

**Endogenous Fertility and Modified
Pareto-Optimality**

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Abstract

In a world of endogenous fertility, the traditional concept of Pareto-optimality does not make sense any longer: the number of individuals present on earth may vary with two allocations to be compared. Therefore, the concept needs some modification in order to give new life to the discussion of economic efficiency. This work introduces and discusses a rectification of the Pareto-principle, and it characterizes economic allocations according to this new idea. The fiscal implications of an application of *Modified Pareto-Optimality* are explained, as well as the consequences for empirical research on dynamic efficiency.

KEYWORDS: Endogenous Fertility, Modified Pareto-Optimality.

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Introduction

In a framework of an overlapping generations model with endogenous fertility, as introduced by Pazner and Razin [1980], the traditional concept of Pareto-optimality is no longer applicable. This simply is the case because the principle requires a fixed set of individual identities to be exposed to different economic allocations for choice. If fertility, too, is subject to choice, then the set of individuals to be confronted with certain allocations may vary with the option we take into consideration.

This work will not make the attempt to pose and solve (or not solve) a social choice problem. We do not consider a setting with all types of admissible preferences to be aggregated, rather the framework here consists of an overlapping generations model with productive capital à la Diamond [1965] with identical individuals but the modification that the number of offspring enters the preferences of their parents, and child-raising causes a certain fixed amount of costs per child.

We shall then introduce a partial ordering on allocations that is closely related to the traditional Pareto-ordering but which takes into account of the new dimension of the problem: the choice of who, or better, how many individuals, should live.

Goods in this framework, from an individual's point of view, are consumption in the first respective second period of life, and the number of children raised. The production technology for consumption goods is given by the typical neoclassical constant-returns-to-scale production function. Children are 'produced' within the family, using a linear technology that requires as input units of the consumption good.¹

Under rather standard assumptions, it is easily possible then to claim existence of Walrasian equilibria without and with lump-sum transfers between generations. These equilibria are characterized, as usual, by the equality of the marginal rate of substitution and the marginal rate of transformation between two different goods.

Now let us assume, for the time being, that the number of children were fixed in each generation, i.e. the set of individuals which constitute the whole intertemporal population were a given. Then it is easy to see that everything can be reduced to the discussion of whether or not the allocation is Pareto-optimal, and the standard methods proposed in Lang [1992a,1994] can be applied to decide this question. It turned out that, while in a Walrasian equilibrium, no generation can be improved without diminishing the resources per head of the following one. It may be possible, however, that the following generations can maintain the same

¹It would be analogous to assume that children can be bought on a market, being offered by a firm that produces them via a linear technology, using consumption goods as intermediate inputs.

level of satisfaction with fewer resources per head available to them, making it possible to achieve a Pareto-improvement.

Returning now to the option to make fertility subject to choice, suppose that there is historically a first generation with a given number of individuals, and consider a Walrasian equilibrium of this infinite-horizon economy. One must then rethink the results obtained for the case of fixed or exogenous fertility and ask again the following question: does this new framework offer a way to improve the individuals of the first generation *without* decreasing the resources per head of their children?

One has, however, first to understand the relevance of this question. The point here is that, as will be shown, the OLG model with endogenous fertility is *scaleable* as is the traditional model with given population growth rate. Scalability thereby means the following: given that we treat all individuals of a generation equally, the set of feasible per-head allocations (i.e. first- and second-period consumption, as well as the number of offspring) over all generations is independent of the initial number of individuals on earth, it only depends on the resources per head at this initial stage.

Therefore, holding resources per head of the children-generation constant implies that, starting with this generation, the set of feasible per-head allocations over all cohorts is kept the same, and consequently, the well-being of the *representative head* in each generation can be kept on the same level. This follows simply from the fact that in the OLG model with endogenous fertility, each stage is an exact *structural replication* of the stage before, the only relevant non-structural difference arises with the potentially different amount of resources per head.

If the answer to the above question were yes, then the improvement of the individuals of the first living generation would be possible without hurting the well-being of the representative head of any of the following generations. This, however, necessarily requires that the number of their children in the Walrasian reference-equilibrium must be altered.

In order to get the point, one should start the argument considering an economy with inelastic prices, i.e. with interest and wage rates not being influenced by the capital-per-head endowment. If in such an economy there is a lump-sum payment to the 'old', financed by contributions of the 'young', then an individual of the parent-generation does not take into account of the following:² an additional child does not only cause costs of raising it, but the lump-sum payment the parents receive when old can be increased by the amount that this newly born child contributes when in its working phase. This does not hurt the potential to derive utility neither of the already existing population of sons and daughters nor of the additional child, nor of future populations, and consequently, one can

²An analogous argument applies if there is a lump-sum tax on the father to be paid in equal parts to each of his children.

require that a subsidy per child on child-raising is being introduced to reveal the true costs and benefits per offspring to their parents.

As easy as the argument reads for an economy with fixed prices, as uneasy is the situation in a world with neoclassical production: an additional child c.p. (i.e. given a certain capital stock) diminishes the marginal product of labour, and therefore wage rates in equilibrium. In order to maintain resources per head of the young, the contribution of each of them must necessarily decrease, making it per se unclear whether the lump-sum payment to each member of the parent-generation can be maintained. But even if it cannot, might the positive (utility-)effect of an additional child offset the loss in transfers?

The present work will solve the puzzle, and it will finally answer to the positive the question posed above of whether the first generation can be improved without decreasing the endowment of resources per head of their children. This finding stands in striking contrast to what we know from traditional overlapping generations models (see Balasko and Shell [1980,1981], Lang [1994]), namely that Walrasian equilibria without externalities are characterized by the so called property of *short-run Pareto-optimality*: improving the well-being of the individuals of the first generation is impossible without diminishing the amount of resources per head of their children.

Therefore, also in the neoclassical model with prices depending on the capital-per-head endowment of the work force, subventions/taxes must be paid to/by parents per additional child in order to signal the true financial benefit/cost of an offspring.

One must emphasize here that the improvement of the parent-generation may have an important effect: an increase in the number of children for the case of transfers from 'young' to 'old', a decrease for transfers in the opposite direction. But, as already mentioned, the well-being of the representative head of all subsequent generations can remain unchanged. Therefore, the *value judgement beyond Paretianism* to be proposed in the framework of this work is simply: if the representative head of a generation can be improved without worsening the representative head of any other generation, then the according allocation should be chosen as qualitatively superior. If such a superiority cannot be achieved, then we shall call an allocation *modified Pareto-optimal*.

In how far is the proposed value judgement in accordance or conflict with the preferences of individual entities? First of all, some individuals might exist for one allocation but do not appear on earth for another. This sentence, however, contains a pure absurdity: there is little evidence that an individual can either exist or not exist, the same way as someone can be in one room of a building or another. So it makes little sense to speak about 'someone' who never was or will be on earth, nor is it hardly possible to attribute to this 'someone' a preference ordering over being or not being, and if being, over consumption bundles hold.³

³We will therefore save the reader an even more bizarre extension of the already bizarre.

Therefore, the value judgement proposed cannot be evaluated in the sense of whether it is in accordance or conflict with the preferences of a 'person' that does never live. If 'someone' is not born, then 'he' cannot be unlucky about that. Only for an ever existing individual it has relevance whether an additional son or daughter is born to him. This relevance is expressed via his preference ordering, and this ordering is respected by the criterion of modified Pareto-optimality.

But how about those ones who live for both allocations we compare? Also this question is not without problem: how can we know that it is the same individual that lives for two alternatives? For example, if we consider whether an actually existing couple should have two or three children, does the alternative with three children consist of the two children of the other alternative plus an additional offspring?⁴ We will *assume* here that this is indeed the case. Comparing two allocations with $1 + n$ respectively $1 + n'$ offspring of a family, we suppose that $\min(1 + n, 1 + n')$ of them are alive for both alternatives. Then the traditional principle of Pareto-optimality can be applied to those ones who live in both cases, and the equity principle forces us to equalize utility over the existing population in each generation for both alternatives.

Equity, however, should not only be applied within but also across generations. Individuals living in different generations cannot be considered per se as unequal only because of the fact that their location in relation to the fourth dimension is not the same. This issue will also be discussed in a later part of this work.

Summarizing, the idea of modified Pareto-optimality formalized and applied in the following can be seen as a simultaneous combination of Pareto-optimality and intragenerational equity, paired with an as-such-irrelevance of birth or non-birth of an individual.

Definitions and Preliminary Results

First, let's have a look at the ingredients of the model. The individual problem of maximizing utility read as

$$\max_{c_t^1, 1+n_t, s_t, c_{t+1}^2} u(c_t^1, 1+n_t, c_{t+1}^2) \quad (1)$$

subject to

$$c_t^1 + (1+n_t)b + s_t \leq w_t \quad (2)$$

$$c_{t+1}^2 \leq (1+r_{t+1})s_t. \quad (3)$$

enough story of a risk-averse individuals' meeting at a pre-natal conference, where nobody knows in advance his position in society after entering the scenario of life, to a meeting where even the finally 'never-to-be-born' are present to fight for their interests.

⁴Of course, the question whether a family should have one more child in addition to some already created is a different one, but we will not discuss this — undoubtedly important — issue.

c_t^1 denotes first-period consumption, $(1+n_t)$ denotes the number of offspring of an individual of generation t , s_t denotes its savings, and c_{t+1}^2 represents consumption in the final period of the individual's life. Thereby we assume that the utility function $u(c_t^1, 1+n_t, c_{t+1}^2)$ has the standard properties of monotonicity, strict quasi-concavity, differentiability, and that it fulfills the typical Inada conditions to guarantee an interior solution to the consumers' problems. w_t and $1+r_{t+1}$ denote the wage rate and the interest factor respectively. Due to these properties of the utility function, all weak inequalities in (1)-(3) are fulfilled as equalities in the individual's optimum, which itself is characterized necessarily and sufficiently by the first-order conditions

$$\frac{u_1(\cdot)}{u_2(\cdot)} = 1 + r_{t+1} \quad (4)$$

$$\frac{u_3(\cdot)}{u_1(\cdot)} = b \quad (5)$$

$$c_{t+1}^2 = (1+r_{t+1})(w_t - c_t^1 - (1+n_t)b). \quad (6)$$

The consumption goods c_t^1 and c_{t+1}^2 , as well as savings s_t , are standard to overlapping generations models, but the number $(1+n_t)$ of children 'consumed' needs some clarification. As in the standard Diamond [1965] model type, in each period, there are three markets: the market for buying consumption goods, the labour market, and finally the market for credit contracts. The only new interpretation required is for the first one, the market for consumption goods. It is assumed here that the suppliers for goods cannot discriminate between the different finalities of use of their goods: they cannot discriminate against the use as capital good nor the use as a consumption good in the traditional sense, and moreover, they are also unable to identify its use as *consumption of children*. Therefore, the decomposition of the aggregate good in each period is purely accomplished by the individuals themselves, and in this interpretation, the aggregate good is only intermediary to *household production*. This interpretation is not completely satisfactory since capital goods and consumption goods look quite different and in reality they are typically not produced within the household, and what looks like 'consumption of children' is — at least to a large extent — nothing else than specialized gear and nutrition for children which, too, is only available on markets but not produced within households. But, as is well known, the neoclassical model with constant-returns-to-scale production is also open for this (more realistic) setting: production of one type of good as well as of different types of goods is compatible with and equivalent to the existence of more than only the representative firm to produce all types and all of one type of good. Therefore, the existence of offsprings as 'consumption goods' does not rise interpretative difficulties.

The (representative) firm's profit maximization problem is quite standard and can be described by the following two equations, given the assumption of perfect

competition:

$$f'(k_t) = 1 + r_t \quad \text{and} \quad (7)$$

$$f(k_t) - k_t f'(k_t) = w_t. \quad (8)$$

Thereby k_t denotes the capital per worker employed in production and $f(k_t)$ stems from the constant-returns-to-scale production function $F(K_t, N_t)$ as $f(k_t) := F(k_t, 1)$ where $k_t := \frac{K_t}{N_t}$. Furthermore, we assume that $F(\cdot)$ fulfills the standard Inada conditions as does the utility function. Regarding (1)–(3) and (7)–(8), it is obvious that we have chosen the price of output as numeraire.

Let us in the following define the set of feasible allocations. Thereby we only consider allocations with the property that within a generation all individuals realize the same individual allocation. This is justified in two ways. First, since all individuals hold the same preferences and initial endowment, and those within a generation are exposed to the same market price system, they realize per se identical allocations from a positive economic point of view. Second, concerning our normative perspective, there is no a priori reason, why a central planner should treat (the equal) individuals of the same generation unequal. Moreover, it also avoids some notational clutter due to an indexation within a generation.

Definition 1 (Feasible Allocations) *Given an economy with initial population N_1 and an initial per-head budget x_1 , a sequence of individual allocations $\{c_t^1, 1 + n_t, s_t, c_{t+1}^2\}_{t=1, \dots, \infty}$ is called feasible iff:*

$$(i) \quad (c_t^1, 1 + n_t, s_t, c_{t+1}^2) \geq 0 \text{ for all } t \geq 1,$$

$$(ii) \quad N_t x_t \geq N_t (c_t^1 + (1 + n_t)b + s_t) \text{ for all } t \geq 1, \text{ whereby } N_t x_t \leq N_t f\left(\frac{N_{t-1} s_{t-1}}{N_t}\right) - N_{t-1} c_t^2 \text{ for all } t \geq 2 \text{ and } N_t = (1 + n_{t-1})N_{t-1}.$$

We denote by S the set of all feasible sequences of individual allocations.

(i) simply requires non-negativity of individual consumption and saving. (ii) imposes the requirement that (aggregate) real expenditures on consumption, child-raising and savings, $N_t(c_t^1 + (1 + n_t)b + s_t)$ do not exceed (aggregate) budget, $N_t x_t$, whereby this (aggregate) budget itself may not exceed total production output, $N_t f\left(\frac{N_{t-1} s_{t-1}}{N_t}\right)$ minus consumption of 'old' generation $t - 1$, $N_{t-1} c_t^2$.

By dividing all inequalities and the equality in (ii) by N_t , we see that (ii) is equivalent to

$$(ii)' \quad x_t \geq c_t^1 + (1 + n_t)b + s_t \text{ for all } t \geq 1, \text{ whereby } x_t \leq f\left(\frac{s_{t-1}}{1 + n_{t-1}}\right) - \frac{c_t^2}{1 + n_{t-1}} \text{ for all } t \geq 2.$$

Therefore, the set S of feasible individual allocations is independent of the level of initial population N_1 and only dependent on the initial per-head budget x_1 .

This observation is familiar from the traditional Diamond [1965] model type and is not so surprising for the present model with endogenous fertility, however, it is worthwhile to be mentioned because it has some important consequences for our concept of modified Pareto-optimality defined in the following.

Having defined the notion of feasibility, let us now turn to the formalization of the concept of *modified Pareto-optimality* for our framework.

Definition 2 (Modified Pareto-Optimality) *Given an economy with initial population N_1 and per-head budget x_1 , let $a := \{c_t^1, 1 + n_t, s_t, c_{t+1}^2\}_{t=1, \dots, \infty} \in S$ be a feasible sequence of individual allocations and denote by $\{u_t\}_{t=1, \dots, \infty}$ the sequence of utilities of the representative head in each generation generated by the individual allocations, i.e. $u_t := u(c_t^1, c_{t+1}^2, 1 + n_t)$. a is called *modified Pareto-optimal (MPO)* iff there exists no other feasible sequence $a^* \in S$ such that $u_t^* \geq u_t$ for all $t \geq 1$ and $u_{t'}^* > u_{t'}$ for at least one $t' \geq 1$.*

The only difference between our concept and traditional Pareto-optimality is that Pareto-optimality requires the number of individuals in all generations to remain the same when comparing two different allocations,⁵ whereas modified Pareto-optimality allows for the possibility to compare allocations with a different number of individuals as members of generations. Of course, we would call an allocation as modified Pareto-inferior if the (identical) individuals of no generation are made worse off, and at least the individuals of one generation are improved upon their original well-being by shifting to another feasible allocation.

As above, the set of MPO allocations does not depend on initial population N_1 : if a feasible allocation a is MPO for a certain N_1 , then there cannot exist a feasible allocation a^* which improves a generation without hurting any other for some other N_1 .

We shall use in our analysis also another concept, whose analogue for models with exogenous fertility is well-known from the literature, namely *modified short-run Pareto-optimality*.

Definition 3 (Modified Short-Run Pareto-Optimality) *Given an economy with initial population N_1 and per-head budget x_1 , let $a := \{c_t^1, 1 + n_t, s_t, c_{t+1}^2\}_{t=1, \dots, \infty} \in S$ be a feasible sequence of individual allocations. Defining $x_t := c_t^1 + (1 + n_t)b + s_t$ for $t \geq 2$ we say: a is *modified short-run Pareto-optimal (MSRPO)* iff there exists no other feasible allocation $a^* \in S$ such that for a $t' \geq 1$ and a $0 \leq \tau < \infty$ we have $(c_t^1, 1 + n_t, s_t, c_{t+1}^2)^* = (c_t^1, 1 + n_t, s_t, c_{t+1}^2)$ for $1 \leq t \leq t' - 1$, $x_{t'}^* = x_{t'}$ and $x_{t'+\tau+1}^* = x_{t'+\tau+1}$, and $u_t^* \geq u_t$ for all $t \in \{t', \dots, t' + \tau\}$ and $u_t^* > u_t$ for at least one $t \in \{t', \dots, t' + \tau\}$.*

⁵Exactly spoken, to stay with the original idea of Pareto-optimality, it is required that the same individuals are exposed to two different consumption allocations, whereby one part of this allocation, namely 'consumption of children', therefore has to remain the same for both allocations.

This definition is very familiar from the literature. It is analogous to the definition given in Balasko and Shell [1980,1981] for the case of an overlapping generations exchange economy. Our version of modified short-run Pareto-optimality now needs some closer explanation. We compare two different feasible allocations where both of them, on the one hand, imply the same initial budget $x_{t'+\tau+1}$ for the individuals of generation $t' + \tau + 1$, however, allowing for different numbers $N_{t'+\tau+1}$ and $N_{t'+\tau+1}^*$ of initial population in $t' + \tau + 1$. This guarantees that, beginning with $t' + \tau + 1$, the same sequences of individual allocations — and therefore the same sequences of utilities $\{u_t\}_{t=t'+\tau+1, \dots, \infty}$ — are feasible in both cases; following simply from our argument made above that the set of feasible allocations does not depend on initial population but only on initial per-head budget and the obvious fact that our model has the property of invariant structure beginning at any point in time. On the other hand, we require for both allocations that individual allocations up to generation $t' - 1$, including fertility, are identical, and initial budget $x_{t'}$ and $x_{t'}^*$ coincide. This has the following consequence: generations' $1, \dots, t' - 1$ individual utilities are the same for both allocations.

Hence, roughly spoken, an allocation fulfills the criterion of modified short-run Pareto-optimality if, given the utility position of the individuals of generations $1, \dots, t' - 1$ and $t' + \tau + 1, \dots, \infty$, there is no way to improve (at least) one of the generations $t', \dots, t' + \tau$ without hurting (at least) another among them — up to here reading like it would be the case for the traditional criterion of short-run Pareto-optimality — however, allowing for that the number of individuals in generations $t' + 1, \dots, \infty$ changes. Put differently: (because of the invariant structure of the economy as time goes by) the idea of modified short-run Pareto-optimality tries to identify allocations which are the 'best' for a finite sequence of generations, given a certain initial per-head budget for the first generation following this sequence, and therefore, a certain set of feasible per-head allocations for the generations including and following this first one.

Very analogously to our above argumentation, it follows that the set of MSRPO allocations, too, does not depend on initial population N_1 .

Let us now proceed with the definition of equilibrium allocations for our model.

Definition 4 (Perfect Foresight Laissez-Faire Equilibrium) *Given an economy with initial population N_1 and per-head budget x_1 , a feasible sequence of individual allocations $\{c_t^1, 1 + n_t, s_t, c_{t+1}^2\}_{t=1, \dots, \infty}$, together with a sequence of prices $\{w_t, 1 + r_t\}_{t=2, \dots, \infty}$, is called a perfect foresight laissez-faire equilibrium (PLE) for the model with initial population N_1 and per-head budget x_1 iff:*

- (i) $(c_t^1, 1 + n_t, s_t, c_{t+1}^2)$ solves the individual utility maximization problem (1)-(3) for prices $(w_t, 1 + r_{t+1})$ for $t \geq 2$, and for w_1 replaced by x_1 for $t = 1$,

$$(ii) \quad N_{t-1}c_t^2 + N_t(c_t^1 + (1+n_t)b + s_t) = N_t f(k_t) \text{ for } t \geq 2, \text{ and } N_1(c_1^1 + (1+n_1)b + s_1) = N_1 x_1,$$

$$(iii) \quad k_t = \frac{N_{t-1}s_{t-1}}{N_t}, \text{ for } t \geq 2, \text{ and}$$

(iv) k_t solves the profit maximizing problem (7)-(8) of the representative firm for $t \geq 2$.

(i) and (iv), respectively, require that the equilibrium allocation solves the individuals' and the representative firms' choice problems, (ii) and (iii), respectively, demand for equality of supply and demand on the good's and the capital market. As is typical, the clearing condition for the third market, the market for labour, can be omitted.

It should be noted here that our definition of equilibrium is restrictive in two senses: first, we concentrate on perfect foresight, implying that individuals and firms can correctly predict the future course of prices in the economy. Second, the definition does not capture the very relevant case of social planners who try to implement certain allocations by using the typical tax tools. The second restriction will be given up in the following section to show that modified Pareto-optimal allocations even require the intervention of a social planner using 'distortive' tax instruments for implementation.

Analogously to our observation for feasible individual allocations, we can make the observation that the set of equilibrium allocations does not depend on initial population N_1 but only on initial budget x_1 : dividing (ii) by N_t and using $N_t/N_{t-1} = 1 + n_{t-1}$ in (iii), we obtain

$$(ii)' \quad \frac{c_t^2}{1+n_{t-1}} + (c_t^1 + (1+n_t)b + s_t) = f(k_t), \text{ and}$$

$$(iii)' \quad k_t = \frac{s_{t-1}}{1+n_{t-1}},$$

making our definition independent of N_1 . Hence, the model with endogenous fertility has the property that feasible individual allocations and equilibrium allocations, respectively, coincide for all sizes of initial population but for the same initial per-head budget. Furthermore, because of (7) and (8), equilibrium prices remain the same as well.

Finally in this section we shall follow the tradition of Galor and Ryder [1989] in giving a sufficient condition for the existence of a unique perfect foresight laissez-faire equilibrium as defined above. We shall neither be deeper involved in the question of existence of a (unique) PLE nor concerned with the existence of a (unique) steady-state equilibrium because these questions are irrelevant to our strand of argumentation.

As in the original Diamond model type considered by Galor and Ryder [1989], the conditions for existence of an equilibrium can be reduced to the question of

whether a unique solution to the equation

$$k_{t+1} = \frac{s\{f(k_t) - k_t f'(k_t), f'(k_{t+1})\}}{(1 + n_t)[f(k_t) - k_t f'(k_t), f'(k_{t+1})]}, \quad (9)$$

for $k_t \geq 0$ and all $t \geq 2$ exists. Here the functions $s[w_t, r_{t+1}]$ and $(1 + n_t)[w_t, r_{t+1}]$ denote individual supply of capital and demand for children, respectively. w_t and r_{t+1} are replaced by their marginal products as given in equations (7)-(8). For $t = 1$, the same equation stands, however, with $f(k_1) - k_1 f'(k_1)$ replaced by x_1 .

The only difference in (9) to Galor and Ryder's [1989] equilibrium condition is the dependence of fertility on equilibrium prices. We can then state the following lemma on the existence of a unique solution of this equation.

Lemma 1 (Temporary Perfect Foresight Laissez-Faire Equilibrium) *Given $k_t \geq 0$ ($x_1 \geq 0$), there exists a unique $k_{t+1} \geq 0$ that is a self-fulfilling expectation, if*

$$\frac{\partial s[w_t, r_{t+1}]}{\partial r_{t+1}} \geq 0 \quad \text{and} \quad \frac{\partial (1 + n)[w_t, r_{t+1}]}{\partial r_{t+1}} \leq 0, \quad (10)$$

i.e. savings are non-decreasing and fertility is non-increasing in the interest rate.

Proof: If savings are non-decreasing in the interest rate, and fertility is non-increasing, then the capital-per-labour ratio does not decrease in r_{t+1} . This means that the right-hand side of (9) is non-increasing in k_{t+1} . Since the left-hand side is strictly increasing in k_{t+1} , a unique temporary perfect foresight laissez-faire equilibrium exists. ■

Here the addition of the word *temporary* only expresses that the solution k_{t+1} of (9) only provides us with a part of the overall equilibrium as defined in Definition 2, namely with $(c_t^1, 1 + n_t, s_t, c_{t+1}^2)$ and the prices w_{t+1} and r_{t+1} . The (overall) perfect foresight laissez-faire equilibrium, of course, then is obtained by recursive application of (9).

In the following section we shall be concerned with the relation between modified short-run Pareto-optimality and modified Pareto-optimality on the one hand and their relation to equilibrium allocations on the other hand.

Modified Short-Run Pareto-Optimality and Decentralization

The first result in this section is to show that modified Pareto-optimality requires modified short-run Pareto-optimality.

Theorem 1 (MPO \Rightarrow MSRPO) *A necessary condition for a feasible allocation a to be modified Pareto-optimal is that it is modified short-run Pareto-optimal.*

Proof: Let us assume, on the contrary, that the allocation is MPO but not MSRPO. Then there exists a feasible allocation a^* such that for a $t' \geq 1$ and a $0 \leq \tau < \infty$ we have $(c_t^1, 1 + n_t, s_t, c_{t+1}^2)^* = (c_t^1, 1 + n_t, s_t, c_{t+1}^2)$ for $1 \leq t \leq t' - 1$, $x_{t'}^* = x_{t'}$ and $x_{t'+\tau+1}^* = x_{t'+\tau+1}$, and $u_t^* \geq u_t$ for all $t \in \{t', \dots, t'+\tau\}$ and $u_t^* > u_t$ for at least one $t \in \{t', \dots, t'+\tau\}$. Therefore, the allocation a^* allows us to keep generations $1, \dots, t' - 1$ with the same allocation as with a — and hence with the same utility. At least some of the generations in $\{t', \dots, t'+\tau\}$ can be improved without hurting any other in this set. Finally, since $x_{t'+\tau+1}^* = x_{t'+\tau+1}$, starting with $t' + \tau + 1$, the same set of feasible individual allocations results, implying that (although the initial number of individuals in $t' + \tau + 1$ might have changed) all individuals in $\{t' + \tau + 1, \dots, \infty\}$ can realize the same utility under a and a^* . Hence, a cannot be MPO according to Definition 2, being a contradiction to our assumption that a were MPO. ■

We will understand later that the converse need not be true, the same way as is the case for the model type with exogenous fertility.

Up to now, everything seems quite familiar from the traditional definitions of Pareto-optimality and short-run Pareto-optimality as well as their relation. Now differences begin. Let us first state the following lemma.

Lemma 2. *Given x_t and x_{t+1} , the problem*

$$\max_{c_t^1, 1+n_t, s_t, c_{t+1}^2} u(c_t^1, c_{t+1}^2, 1+n_t) \quad (11)$$

subject to

$$c_t^1 + (1+n_t)b + s_t \leq x_t \quad (12)$$

$$\frac{c_{t+1}^2}{1+n_t} - f\left(\frac{s_t}{1+n_t}\right) \leq -x_{t+1}, \quad (13)$$

has a unique solution, characterized by the following necessary and sufficient first-order conditions:

$$\frac{u_1(\cdot)}{u_2(\cdot)} = f'(\cdot) \quad (14)$$

$$\frac{u_3(\cdot)}{u_1(\cdot)} = b - \frac{c_{t+1}^2 - s_t f'(\cdot)}{f'(\cdot)} \quad (15)$$

$$c_{t+1}^2 = (1+n_t)f\left(\frac{x_t - c_t^1 - (1+n_t)b}{1+n_t}\right) - (1+n_t)x_{t+1}. \quad (16)$$

We can write $v(x_t, x_{t+1})$ as the indirect utility function, having the properties:

$$\frac{dv(x_t, x_{t+1})}{dx_t} > 0 \quad (17)$$

$$\frac{dv(x_t, x_{t+1})}{dx_{t+1}} < 0. \quad (18)$$

Proof: $u(c_t^1, c_{t+1}^2, 1 + n_t)$ is strictly quasi-concave by assumption. The left-hand side of (12) is quasi-convex because linear. Concerning (13), $\frac{c_{t+1}^2}{1+n_t}$ and $\frac{s_t}{1+n_t}$ are quasi-convex and quasi-concave at the same time. Since f (the per-capita production function) is monotonically increasing, $-f\left(\frac{s_t}{1+n_t}\right)$ is quasi-convex, and therefore the left-hand side of (13), as the sum of two quasi-convex functions, too. This guarantees the existence of a unique solution to the maximization problem. (14)–(18) then follow simply by using the method of Lagrange. ■

It must be noted here that the right-hand side of (15) is indeed positive. Since in optimum (12) and (13) are fulfilled with equality, we can write

$$\begin{aligned} \frac{dc_{t+1}^2}{d(1+n_t)} &= f\left(\frac{s_t}{1+n_t}\right) - \frac{s_t}{1+n_t} f'\left(\frac{s_t}{1+n_t}\right) - b f'\left(\frac{s_t}{1+n_t}\right) - x_{t+1} \\ &= \frac{c_{t+1}^2}{1+n_t} - \frac{s_t}{1+n_t} f'\left(\frac{s_t}{1+n_t}\right) - b f'\left(\frac{s_t}{1+n_t}\right), \end{aligned} \quad (19)$$

with $s_t := \frac{x_t - c_t^1}{1+n_t} - b$. In optimum this derivative must be negative, as the following discussion shows, and hence our claim follows.

Having a closer look at the first line of (19), one realizes the costs and benefits of an additional child from the point of view of their parents in terms of the latter's second-period consumption: $f\left(\frac{s_t}{1+n_t}\right)$ denotes the additional output of productive activities due to one additional offspring. $-\frac{s_t}{1+n_t} f'\left(\frac{s_t}{1+n_t}\right)$ takes account of the fact that an additional child decreases the capital-per-head ratio and hence output per head. $-b f'\left(\frac{s_t}{1+n_t}\right)$ reflects the reduction in capital formation due to the increased expenditures on child-rearing. Finally, $-x_{t+1}$ represents the initial endowment of the additional child.

The sum of all of these factors must necessarily be negative. If it were not then an optimum would not have been achieved: increasing fertility would be of advantage to parents, directly via an immediate utility increase ($1 + n_t$ enters utility), and indirectly via a higher second-period consumption, thereby holding the initial endowment of their children constant.

From Lemma 2 we can infer the following theorem.

Theorem 2 *A feasible allocation a^* is modified short-run Pareto-optimal iff for all $t \geq 1$, $(c_t^1, 1 + n_t, s_t, c_{t+1}^2)^*$ solves (11) – (13) for $x_t := c_t^1 + (1 + n_t)^* b + s_t^*$ and $x_{t+1} := c_{t+1}^1 + (1 + n_{t+1})^* b + s_{t+1}^*$.*

Proof: 1. Assume that a^* is MSRPO. Then there exists no other feasible allocation a such that for t and $t + 1$ we have $x_t = x_t^*$ and $x_{t+1} = x_{t+1}^*$, and $u_t > u_t^*$. Hence, given x_t and x_{t+1} , $(c_t^1, 1 + n_t, s_t, c_{t+1}^2)^*$ maximizes utility for generation t among all feasible $(c_t^1, 1 + n_t, s_t, c_{t+1}^2)$ for x_t and x_{t+1} . Therefore, $(c_t^1, 1 + n_t, s_t, c_{t+1}^2)^*$ solves (11)–(13).

2. Assume that an allocation a^* solves (11)–(13) for all $t \geq 1$, given $x_t := c_t^1 + (1 + n_t^*) b + s_t^*$ for all $t \geq 2$, but suppose that it were not MSRPO.

Then there exists a feasible allocation a such that for a $t' \geq 1$ and a $0 \leq \tau < \infty$ we have $(c_t^1, 1 + n_t, s_t, c_{t+1}^2) = (c_t^1, 1 + n_t, s_t, c_{t+1}^2)^*$ for $1 \leq t \leq t' - 1$, $x_{t'} = x_{t'}^*$ and $x_{t'+\tau+1} = x_{t'+\tau+1}^*$, and $u_t \geq u_t^*$ for all $t \in \{t', \dots, t' + \tau\}$ and $u_t > u_t^*$ for at least one $t \in \{t', \dots, t' + \tau\}$.

Since $u_{t'} \geq u_{t'}^*$, it follows that $x_{t'+1} \leq x_{t'+1}^*$, whereby strict inequality for the latter expression applies if it holds for the first one: this follows simply from (17) and (18) using a simple duality argument. Continuing with generations $t' + 1, t' + 2, \dots, t' + \tau$, we conclude that necessarily $x_{t'+\tau+1} > x_{t'+\tau+1}^*$ since at least one generation in $t', t' + 1, t' + 2, \dots, t' + \tau$ is strictly improved. This, however, contradicts our supposition. ■

Theorem 2 gives a complete characterization of modified short-run Pareto-optimal allocations in our economy. It thereby appears that such allocations must fulfill conditions (14) – (16) of Lemma 2. These conditions therefore give a hint on how to decentralize MSRPO allocations:

1. Set the interest rate $1 + r_{t+1} = f'(\cdot)$.
2. Set the price per child $p_t^c = b - \frac{c_{t+1}^2 - s_t f'(\cdot)}{f'(\cdot)} = b - \frac{a_{t+1}}{1 + r_{t+1}} =: b - q_t$.
3. Set the wage rate $w_t = f(\cdot) - s_{t-1} f'(\cdot)$.
4. Finally, normalize the price of the consumption good to one.

The individuals' budget constraints which induce it to select the MSRPO allocation to decentralize then are

$$c_t^1 + (1 + n_t)(b - q_t) + s_t = w_t - a_t \quad (20)$$

$$c_{t+1}^2 = (1 + r_{t+1})s_t + d_{t+1}, \quad (21)$$

where

$$a_t - (1 + n_t)q_t - \frac{d_t}{1 + n_{t-1}} = 0. \quad (22)$$

The latter condition simply indicates that the subsidy on child-raising plus lump-sum transfers to the 'old' must be financed out of lump-sum taxes on the young (of course, a_t , q_t and d_t can be negative).

The important thing here is that individuals are not only price-takers, but that they take the tax-parameters a_t , q_t and d_t as well as given, believing to have no influence on the budget-equation (22). If this were not the case, they would start to think strategically and infer, for example, that a decrease in fertility would lower their tax burden a_t in case that the subsidy q_t and a_t are positive, and given d_t .

From Lemma 2 and Theorem 2 we can infer the following statement on the efficiency of the laissez-faire perfect-foresight equilibrium.

Corollary 1 *The laissez-faire perfect-foresight equilibrium according to Definition 4 is modified short-run Pareto-optimal.*

Proof: Since in PLE we have $c_{t+1}^2 = s_t(1 + r_{t+1}) = s_t f'(\cdot)$, the validity of (5) implies that (15) holds. (4) directly coincides with (14). Finally,

$$\begin{aligned} c_{t+1}^2 &= (1 + r_{t+1})(w_t - c_t^1 - (1 + n_t)b) \\ &= (1 + f'(\cdot))s_t \\ &= (1 + n_t)f\left(\frac{x_t - c_t^1 - (1 + n_t)b}{1 + n_t}\right) - (1 + n_t)x_{t+1}, \end{aligned}$$

since $x_t = c_t^1 + s_t + (1 + n_t)b = w_t$ for all $t \geq 1$ in PLE, i.e. (6) implies (16). ■

Still the PLE is modified short-run Pareto-optimal, as is the corresponding equilibrium in Balasko and Shell [1980,1981], however, comparison of (5) and (15) shows that not only equilibria where private relative costs of child-raising equal to the marginal rate of substitution between the respective good and children are modified short-run Pareto-optimal: the second term on the right-hand side of (15) indicates that, as soon as the 'old' generations' consumption in the final period of life differs from interest income, the (discounted) difference must diminish the real costs b of raising children. What is the reason for that?

The reason is simply that, as long as we hold the young generation's budget x_{t+1} constant, an increase in fertility does not only bear the costs b on parents, but, on the other hand, benefits them by enlarging the amount of resources to be transferred from the young generation as a whole — without altering the latter's perspectives for well-being. Therefore, this additional benefit of children — besides the joy from making and rearing them — must be taken into account.⁶

Note here that this fact is by far not trivial as it would be for a model with fixed prices (i.e. a constant marginal product) of all factors. In such a case, increasing fertility has no effect on the equilibrium wage rate, and therefore, if there are transfers from 'young' to 'old', an increasing fertility does not hurt the young per se. However, with neoclassic production, a higher fertility means a lower marginal product of labour, given a certain capital stock, and therefore, a policy which increases fertility requires compensation for the young generation due to this loss in wage income. Just this effect makes it per se unclear whether an increase in fertility, being favourable to the 'old' by increasing their old-age income, via transfers from the young, and by increasing the interest rate on their savings, can leave the utility position of future generations unchanged.

⁶An analogous argument, of course, applies if parents consume less than interest income in old age. In that case the burden of children is higher than only the costs b of raising them, namely increased by the (discounted) loss of income due to a higher transfer to the larger number of children.

Characterization of Modified Pareto-Optimality

We have learned in the previous section which conditions must be fulfilled to guarantee modified short-run Pareto-optimality of an allocation and we know that these conditions must also be fulfilled for an allocation to be Modified Pareto-optimal. As this section will show, a MSRPO allocation need not be MPO. In order to decide on MPO of an allocation it is first necessary to develop a characterization of this property.

For this purpose let us first consider the dual problem to the one addressed in Lemma 2, that is

$$\max_{c_t^1, c_{t+1}^2, 1+n_t, s_t, x_{t+1}} x_{t+1} \quad (23)$$

subject to

$$-u(c_t^1, c_{t+1}^2, 1+n_t) \leq -u \quad (24)$$

$$c_t^1 + (1+n_t)b + s_t \leq x_t \quad (25)$$

$$\frac{c_{t+1}^2}{1+n_t} + x_{t+1} - f\left(\frac{s_t}{1+n_t}\right) \leq 0, \quad (26)$$

yielding the side conditions (24)–(26) with equality and (14)–(15) as necessary and sufficient first order conditions to determine the solution. This problem reflects the necessity for an allocation to be MSRPO that a given utility-level for a generation must be achieved at the lowest costs for future generations.

Let us denote by $g(x_t, u)$ the maximal x_{t+1} of the above problem, where x_t and u play the role of parameters of the model. Using the techniques of the *Envelope-Theorem*, one obtains the following comparative-static results:

$$\frac{dg(x_t, u)}{dx_t} = \left. \frac{dx_{t+1}}{dx_t} \right|_{u \text{ const.}} = \frac{f'(\cdot)}{1+n_t} > 0 \quad (27)$$

$$\frac{dg(x_t, u)}{du} = \left. \frac{dx_{t+1}}{du} \right|_{x_t \text{ const.}} = -\frac{1}{(1+n_t)u_3(\cdot)} < 0. \quad (28)$$

Moreover, $g(x_t, u)$ is strictly concave in x_t .⁷

For considerations on modified Pareto-optimality both equations represent the foundation of a characterization. If an allocation is not MPO, then the utility of a generation can be increased without hurting any other. Given that the allocation

⁷Consider e.g. Theorem 21.23 in Simon and Blume [1994] which says that the maximal value of a problem with concave objective function and convex constraints is concave in the parameters. It is rather straightforward to prove that, if some of the constraints are only quasi-convex, the maximal value remains concave in the parameters belonging to the convex constraints. Therefore the maximal value $g(x_t, u)$ need not be concave in u , but it is so in x_t .

is MSRPO, according to (28), an improvement of a generation t is only possible by decreasing x_{t+1} . Holding all of the following generations at least indifferent then requires compensation according to (27), whereby this marginal expression gives a lower bound for the actual non-marginal restitution — following simply from the fact that the utility function is strictly quasi-concave and the budget set convex.

Therefore (by replacing the marginal product of capital in (27) by the interest rate), we can state immediately the following theorem.

Theorem 3 *A necessary condition for an allocation to fulfill MSRPO but to fail MPO is as follows: $\exists \Delta x_2 < 0 : \forall t \geq 2$:*

$$x_t + \prod_{i=2}^{t-1} \frac{1+r_{i+1}}{1+n_i} \cdot \Delta x_2 \geq 0. \quad (29)$$

This condition coincides with the necessary condition for an allocation to fail traditional Pareto-optimality in the framework of the Diamond [1965] model-type with exogenous fertility, developed in Lang [1992a,1994 cond. (1.28)]. Therefore, all sufficient conditions for traditional Pareto-optimality developed from (1.28) in Lang [1994] carry over to the present framework with endogenous fertility.

Let us now turn to a sufficient characterization in terms of equilibrium prices for an allocation to fail MPO. We will thereby follow the lines developed in Lang [1994] for the traditional model with exogenous (but not necessarily constant) population development.

Theorem 4 *Let $\{c_t^1, 1+n_t, s_t, c_{t+1}^2\}_{t=1, \dots, \infty}$ be a MSRPO allocation. The allocation is not MPO iff: $\exists \Delta x_2 < 0 : \forall t \geq 2 : \exists a_t > 1$:*

$$|\Delta x_{t+1}| := \prod_{i=2}^t \frac{(1+r_{i+1})a_{i+1}}{1+n_i} \cdot |\Delta x_2| \leq x_{t+1}, \quad (30)$$

and

$$v\left(x_t + \Delta x_t, x_{t+1} + \frac{(1+r_{t+1})a_{t+1}}{1+n_t} \Delta x_t\right) \geq v(x_t, x_{t+1}). \quad (31)$$

Proof: (\Leftarrow) $\Delta x_2 < 0$ means that generation 1's utility position can be improved. All following generations then receive compensation along the feasible procedure (30) – feasible because of the right-hand side weak inequality. The utility position of each generation then can be at least compensated, (31). Therefore, generation $t = 1$ can be improved without worsening any of the following generations, and the allocation cannot be MPO.

(\Rightarrow) If the allocation is not MPO then there exists a feasible sequence $\{\Delta x_t\}_{t=2, \dots, \infty}$, $\Delta x_2 < 0$, such that $v(x_t + \Delta x_t, x_{t+1} + \Delta x_{t+1}) \geq v(x_t, x_{t+1})$. We can then define.

$a_{t+1} := \frac{\Delta x_{t+1}(1+n_t)}{\Delta x_t(1+r_{t+1})} > 1$ to see that there exists a $\Delta x_2 < 0$ such that for all $t \geq 2$ there exist $a_t > 1$ such that (30) and (31) are fulfilled. ■

In the following we will go on to characterize stationary allocations and allocations which converge to stationary ones, only using observable information, namely population dynamics and interest rates.

Stationary Allocations

Because of the strict concavity and monotonicity of $g(x, u)$ in x , $g(x, u) = x$ can only have two solutions at maximum.

As one can see rather easily, for the case of one solution, two types of behaviour of $g(x, u)$ are possible (see Figure 1). Either $g(x, u)$ starts below the diagonal, crosses it, and stays above it thereafter, (a), or it starts below, touches it, and then returns to a location below, (b).⁸

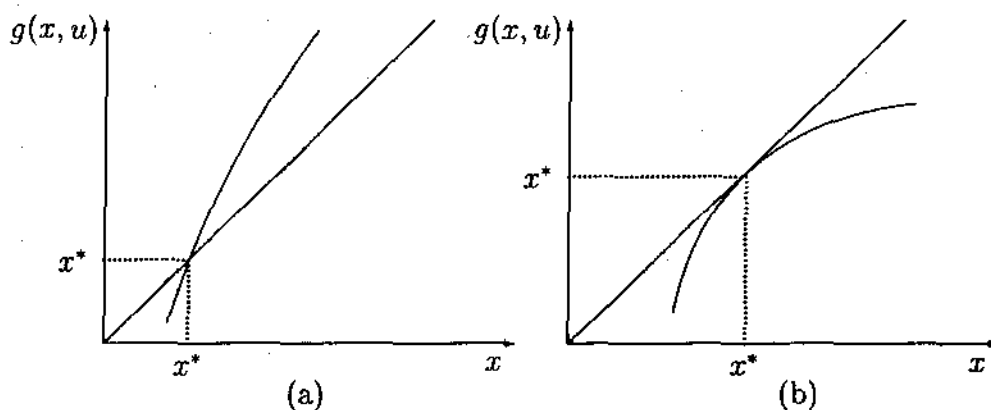


Figure 1: The case of only one steady-state.

It follows that

$$\frac{dg(x, u)}{dx} = \frac{f'(\cdot)}{1+n} \geq 1, \quad (32)$$

i.e. $r \geq n$ for this case. It is also easy to see that the stationary allocations described in Figure 1 are modified Pareto-optimal: starting with $x_1 = x^*$, increasing the utility of one generation requires a downward shift of the $g(x, u)$ curve. Therefore, $x_2 < x^*$, and the induced sequence $x_{t+1} = g(x_t, u)$ then becomes infeasible, meaning that starting with $t = 2$ not all generations can at least enjoy utility u .

⁸ $g(x, u)$ cannot start somewhere strictly above the diagonal since $g(0, u) \leq 0$ for all u .

For the case of two solutions to $g(x, u) = x$, Figure 2 gives an illustration. As mentioned before, $g(x, u)$ cannot start above the diagonal. Therefore, the figure shows the unique way to have two steady-states.

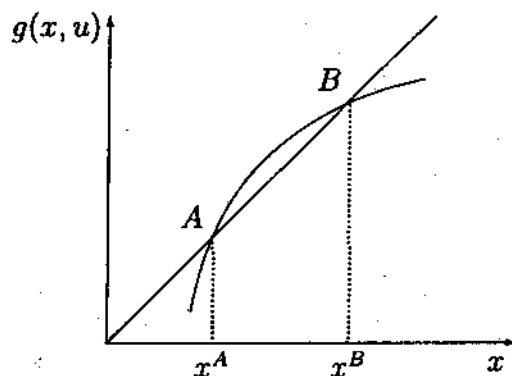


Figure 2: The case of two steady-states.

Point A clearly corresponds to case (a) in Figure 1. We have $r > n$, and the steady-state is modified Pareto-efficient. Point B , however, is distinct in that it represents an inefficient steady-state: it is possible to increase the utility of the first generation without hurting any of its successors (for the following see Figure 3).

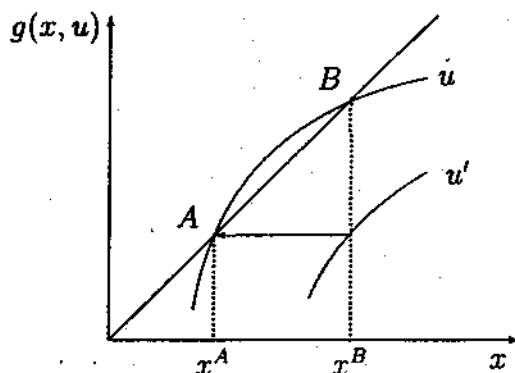


Figure 3: The inefficient steady-state.

We can increase the utility of the first generation up to a utility level u' such that $x_2 = x^A$. Starting with this x_2 , utility u still remains feasible for all generations following the first one — who is better off. Therefore, point B corresponds to an inefficient steady-state. Moreover, we can find $f'(\cdot)/(1+n) = r/n < 1$, i.e. $r < n$.

These findings on stationary modified short-run Pareto-optimal allocations can be gathered in the following theorem.

Theorem 5 Consider a MSRPO stationary allocation $\{c^1, 1+n, s, c^2\}_{t=1, \dots, \infty}$. The allocation is MPO iff $n \leq r$.

It turns out that the conditions for traditional Pareto-optimality and for the new concept of modified Pareto-optimality coincide for stationary allocations.

Allocations Converging to Steady-States

Let us first consider the case that the MSRPO allocation converges to an efficient steady-state with $n < r$. Denoting the steady-state values of x and u by x' and u' , and the converging sequences' values by x_t and u_t , we can observe that

$$\frac{dg(x_t, u_t)}{dx} = \frac{1+r_{t+1}}{1+n_t} \rightarrow \frac{1+r}{1+n} = \frac{dg(x', u')}{dx} \quad (33)$$

since $x_t \rightarrow x'$ and $u_t \rightarrow u'$, and $g(x, u)$ is continuous in its arguments.

Therefore, for an $\epsilon > 0$, a $T \in \mathbb{N}$, and all $t \geq T$, we have $\frac{1+r_{t+1}}{1+n_t} > 1 + \epsilon$. Since $g(x, u)$ is strictly concave,

$$\Delta x_{t+1} := g(x_t + \Delta x_t, u_t) - g(x_t, u_t) < \frac{dg(x_t, u_t)}{dx} \cdot \Delta x_t, \quad (34)$$

for $\Delta x_t \neq 0$, implying that the necessary compensation for a decrease in x_t by decreasing x_{t+1} leads – as in Theorem 3 – to the sequence

$$\Delta x_t = \prod_{i=2}^{t-1} \frac{1+r_{i+1}}{1+n_i} \cdot \Delta x_2 \quad (35)$$

that under-estimates the actual burden. As $\frac{1+r_{t+1}}{1+n_t} > 1 + \epsilon$ for $t \geq T$, we see that $\Delta x_t \rightarrow \infty$ as t increases over all boundaries. Since the sequence $\{x_t\}_{t=1, \dots, \infty}$ is bounded by the mere fact that it converges, the compensation process for an increase of the utility of generation 1 – requiring $\Delta x_2 < 0$ – turns out to be infeasible.

Therefore, a MSRPO allocation that converges to an MPO steady-state with $n < r$ is MPO, too, and the corresponding sequence of interest rates fulfills

$$\lim_{t \rightarrow \infty} \prod_{i=2}^t \frac{1+r_{i+1}}{1+n_i} = \infty. \quad (36)$$

Let us now turn to the case that a MSRPO allocation converges to an inefficient steady-state with $n > r$. Then the corresponding sequence of interest rates

$\{r_t\}_{t=1,\dots,\infty}$ and population growth rates $\{n_t\}_{t=1,\dots,\infty}$ fulfills the condition that for an $\epsilon > 0$ there exists a $T \in \mathbb{N}$ such that for all $t \geq T$ we have

$$\frac{1 + r_{t+1}}{1 + n_t} < 1 - \epsilon. \quad (37)$$

Let $\bar{x}(u')$ represent this steady-state, i.e. $\bar{x}(u') = g(\bar{x}(u'), u')$ is the larger of the two existing solutions to $x = g(x, u')$. Consider the function $g(x, u)$ for given u .

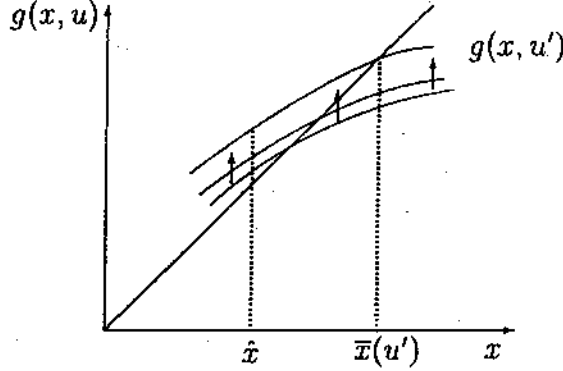


Figure 4: Convergence to a non-MPO steady-state.

Then there exists a \hat{x} with $0 < \hat{x} < \bar{x}(u')$ such that $g(\hat{x}, u) > \hat{x}$ (see Figure 4). Since $g(x, u)$ is continuous in u for all x , $g(\hat{x}, u_t) \rightarrow g(\hat{x}, u')$ for $u_t \rightarrow u'$. Therefore, for all $\epsilon > 0$ there exists a $T \in \mathbb{N}$ such that for all $t \geq T$: $g(\hat{x}, u_t) > g(\hat{x}, u') - \epsilon$. ϵ then can be chosen such that $g(\hat{x}, u') - \epsilon > \hat{x}$.

This implies that, starting at any $\hat{x}_T \geq \hat{x}$, the sequence $\{\hat{x}_T, \hat{x}_{T+1}, \dots\}$ given by

$$\hat{x}_{t+1} = g(\hat{x}_t, u_t), \quad t \geq T, \quad (38)$$

fulfills $\hat{x}_t \geq \hat{x}$ for all $t \geq T$.

Now turn to the sequence $\{x_t\}_{t \geq 1}$. Because of its convergence, for all $\delta > 0$ there exists a $T' \in \mathbb{N}$ such that for all $t \geq T'$ we have

$$x_t > \bar{x}(u') - \delta. \quad (39)$$

We can choose δ such that $\bar{x}(u') - \delta > \hat{x}$.

For $T'' = \max(T, T')$, we then proceed in the following way: Since $x_{T''} > \bar{x}(u') - \delta > \hat{x}$, we can increase the well-being of generation $T'' - 1$ such that $\hat{x}_{T''} := (x_{T''-1}, \hat{u}_{T''-1}) = \hat{x} < \bar{x}(u') - \delta$, i.e. $\hat{u}_{T''-1} > u_{T''-1}$. Since this can be done without touching utilities $u_1, \dots, u_{T''-2}$, generations $1, \dots, T'' - 2$ stay indifferent, and, as already mentioned, $\hat{u}_{T''-1} > u_{T''-1}$. Because $\hat{x}_{T''} = \hat{x}$, the sequence

$\{\hat{x}_{T''}, \hat{x}_{T''+1}, \dots\}$ generated by (38), with T replaced by T'' , fulfills $\hat{x}_t \geq \hat{x}$ for all $t \geq T''$. Therefore it is possible to keep generations $T'', T'' + 1, \dots$ indifferent (because of the appearance of u_t in (38)).

This way we have found a Pareto-improvement. So MSRPO allocations which converge to an inefficient steady-state with $n > r$ are also non-MPO, and given (37),

$$\lim_{t \rightarrow \infty} \prod_{i=2}^t \frac{1+r_{i+1}}{1+n_i} = 0. \quad (40)$$

It remains in the following to turn to the examination of the singular case that allocations converge to the MPO Golden Rule with $n = r$.

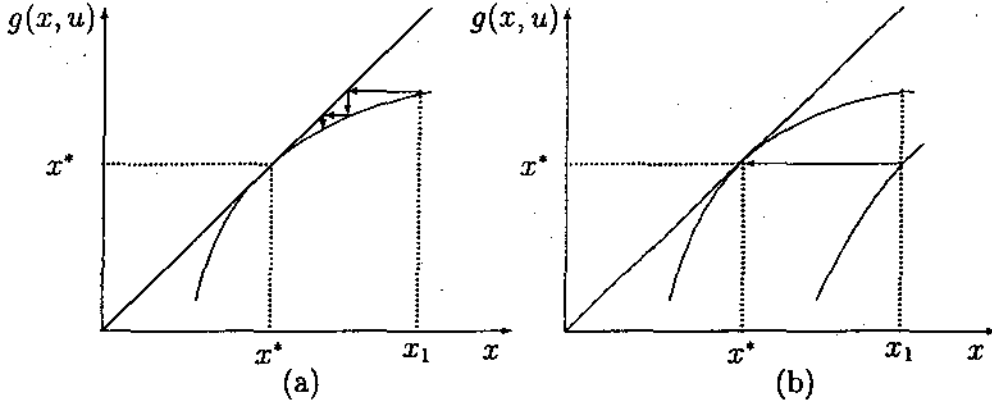


Figure 5: Convergence to the 'modified' Golden Rule: inefficient case.

Figure 5 represents a MSRPO allocation that converges to the 'modified' Golden Rule with $n = r$. As one observes,

$$\frac{dg(x_t, u_t)}{dx} = \frac{1+r_{t+1}}{1+n_t} \rightarrow 1, \quad (41)$$

monotonically from below. The allocation is inefficient, as (b) shows, but $\frac{1+r_{t+1}}{1+n_t}$ can converge to 1 so 'fast' that $\lim_{t \rightarrow \infty} \prod_{i=2}^t \frac{1+r_{i+1}}{1+n_i}$ is bounded away from 0. Therefore, (40) is not a necessary condition for a MSRPO allocation to be non-MPO.

Figure 6 refers a case with a MSRPO allocation that is also MPO, but $\lim_{t \rightarrow \infty} \prod_{i=2}^t \frac{1+r_{i+1}}{1+n_i} < 1$, i.e. (36) is not necessary for the MPO of a MSRPO allocation. The findings of this section can be stated in the following theorem.

Theorem 6 Consider the set of MSRPO allocations that converge to a stationary allocation. Then

$$\lim_{t \rightarrow \infty} \prod_{i=2}^t \frac{1+r_{i+1}}{1+n_i} = 0 \quad (42)$$

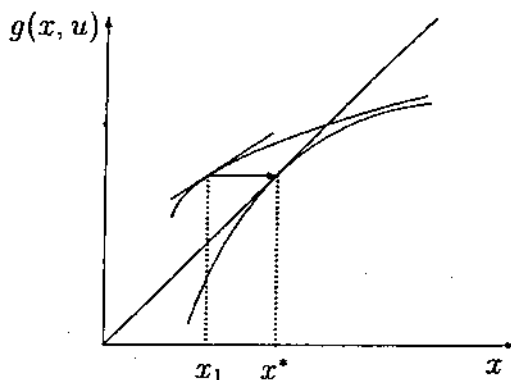


Figure 6: Convergence to the 'modified' Golden Rule: efficient case.

is generically necessary and sufficient for the non-MPO of an allocation. Only for allocations converging to the 'modified' Golden Rule, (42) need not be an indicator for MPO or non-MPO.

Intergenerational Equity and Efficiency

As mentioned in the introduction, equity should not only be applied within generations but also across generations. If existing individuals, may they exist today or in a thousand years, are equipped with the same **cardinal** preferences, then they ought to be treated as equal.⁹ If we only make individuals unequal by the mere fact that they live sooner or later in time, then we can also claim that simultaneously living individuals cannot be subject to equal treatment by the mere fact that they do not live at the same place — making any discussion on equity obsolete. Of course, in reality we know little about the future, we do know little about our descendants, but in a world of models one should first begin with complete knowledge of everything before proceeding to settings where one does not know anything. In this sense, everything in the model, the economic and technological mechanisms, and especially all individuals to ever live, are very familiar to us, and the idealized setting of the model used in this work gives us no other choice than to make all individuals to ever exist subject to equal treatment.

At the same time, one would also like to guarantee efficiency. But this is not always possible as can be derived from the findings in the preceding section.

⁹Equal treatment need not apply, however, for individuals with the same ordinal preferences but a different ability to generate satisfaction from consumption bundles. This, of course, requires a cardinal utility concept, but we avoid a discussion of this issue by assuming that the utility function $u(c_t^1, c_{t+1}^2, 1 + n_t)$ represents the same cardinal preferences of all individuals to exist.

Equal treatment demands for equal utility for equal individuals. The stationary allocations considered in the preceding section are therefore not candidates for *equitable* allocations. Equitability is thereby equivalent to possessing the properties of equity and efficiency (or better MPO) at the same time. Having a look at Figure 1 in the last section, one realizes indeed that x^* represents an equitable allocation, in (a) as well as in (b). On the other hand, referring to Figures 2 and 3, only the stationary allocation A is equitable. B is characterized by equity but, as Figure 3 shows, the first generation can be improved to u' , leaving all other generations with the same utility u as in B . Hence, although representing an equal allocation, B is not equitable because it fails MPO.

Given a certain initial stock x_1 of resources per head, construction of an equal allocation is always possible. One can in any case choose u so as to guarantee $x_1 = g(x_1, u)$. But it depends on x_1 and the functional form of $g(x, u)$ whether one gets a steady-state like A or B .

Hence, we end up in the dilemma that the desire for both, equality and efficiency, cannot always be fulfilled at the same time.

Conclusion

Some important conclusion, besides social choice implications, must be drawn from the findings in this work.

The first one concerns the necessity to reform the typical systems of inter-generational redistribution in Western Economies which pay benefits to retired individuals, independently of the number of children they created. As we have learned, for the case of transfers from 'young' to 'old', a subvention on the real costs of child-raising must be paid. This subvention must just equal the transfer per head of the children-generation. One fiscally neutral way — in the sense that each generation finances the subvention to itself — were to levy a lump-sum tax. Another possibility were to make benefits payments dependent on the number of children, i.e. to grant a 'pension' equal to the contribution per child times the number of children.

Neither fiscal neutrality nor the second possibility, however, suggest that the children-generation cannot be hurt by such a reform. The negative general-equilibrium effects of a subsidy on child-raising can be twofold:

- (i) If the reform leads to higher fertility rates, then, c.p., the marginal product of labour declines, and therefore, in the new equilibrium, the wage rate sinks.
- (ii) The reform may generate lower savings, having, c.p., the same consequence as in (i).

Hence, the young generation has to be compensated by lower contributions to the pay-as-you-go system for a decrease in the wage rate. This might be necessary,

but (i) and (ii) need not necessarily occur: one can easily imagine that the relative decrease in the (net) price of child-raising leads only to an increase in second-period consumption, but 'consumption' of children and consumption in the first period of life are kept constant, thereby implying even a positive effect on the capital-per-head endowment of the young, and hence, on the wage rate. The design of a reform depends, as so often, on the specification of preferences and production technology.

On the other hand, one should not forget that also costs of child-raising, in particular education expenses, are not directly charged to parents per each additional child they have. Schools and universities are typically financed from the general government budget, thereby withholding to parents the expenditures necessary to educate an additional offspring.

Education expenses can be seen as transfers to children, in contrast to payments received from the pay-as-you-go system, which are transfers to parents. Without further empirical investigation it is per se unclear which component is dominant, making it impossible to decide whether the necessary disincentive to create children (due to anticipated costs of education) outweighs the necessary incentive to get them (anticipated financial benefits via a pay-as-you-go system, or, indirectly, additional taxpayers who help to retire government debt). This question undoubtedly is of much interest for further investigation.

It is also worth to mention that the setting here does not require child-allowances to be paid retrospectively (i.e. after the period of fertility terminates), if they are not foreseen, and hence do not have an incentive effect. Of course, if individuals behave strategically the currently applied intergenerational transfer systems make free-riding possible, since there are typically no (or not high enough) subventions on child-raising. If, however, the corrective subvention is introduced, then no one can skim off the benefits of others who have created children. This also applies in a heterogenous framework with individuals holding different preferences for the number of offspring. Hence, the frequent complaints of some families with many children, of not receiving higher pensions (or subventions) than pensioners with fewer offspring, can only refer to the above free-riding problem.

Of course, if one introduces positive externalities, then subventions on child-raising must be paid even if there are no transfers between generations: for example in some settings of the newer growth theory (see the book by Barro and Sala-I-Martin [1995] for an excellent survey), the rate of invention depends on the number of individuals present in an economy. Hence, transferring this mechanism to an overlapping generations model with endogenous fertility, child-creation must be subsidized due to beneficial effects on the number of discoveries made.

In general, it is, however, quite questionable whether children entail only positive externalities and who is free-riding on whom: a quite typical example of

a negative externality of having children is the waste of time imposed on others while paying at the check-out of a supermarket, trying, at the same time, not to lose control of the kids. Another example is the regulative instrument of speed limits imposed for cars to protect the health of unexperienced children, thereby distorting their own parents incentives to polish them.

A final conclusion is related to the empirical verifiability of dynamic efficiency. As is well-known, Abel et al. [1989] use the cash flow criterion to check real data for the occurrence of capital overaccumulation. Their finding is that Western Economies, with huge amounts of government debt and other forms of intergenerational redistribution, are indeed on efficient growth paths. The cash flow criterion, applied to the original Diamond [1965] framework, reduces to a simple comparison between population growth and interest rate: in steady state, an economy is dynamically efficient iff $n \leq r$.

As was derived in this work, exactly the same condition fully characterizes modified dynamic efficiency in steady state. Therefore, one could basically make use of the same historical data to investigate the question whether or not Western Economies appear as modified dynamically efficient. But the exact correspondence of the conditions for dynamic efficiency and modified dynamic efficiency is only valid in a world where the corrective subvention/tax on child-raising exists. This probably has not been the case in past. Therefore, referring to the first conclusion drawn above, fertility, as well as capital accumulation, might have been lower than if the appropriate corrective instrument were utilized. As above, without a closer specification of preferences and production technology, it is not possible to draw any conclusions from the data utilized in the work cited above, and empirical investigations guided by the traditional paradigm of exogenous fertility have become obsolete.

The notion of Modified Pareto-Optimality opens a wide opportunity for theoretical as well as empirical research on problems of intergenerational resource sharing.

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