.*

Moreover, given any sequence of real numbers $\lambda_1 > \cdots > \lambda_r$ and a sequence of positive integers $q_1, \ldots, q_r$, there exists $A \in \mathcal{H}(T)$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ and such that $m_A(\lambda_i) = q_i$ if and only if $q_1, \ldots, q_r$ verify (a)-(c), *that is* $(q_1, \ldots, q_r) \in \mathcal{L}_o(T)$.

A *double generalized star* is a tree resulting from joining the central vertices of two generalized stars T_1 and T_2 by an edge. Such a tree will be denoted by $D(T_1, T_2)$. Note that, if T_1 or T_2 is a path having more than one vertex, the double generalized star based on T_1 and T_2 depends on the selected central vertex in the path. When we write $D(T_1, T_2)$ we are supposing that the central vertices were previously fixed.

The set $\mathcal{L}_o(D(T_1, T_2))$ was described in [**JLS2**] (see also [**Sa**]), via the so-called "upward multiplicities" of T_1 and T_2. Given a vertex v of a tree T and an eigenvalue λ of a matrix $A \in \mathcal{H}(T)$, we say that λ is an *upward eigenvalue* of A at v if $m_{A(v)}(\lambda) = m_A(\lambda) + 1$, and we call to $m_A(\lambda)$ an upward multiplicity of A at v. If $q = q(A) = (q_1, \ldots, q_r)$ is the list of ordered multiplicities of A, we define the *list of upward multiplicities* of A at v, denoted by $\hat{q}$, as the list with the same entries as q but in which any upward multiplicity q_i of A at v, is marked as $\hat{q}_i$ in $\hat{q}$. Given a generalized star T with central vertex v, we denote by $\hat{\mathcal{L}}_o(T)$ the set of all lists of upward multiplicities at v occurring among matrices in $\mathcal{H}(T)$.

Note that, when T is a generalized star, by theorem 2.1, any multiplicity greater than one is an upward multiplicity but also some of the simple multiplicities may be upwards. Nevertheless, if $q = (q_1, \ldots, q_r) \in \mathcal{L}_o(T)$, some of the q_i's (namely q_1 and q_r) are never upward multiplicities. From theorem 4.4, it is not difficult to obtain the description of the set $\hat{\mathcal{L}}_o(T)$, T a generalized star [**JLS2**, theorem 15].

THEOREM 4.5. *Let T be a generalized star on n vertices with central vertex v of degree k and arm lengths $l_1 \geq \cdots \geq l_k$. Let $\lambda_1 < \cdots < \lambda_r$ be any sequence of real numbers.*

Then there exists a matrix A in $\mathcal{H}(T)$ with distinct eigenvalues $\lambda_1 < \cdots < \lambda_r$ and list of upward multiplicities $\hat{q} = \widehat{(q_1, \ldots, q_r)}$ if and only if $\hat{q}$ satisfies the following conditions:

(a) $\sum_{i=1}^{r} q_i = n$;
(b) *if q_i is an upward multiplicity in $\hat{q}$ then $1 < i < r$ and neither q_{i-1} nor q_{i-1} is an upward multiplicity in $\hat{q}$;*
(c) $(q_{i_1} + 1, q_{i_2} + 1, \ldots, q_{i_h} + 1)_e \preceq (l_1, l_2, \ldots, l_k)^*$, *in which* $q_{i_1} \geq q_{i_2} \geq \cdots \geq q_{i_h}$ *are the upward multiplicities of $\hat{q}$.*

Now the next results allow us to describe the set $\mathcal{L}_o(D(T_1, T_2))$ [**JLS2**, theorems 24 and 26].

THEOREM 4.6 (Superposition Principle). *Let $D(T_1, T_2)$ be a double generalized star, $\hat{b} = \widehat{(b_1, \ldots, b_{s_1})} \in \hat{\mathcal{L}}_o(T_1)$ and $\hat{c} = \widehat{(c_1, \ldots, c_{s_2})} \in \hat{\mathcal{L}}_o(T_2)$. Construct any $b^+ = (b_1^+, \ldots, b_{s_1+t_1}^+)$ and $c^+ = (c_1^+, \ldots, c_{s_2+t_2}^+)$ subject to the following conditions:*

0. $t_1, t_2 \in \mathbb{N}_0$ *and* $s_1 + t_1 = s_2 + t_2$;
1. *b^+ (resp. c^+) is obtained from $\hat{b}$ (resp. $\hat{c}$) by inserting t_1 (resp. t_2) 0's;*
2. *b_i^+ and c_i^+ cannot both be 0; and*
3. *if $b_i^+ > 0$ and $c_i^+ > 0$, then at least one of the b_i^+ or c_i^+ must be an upward multiplicity of $\hat{b}$ or $\hat{c}$.*

Then we have $b^+ + c^+ \in \mathcal{L}_o(D(T_1, T_2))$. Moreover, $a \in \mathcal{L}_o(D(T_1, T_2))$ if and only if there are $\hat{b} \in \hat{\mathcal{L}}_o(T_1)$, $\hat{c} \in \hat{\mathcal{L}}_o(T_2)$ such that $a = b^+ + c^+$.

The above results may be extended to a general tree T by associating with T either a generalized star or a double generalized star according to the following definition.

DEFINITION 4.7. Let G be a tree. Let v be a vertex of G of degree k and $G_1, \ldots, G_k$ be the components of $G - v$ having size $l_1, \ldots, l_k$, respectively. To the tree G associate the generalized star, $S_v(G)$, with central vertex v of degree k, and with arms $T_1, \ldots, T_k$ of lengths $l_1, \ldots, l_k$, respectively.

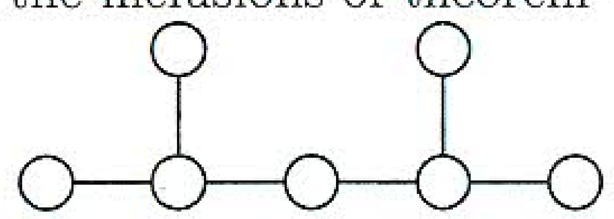
Let u_1 and u_2 be adjacent vertices of G. Denote by G_{u_1} the connected component of $G - u_2$ that contains u_1 and by G_{u_2} the connected component of $G - u_1$ that contains u_2. Put $S_1 = S_{u_1}(G_{u_1})$ and $S_2 = S_{u_2}(G_{u_2})$. Now, to the tree G associate the double generalized star, denoted by $D_{u_1,u_2}(G)$, $D(S_1, S_2)$.

Although not explicitly stated in [**JLS2**] is not difficult to see that the following result holds (see also [**JLS2**, theorem 10] for a corresponding result for the GIEP).

THEOREM 4.8. *Let T be a tree, v be a vertex of T and v_1, v_2 be adjacent vertices of T. Then*

1. $\mathcal{L}_u(S_v(T)) \subseteq \mathcal{L}_u(T)$, $\mathcal{L}_o(S_v(T)) \subseteq \mathcal{L}_o(T)$;
2. $\mathcal{L}_u(D_{v_1,v_2}(T)) \subseteq \mathcal{L}_u(T)$, $\mathcal{L}_o(D_{v_1,v_2}(T)) \subseteq \mathcal{L}_o(T)$.

Note that there are many different possibilities to associate either a generalized star or a double generalized star to a given tree T in the above described manner, and so theorem 4.6 provides many possible lists for $\mathcal{L}(T)$. The natural question is to ask whether all the elements of $\mathcal{L}(T)$ can be obtained in this manner. The answer is no! It suffices to note that the path cover number of T will be, in general, strictly greater than that of either of $S_v(T)$ and $D_{v_1,v_2}(T)$ for any possible choice of v, v_1 and v_2. So any lists for which the maximum multiplicity occurs cannot generally be obtained from the inclusions of theorem 4.8. For example, the tree T



has path cover number 3 which is strictly greater than the maximum path cover number, 2, of any generalized star or double generalized star associated with T.

The next class of trees which seems natural to consider is the trees obtained from two generalized stars by joining their central vertices by a path. We give the following definitions.

DEFINITION 4.9. A *separated double generalized star* is a tree with exactly two vertices of degree greater than 2 which are connected by a path. Any such tree can be obtained from two generalized stars T_1 and T_2 whose central vertices are connected by a path of length s ($s \geq 1$). Such a tree will be denoted by $SD(T_1, T_2, s)$. When $s = 1$ we call $SD(T_1, T_2, s)$ a *singly separated double generalized star.*

These kind of trees were studied in [**Sh**] and, for a singly separated double generalized star, an algorithm that generalizes the superposition principle was designed, which describes the list of multiplicities for such trees [**Sh**, theorem 6.6].

4.3. Vines. The following class of trees were introduced in [**JSW**].

DEFINITION 4.10. A *binary tree* is a tree in which no vertex has degree greater than 3.

A *vine* is a binary tree in which every degree 3 vertex is adjacent to at least one vertex of degree 1 and no two vertices of degree 3 are adjacent.

By using the implicit function theorem technique referred in Section 3.5, the set $\mathcal{L}_u(T)$, in which T is a vine has described in [**JSW**, theorem 5.3].

THEOREM 4.11. *Let T be a vine on n vertices. The set $\mathcal{L}_u(T)$ consists of all sequences that are majorized by the sequence $s = (P(T), 1, 1, \ldots, 1)$ (s being a partition of n).*

4.4. Cycles. Very little is known about the eigenvalue multiplicities for graphs other than trees. The next graph which is natural to consider is the *cycle*: a connected graph in which every vertex has degree 2. For a cycle C, the eigenvalues of a matrix $A \in \mathcal{S}(C)$ with positive off-diagonal entries were studied in [**Fe**] and, in fact, for these matrices the IEP problem was solved.

Note that, if C is a cycle on n vertices and we delete a vertex of C a path remains. So it follows immediately from interlacing (and the fact that the only multiplicity list for a path is all 1's) that the maximum multiplicity of an eigenvalue occurring for any $A \in \mathcal{S}(C)$ is not greater than 2, and, by [**Fe**], such multiplicities do occur. Moreover, the maximum number of 2's occurring in $\mathcal{L}_u(C)$ is $[\frac{n-1}{2}]$ and also for any $s \leq [\frac{n-1}{2}]$ there is a list in $\mathcal{L}_u(C)$ with s 2's.

5. Multiplicity Lists for Trees with Fewer than 11 Vertices

We conclude with tables giving all (unordered) multiplicity lists (M-Lists) for all trees on fewer than 11 vertices. These can be helpful to those interested in making or checking conjectures of more general results. A large majority of these follow from results about general classes of trees, such as those reported in the prior section, while several result from painstaking constructions coupled with known necessary conditions. (Most 11 vertex trees could be resolved similarly.) The smaller number of vertices have been reported in references, while the 9-vertex trees are from [**Sa**] (see also [**JoSa3**]), corrected, and the 10-vertex trees are from [**Sh**], corrected. We have chosen a format for this information to compress it into minimal space, and (arbitrarily) begun the lists with 5-vertex trees (as smaller numbers are trivial).

The trees are numbered according to the list appearing in the appendix to [**CDS**] (which contains a few easily corrected errors). All multiplicity lists for a particular tree are given, except that in each list all 1's are omitted, so that the list of all 1's (occurring for every graph) does not appear. The actual number of 1's may be inferred from the number of vertices and the given higher multiplicities. Individual multiplicity lists are separated with a semicolon and space, while the higher multiplicities of a list are simply given in nonincreasing order by juxtaposition (which is unambiguous, as all multiplicities are single digit). For example, tree number 2.65 has 9 vertices and the 6 possible multiplicity lists: $(4,1,1,1,1,1)$, $(3,2,1,1,1,1)$, $(3,1,1,1,1,1,1)$, $(2,2,1,1,1,1,1)$, $(2,1,1,1,1,1,1,1)$, $(1,1,1,1,1,1,1,1,1)$, which is abbreviated unambiguously to

2.65	4 ; 32 ; 3 ; 2^2 ; 2

TREES ON 5 VERTICES

Tree	M-Lists	Tree	M-Lists	Tree	M-Lists
2.5	3 ; 2	**2.6**	2	**2.7**	

TREES ON 6 VERTICES

Tree	M-Lists	Tree	M-Lists	Tree	M-Lists
2.8	4 ; 3 ; 2	**2.10**	2^2 ; 2	**2.12**	2
2.9	3 ; 2	**2.11**	2	**2.13**	

TREES ON 7 VERTICES

Tree	M-Lists	Tree	M-Lists	Tree	M-Lists	Tree	M-Lists
2.14	5 ; 4 ; 3 ; 2^2 ; 2	**2.17**	3 ; 2	**2.20**	2^2 ; 2	**2.23**	2^2 ; 2
2.15	4 ; 3 ; 2^2 ; 2	**2.18**	3 ; 2^2 ; 2	**2.21**	2	**2.24**	
2.16	32 ; 3 ; 2^2 ; 2	**2.19**	3 ; 2^2 ; 2	**2.22**	2		

Trees on 8 vertices

Tree	M-Lists	Tree	M-Lists	Tree	M-Lists
2.25	$6;5;4;32;3;2^2;2$	**2.33**	$32;3;2^3;2^2;2$	**2.41**	$2^3;2^2;2$
2.26	$5;4;32;3;2^2;2$	**2.34**	$3;2$	**2.42**	$2^2;2$
2.27	$42;4;32;3;2^3;2^2;2$	**2.35**	$32;3;2^3;2^2;2$	**2.43**	2
2.28	$4;3^2;32;3;2^2;2$	**2.36**	$3;2^2;2$	**2.44**	2
2.29	$4;3;2^2;2$	**2.37**	$32;3;2^2;2$	**2.45**	2
2.30	$4;32;3;2^2;2$	**2.38**	$3;2^2;2$	**2.46**	$2^2;2$
2.31	$4;32;3;2^2;2$	**2.39**	$3;2^2;2$	**2.47**	
2.32	$32;3;2^2;2$	**2.40**	$2^2;2$		

Trees on 9 vertices

Tree	M-Lists	Tree	M-Lists
2.48	$7;6;5;42;4;3^2;32;3;2^2;2$	**2.72**	$3;2$
2.49	$6;5;42;4;33;32;3;2^2;2$	**2.73**	$32;3;2^3;2^2;2$
2.50	$52;5;42;4;32^2;32;3;2^3;2^2;2$	**2.74**	$32^2;32;3;2^3;2^2;2$
2.51	$5;4;32;3;2^2;2$	**2.75**	$3;2^2;2$
2.52	$5;42;4;3^2;32;3;2^2;2$	**2.76**	$3;32;2^3;2^2;2$
2.53	$5;43;42;4;3^2;32^2;32;3;2^3;2^2;2$	**2.77**	$3;2^2;2$
2.54	$5;42;4;32;3;2^3;2^2;2$	**2.78**	$32;3;2^2;2$
2.55	$42;4;32;3;2^3;2^2;2$	**2.79**	$3^2;32;3;2^2;2$
2.56	$42;4;32^2;32;3;2^3;2^2;2$	**2.80**	$3;2^2;2$
2.57	$5;4;3^2;32;3;2^2;2$	**2.81**	$3;2^2;2$
2.58	$4;3;2^2;2$	**2.82**	$3;2^2;2$
2.59	$4;3^2;32^2;32;3;2^3;2^2;2$	**2.83**	$2^2;2$
2.60	$4;32;3;2^2;2$	**2.84**	$32;3;2^3;2^2;2$
2.61	$42;4;3^2;32;3;2^2;2$	**2.85**	$3;2^2;2$
2.62	$42;4;32^2;32;3;2^3;2^2;2$	**2.86**	$2^3;2^2;2$
2.63	$3^2;32^2;32;3;2^3;2^2;2$	**2.87**	$2^2;2$
2.64	$4;32;3;2^2;2$	**2.88**	$2^3;2^2;2$
2.65	$4;32;3;2^2;2$	**2.89**	2
2.66	$4;32;3;2^3;2^2;2$	**2.90**	2
2.67	$32;3;2^2;2$	**2.91**	2
2.68	$32^2;32;3;2^3;2^2;2$	**2.92**	$2^2;2$
2.69	$32;3;2^3;2^2;2$	**2.93**	$2^2;2$
2.70	$32;3;2^3;2^2;2$	**2.94**	
2.71	$32^2;32;3;2^3;2^2;2$		

Trees on 10 vertices

Tree	M-Lists	Tree	M-Lists
2.95	$8;7;6;52;5;43;42;4;3^2;32;3;2^3;2^2;2$	**2.148**	$32;3;2^2;2$
2.96	$7;6;52;5;43;42;4;3^2;32;3;2^3;2^2;2$	**2.149**	$3^2;32^2;32;3;2^4;2^3;2^2;2$
2.97	$62;6;52;5;42^2;42;4;3^22;3^2;32^2;32;3;2^3;2^2;2$	**2.150**	$4;32;3;2^3;2^2;2$
2.98	$6;5;42;4;3^2;32;3;2^3;2^2;2$	**2.151**	$32^2;32;3;2^3;2^2;2$
2.99	$6;52;5;43;42;4;3^2;32;3;2^3;2^2;2$	**2.152**	$32;3;2^3;2^2;2$
2.100	$6;53;52;5;43;42;4;3^22;3^2;32^2;32;3;2^3;2^2;2$	**2.153**	$42;4;3^2;32^2;32;3;2^3;2^2;2$
2.101	$6;5;4^2;43;42^2;42;4;3^2;32^2;32;3;2^4;2^3;2^2;2$	**2.154**	$32;3;2^3;2^2;2$
2.102	$6;52;5;42;4;3^2;32^2;32;3;2^3;2^2;2$	**2.155**	$32;3;2^3;2^2;2$
2.103	$52;5;42;4;32^2;32;3;2^3;2^2;2$	**2.156**	$32^2;32;3;2^3;2^2;2$
2.104	$52;5;42^2;42;4;3^22;3^2;32^2;32;3;2^3;2^2;2$	**2.157**	$32^2;32;3;2^4;2^3;2^2;2$
2.105	$5;4;32;3;2^2;2$	**2.158**	$32^2;32;3;2^3;2^2;2$
2.106	$5;42;4;3^2;32;3;2^3;2^2;2$	**2.159**	$3^22;3^2;32^2;32;3;2^3;2^2;2$
2.107	$52;5;43;42;4;3^2;32;3;2^3;2^2;2$	**2.160**	$4;32;3;2^3;2^2;2$
2.108	$6;5;43;42;4;3^2;32^2;32;3;2^3;2^2;2$	**2.161**	$3;2$
2.109	$5;43;42^2;42;4;3^2;32^2;32;3;2^4;2^3;2^2;2$	**2.162**	$32;3;2^3;2^2;2$
2.110	$5;43;42;4;3^22;3^2;32^2;32;3;2^3;2^2;2$	**2.163**	$32^2;32;3;2^4;2^3;2^2;2$
2.111	$52;5;42^2;42;4;32^2;32;3;2^4;2^3;2^2;2$	**2.164**	$32;3;2^3;2^2;2$
2.112	$43;42^2;42;4;3^22;3^2;32^2;32;3;2^4;2^3;2^2;2$	**2.165**	$32;3;2^3;2^2;2$
2.113	$5;42;4;32;3;2^3;2^2;2$	**2.166**	$32^2;32;3;2^4;2^3;2^2;2$
2.114	$52;5;42;4;3^22;3^2;32^2;32;3;2^3;2^2;2$	**2.167**	$3;2^2;2$
2.115	$5;42;4;32;3;2^3;2^2;2$	**2.168**	$32;3;2^3;2^2;2$
2.116	$5;42;4;3^2;32^2;32;3;2^3;2^2;2$	**2.169**	$3;2^2;2$
2.117	$43;42^2;42;4;3^22;3^2;32^2;32;3;2^4;2^3;2^2;2$	**2.170**	$32^2;32;3;2^4;2^3;2^2;2$
2.118	$42;4;32;3;2^3;2^2;2$	**2.171**	$32^2;32;3;2^4;2^3;2^2;2$
2.119	$42^2;42;4;32^2;32;3;2^4;2^3;2^2;2$	**2.172**	$32;3;2^3;2^2;2$
2.120	$42;4;32^2;32;3;2^3;2^2;2$	**2.173**	$32;3;2^3;2^2;2$
2.121	$42;4;32^2;32;3;2^3;2^2;2$	**2.174**	$32;3;2^2;2$
2.122	$42^2;42;4;3^22;3^2;32^2;32;3;2^3;2^2;2$	**2.175**	$3^2;32;3;2^3;2^2;2$
2.123	$5;4;3^2;32;3;2^2;2$	**2.176**	$3;2^2;2$
2.124	$5;42;4;3^2;32^2;32;3;2^3;2^2;2$	**2.177**	$3;2^2;2$
2.125	$4;3;2^2;2$	**2.178**	$3;2^2;2$
2.126	$4;3^2;32^2;32;3;2^3;2^2;2$	**2.179**	$3;2^2;2$
2.127	$4;32;3;2^3;2^2;2$	**2.180**	$32;3;2^3;2^2;2$
2.128	$4;32;3;2^2;2$	**2.181**	$2^2;2$
2.129	$42;4;3^22;3^2;32^2;32;3;2^3;2^2;2$	**2.182**	$3;2^2;2$
2.130	$4;3^2;32^2;2^4;2^3;2^2;2$	**2.183**	$32;3;2^3;2^2;2$
2.131	$42;4;3^2;32;3;2^3;2^2;2$	**2.184**	$3;2^2;2$
2.132	$43;42;4;3^2;32;3;2^3;2^2;2$	**2.185**	$2^3;2^2;2$
2.133	$42;4;32^2;32;3;2^3;2^2;2$	**2.186**	$2^2;2$
2.134	$42;4;3^2;32^2;32;3;2^3;2^2;2$	**2.187**	$2^3;2^2;2$
2.135	$42;4;3^2;32^2;32;3;2^4;2^3;2^2;2$	**2.188**	$2^2;2$
2.136	$42^2;42;4;32^2;32;3;2^4;2^3;2^2;2$	**2.189**	$32;3;2^3;2^2;2$
2.137	$42;4;32^2;32;3;2^3;2^2;2$	**2.190**	$2^3;2^2;2$
2.138	$42;4;32^2;32;3;2^4;2^3;2^2;2$	**2.191**	$2^3;2^2;2$
2.139	$3^2;32^2;32;3;2^4;2^3;2^2;2$	**2.192**	$2^4;2^3;2^2;2$
2.140	$3^22;3^2;32^2;32;3;2^4;2^3;2^2;2$	**2.193**	2
2.141	$4;32;3;2^2;2$	**2.194**	2
2.142	$4;32;3;2^2;2$	**2.195**	2
2.143	$4;32;3;2^2;2$	**2.196**	2
2.144	$4;32;3;2^3;2^2;2$	**2.197**	$2^3;2^2;2$
2.145	$42;4;32^2;32;3;2^3;2^2;2$	**2.198**	$2^2;2$
2.146	$4;32;3;2^3;2^2;2$	**2.199**	$2^2;2$
2.147	$42^2;42;4;32^2;32;3;2^4;2^3;2^2;2$	**2.200**	

References

[BF] F. Barioli and S. M. Fallat. On two conjectures regarding an inverse eigenvalue problem for acyclic symmetric matrices. *Electronic Journal of Linear Algebra* 11:41–50 (2004).

[B L-D] A. Bento and A. Leal-Duarte. On Fidler's characterization of tridiagonal matrices over arbitrary fieds. *Linear Algebra and its Applications* 401:467–481 (2005).

[BG] D. Boley and G. H. Golub. A survey of inverse eigenvalue problems. *Inverse Problems* 3:595–622 (1987).

[CL] C. Chartrand and L. Lesniak. *Graphs & Digraphs*. Chapman & Hall, London, 1996.

[Ch] M. T. Chu. Inverse eigenvalue problems. *SIAM Review* 40:1–39 (1998).

[CDS] D. Cvetković, M. Doob and H. Sachs. *Spectra of Graphs*. Johann Ambrosius Barth Verlag, 1995.

[FP] K. Fan and G. Pall. Imbedding conditions for Hermitian and normal matrices. *Canadian Journal of Mathematics* 9:298–304 (1957).

[Fe] W. Ferguson. The construction of Jacobi and periodic Jacobi matrices with prescribed spectra. *Mathematics of Computation* 35:1203–1220 (1980).

[Fi] M. Fiedler. A characterization of tridiagonal matrices. *Linear Algebra and Applications* 2:191–197 (1969).

[GM] J. Genin and J. Maybee. Mechanical vibration trees. *Journal of Mathematical Analysis and Applications* 45:746–763 (1974).

[HoJo] L. Hogben and C. R. Johnson. On the number of minimum path covers. Manuscript

[HJ] R. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, New York, 1985.

[JL1] C. R. Johnson and A. Leal-Duarte. The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree. *Linear and Multilinear Algebra* 46:139–144 (1999).

[JL2] C. R. Johnson and A. Leal-Duarte. On the minimum number of distinct eigenvalues for a symmetric matrix whose graph is a given tree. *Mathematical Inequalities and Applications* 5(2):175–180(2002).

[JL3] C. R. Johnson and A. Leal-Duarte. On the possible multiplicities of the eigenvalues of an Hermitian matrix whose graph is a given tree. *Linear Algebra and its Applications* 348:7–21 (2002).

[JL4] C. R. Johnson and A. Leal-Duarte. Converse to the Parter-Wiener theorem: the case of non-trees. *Discrete Mathematics*, to appear.

[JLSSW] C. R. Johnson, A. Leal-Duarte, C. M. Saiago, B. D. Sutton and A. J. Witt. On the relative position of multiple eigenvalues in the spectrum of an Hermitian matrix with a given graph. *Linear Algebra and its Applications* 363:147–159 (2003).

[JLS1] C. R. Johnson, A. Leal-Duarte and C. M. Saiago. The Parter-Wiener theorem: refinement and generalization. *SIAM Journal on Matrix Analysis and Applications* 25(2):352–361 (2003).

[JLS2] C. R. Johnson, A. Leal-Duarte and C. M. Saiago. Inverse eigenvalue problems and lists of multiplicities of eigenvalues for matrices whose graph is a tree: the case of generalized stars and double generalized stars. *Linear Algebra and its Applications* 373:311–330 (2003).

[JoLoSm] C. R. Johnson, R. Loewy and P. Smith. The graphs for which the maximum multiplicity of an eigenvalue is two. Manuscript.

[JoSa1] C. R. Johnson and C. M. Saiago. Estimation of the maximum multiplicity of an eigenvalue in terms of the vertex degrees of the graph of a matrix. *Electronic Journal of Linear Algebra* 9:27–31 (2002).

[JoSa2] C. R. Johnson and C. M. Saiago. The trees for which maximum multiplicity implies the simplicity of other eigenvalues. *Discrete Mathematics*, to appear.

[JoSa3] C. R. Johnson and C. M. Saiago. Branch Duplication for the construction of multiple eigenvalues in an Hermitian matrix whose graph is a tree. *Linear and Multilinear Algebra*, to appear.

[JoSu] C. R. Johnson and B. D. Sutton. Hermitian matrices, eigenvalue multiplicities, and eigenvector components. *SIAM Journal on Matrix Analysis and Applications* 26(2):390–399 (2004).

[JSW] C. R. Johnson, B. D. Sutton and A. Witt. Implicit construction of multiple eigenvalues for trees. Preprint.

[Le1] A. Leal-Duarte. Construction of acyclic matrices from spectral data. *Linear Algebra and its Applications* 113:173–182 (1989).

[Le2] A. Leal-Duarte. *Desigualdades Espectrais e Problemas de Existência em Teoria de Matrizes*. Dissertação de Doutoramento, Coimbra, 1992.

[MODW] J. S. Maybee, D. D. Olesky, P. Van Den Driessche and G. Wiener. Matrices, digraphs, and determinants. *SIAM Journal on Matrix Analysis and Applications* 10(4):500–519 (1989).

[Ny] P. Nylen. Minimum-rank matrices with prescribed graph. *Linear Algebra and its Applications* 248:303–316 (1996).

[Pa] S. Parter. On the eigenvalues and eigenvectors of a class of matrices. *Journal of the Society for Industrial and Applied Mathematics* 8:376–388 (1960).

[Sa] C. M. Saiago. *The Possible Multiplicities of the Eigenvalues of an Hermitian Matrix whose Graph is a Tree*. Dissertação de Doutoramento, Universidade Nova de Lisboa, 2003.

[Sh] D. Sher (Advisor: C.R. Johnson). Observations on the multiplicities of the eigenvalues of an Hermitian matrix with a tree graph. William and Mary, Research Experiences for Undergraduates program, summer 2004.

[Ry] H. J. Ryser. *Combinatorial Mathematics*. The Carus Mathematical Monographs, n. 14, M. A. A., 1963.

[Wi] G. Wiener. Spectral multiplicity and splitting results for a class of qualitative matrices. *Linear Algebra and Applications* 61:15–29 (1984).

Department of Mathematics, College of William and Mary, P.O. Box 8795, Williamsburg, VA 23187-8795, USA

E-mail address: crjohnso@math.wm.edu

Departamento de Matemática, Universidade de Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal

E-mail address: leal@mat.uc.pt

Departamento de Matemática, Faculdade de Ciências e Tecnologia da Universidade Nova de Lisboa, 2829-516 Quinta da Torre, Portugal

E-mail address: cls@fct.unl.pt

Department of Mathematics, 404 Krieger Hall, 3400 N. Charles St., Johns Hopkins University, Baltimore, MD 21218, USA

E-mail address: dsher1@jhu.edu