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A NOTE ON THE EXISTENCE OF A RATIONAL PREFERENCE STRUCTURE

The sufficient condition for WARP

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A NOTE ON THE EXISTENCE OF A RATIONAL PREFERENCE STRUCTURE
The sufficient condition for WARP

by

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Abstract

By exploring the relationship between the two approaches to modelling individual choice behavior, we examine a case under which the sufficient condition for WARP is also necessary. Moreover, it extends the 2-goods world case's result.

1. INTRODUCTION

One of the goals of applied demand analysis is to make welfare evaluations for policy purposes. In order to evaluate changes in the level of the individual consumer's well-being, one needs to know his expenditure function. In the context of the choice-based approach, the researcher is confronted with the data available, either market observed (RP) or generated by hypothetical surveys (SP). Therefore, the first question to be addressed in any empirical analysis should be how to legitimate the use of the data to secure welfare significance. This requires testing for the existence of an underlying rational preference structure.

A differentiable walrasian demand function $x(p,w)$, satisfying Walras' law and homogeneity of degree zero in prices ($p>>0$) and income (w), is generated from the maximization of a quasi-concave utility function if and only if the matrix of the substitution effects $S(p,w)$ is symmetric (S) and negative semi-definite (NSD), for all (p,w). Therefore, following the preference-based approach, the researcher has to test for S and NSD of $S(p,w)$. However, assuming S, we show that, under certain conditions, the choice-based approach, based on the Weak Axiom of Revealed Preference (WARP) provides a simpler alternative procedure, at least from an empirical perspective, as the number of constraints to be checked is considerably reduced. Moreover, the result obtained in the 2-goods world case is a particular case of the more general N-goods world case.

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In a choice-based approach, testing for the existence of an underlying rational preference structure is equivalent to testing for the WARP plus S. A necessary condition for WARP is the negative semi-definiteness of $S(p,w)$. However, it is not sufficient. A sufficient condition is that $S(p,w)$ is negative definite (ND) for all vectors not proportional to $p>>0$ [see Kihlstrom et al. (1976)]. Assuming S, two different cases can be distinguished: (i) the sufficient condition is also necessary, and (ii) the sufficient condition is not necessary. While in case (i) testing for the sufficient condition is enough to conclude about the existence of an underlying rational preference structure, in case (ii) this is not true.

2. THE SUFFICIENT CONDITION FOR WARP IS ALSO NECESSARY

In this section, we examine the 2-goods world case, and we show that this result can be extended to more than two goods.

2.1 2-goods world

In Proposition 1, we show that there is only one direction (p) for which $p^T S(p,w)\, p = 0$. That is, the WARP is equivalent to NSD and to ND.

Proposition 1: In a 2-goods world, given S,[1] negative semi-definiteness (NSD) of $S(p,w)$ guarantees that for all vectors not proportional to $p>>0$, the matrix is automatically negative definite (ND). Therefore, NSD of $S(p,w)$, for all (p,w), is equivalent to ND of $S(p,w)$, for all vectors not proportional to $p>>0$.

Proof:[2]

The Slutsky matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is negative definite for all vectors not proportional to $p=\begin{bmatrix} p_1 & p_2 \end{bmatrix}$ if and only if

$\begin{bmatrix} \alpha p_1 & \beta p_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \alpha p_1 \\ \beta p_2 \end{bmatrix} \langle\, 0$, for all α, $\beta > 0$ and $\alpha \neq \beta$. Equivalently, we can rewrite this condition as follows:

$$\alpha^2 p_1^2 a_{11} + 2\alpha\beta p_1 p_2 a_{12} + \beta^2 p_2^2 a_{22} \langle\, 0 \tag{1}$$

[1] Symmetry is implied by $p^T(p,w)=0$ and $S(p,w)p=0$.
[2] Throughout the proof, all quantities (x_1, x_2) are assumed to be optimal choices.

Given that $a_{11} = s_{11}\frac{x_1}{p_1}$, $a_{22} = s_{22}\frac{x_2}{p_2}$, and $a_{12} = a_{21} = s_{12}\frac{x_1}{p_2}$, where s_{ij}, for i=1,2 and j=1,2, represents the compensated demand elasticity of substitution between good i and the price of good j, inequality (1) can be restated as:

$$\alpha^2 p_1 s_{11} x_1 + 2\alpha\beta p_1 s_{12} x_1 + \beta^2 p_2 s_{22} x_2 \langle 0 \tag{2}$$

Given that $p^T S(p,w)\, p = 0$, we have:

$$p_1 s_{11} x_1 + 2p_1 s_{12} x_1 + p_2 s_{22} x_2 = 0 \quad \text{or} \quad 2p_1 s_{12} x_1 = -\, p_1 s_{11} x_1 \;-\; p_2 s_{22} x_2 \tag{3}$$

Inserting (3) into (2), and dividing all terms by w, we obtain:

$$\alpha^2 s_{11}\eta_1 - \alpha\beta(s_{11}\eta_1 + s_{22}\eta_2) + \beta^2 s_{22}\eta_2 \;\langle\; 0 \tag{4}$$

where η_i, for i=1,2, represents the share of good i in w. If $s_{11}\eta_1 = s_{22}\eta_2 = C$, then inequality (4) becomes:

$$\alpha^2 C - 2\alpha\beta C + \beta^2 C \langle 0 \iff C(\alpha - \beta)^2 \;\langle\; 0 \tag{5}$$

(i) NSD $\Rightarrow$ ND:

For $\alpha \neq \beta$, (5) is always true as far as $c<0$. Given that symmetry holds (ie, $a_{12}=a_{21}$), we can show that

$$s_{12}\eta_1 = s_{21}\eta_2 \tag{6}$$

Using the Cournot aggregation properties, ($s_{1j}\eta_1 + s_{2j}\eta_2 = 0$, for j=1,2), (6) is equivalent to:

$s_{11}\eta_1 = s_{22}\eta_2$. As in a 2-goods world, the Slutsky matrix is negative semidefinite if and only if the diagonal entries are negative (a_{11}, $a_{22} < 0$), the constant c is always negative.

(ii) ND $\Rightarrow$ NSD:

By Theorem M.D.4 [see Mas-Colell et al. (1995, p. 939)] a_{11}, $a_{22} < 0$. As, in addition, $S(p,w)$ is singular,[3] $S(p,w)$ is NSD.□

Remark: From the above mentioned Cournot aggregation properties, the off-diagonal entry a_{12} has to be positive, that is, when $N=2$, the goods have to be net substitutes, as it is well known.

2.2 N-goods world

Proposition 2: In a world of N net substitute goods, negative semi-definiteness (NSD) of a symmetric $S(p,w)$ guarantees that for all vectors not proportional to $p>>0$, the matrix is automatically negative definite (ND). Therefore, NSD of $S(p,w)$ for all (p,w) is equivalent to ND of $S(p,w)$, for all vectors not proportional to $p>>0$.

Proof: We focus on $N=3$. The proof can be easily extended to $N>3$.

[3] Singularity of $S(p,w)$ results from homogeneity of degree zero and satisfaction of Walras' law.

(i) NSD and $a_{ij} > 0$, for all i,j=1,2,3 $\Rightarrow$ ND:

If $a_{ij} > 0$, for all i,j=1,2,3, then, by the Cournot aggregation properties, $a_{ij} < 0$, for all i=j. Therefore, all principal minors of $S(p,w)$ with order 1 are negative. Moreover, singularity of $S(p,w)$ imposes that all principal minors with order 2 are positive. But, by Theorem M.D.4 [see Mas-Colell et al. (1995, p. 939)], these two conditions together are equivalent to ND of $S(p,w)$ for all vectors not proportional to $p >> 0$.

(ii) ND and $a_{ij} > 0$, for all i,j=1,2,3 $\Rightarrow$ NSD:

ND together with singularity of $S(p,w)$ $\Rightarrow$ NSD.□

When testing for the existence of an underlying rational preference structure , the researcher only needs to check for $a_{ij} > 0$, for all i≠j, besides S, homogeneity of degree zero and satisfaction of Walras' law.

3. CONCLUSIONS

By exploring the relationship between the two distinct approaches to modelling individual choice behavior, that is, the preference-based approach and the choice-based approach, we show that, under certain conditions, the 2-goods world case result is just a particular case of the more general N-goods world one. Therefore, from an empirical perspective, this is an important result. If we are in position to accept the hypothesis that all goods are net substitutes, there exists some preferences that rationalize the individual choice rule, $x(p,w)$.

4. REFERENCES

[1] Kihlstrom, R., A. Mas-Colell and H. Sonnenschein, 1976, The Demand Theory of the Weak Axiom of the Revealed Preference, Econometrica, 44, 971-78.

[2] Mas-Colell, A., M.D. Winston and J.R. Green, 1995, Microeconomic Theory (Oxford University Press).