DEPARTMENT OF MATHEMATICS

AITP METHODS FOR MAXIMIZING GENERALITY AND TRACTABILITY OF CLASSES OF ALGEBRAS

GONÇALO GOMES ARAÚJO

Master in Mathematics and Applications

DOCTORATE IN MATHEMATICS

NOVA University Lisbon
September, 2025

DEPARTMENT OF MATHEMATICS

AITP METHODS FOR MAXIMIZING GENERALITY AND TRACTABILITY OF CLASSES OF ALGEBRAS

GONÇALO GOMES ARAÚJO

Master in Mathematics and Applications

Adviser: João Jorge Ribeiro Soares Gonçalves de Araújo
Full Professor, NOVA University of Lisbon

Examination Committee

Chair: Maria Fernanda de Almeida Cipriano Salvador Marques
Full Professor, NOVA University of Lisbon

Rapporteurs: Mikoláš Janota
Senior Researcher, Czech Technical University

Conceição Veloso Nogueira
Adjunct Professor, Polytechnic University of Leiria

Adviser: João Jorge Ribeiro Soares Gonçalves de Araújo
Full Professor, NOVA University of Lisbon

Members: Michael Rawson
Researcher, University of Southampton

Luís Fernando Rodrigues de Sequeira
Assistant Professor, University of Lisbon

DOCTORATE IN MATHEMATICS

NOVA University Lisbon
September, 2025

AITP Methods for Maximizing Generality and Tractability of Classes of Algebras

This document was created with the (pdf/Xe/Lua)LaTeX processor and the NOVAthesis template (v7.3.9) .

J. M. Lourenço. *The NOVAthesis LaTeX Template User's Manual*. NOVA University Lisbon. 2021. URL: https://github.com/joaomlourenco/novathesis/raw/main/template.pdf.

Acknowledgements

I would like to thank my advisor Prof. João Araújo for the huge help that he provided during all these years. He taught me several things, not only academic but also regarding personal life. He was always available to talk and help me with all my work. I also want to thank Prof. Josef Urban and Prof. Mikolas Jánota for all the support they provided regarding the work I have made in Prague, giving me all the necessary resources to do my research.

Moreover, this work is funded by national funds through the FCT – Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 (https://doi.org/10.54499/UIDB/00297/2020) and UIDP/00297/2020 (https://doi.org/10.54499/UIDP/00297/2020) (Center for Mathematics and Applications), and under the scope of the project UI/BD/154405/2023. I also want to thank Nova Math for providing all the resources to assure that I had everything that I needed to properly do my research.

At last, but not least important, I want to thank my family, especially my brother Gustavo, my mother Ana and my father Paulo, for the support, encouragement and kindness that they gave me through all these years, even in those days where my motivation was smaller and I was feeling sad. To Prof. Elvira who was always there for me to help me regarding any academic or personal matter. To my friends that were always by my side, telling me to never give up and giving me hope that I would succeed. To my beautiful soulmate, Ana Rita, for all the love, kindness, encouragement and happiness she gave me every single day. She is the sunshine of my heart, and no words can describe the love I feel for her.

Abstract

This thesis was motivated by the search for new classes of algebras that maximize both generality (the class contains many important examples) and tractability (the class allows for deep structural theorems), combining algebra, automated reasoning, and large-scale computation. A database of nearly 500 classes of algebras was autoformalized from the literature and organized as a partially ordered set of inclusions. More than 80,000 theorems were proved with automated theorem provers, many requiring non-trivial proofs. The work also produced dozens new GAP packages.

In the process, group-theoretic constructions such as verbal and marginal subgroups were extended to inverse semigroups and related structures, yielding new theorems.

The methods developed show how autoformalization and database techniques can generate new algebraic classes and enable property transfer between them. The work toward the original goal—identifying classes that maximize generality and depth—is more than half complete, and we expect that this goal will soon be achieved, opening one or more new areas of mathematics.

Keywords: Marginal Semigroups, Classes of Algebras, Artificial Intelligence, ArXiv Extraction.

Resumo

Esta tese foi motivada pela procura de novas classes de álgebras que maximizem tanto a generalidade (a classe contém muitos exemplos importantes) como a tratabilidade (a classe permite teoremas estruturais profundos), combinando álgebra, raciocínio automatizado e computação em grande escala. Uma base de dados com quase 500 classes de álgebras foi autoformalizada a partir da literatura e organizada como um conjunto parcialmente ordenado de inclusões. Foram demonstrados mais de 80 000 teoremas com recurso a demonstradores automáticos, muitos dos quais exigiram provas não triviais. O trabalho produziu também dezenas de novas bibliotecas GAP.

No processo, construções de teoria de grupos, tais como subgrupos verbais e marginais, foram estendidas a semigrupos inversos e estruturas relacionadas, dando origem a novos teoremas.

Os métodos desenvolvidos mostram como a autoformalização e as técnicas de bases de dados podem gerar novas classes algébricas e permitir a transferência de propriedades entre elas. O trabalho em direção ao objetivo inicial (identificar classes que maximizem a generalidade e a profundidade) está mais de metade concluído, e espera-se que este objetivo seja em breve alcançado, abrindo uma ou mais novas áreas da matemática.

Palavras-Chave: Semigrupos Marginais, Classes de Álgebras, Inteligência Artificial, Extração do ArXiv.

Contents

List of Figures

List of Algorithms

1

Introduction

The theory of groups led to some of the deepest and most beautiful results in mathematics. The structure theory of finite groups, the classification of finite simple groups, connections with geometry, topology, analysis, and number theory. All testify that. Yet, as powerful as group theory is, it is a *specialized* theory. The axioms of invertibility make the class of groups narrower, thereby excluding as particular examples many natural algebraic structures of mathematics.

Semigroups, on the other hand, are a very general framework: they only require the ubiquitous associativity: it is satisfied by the composition of functions, and many other object across mathematics. For a long time, however, this very generality was seen as an obstacle. The prevailing view was that semigroup theory could not possibly admit deep structural theorems beyond the basic tools of universal algebra (isomorphism theorems, homomorphism theorems, congruence theory, and so forth). The perception was that the price of generality was the impossibility of depth.

This perception was spectacularly overturned by the work of Green [27] and Rees [61]. They showed that profound structure theorems are indeed possible in semigroup theory. Green's relations and Rees' theorem provided tools as deep and influential for semigroups as Sylow theory and Jordan-Holder theory are for groups. With these breakthroughs the subject exploded into a flourishing field, proving that it is indeed possible to combine generality with depth.

The goal of the present thesis is inspired by this historical trajectory. The aim was to identify new contexts analogous to semigroups: algebraic theories which are broad enough to contain many important classes of algebras as particular cases, but structured enough to permit the existence of deep and general theorems.

Consider, for example, the class of *magmas*, that is, sets with a binary operation and no further axioms. Virtually all of mathematics can be encoded in this framework, because whenever there is a binary operation one can speak of a magma. However, outside of the general results of universal algebra, it is not clear which kinds of profound theorems could ever be obtained about magmas. Classes slightly less general, such as semigroups or loops, may admit more structure and more powerful theorems. The challenge, then, is to identify the gems: theories that maximize tractability and generality.

In order to achieve this goal, three main steps are required:

1. The construction of a very large database of algebraic classes, extracted from the literature, formalized, and organized systematically.

2. The construction of a large database of theorems, each valid in some particular class of algebras, and each linked to its algebraic context.

3. A large-scale analysis to find which classes allow the largest number of theorems from the database to be generalized to them.

It became clear during the course of this work that such an ambitious program was incompatible with the timeframe of a two and a half year doctoral thesis. A different project will be necessary to realize the full vision. Nonetheless, what has been accomplished already opens the door to this final objective and provides solid foundations for it.

The contributions of the present thesis are as follows:

1. A first research article in mathematics was written with the dual purpose of proving interesting new theorems and of acquiring advanced skills in automated theorem proving (ATP). This article, (See Appendix A) which was published in a peer-reviewed journal, provided the training ground to learn the tools of ATP, to formulate conjectures, to control ATP through Python scripts to perform massive automated tests, to analyze automated proofs, and to humanize them into publishable mathematics.

2. The construction of a large database of algebraic classes, a major project in the domain of auto-formalization. This is a field still very much open, and the present work contributes to it. The database was built over a year of research conducted in Prague, a leading center for auto-formalization that has secured multiple ERC grants in the area. The database includes not only formal definitions of algebraic classes, but also metadata: the first paper where the class appears, the number of citations of that paper, the AMS Mathematics Subject Classification code, and more. All this allows to get an idea of how important will be the class that generalizes many of the classes in the database.

3. The construction of the partially ordered set (poset) of algebraic classes, enabling the identification of maximal elements and relationships of containment between theories.

4. The introduction of new natural theories, obtained by extending the language of existing classes with additional commuting automorphisms. This provided a method for producing new classes which are both novel and naturally linked to established algebraic structures. This is a project in data augmentation.

5. Whenever an object can be defined in a class of algebras of some type (say, magmas) and in the class C that object has property P, we can check if that object has property P in any other class of algebras of the same type. This idea opens the gate for many new familiar theorems. We provide many concrete examples of the application of this idea.

The structure of the thesis reflects this trajectory. In the first chapter we describe the results obtained on marginal semigroups, which constituted the "training chapter". The second chapter explains in detail the process of auto-formalization of algebraic classes extracted from the arXiv. The third chapter constructs the poset of classes of algebras. The fourth chapter discusses data augmentation through the introduction of new classes. The fifth chapter illustrates several applications of the library to generate new theorems.

What remains to be done in order to complete the project? The most demanding task is the auto-formalization of a large corpus of theorems in the classes of algebras already collected. This is a much more complex challenge than the auto-formalization of algebraic classes alone. Once this is achieved, the next step will be to generalize these theorems to all the overclasses where they are true. Then it will be necessary to define and compute metrics that measure both *generality* (the number of subclasses contained) and *tractability* (the number of theorems that can be generalized to the class). The ultimate goal is to identify candidates for the twenty-first century analogous to what semigroup theory was in the twentieth: a field of extreme generality where deep and far-reaching theorems can nonetheless be proved.

2

Marginal Subsemigroups

2.1 Initial Considerations

In his celebrated work, Garrett Birkhoff introduced the concept of *varieties of universal algebras*: classes of algebras of a fixed type closed under subalgebras, direct products, and homomorphic images. He then proved a cornerstone theorem of twentieth-century algebra: a class of algebras is a variety if and only if it is axiomatizable by identities [8].

This result launched modern universal algebra, theoretical computer science and shaped later advances in automated reasoning. A landmark event is McCune's use of the Equational Theorem Prover (ETP) to solve Robbins' problem in lattice theory, a domain to which Birkhoff made deep contributions [46], and the first time a machine managed to solve a mathematical problem that outperformed several mathematicians (including Alfred Tarski).

Since groups are universal algebras, one may speak of *varieties of groups*. For example, the variety of abelian groups is defined by the identities $(xy)z = x(yz)$, $x1 = x$, $xx' = 1$ and $xy = yx$. Because inversion is available, any identity $p(x_1, \dots, x_k) = q(x_1, \dots, x_k)$ in groups is equivalent to

$$(q(x_1, \dots, x_k))^{-1} p(x_1, \dots, x_k) = 1.$$

Hence, assuming $= 1$ implicitly, a variety of groups can be specified by a set of words. For instance, in groups, the list $[\, x^{-1}y^{-1}xy,\ x^3 \,]$ defines the variety of groups satisfying $x^{-1}y^{-1}xy = 1$ and $x^3 = 1$.

Saying that a group G satisfies the word $x^{-1}y^{-1}xy$ means $a_1^{-1}a_2^{-1}a_1a_2 = 1$ for all $a_1, a_2 \in G$. If G does not belong to that variety, the set

$$S = \{\, a_1^{-1}a_2^{-1}a_1a_2 \mid a_1, a_2 \in G \,\}$$

contains nontrivial elements, and thus the subgroup $\langle S \rangle$ gives an idea of how far G is from being abelian. This already hints at a structural role of this subgroup linked to characteristic subgroups. Recall that an automorphism of G is a bijection $f : G \to G$ with $f(ab) = f(a)f(b)$ and $f(a^{-1}) = (f(a))^{-1}$. A subgroup $H \leq G$ is *characteristic* if $f(H) = H$ for every automorphism f of G.

The following result is a well known theorem from Group theory.

Theorem 2.1.1. *Let G be a group and let $w(x_1, \ldots, x_n)$ be a group word. Then*

$$H = \left\langle\, w(a_1, \ldots, a_n) \;\middle|\; a_1, \ldots, a_n \in G \,\right\rangle$$

is characteristic in G.

Proof. Let $f \in \mathrm{Aut}(G)$. For all $a_1, \ldots, a_n \in G$,

$$f\big(w(a_1, \ldots, a_n)\big) = w\big(f(a_1), \ldots, f(a_n)\big) \in H,$$

so $f(H) \subseteq H$. Applying the same argument to f^{-1} gives $H \subseteq f(H)$. Hence $f(H) = H$, and therefore H is characteristic. □

Since every characteristic subgroup is normal, this construction offers a canonical, variety-oriented way, to produce normal subgroups. In this setting, P. Hall's brief 1940 note introduced two dual notions that became standard: *verbal subgroups* and *marginal subgroups* [29] [62].

2.1.1 Verbal and marginal subgroups

Let X be a non-empty set. Given a set W of words in the free group $F(X)$ and a group G, the *verbal subgroup* is defined as

$$W(G) := \left\langle\, w(g_1, \ldots, g_n) \;\middle|\; w \in W,\; g_i \in G \,\right\rangle.$$

The *marginal subgroup* $M_W(G)$ consists of elements $a \in G$ such that inserting a into any coordinate of any $w \in W$ never changes the value:

$$w(g_1, \ldots, ag_i, \ldots, g_n) = w(g_1, \ldots, g_i, \ldots, g_n) = w(g_1, \ldots, g_i a, \ldots, g_n)$$

for all choices of $g_1, \ldots, g_n \in G$ and each position i. Heuristically, $W(G)$ captures the *active content* of the variety-defining laws, while $M_W(G)$ collects the *silent elements* for those laws. Both mentioned subgroups are fully characteristic (that is, they are invariant under every endomorphism) [50, 41].

Example 2.1.2. (The commutator word) Let $w(x, y) = [x, y] = x^{-1}y^{-1}xy$. Then

$$w(G) = \langle [g, h] \mid g, h \in G \rangle = G^{(1)}, \qquad M_w(G) = Z(G).$$

Example 2.1.3. (The pth power word) Fix a prime p and let $w(x) = x^p$. Then

$$w(G) = \langle g^p \mid g \in G \rangle = G^p,$$

and $M_w(G)$ is the set of all $a \in G$ with $(ga)^p = g^p$ for every $g \in G$.

2.2 Extensions to Inverse Semigroups

Many group-theoretic ideas extend naturally to inverse semigroups, often with subtle but illuminating adjustments. Conjugation offers a familiar example: replacing global symmetries by partial ones preserves much of the structure and leads to a rich theory in inverse semigroups [42, 58, 2, 30, 33].

An *inverse semigroup* is a semigroup S in which each $a \in S$ has a unique inverse a^{-1} with $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. As in groups, one can form words in the signature $\langle \cdot, {}^{-1} \rangle$ and ask for the structures generated by their elements, as well as for the elements that are *silent* with respect to those words. This is exactly Hall's program transposed to the world of partial symmetries and an idempotent-rich algebra as are inverse semigroups [29].

2.2.1 Marginal Subsemigroups in Inverse Semigroups

2.2.1.1 Definitions

Fix an inverse semigroup S and a word $\omega(x_1, \ldots, x_n)$ in $\langle \cdot, {}^{-1} \rangle$. For each position i, an element $a \in S$ is called *ith partial marginal* for ω if there exist idempotents $e, f \in E(S)$ such that, for all choices of variables,

$$\omega(x_1, \ldots, ax_i, \ldots, x_n) = \omega(x_1, \ldots, ex_i, \ldots, x_n),$$

$$\omega(x_1, \ldots, x_i a, \ldots, x_n) = \omega(x_1, \ldots, x_i f, \ldots, x_n).$$

The *ith partial margin* $M_i^\omega(S)$ is the set of such elements, and the *margin* $M^\omega(S) = \bigcap_{i=1}^n M_i^\omega(S)$ consists of the *marginal* elements.

2.2.1.2 Main results

The first goal of this thesis was to extend to inverse semigroups the idea of Marginal groups. This was done as a goal in itself, but also as a way to acquire advanced skills using automated reasoning tools, in our case ProverX [1].

Our two main structural theorems mirror the group case while revealing the role of idempotents.

- *Full inverse-subsemigroup property.* For every i, the set $M_i^\omega(S)$ is a full inverse subsemigroup of S (it contains all idempotents and is closed under product and inversion). Consequently, the margin $M^\omega(S)$ is also a full inverse subsemigroup.

- *Commutator case and the metacenter.* For the commutator word $\omega(x, y) = [x, y] = x^{-1}y^{-1}xy$, the marginal inverse subsemigroup coincides with the *metacenter*

$$Z(S) = \{ a \in S \mid aa^{-1}xa = axa^{-1}a \text{ for all } x \in S \}.$$

  Thus $M^\omega(S) = Z(S)$, which is a normal inverse subsemigroup.

These results extend the classic ones about groups. Moreover, the proofs of the mentioned theorems were obtained with the help of ProverX, but not directly, as we cannot feed into to machine general words of the form $w(x_1, \ldots, x_n)$. It was necessary to work with some particular words and some small n, say $n = 2, 3$; then humanize the proofs, another skill important to master for the subsequent work; and, finally, try to infer the general rule (or proof) from those partial proofs.

Another instance of the use of the machine, now in a much more direct way, was the proof of many results about commutators in groups that also hold for Cillford semigroups (inverse semigroups with central idempotents). Our main result is the following.

Theorem 2.2.1. *Let S be a Clifford semigroup and write $[x, y] := x^{-1}y^{-1}xy$. Then, for all $x, y, z, u, v, w \in S$, the following identities hold:*

1. $[z^{-1}xz,\ z^{-1}yz] \ = \ z^{-1}[x, y]z$;
2. $[xu,\ xvxw] \ = \ [xu,\ xw]\,(xw)^{-1}[xu,\ xv]\,xw$;
3. $y^{-1}xy \ = \ x\,[x, y]$;
4. $[x,\ zy] \ = \ [x, y]\,y^{-1}[x, z]\,y$;
5. $[x,\ y^{-1}] \ = \ y\,[y, x]\,y^{-1}$;
6. $y^{-1}[[x,\ y^{-1}],\ z]\,y\ \ z^{-1}[[y,\ z^{-1}],\ x]\,z\ \ x^{-1}[[z,\ x^{-1}],\ y]\,x \ \in \ E(S)$;
7. $(xy)^2 \ = \ x^2y^2\,[y, x]\ [[y, x], y]$.

Too see more regarding our work on Marginal Subsemigroups, please see Appendix A.

2.3 Open Problems and Perspectives

Normality and congruences for margins: In groups, marginal subgroups are normal and directly induce congruences. In inverse semigroups, one may first define a *marginal equivalence* by pointwise invariance of a word's value and then pass to the least congruence containing it. A central question asks when this extra step is unnecessary: does the marginal equivalence already define a congruence, and is $M^{\omega}(S)$ automatically normal, beyond the commutator case? Power words $\omega(x) = x^n$ are natural test cases. A positive answer would place inverse-semigroup margins on the same firm footing as their group counterparts and clarify how margins behave under standard constructions (e.g., Rees quotients) and decompositions.

Extending the framework across varieties: Hall's method studies how words interact with algebraic structure. The same viewpoint can be used beyond groups, in semigroup settings where the signature and the available words change. Examples include completely regular semigroups, regular $*$-semigroups, and monoids with extra operations. The aim is to find the *verbal* and *silent* elements for key identities and to see how these identities shape Green's relations, idempotent-separating congruences, and semilattice-of-groups decompositions. In spirit, this mirrors what verbal and marginal subgroups already do in group theory.

3

ArXiv Extraction

3.1 Formalizing Mathematics: Some notes on the history, systems, milestones, and future

By *formalizing mathematics* one means encoding definitions, statements, and proofs in a precise logical language, so that a computer can check the derivations from axioms and rules of inference.

The roots of formalization predate computers by many decades. Frege's *Begriffsschrift* [22] advanced a formal language for logic and function-argument analysis; Whitehead and Russell's *Principia Mathematica* [71] attempted to derive large parts of mathematics from logical axioms; Hilbert's program sought finitary consistency proofs for formal systems [31]; and Gödel's incompleteness theorems [24] showed inherent limitations of formal axiomatic methods.

With digital computers, the idea of *mechanically checking* proofs took concrete form. N. G. de Bruijn's *Automath* [10] pioneered a typed framework in which entire theories could be written so that a machine checks correctness line by line. Automath stimulated long-lasting ideas: compact encodings of binders (de Bruijn indices), book-like organization of developments, and the insight that rigorous type disciplines can enforce correctness of derivations at scale [23] .

A subsequent strand, the *LCF approach* (that is, an interactive automated theorem prover), emphasized small trusted kernels and user-extensible automation. This style strongly influenced later systems in the HOL family and Isabelle (i.e. interacting theorem proving systems). In parallel, the proof-as-programs perspective (the Curry-Howard correspondence) encouraged systems where proofs are typed lambda-terms and types are propositions, laying foundations for type-theoretic assistants.

Beginning in the 1970s, Andrzej Trybulec's Mizar project sought a human-oriented declarative proof language supported by automated checking and a curated, steadily growing library: the Mizar Mathematical Library (MML) [6]. The Mizar style emphasizes textbook-like proofs with strong global organization and a journal-like workflow for library contributions.

The journal *Formalized Mathematics*—established in 1990—has long served as a dedicated venue for Mizar articles and for consolidating formalized material. Its electronic portal lists volumes from 1990 onward and clarifies its editorial relationship to the *Journal of Formalized Mathematics*, an electronic counterpart aligned with the evolving MML [20]. Formalizing mathematics became a mathematical topic in itself [36].

Isabelle/HOL and the HOL tradition

Systems in the HOL lineage implement Church's simple type theory and emphasize small kernels, robust automation, and integration with SAT/SMT and first-order provers. Isabelle/HOL supports a declarative, human-readable proof language (Isar) and a journal-like repository of articles, the *Archive of Formal Proofs* (AFP), founded in 2004 [4], which is refereed and continuously checked. HOL Light, designed with an emphasis on simplicity and reliability, has powered several milestone projects [51].

Coq and dependent type theory

Coq implements the (Calculus of) Inductive Constructions, marrying dependent type theory with inductive definitions and program extraction. Over three decades it has grown into a mature platform for both mathematics and verified software, supported by books, extensive libraries, and active tool development [13] [67].

Lean and mathlib

Lean began as a dependently typed system with a small kernel and extensible metaprogramming. Lean 4 (2021) re-engineered the prover and language, powering the rapid growth of the community-driven *mathlib* library [49] [5] .

Agda, ACL2, PVS, Metamath

Agda advances dependently typed programming and interactive proving; ACL2 (with industrial roots in Boyer–Moore logic) emphasizes automated reasoning about recursive definitions and finite structures; PVS combines expressive higher-order specification with decision procedures and model checking; Metamath features a tiny kernel and large repositories entirely in a minimal proof language [52] [39] [54] [47] .

3.1.1 Landmark case studies

The Four Color Theorem. The 2008 article *Formal Proof—The Four-Color Theorem* reported a fully formalized proof in Coq, offering an end-to-end, machine-checked account of a theorem historically verified with substantial computation but without proof assistants [26].

The Odd Order Theorem (Feit–Thompson). A six-year collaboration culminated in 2012 with a complete constructive formalization in Coq of the Feit–Thompson theorem that every finite group of odd order is solvable. The project integrated domain-specific proof engineering (notably SSReflect), large algebraic libraries, and significant automation [25].

The Kepler Conjecture (Flyspeck). The Flyspeck project, completed in 2014 and documented in 2015, delivered a formal proof of the Kepler Conjecture using HOL Light and Isabelle. It combined geometric analysis, linear programming, and large computational verifications into a fully checked account, later published in *Forum of Mathematics, Pi* [28].

Fermat's Last Theorem. It has been announced that Kevin Buzzard (Imperial) and his team are now formalizing Fermat's last Theorem.

3.1.2 Where formalization meets mathematics

The mathematical payoff is twofold. First, *verification at scale*: difficult arguments can be refactored into smaller lemmas with precise interfaces, then checked end-to-end (as in the Odd Order and Kepler case studies). Second, *new mathematics of formalization*: the discipline of encoding statements uncovers implicit conventions, strengthens invariants, and clarifies boundaries between computational and conceptual parts of proofs.

The resulting artifacts are not merely archives. They are executable knowledge bases: one can search for lemmas, inspect dependencies, migrate results between theories, and build new theorems on a stable foundation.

3.1.3 Observations

Formalizing mathematics is now a sustained scholarly activity: grounded in strong foundations, implemented in robust systems, supported by curated libraries, and published in journals that recognize its distinct contributions. It should be observed that formalizing a theorem can range from a task for a Masters thesis or a PhD Dissertation to a very demanding project that involves a large team of top mathematicians for a long period of time.

In the future, all the work done in the formalization of mathematics will be used to create the autonomous referee first; then the referee that also fixes problems when it finds one (if they can be fixed); it will be used to train AI agents that will be able to prove results as machines could never before. It is important to observe that the first goal of this area is to be able to get a Gold Medal in the International Mathematics Olympiads.

3.2 Autoformalization: History, systems, milestones, and prospects

3.2.1 Opening context

An ambitious collaborative project led by Kevin Buzzard at Imperial College London is presently formalizing Wiles's proof of Fermat's Last Theorem in the Lean theorem prover [11, 21, 34]. This effort combines large-scale library engineering, number-theoretic background (elliptic curves, modular forms, Galois representations), and community infrastructure.

In this context, my supervisor, Professor João Araújo, conjectured at the AI Theorem Proving Conference (Aussois, 2023), in the presence of Kevin Buzzard, that Fermat's Last Theorem will be the last major theorem to be formalized primarily by humans; after that, *autoformalization* will take over.

3.2.2 What is autoformalization?

By *autoformalization* we mean the semi or fullyautomatic transformation of informal mathematical content (typically LaTeX prose, textbook statements, or arXiv articles) into checked formal statements and proofs in a proof assistant. Concretely, the pipeline seeks to (i) parse informal text into a precise logical representation, (ii) resolve symbols, types, and overloading against a formal library, (iii) select relevant background material (premise selection), and (iv) construct or search for a proof (tactics or low-level inference). The end product is a machine-verified artifact integrated into a growing library.

Two complementary streams have converged here. The first is *large-theory automated reasoning* over existing formal corpora (Mizar, HOL Light, Isabelle/HOL, Lean), using learning to guide premise selection and proof search. The second is *text-to-formal* translation, which aims to digest the informal layer itself, mapping natural LaTeX mathematics to a formal vernacular.

3.2.3 Prehistory and enabling ideas

The prehistory of autoformalization is the emergence of methods that scaled automation on *existing* formal libraries. Urban's MPTP program systematized exports from Mizar to ATP (automated theorem prover) formats, enabling cross-fertilization between interactive and automated provers at library scale [69]. Building on this, MaLARea introduced an iterative loop between ATP search and machine learning to prioritize relevant lemmas in *large theories* [68]. In parallel, Kaliszyk and Urban integrated learning and ATPs with HOL Light, demonstrating that a substantial fraction of a major corpus (Flyspeck) can be re-proved automatically via model- and data-driven premise selection [38]. The HOL(y)Hammer service subsequently delivered this capability as a robust, user-facing bridge between interactive proving and strong automation [37].

3.2.4 From large-theory automation to text-to-formal

The mid-2010s saw deep learning applied to premise selection: DeepMath showed that sequence models trained on Mizar statements improve retrieval of useful premises for ATPs [35]. This line continued with richer features and benchmarks, and with higher-order environments such as HOList, which framed HOL Light proving as a reinforcement-learning task and provided a reproducible sandbox for data-driven proof search [7]. In parallel, the Metamath-based GPT-f work explored transformer-guided tactic generation for fully formal proofs [59].

On the text-to-formal front, a direct attack on translation from LaTeX sentences into Mizar statements was explored by Wang, Brown, Kaliszyk, and Urban, including type elaboration and supervised/unsupervised machine translation setups [70]. The goal is not only syntactic parsing but also semantic alignment with an existing library, a central difficulty when the same informal symbol may denote distinct formal entities.

3.2.5 Benchmarks, datasets, and Lean-era developments

Community benchmarks have played a key role in measuring progress. The *miniF2F* dataset aggregates olympiad-level problems formalized across systems and became a standard target for learned theorem provers [73]. The *IsarStep* benchmark mined high-level proof steps from Isabelle's Archive of Formal Proofs, promoting models that reason at human-readable granularity [43]. In Lean, OpenAI's experiments on olympiad problems demonstrated LLM-guided tactic search and curriculum learning for statement understanding [53, 60]. The LeanDojo framework further improved reproducibility and retrieval-augmented proving on mathlib, releasing open toolchains and datasets that lowered the barrier for controlled experimentation [72].

3.2.6 Landmark moments and contributions

Several milestones delineate the field.

- *Library-scale exports and re-proving.* MPTP and MaLARea established scalable exports and learning-guided ATP loops for Mizar; HOL(y)Hammer and Flyspeck experiments extended this to HOL Light, with strong auto-reproving rates [69, 68, 38, 37].
- *Neural premise selection.* DeepMath and successors demonstrated that deep sequence models improve premise selection on large formal corpora [35].
- *Higher-order learning environments.* HOList framed tactic-level proving as RL, enabling controlled comparisons of architectures and search strategies [7].
- *LLM-guided proving.* GPT-f, OpenAI's Lean work on olympiad problems, and curriculum approaches showed that large language models can guide or synthesize steps that verifiers accept, especially when coupled with retrieval [59, 53, 60, 72].

- *Text-to-formal prototypes.* Direct translation from informal LaTeX to Mizar built the first end-to-end autoformalization pipeline components, including domain-specific type elaboration [70].

A unifying thread across these results is the leadership of Josef Urban and collaborators in designing exports, learning loops, and evaluation regimes for large-theory reasoning; this work seeded much of today's autoformalization ecosystem. Urban has been awarded several major grants, including ERC, to carry on the autoformalization project.

3.2.7 Open challenges

Despite rapid progress, several bottlenecks remain.

- *Robust symbol grounding.* Mapping a human's overloaded, context-rich notation to a specific formal constant is still brittle. Better joint representations of informal tokens and formal symbols are needed for stable disambiguation.

- *Coverage of background knowledge.* Many informal texts rely on folklore results or implicit lemmas. Autoformalizers must learn to *ask* for missing lemmas, or synthesize them, not merely translate surface form.

- *Long-range structure.* Scaling to monographs and classification-scale proofs requires models that reason over dependency graphs and propose modular decompositions, not only local steps.

3.2.8 Prospects

With the autoformalization dream fulfilled, hopefully within short term, the corpus of formalized mathematics will be enormous, eventually all mathematics produced so far will be formalized, and hence we will have an enormous amount of material to train AI agents. If Araújo's conjecture comes to pass, it will not be due to a single leap, but to the steady accumulation of models, tools, and libraries that let machines shoulder an ever larger share of translation and search, while humans architect theory and certify meaning.

Recently, an apparent breakthrough has been announced with the proof of the string prime theorem autoformalized (see also Szegedy's recent note [64]).

3.2.9 Our autoformalization project: scope and objectives

As discussed previously, the *autoformalization* project is ongoing. Its long–term objective is to design pipelines that can transform arbitrary mathematical usual writing into fully checked formal artifacts, i.e., definitions, statements, and proofs that are accepted by a proof assistant's kernel. Concretely, the target is to deal effectively with the full spectrum of obstacles present in informal texts: overloading and ambiguous notation; implicit hypotheses and lemmas; linguistic variability and ellipsis; and genuine gaps that must be bridged by auxiliary results or explicit justifications. In domain-restricted settings with stabilized notation and mature libraries, near-fully-automatic workflows are more realistic in the short term, and serve as stepping stones toward the general goal.

Within this broader agenda, our specific aim is to construct the largest feasible, curated knowledge base of algebraic classes in the sense of universal algebra, with emphasis on *varieties* and *quasivarieties*. For each class of algebras $\mathcal{K}$ we record the following data:

- the underlying signature and a canonical axiomatisation (identities or quasi–identities);
- authoritative bibliographic sources (e.g., the original paper establishing $\mathcal{K}$), together with citation data and AMS subject classifications;
- structural relationships: classes contained in $\mathcal{K}$, classes containing $\mathcal{K}$, and known isomorphisms or equivalences between different presentations;
- links to formal counterparts when available (entries in major proof–assistant libraries), enabling cross–navigation between informal and formal artefacts.

The resulting object is a directed, typed graph whose vertices are algebraic classes and whose edges encode inclusion.

To pursue this at scale, I joined Josef Urban's group in Prague. The work there combined export and alignment tools for existing formal corpora with data models that can ingest and normalise heterogeneous mathematical sources (articles, monographs, surveys) into a coherent, queryable repository tailored to varieties and quasivarieties.

We will now describe what we have done in order to get the full library and to get the partial ordered set of inclusions of those classes of algebras.

In order to gather almost 500 non-isomorphic classes of algebras, it was necessary to complete some tasks. Firstly, we had to extract the whole ar5iv database, which is a website that contains almost all arXiv documents, but written in HTML instead of the usual PDF format. After that, we filtered the mathematical definitions that satisfied specific criteria and, following that, we built a prompt and we used an LLM, like ChatGPT, to convert the axioms of those classes of algebras written in LaTeXto axioms written in first order logic, that is, ProverX syntax [1].

Finally, we separated each class of algebras by its type, for example $(2, 2, 1, 0)$ which means two binary operations, one unary operation and one constant, and then we built the partial ordered set, by inclusion, using ProverX, of all those sets. We present now a more detailed explanation of each step.

3.2.10 ArXiv Extraction

Regarding the ar5iv acquisition, it is a fact that arXiv documents use PDF format due to its flexibility and reliability, however, they are more complicated to analyze and parse using a programming language. Therefore, in order to overcome that restriction, we had to download the whole ar5iv database. We extracted approximately two million documents so, taking into account the huge amount and space required to download all those papers, we had to use a remote server instead of a local machine.

3.2.10.1 Parsing Varieties and Quasi-Varieties

Regarding extracting and filtering mathematical definitions, we created a filtering process that was used to gather documents relevant to the mathematical domain from the mentioned dataset. This step involved not only parsing the HTML content of each scientific paper and consider only documents with mathematical definitions, but also apply the filter to each definition in order to get only varieties and quasi-varieties.

3.2.10.2 First Order Logic Conversion

Regarding the conversion to first order logic, the axioms of the mathematical definitions were converted to first-order-logic (FOL) terms. The reason for this step is because ProverX, uses its syntax in FOL. This conversion algorithm was created using the OpenAI API where, after we provide it with a prompt explaining how we wanted the conversion and providing some examples, we were able to translate the LaTeX axioms of the definitions into their corresponding FOL representations. It is important to observe that, in this step, we had to confirm each conversion one by one due to ChatGPT's limitations.

3.2.10.3 Building the Partial Ordered Set

Regarding the construction of the partial ordered set (poset), the final step was to separate all the sets by their algebraic type and consider all the possible combinations to check if a class contains or is contained in other class of algebras. Then we organized, using directed graphs, all those classes of algebras into partially ordered sets (Posets), where each node represents a class of algebras and an arrow from the class A to the class B means that the defining axioms of A imply the defining axioms of B. In the figure below, we show a diagram that illustrates the most important steps throughout our work.

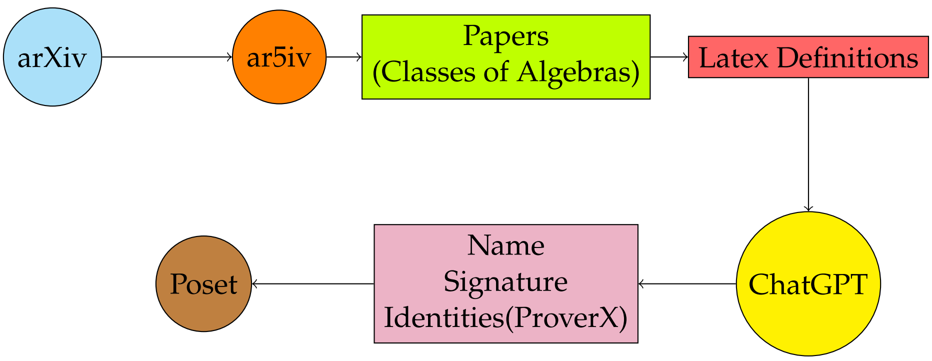


Figure 3.1: Diagram Illustrating all the Necessary Procedures

4

The poset of inclusions on our database

4.0.1 Background and Motivation

Inverse semigroups and completely regular semigroups are two classical success stories of semigroup theory. They share deep structural properties and enjoy elegant classification theorems that connect them to group theory, order theory, and geometry. For decades, semigroup theorists wondered whether there might be a "third family" of regular semigroups with equally strong structural properties. The folklore consensus was pessimistic: no such family exists, yet no rigorous explanation for this conviction had ever been published.

Araújo, Kinyon, and Robert [3] undertook the ambitious task of turning this metamathematical belief into a theorem. The guiding analogy is Arrow's impossibility theorem in social choice: Arrow listed axioms for voting systems and then proved that no voting system with 3 or more candidates could satisfy them all simultaneously in every scenario. Similarly, Araújo–Kinyon–Robert listed the desirable shared properties of inverse and completely regular semigroups, and then proved that under suitable constraints, any variety of regular semigroups with uniquely defined inversion satisfying these properties collapses into one of the two known classes.

4.0.2 Methodology and Scale

The proof method was exhaustive. The authors systematically listed all possible varieties of regular semigroups with uniquely defined inversion, under clearly stated constraints. Each candidate variety was then tested against the desirable properties. At first sight, many appeared promising. But after detailed inspection, each turned out to coincide with (or be contained in) either the class of inverse semigroups or the class of completely regular semigroups.

This approach required the development of *thousands of lemmas*, far beyond the capacity of human verification alone. The project was only possible thanks to the extensive use of automated reasoning, specifically the system *ProverX*.

4.0.3 Workflow of Results

As in the other projects described below, the workflow can be divided into three layers:

- **Rapid results.** Many candidate inclusions or identities could be settled instantly by ProverX, producing quick positive proofs or small counterexamples.
- **Delayed results.** For some identities, proofs or counterexamples emerged only after machines were given more time, often requiring larger search bounds or more complex strategies.
- **Human intervention.** The hardest cases required human mathematicians to analyze why automation stalled, to devise new strategies, or to combine automated fragments into a rigorous whole.

4.0.4 Outcomes

The paper provided the first rigorous mathematical justification of the long-standing conviction that no "third family" exists. The proofs in the paper filled more than one million pages, and there were countless counter-examples.

A key observation was this pattern: many cases treated easily; some cases treated with more computer effort; a few last examples unbeatable by the machine and requiring human intervention.

4.1 The Equational Theories Project (Tao, 2024)

4.1.1 Overview

Terence Tao's *equational theories project* [65, 66] emerged from a similar vision: to study algebraic structures at scale using human–machine collaboration. The object of study here was the class of *magmas*, that is, sets with a single binary operation, together with equational identities of bounded complexity. The project enumerated thousands of candidate axioms and organized them into a poset under logical implication.

4.1.2 Motivation

Unlike the semigroup case, where the departure point was very important classes of semigroups, here the classes were artificial, generated systematically from short equational axioms. This artificiality was intentional: it provided a clean benchmark for testing new workflows, machine provers, and collaborative methods. By creating a dense web of small questions, the project forced researchers and machines alike to scale their effort to thousands of micro-problems.

4.1.3 Challenges

The project thus faced several challenges such as:

- **Massive scale.** Tens or hundreds of thousands of implication queries had to be decided.
- **Demanding results.** Some proofs or counterexamples required more time and advanced techniques and heuristics.
- **Infinite counterexamples.** Some non-implications hold only because a counterexample exists in an infinite algebra. They were found by humans, but this project prompted work by Mikoláš Janota (Prague), showing how automated theorem provers can now *find infinite counterexamples*, a milestone for the field.

4.1.4 Workflow of Results

The project again exhibited a three-layered workflow:

- **Rapid results.** Many implications or non-implications could be settled quickly by ProverX/Mace4, often in seconds.
- **Delayed results.** Some queries required much more time, with machines eventually succeeding after extended searches.
- **Human intervention.** The most difficult cases required human insight.

4.1.5 Outcomes

The project produced a partially ordered set of equational axioms, fully decided, a new benchmark dataset for automated reasoning, and a proof-of-concept for large-scale collaborative workflows.

4.2 Extracting a Database of Algebraic Classes from the arXiv

4.2.1 Motivation and Contrast

After extracting hundreds of classes of algebras from ArXiv and building our database, we had to find (as in Tao's project) the full partial ordered set of inclusions. This project started long before Tao's, but applies the same paradigm to a different data source.

4.2.2 Challenges

This project faced many of the same challenges as the ones mentioned previously, but with an added difficulty: different papers often define the same class with different primitive operations.

- **Massive number of questions.** Each pair of classes raised two inclusion questions.
- **Many proved and many disproved.** Automated tools solved large numbers of these questions, sometimes producing rapid results, and the other times only after long computations.
- **Undecided cases.** Several implications remain undecided due to automated tools limitations.
- **Operational ambiguity.** To handle different naming conventions, a canonical dictionary was introduced (`unary1`, `unary2`, `binary1`, `binary2`, etc.). Yet even this was insufficient: e.g., rings could be encoded with + as `binary1` in one paper and as `binary2` in another. This forced us to test many permutations of operations to ensure completeness.

4.3 Poset described

Regarding the construction of the partial ordered set (poset), we had to separate all the sets by their algebraic type and consider all the possible combinations to check if a class contains or is contained in other class of algebras. Then we organized, using directed graphs, all those classes of algebras into partially ordered sets (Posets), where each node represents a class of algebras, and an arrow from the class A to the class B means that the defining axioms of A imply the defining axioms of B. In order to access this poset, one has to access the link https://www.proverx.dm.fct.unl.pt/app.php, and then click on "Enter as Guest", then click on the folder Class_Database, then click compare_classes.py and run the script. After that, users will be able to introduce their classes of algebras with the corresponding axioms, and the program will compute all the classes in the database that contain and are contained in the given class. The following image shows on of the classes of algebras contained in our database, in this case, Separative Semigroups.

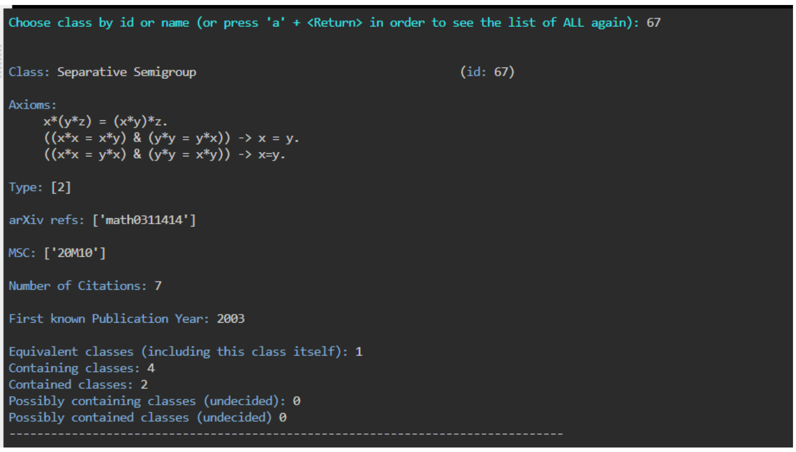


Figure 4.1: Extracted Metadata for Separative Semigroups

It is also possible to check our metadata for those (almost 500) classes of algebras in this github link: https://github.com/gg-araujo/Thesis/tree/main/500

4.3.1 Statistics

In the histogram presented below, we can see the relation between the number of all theorems (inclusions between classes of algebras) that we were able to prove using ProverX, and the maximum length of each proof. That is, if we proved that the defining axioms of A imply the defining axioms of B, assuming for example that the class B is defined by 3 axioms, then if ProverX proved the first axiom in 10 lines, the second in 40 and the last in 20 lines our maximum length of proof would be 40. It is important to observe that the majority of the theorems are really simple or trivial to prove, however, we got 23 theorems, out of 324, that required more than 20 lines to prove.

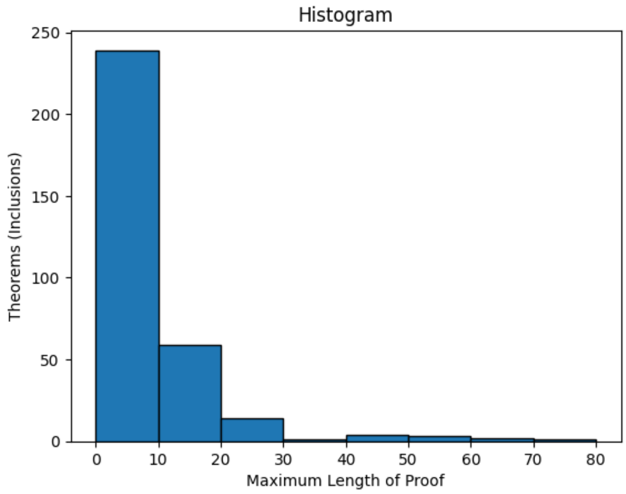


Figure 4.2: Histogram Showing the Relation Between the Number of Theorems and the Length of their Proofs

In the table below, we can see the number of classes extracted, as well as the number of classes that are contained or contain another class of algebras, the total of theorems gathered and the number of trivial and non-trivial theorems. For our purposes, we chose the number 20 to distinguish between trivial and non-trivial proof because ProverX uses some lines for the assumptions and to do simple observations.

Table 4.1: Statistics of the Algebras and their Theorems

Properties	**Values**
Number of Non-Isomorphic Classes of Algebras Extracted	463
Number of Classes of Algebras Extracted with at least 1 Theorem	269
Number of Theorems	324
Number of Trivial Theorems (Maximum Length of Proof $\leq$ 20)	301
Number of Non-Trivial Theorems (Maximum Length of Proof > 20)	23

In the table below, we have the 7 graphs with the highest number of classes of algebras. We present the number of non-trivial implications on each graph, as well as the average of the outdegree of each graph (number of implications divided by the number of classes that are contained in other classes), and also the average of the indegree of each graph (which is the number of implications divided by the number of classes of algebras that contain other classes).

Table 4.2: Statistics of the Graphs (Poset) with most Algebras

	Nº Alg	Non-trivial ⇒	Indegree (Average)	Outdegree (Average)
G3	49	2	1.585	2.097
G9	13	4	1.2	2.4
G10	48	1	2.325	3.207
G11	13	0	1.714	1.714
G14	10	0	1.571	1.833
G15	26	3	1.8	2.4
G32	23	1	1.5	1.5

In the figure below, we see the graph that we obtained for the classes of algebras of type (2), that is the classes defined just by one binary operation. An arrow from a node A to a node B means that the defining axioms of A imply the defining axioms of B, which implies that on the top of the graph we have the maximal classes of algebras and on the bottom we have the minimal classes of algebras, by inclusion.

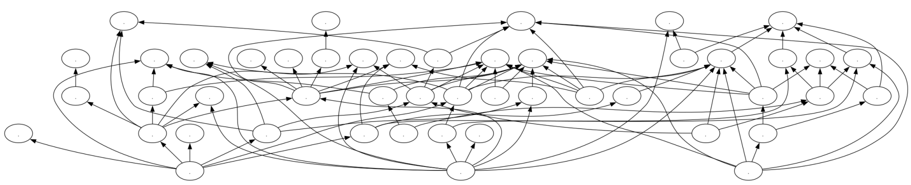


Figure 4.3: Graph with Classes of Algebras of Type (2)

4.4 Implemented Algorithms for the Extraction

In the algorithm presented below, we describe how we implemented the procedure of parsing HTML documents and extracting classes of algebras defined by varieties or quasi-varieties.

Algorithm 1 Extracting and Filtering Mathematical Definitions

```
Require: htmlDocument, filterCriteria
Ensure: filteredDefinitions
Step 1: Load the HTML Document
htmlTree ← Parse HTML
Step 2: Extract Mathematical Definitions
definitions ← []
for all element in htmlTree do
    if IsMathDefinition(element) then
        definitions.append(element)
    end if
end for
Step 3: Apply Filtering Criteria
filteredDefinitions ← []
for all definition in definitions do
    if MeetsCriteria(definition, filterCriteria) then
        filteredDefinitions.append(definition)
    end if
end for
Step 4: Return Filtered Definitions
return filteredDefinitions
```

In the algorithm presented below, we describe how we implemented the procedure of converting LaTeXaxioms to ProverX syntax.

Algorithm 2 Converting LaTeX Axioms to First-Order Logic Axioms

```
Require: prompt, apiKey
Ensure: convertedAxioms
Step 1: Prepare the API Request
apiEndpoint ← "https://api.openai.com/v1/engines/davinci-codex/completions"
headers ← { "Content-Type": "application/json", "Authorization": "Bearer " + apiKey }

data ← { "prompt": prompt, "maxtokens": 1024, "temperature": 0.5 }
Step 2: Send the Request to OpenAI API
response ← SendPostRequest(apiEndpoint, headers, data)
Step 3: Parse the API Response
responseData ← ParseJSON(response)
convertedText ← responseData["choices"][0]["text"]
Step 4: Extract and Format First-Order Logic Axioms
convertedAxioms ← ExtractFOLAxioms(convertedText)
Step 5: Return the Converted Axioms
return convertedAxioms
```

In the final algorithm presented below, we describe how we separated each class of algebras by its signature, used ProverX to prove all inclusions and then created the Posets, separated by their algebraic type.

Algorithm 3 Creating the Poset of all Classes of Algebras

```
Require: algebras, signatures
Ensure: poset
Step 1: Separate Algebras by Algebraic Signature
separatedAlgebras ← SeparateBySignature(algebras, signatures)
Step 2: Generate Inclusion Relationships Using ProverX
inclusionRelations ← [ ]
for all signature in signatures do
   for all algebra1 in separatedAlgebras[signature] do
      for all algebra2 in separatedAlgebras[signature] do
         if algebra1 ≠ algebra2 then
            result ← UseProverX(algebra1, algebra2)
            if result == "algebra1 includes algebra2" then
               inclusionRelations.append((algebra1, algebra2))
            end if
         end if
      end for
   end for
end for
Step 3: Create Partially Ordered Set (Poset)
poset ← CreatePoset(inclusionRelations)
Step 4: Return the Poset
return poset
```

4.4.1 Outcomes and Open Work

This ordered database now offers many opportunities:

1. When a new class of algebras is introduced, it can immediately be checked which existing classes contain it and which classes are contained in it.

2. Given a theorem in a specific class of algebras, one can ask the machine to determine in which of those classes the theorem still holds.

3. When a theorem appears in two different classes of algebras (even if the proofs are very different), one may search in the poset for a class containing both and check whether a common generalization exists.

4. It is possible to introduce new notions of equivalence between classes of algebras (for instance, term equivalence) and to verify, ideally in an automated way, which classes are equivalent under the chosen concept.

Yet the open cases remain significant. They represent both a challenge and an opportunity: to refine automated reasoning, possibly to improve the ways of finding infinite models, to integrate the lessons from the other projects, and to deepen our human understanding to tackle the cases the machine cannot handle.

4.5 Concluding Perspective

Taken together, these three projects—Araújo–Kinyon–Robert's impossibility theorem for regular semigroups, Tao's equational theories project, and the extraction of algebraic classes from the arXiv—illustrate a new research paradigm in algebra:

- Begin with thousands of classes or axioms.
- Pose a uniform property (existence of inversion, equational implication, class inclusion).
- Attack at scale using automated provers and counterexample finders.
- Accept a layered workflow: rapid machine results, delayed machine results, and essential human intervention.
- Exploit new techniques for constructing large and even infinite counterexamples, as pioneered in part by Mikoláš Janota and others.

5

Engineering Classes That Contain Algebras from Important Classes

Our first objective was to *gain fluency* in the use of Automated Theorem Proving (ATP) tools on algebraic structures. Work on marginal semigroups served precisely this purpose.

At a second stage we built a pipeline to extract algebraic classes from the literature and convert their LaTeX axioms into first-order logic suitable for ATP. To this end, we scraped the *ar5iv* HTML mirror of arXiv (about two million documents), filtered for definition environments that specify varieties or quasi-varieties, and converted the axioms via an LLM prompt carefully tuned to ProverX syntax. A guiding principle was *noiselessness*: we keep primarily classes that have generated at least one paper, so that every subsequent implication or representation theorem carries mathematical weight.

The third stage was to group the classes by signature; for each signature we attempted all pairwise implications and recorded the resulting directed graph (edge $A \to B$ means axioms of A imply axioms of B).

Now we arrived to an application we were looking for and that required the introduction of all the previous machinery.

5.0.1 Strategic aim

We now seek *general classes* that contain, as special cases, algebras from multiple important classes extracted from the literature. Unlike Tao's project, our classes will not be artificial or generated randomly. The goal is to generate classes that contain algebras from other classes but in a rich algebraic way.

5.0.2 Design principles

- Controlled strengthening: add just enough structure (e.g., definable unary operations or weak distributivity laws) to render deep statements provable while retaining wide coverage.

- **R**epresentation-aware: design the general class so that known important classes appear via systematic *realization* schemes.
- **A**TP-friendly: keep axioms short and oriented for saturation-based provers; prefer identities that canonically orient under LPO/KBO to accelerate search.

5.0.3 Conceptual move

Classical containment $\mathcal{A} \subseteq \mathcal{B}$ is often too rigid. We therefore consider an *extended inclusion*: a class C *is realized inside* a class $\mathcal{T}$ if, for algebras $T \in \mathcal{T}$ equipped with a fixed family of commuting automorphisms $f, g, \ldots$ that preserve the operations of T, there are term-definitions of operations $\oplus, \ldots$ in the language of $\mathcal{T} \cup \{f^{\pm 1}, g^{\pm 1}, \ldots\}$ such that $(T, \oplus, \ldots) \in C$. Informally: *$C$ is a definable shadow of $\mathcal{T}$ modulo commuting automorphisms*.

5.0.4 A generic ATP workflow for discovering general classes

The procedure for generating candidate general classes can be summarized as follows.

Step 1. Fix a target class $\mathcal{T}$. Choose a well-studied class as the starting point. *Example:* Let $\mathcal{T}$ be the class of inverse semigroups $(S, \cdot, ')$.

Step 2. Extend the language with commuting automorphisms. Introduce two formal symbols f, g representing commuting automorphisms that preserve all primitive operations of $\mathcal{T}$.

Example: For inverse semigroups, we stipulate that f, g are bijections and that $f(xy) = f(x)f(y)$, $g(xy) = g(x)g(y)$, and $f(x') = f(x)'$, $g(x') = g(x)'$.

Step 3. Enumerate algebraic words. Systematically build algebraic terms in the extended language. These involve the original primitives and the new automorphisms (with inverses).

Example (inverse semigroups):

$$f(x)g(y), \qquad g^{-1}(f(x')y), \qquad f^{-1}(xg(y')).$$

Step 4. Define candidate operations. Interpret these terms as new primitive symbols, forming the signature of a putative general class.

Example (inverse semigroups): Define

$$x + y := f(x)g(y), \quad x\backslash y := g^{-1}(f(x')y), \quad x/y := f^{-1}(xg(y')).$$

Step 5. Run goal-free ATP. Apply ProverX in *goal-free* mode with LPO priorities (that is, putting the generated axioms defined by the new operations on the top), allowing the prover to generate consequences freely. After a fixed time budget (e.g., 60s), extract the identities it discovered that involve only the new symbols.

Example: In the inverse semigroup setting, one quickly obtains:

$$((x+y)/y)+y=x+y, \qquad x\backslash(x+(x\backslash y))=x\backslash y.$$

Step 6. Consolidate a candidate axiom set. Repeat the process until a compact and stable set of axioms emerges (about ten axioms is a good compromise between potential richness and time taken to find them).

Step 7. Test containments. Compare the newly defined class against the database of important classes.

This workflow has the advantage of being largely systematic: once the target $\mathcal{T}$ is fixed, the generation of words, the definition of candidate operations, and the extraction of axioms are all automatable, while the testing against the database ensures immediate mathematical relevance.

5.1 Classes of Algebras Created with this Process

Below are representative families distilled from this process. We chose these classes of algebras due to the fact that they had the highest length of proof when we used ProverX to prove them.

5.1.1 Generalized Inverse Semigroups

Definition 5.1.1. Given a non-empty set A, we say that A is a Generalized Inverse Semigroup (GIS) of type $(2,2,2)$ if and only if there exist three binary operations $+, \backslash, /: A\times A \to A$ such that the following axioms hold:

1. $x\backslash((y/(x\backslash y))+(x\backslash y))=x\backslash y;$
2. $((x+y)/(x\backslash(x+y)))+y=x+y.$

Theorem 5.1.2. *Generalized Inverse Semigroup of type* $(2,2,2)$ *contains the following classes of algebras: Quasigroups [[16], Definition 1] and Nelson algebras [[15], Definition 2.2].*

Proof. If $A(\cdot, \backslash_1, /_1)$ is a Quasigroup, then $A(+, \backslash_2, /_2)$ is a GIS of type $(2,2,2)$ where the operations $+, \backslash_2, /_2$ are defined as

$$x+y:=xy, \quad x\backslash_2 y:=x\backslash_1 y, \quad x/_2 y:=x/_1 y.$$

If $A(\wedge, \vee, \to, \sim, 1)$ is a Nelson algebra, then $A(+, \backslash, /)$ is a GIS of type $(2,2,2)$ where the operations $+, \backslash, /$ are defined as

$$x+y:=x\vee y, \quad x\backslash y:=x\to y, \quad x/y:=x\wedge y.$$

$\square$

Theorem 5.1.3. *Let* $(I, \cdot, ')$ *be an inverse semigroup [45, Definition 2.1], and let* f, g *be two commuting automorphisms of* I. *Then* $(I, +, /, \backslash)$ *is a GIS of type* $(2, 2, 2)$, *where the operations* $+$, $\backslash$ *and* $/$ *are defined as follows:*

1. $x + y := f(x)g(y)$;

2. $x \backslash y := g^{-1}(f(x')y)$;

3. $x/y := f^{-1}(xg(y'))$.

5.1.2 Generalized Boolean Algebras

Definition 5.1.4. Given a non-empty set A, we say that A is a GBA of type (2,1) if and only if there exist one binary operation $+ : A \times A \to A$ and one unary operation $b : A \to A$ such that the following axiom hold:

1. $b(b(b(x))) + b(x) = b(b(b(y))) + b(y)$.

Theorem 5.1.5. *GBA of type* $(2, 1)$ *contains the following classes of algebras: Nelson algebras [[15], Definition 2.2].*

Proof. If $A(\wedge, \vee, \to, \sim, 1)$ is a Nelson algebra, then $A(+, b)$ is a GBA of type $(2, 1)$ where the operations $+, b$ are defined by

$$x + y := x \to y, \quad b(x) := \sim x.$$

□

Theorem 5.1.6. *Let* $(B, \cdot, +, a, 0, 1)$ *be a Boolean Algebra [[55], Definition 1] and let* f, g *be two commuting automorphisms of* B. *Then* $(I, \vee, b)$ *is a GBA, where the operations* $\vee, b$ *are defined as follows:*

1. $x \vee y = f(f(x))a((yy))$;

2. $b(x) = a((g^{-1}(x)a(f^{-1}(x))))$.

5.1.3 Generalized Abelian Groups

Definition 5.1.7. Given a non-empty set A, we say that A is a GAG of type (2,1,1) if and only if there exist one binary operation $+ : A \times A \to A$ and two unary operations $b, c : A \to A$ such that the following axiom hold:

1. $c(x + b(x)) = c(x) + b(c(x))$.

Theorem 5.1.8. *GAG of type* $(2, 1, 1)$ *contain the following classes of algebras: Groups.*

Proof. If $A(\cdot, ', 1)$ is a Group, then $A(+, b, c)$ is a GAG of type $(2, 1, 1)$ where the operations $+, b, c$ are defined by

$$x + y := xy, \quad b(x) = c(x) = x'.$$

□

Theorem 5.1.9. *Let $(G, \cdot, +, a, 0)$ be an Abelian L-Group [[48], Definition 11] and let f, g be two commuting automorphisms of G. Then $(G, \vee, b, c)$ is a GAG, where the operations $\vee, b, c$ are defined as follows:*

1. $x \vee y = a(xx) \cdot a(g^{-1}(y))$;
2. $b(x) = f^{-1}(a(x)) \cdot g(x)$;
3. $c(x) = f(a(x)) + a(g^{-1}(x))$.

5.1.4 Generalized Distributive Monoids

Definition 5.1.10. Given a non-empty set A, we say that A is a GDM of type (2,2,1,1) if and only if there exist two binary operations $+, \cdot : A \times A \to A$ and two unary operations $b, c : A \to A$ such that the following axioms hold:

1. $c(b(x)) = (b(c(x)) \cdot b(c(x))) \cdot b(c(x))$;
2. $(x + x) \cdot ((x + x) \cdot (x + x)) = x + (x \cdot x)$.

Theorem 5.1.11. *GDM of type $(2, 2, 1, 1)$ contain the following classes of algebras: Boolean Algebras [[55], Definition 1], Weakly Orthomodular Ortholattice [[56], Definition 2.8].*

Proof. If $A(\wedge, \vee, \sim, 0, 1)$ is a Boolean algebra, then $A(\cdot, +, b, c)$ is a GDM of type $(2, 2, 1, 1)$ where the operations are defined by

$$xy := x \wedge y, \quad x + y := x \vee y, \quad b(x) = c(x) = \sim x.$$

If $A(\cup, \cap, ', 1)$ is a Weakly Orthomodular Ortholattice, then $A(\cdot, +, b, c)$ is a GDM of type $(2, 2, 1, 1)$ where the operations are defined by

$$xy := x \cap y, \quad x + y := x \cup y, \quad b(x) = c(x) = x'.$$

□

Theorem 5.1.12. *Let $(M, \cdot, +, 1)$ be a Distributive Monoid [[32], Definition 2.19] and let f, g be two commuting automorphisms of M. Then $(M, \vee, \backslash, b, c)$ is a GDM of type $(2, 2, 1, 1)$, where the operations $\vee, \backslash, b, c$ are defined as follows:*

1. $x \vee y = g^{-1}(f(y)) \cdot (x + y)$;
2. $x \backslash y = x + y$;
3. $b(x) = g^{-1}(x + x) + g^{-1}(x)$;
4. $c(x) = f^{-1}(f^{-1}(x)) \cdot (x + x)$.

5.1.5 Generalized Involutive Bisemilattice

Definition 5.1.13. Given a non-empty set A, we say that A is a GIB of type $(2,1,1)$ if and only if there exist one binary operation $\vee : A \times A \to A$ and two unary operations $b, c : A \to A$ such that the following axiom hold:

1. $c(x \vee y) = c(y) \vee c(x)$;

2. $b(b(x \vee x)) = b(b(x) \vee b(x))$.

Theorem 5.1.14. *GIB of type* $(2,1,1)$ *contain the following classes of algebras: Groups, Assemblies [[17], Definition 3.1] and Bar Inverse Monoids [[57], Definition 2.2]*

Proof. If $A(\cdot, {}', 1)$ is a Group , then $A(\vee, b, c)$ is a GIB of type $(2,1,1)$ where the operations are defined by

$$x \vee y := xy, \quad b(x) = c(x) = x'.$$

If $A(+, e, s)$ is an Assembly , then $A(\vee, b, c)$ is a GIB of type $(2,1,1)$ where the operations are defined by

$$x \vee y := x + y, \quad b(x) = e(x), \quad c(x) = s(x).$$

If $A(\cdot, {}^*, [], 0, 1)$ is a Bar Inverse Monoid , then $A(\vee, b, c)$ is a GIB of type $(2,1,1)$ where the operations are defined by

$$x \vee y := xy, \quad b(x) = c(x) = x^*.$$

□

Theorem 5.1.15. *Let* $(I, \cdot, +, a, 0, 1)$ *be an Involutive Bisemilattice [[9], Definition 2.7] and let* f, g *be two commuting automorphisms of* I. *Then* $(I, \vee, b, c)$ *is a GIB of type* $(2,1,1)$, *where the operations* $\vee, b, c$ *are defined as follows:*

1. $x \vee y = g(f^{-1}(y) + a(f^{-1}(x)))$;

2. $b(x) = g^{-1}(x + x) \cdot a(x + x)$;

3. $c(x) = f(x \cdot x) + f(a(x))$.

Definition 5.1.16. Given a non-empty set A, we say that A is a GIB of type $(2,1)$ if and only if there exist one binary operation $\vee : A \times A \to A$ and one unary operation $c : A \to A$ such that the following axiom hold:

1. $c(x \vee y) = c(x) \vee c(y)$.

Theorem 5.1.17. *GIB of type* $(2,1)$ *contain the following classes of algebras: f-Quandles [[12], Definition 2.1].*

Proof. If $A(\cdot, f)$ is an f-Quandle , then $A(\vee, c)$ is a GIB of type $(2, 1)$ where the operations are defined by

$$x \vee y := xy, \quad c(x) = f(x).$$

□

Theorem 5.1.18. *Let $(I, \cdot, +, a, 0, 1)$ be an Involutive Bisemilattice [[9], Definition 2.7] and let f, g be two commuting automorphisms of I. Then $(I, \vee, \backslash, b, c)$ is a GIB of type $(2, 1)$, where the operations $\vee, \backslash, b, c$ are defined as follows:*

1. $x \vee y = g^{-1}(a(y + x) \cdot (x + y))$;

2. $c(x) = g((a(g^{-1}(x)) \cdot (x + x)))$.

5.1.6 Generalized De Morgan Algebras

Definition 5.1.19. Given a non-empty set A, we say that A is a GMA of type $(2, 2, 1, 1)$ if and only if there exist two binary operations $\vee, \backslash : A \times A \to A$ and two unary operations $b, c : A \to A$ such that the following axiom hold:

1. $x \vee b(x) = x$;

2. $c(x) \vee x = c(x)$;

3. $c(b(x)) = b(c(x))$;

4. $x \vee (b(x)\backslash y) = x$.

Theorem 5.1.20. *Let $(M, \cdot, a, 0)$ be a De Morgan Algebra [[14], Definition 2.2] and let f, g be two commuting automorphisms of M. Then $(M, \vee, \backslash, b, c)$ is a GMA of type $(2, 2, 1, 1)$, where the operations $\vee, \backslash, b, c$ are defined as follows:*

1. $x \vee y = a((g^{-1}((y \cdot y)) \cdot a(x)))$.

2. $x\backslash y = g^{-1}((g(x) \cdot a((y \cdot x))))$.

3. $b(x) = g(a((a(x) \cdot (x \cdot x))))$.

4. $c(x) = a(f^{-1}(g^{-1}(f((x \cdot x)))))$.

Definition 5.1.21. Given a non-empty set A, we say that A is a GMA of type $(2, 1, 1)$ if and only if there exist one binary operation $\vee : A \times A \to A$ and two unary operations $b, c : A \to A$ such that the following axiom hold:

1. $b(c(x) \vee x) = b(c(x))$;

2. $c(x \vee b(x)) = c(x)$.

Theorem 5.1.22. *GMA of type* $(2,1,1)$ *contain the following classes of algebras: Assemblies [[17], Definition 3.1].*

Proof. If $A(+,e,s)$ is an Assembly , then $A(\vee,b,c)$ is a GMA of type $(2,1,1)$ where the operations are defined by

$$x \vee y := x + y, \quad b(x) = e(x), \quad c(x) = s(x).$$

□

Theorem 5.1.23. *Let* $(M,\cdot,a,0)$ *be a De Morgan Algebra [[14], Definition 2.2] and let* f,g *be two commuting automorphisms of* M. *Then* $(M,\vee,b,c)$ *is a GMA of type* $(2,1,1)$, *where the operations* $\vee,b,c$ *are defined as follows:*

1. $x \vee y = g^{-1}(g(x)\cdot a(y\cdot x))$;
2. $b(x) = g(a(a(x)\cdot(x\cdot x)))$;
3. $c(x) = a(g^{-1}(x\cdot x))$.

5.1.7 Generalized Orthomodular Ortholattice

Definition 5.1.24. Given a non-empty set A, we say that A is a GDO of type $(2,2)$ if and only if there exist two binary operations $\vee,\backslash : A\times A \to A$ such that the following axiom hold:

1. $x\backslash((x\backslash x)\vee y) = x\backslash(x\backslash x)$.

Theorem 5.1.25. *GDO of type* $(2,2)$ *contain the following classes of algebras: Weakly Orthomodular Ortholattice [[56], Definition 2.8].*

Proof. If $A(\cup,\cap,{}',1)$ is a Weakly Orthomodular Ortholattice, then $A(\backslash,\vee)$ is a GDO of type $(2,2)$ where the operations are defined by

$$x\backslash y := x\cap y, \quad x\vee y := x\cup y.$$

□

Theorem 5.1.26. *Let* $(D,\cdot,+,a)$ *be a Distributive Ortholattice [[56], Definition 3.2] and let* f,g *be two commuting automorphisms of* D. *Then* $(D,\vee,\backslash)$ *is a GDO of type* $(2,2)$, *where the operations* $\vee,\backslash$ *are defined as follows:*

1. $x\vee y = (f(g(x))\cdot g^{-1}(a(y)))$;
2. $x\backslash y = (g(f(x))\cdot a((x\cdot y)))$.

With this method we were able to generate approximately 80000 Theorems, as the reader can check in this github link: https://github.com/gg-araujo/Thesis/tree/main/80000%20Theorems, and we were able to prove that Central Groupoids (defined by one binary operation) are equivalent to structures with two binary operations. We present now those results and their corresponding proofs.

5.1.8 New Characterizations of Central Grupoids

Definition 5.1.27. We say that a non-empty set A is a Central Grupoid if there exists a binary operation $\cdot : A \times A \to A$ such that for all $x, y, z \in A$:

$$(xy)(yz) = y.$$

Definition 5.1.28. We say that a non-empty set A is a GCG of type $(2, 2)$ if there exist two binary operations $\backslash, /: A \times A \to A$ such that for all $x, y, z, w \in A$:

1. $((x/y)\backslash(y/z))/(y/w) = y$;
2. $(x/(y\backslash z))/(y/z) = y\backslash z$.

The following proof was obtained using XPlain, a program that translates ProverX proofs into LaTeX.

Theorem 5.1.29. *Let $(A, \cdot)$ be a Central Grupoid and let f, g be two commuting automorphisms of A. Then $(A, \vee, /)$ is a GCG of type $(2, 2)$ where the operations $\vee$ and $/$ are defined as*

$$x \vee y := f(g(x)) \cdot (x \cdot y), \quad x/y := xy.$$

Proof. Given the following propositions:

$$(x \cdot y) \cdot (y \cdot z) = y \tag{A1}$$

$$f1(x \cdot y) = f1(x) \cdot f1(y) \tag{A2}$$

$$g1(x \cdot y) = g1(x) \cdot g1(y) \tag{A3}$$

$$x \vee y = f1(g1(x)) \cdot (x \cdot y) \tag{A4}$$

$$x/y = x \cdot y \tag{A5}$$

we want to prove:

$$((x/y) \vee (y/z))/(y/u) = y. \tag{G}$$

The following lines are negations of goal:

$$((c_1/c_2) \vee (c_2/c_3))/(c_2/c_4) \neq c_2. \tag{N1}$$

We can now proceed to prove the above axiom

Proof.

$$\underbrace{\overline{c_2}}_{y} = \overline{(\underbrace{(f1(g1(c_1)) \cdot f1(g1(c_2)))}_{x} \cdot \underbrace{c_2}_{y}) \cdot (\underbrace{c_2}_{y} \cdot \underbrace{c_4}_{z})} \tag{5.1}$$

$$= \overline{\underbrace{((f1(g1(c_1)) \cdot f1(g1(c_2))) \cdot c_2)}_{x} \cdot \underbrace{(c_2 \cdot c_4)}_{y}} \tag{5.2}$$

$$=\overline{\underbrace{((f1(g1(c_1))\cdot f1(g1(c_2)))\cdot c_2)}_{x}/\underbrace{(c_2\cdot c_4)}_{y}} \tag{5.3}$$

$$=((f1(g1(c_1))\cdot f1(g1(c_2)))\cdot c_2)/\overline{(\underbrace{c_2}_{x}\cdot \underbrace{c_4}_{y})} \tag{5.4}$$

$$=((f1(g1(c_1))\cdot f1(g1(c_2)))\cdot c_2)/\overline{(\underbrace{c_2}_{x}/\underbrace{c_4}_{y})} \tag{5.5}$$

$$=((f1(g1(c_1))\cdot f1(g1(c_2)))\cdot \overline{\underbrace{c_2}_{y}})/(c_2/c_4) \tag{5.6}$$

$$=((f1(g1(c_1))\cdot f1(g1(c_2)))\cdot \overline{((\underbrace{c_1}_{x}\cdot \underbrace{c_2}_{y})\cdot(\underbrace{c_2}_{y}\cdot \underbrace{c_3}_{z}))})/(c_2/c_4) \tag{5.7}$$

$$=\overline{((f1(\underbrace{g1(c_1)}_{x})\cdot f1(\underbrace{g1(c_2)}_{y})))}\cdot((c_1\cdot c_2)\cdot(c_2\cdot c_3)))/(c_2/c_4) \tag{5.8}$$

$$=(\overline{f1(\underbrace{g1(c_1)}_{x}\cdot \underbrace{g1(c_2)}_{y})})\cdot((c_1\cdot c_2)\cdot(c_2\cdot c_3)))/(c_2/c_4) \tag{5.9}$$

$$=(f1(\overline{g1(\underbrace{c_1}_{x})\cdot g1(\underbrace{c_2}_{y})}))\cdot((c_1\cdot c_2)\cdot(c_2\cdot c_3)))/(c_2/c_4) \tag{5.10}$$

$$=(f1(\overline{g1(\underbrace{c_1}_{x}\cdot \underbrace{c_2}_{y})}))\cdot((c_1\cdot c_2)\cdot(c_2\cdot c_3)))/(c_2/c_4) \tag{5.11}$$

$$=\overline{(f1(g1(\underbrace{c_1\cdot c_2}_{x}))\cdot(\underbrace{(c_1\cdot c_2)}_{x}\cdot \underbrace{(c_2\cdot c_3)}_{y}))}/(c_2/c_4) \tag{5.12}$$

$$=\overline{(\underbrace{(c_1\cdot c_2)}_{x}\vee \underbrace{(c_2\cdot c_3)}_{y})}/(c_2/c_4) \tag{5.13}$$

$$=((c_1\cdot c_2)\vee \overline{(\underbrace{c_2}_{x}\cdot \underbrace{c_3}_{y})})/(c_2/c_4) \tag{5.14}$$

$$=((c_1\cdot c_2)\vee \overline{(\underbrace{c_2}_{x}/\underbrace{c_3}_{y})})/(c_2/c_4) \tag{5.15}$$

$$=(\overline{(\underbrace{c_1}_{x}\cdot \underbrace{c_2}_{y})}\vee(c_2/c_3))/(c_2/c_4) \tag{5.16}$$

$$=(\overline{(\underbrace{c_1}_{x}/\underbrace{c_2}_{y})}\vee(c_2/c_3))/(c_2/c_4) \tag{5.17}$$

$$=((c_1/c_2)\vee(c_2/c_3))/(c_2/c_4) \tag{5.18}$$

$$\neq c_2 \tag{5.19}$$

The previous line is an instance of $x \neq x$. □

Given the following propositions:

$$(x\cdot y)\cdot(y\cdot z)=y \tag{A1}$$

$$x\vee y=f1(g1(x))\cdot(x\cdot y) \tag{A2}$$

$$x/y=x\cdot y \tag{A3}$$

we want to prove:

$$(x/(y \vee z))/(y/z) = y \vee z. \tag{G}$$

The following lines are negations of goal:

$$c_2 \vee c_3 \neq (c_1/(c_2 \vee c_3))/(c_2/c_3). \tag{N1}$$

We can now proceed to prove the second axiom

Proof.

$$\overline{f1(g1(\underbrace{c_2}_{x})) \cdot (\underbrace{c_2}_{x} \cdot \underbrace{c_3}_{y})} = \overline{\underbrace{c_2}_{x} \vee \underbrace{c_3}_{y}} \tag{5.20}$$

$$= c_2 \vee c_3 \tag{5.21}$$

$$\neq (c_1/(c_2 \vee c_3))/(c_2/c_3) \tag{5.22}$$

$$= (c_1/\overline{(\underbrace{c_2}_{x} \vee \underbrace{c_3}_{y})})/(c_2/c_3) \tag{5.23}$$

$$= (c_1/\overline{(f1(g1(\underbrace{c_2}_{x})) \cdot (\underbrace{c_2}_{x} \cdot \underbrace{c_3}_{y}))})/(c_2/c_3) \tag{5.24}$$

$$= \overline{(\underbrace{c_1}_{x}/\underbrace{(f1(g1(c_2)) \cdot (c_2 \cdot c_3)))}_{y}}/(c_2/c_3) \tag{5.25}$$

$$= \overline{(\underbrace{c_1}_{x} \cdot \underbrace{(f1(g1(c_2)) \cdot (c_2 \cdot c_3)))}_{y}}/(c_2/c_3) \tag{5.26}$$

$$= (c_1 \cdot (f1(g1(c_2)) \cdot (c_2 \cdot c_3)))/\overline{(\underbrace{c_2}_{x}/\underbrace{c_3}_{y})} \tag{5.27}$$

$$= (c_1 \cdot (f1(g1(c_2)) \cdot (c_2 \cdot c_3)))/\overline{(\underbrace{c_2}_{x} \cdot \underbrace{c_3}_{y})} \tag{5.28}$$

$$= \overline{\underbrace{(c_1 \cdot (f1(g1(c_2)) \cdot (c_2 \cdot c_3)))}_{x}/\underbrace{(c_2 \cdot c_3)}_{y}} \tag{5.29}$$

$$= \overline{\underbrace{(c_1 \cdot (f1(g1(c_2)) \cdot (c_2 \cdot c_3)))}_{x} \cdot \underbrace{(c_2 \cdot c_3)}_{y}} \tag{5.30}$$

$$(x \cdot (y \cdot z)) \cdot \overline{z} = (x \cdot (y \cdot z)) \cdot \overline{((\underbrace{y}_{x} \cdot z) \cdot (z \cdot w))} = (x \cdot \underbrace{(y \cdot z)}_{y}) \cdot (\underbrace{(y \cdot z)}_{y} \cdot \varphi) = \underbrace{y \cdot z}_{y} \tag{5.31}$$

with $\varphi = z \cdot w$. Using the previous equality

$$\overline{(\underbrace{c_1}_{x} \cdot \underbrace{(f1(g1(c_2))}_{y} \cdot \underbrace{(c_2 \cdot c_3)))}_{z} \cdot \underbrace{(c_2 \cdot c_3)}_{z}} = \underbrace{f1(g1(c_2))}_{y} \cdot \underbrace{(c_2 \cdot c_3)}_{z}$$

with the previous calculations

$$\overline{(c_1 \cdot (f1(g1(c_2)) \cdot (c_2 \cdot c_3))) \cdot (c_2 \cdot c_3)} \neq f1(g1(c_2)) \cdot (c_2 \cdot c_3)$$

gives

$$F \tag{5.32}$$

□

□

Theorem 5.1.30. *Let $(A, +, \cdot)$ be a GCG of type $(2, 2)$. In this conditions we can assure that $(A, \cdot)$ is a Central Grupoid.*

Proof. Let us suppose that $(A, +, \cdot)$ is a GCG of type $(2, 2)$. In these conditions we can assure that

$$((uy) + (yz))(yw) = y; \tag{5.33}$$

$$(u(y + z))(yz) = y + z. \tag{5.34}$$

Using equation 5.33 and replacing y by xy we get that

$$((u(xy)) + ((xy)z))((xy)w) = xy,$$

and replacing u by $((xx) + (xx))$ and using 5.33 once again we get that

$$(x + ((xy)z))((xy)w) = xy. \tag{5.35}$$

Moreover, replacing y by $y(z + u)$ and u by x in 5.33 we get that

$$((x(y(z + u))) + ((y(z + u))v))((y(z + u))w) = y(z + u),$$

therefore replacing v by zu and using 5.34 we can see that

$$((x(y(z + u))) + (z + u))((y(z + u))w) = y(z + u). \tag{5.36}$$

Replacing y by $y + z$ in 5.35 we get that

$$(x + ((x(y + z))u))((x(y + z))u) = x(y + z),$$

and replacing u by yz and applying 5.34, we can assure that

$$(x + (y + z))((x(y + z))u) = x(y + z). \tag{5.37}$$

Using the previous equation with $u = yz$ and applying 5.34 we get that

$$(x + (y + z))(y + z) = x(y + z). \tag{5.38}$$

Changing the name of the variables in the previous equation, it is clear that

$$(y + (z + u))(z + u) = y(z + u),$$

and replacing y by $(x(y(z + u)))$ we obtain

$$((x(y(z + u))) + (z + u))(z + u) = (x(y(z + u)))(z + u)$$

which implies, applying both previous equations, that

$$y(z + u) = (x(y(z + u)))(z + u). \tag{5.39}$$

Replacing y by $y(z+u)$ and z by $z+u$ in 5.35 we can assure that

$$(x+((x(y(z+u)))(z+u)))((x(y(z+u)))w) = x(y(z+u)),$$

and applying 5.39 we get that

$$(x+(y(z+u)))((x(y(z+u)))w) = x(y(z+u)). \tag{5.40}$$

Using equation 5.40 with $w = z+u$ and using 5.39, we can guarantee that

$$(x+(y(z+u)))(y(z+u)) = x(y(z+u)). \tag{5.41}$$

Using 5.33 with $w = z+u$, we get that

$$((xy)+(yv))(y(z+u)) = y,$$

and then using $v = z+u$ it is clear that

$$((xy)+(y(z+u)))(y(z+u)) = y,$$

which implies using 5.41 that

$$(xy)(y(z+u)) = y. \tag{5.42}$$

By equation 5.35 we know that

$$(x+((xy)w))((xy)z) = xy,$$

and replacing w by $y(z+u)$ and using 5.42 we can assure that

$$(x+y)((xy)z) = xy. \tag{5.43}$$

Using $z = y(z+u)$ on the previous equation and considering 5.42, we get that

$$(x+y)y = xy. \tag{5.44}$$

Using 5.33 with $w = z$ and $u = x$, it is clear that

$$((xy)+(yz))(yz) = y,$$

therefore applying the previous equation and 5.44, we conclude that

$$(xy)(yz) = y$$

as we wanted to prove. □

Definition 5.1.31. We say that a non-empty set A is a $GCG2$ of type $(2,2)$ if there exist two binary operations $\vee, \backslash : A \times A \longrightarrow A$ such that for all $x, y, z \in A$:

1. $(x\backslash y)\backslash(y \vee z) = y$;

2. $x\backslash((x\backslash y)\backslash z) = x\backslash y$.

In the following two results we will see a different way of characterizing Central Grupoids.

Theorem 5.1.32. *Let $(A, \cdot)$ be a Central Grupoid. Then $(A, /, \vee)$ is a $GCG2$ of type $(2, 2)$ where the operations $/$ and $\vee$ are defined as*

$$x/y := yx, \quad x \vee y := (xy)x.$$

Proof. Given the following propositions:

$$(x \cdot y) \cdot (y \cdot z) = y \tag{A1}$$

$$x/y = y \cdot x \tag{A2}$$

$$x \vee y = (x \cdot y) \cdot x \tag{A3}$$

we want to prove:

$$(x/y)/(y \vee z) = y. \tag{G}$$

The following lines are negations of goal:

$$(c_1/c_2)/(c_2 \vee c_3) \neq c_2. \tag{N1}$$

We can now proceed to prove the first axiom:

Proof.

$$\overline{\underbrace{c_2}_{y}}=\overline{(\underbrace{(c_2 \cdot c_3)}_{x} \cdot \underbrace{c_2}_{y}) \cdot (\underbrace{c_2}_{y} \cdot \underbrace{c_1}_{z})} \tag{5.45}$$

$$= \overline{\underbrace{((c_2 \cdot c_3) \cdot c_2)}_{y} \cdot \underbrace{(c_2 \cdot c_1)}_{x}} \tag{5.46}$$

$$=\overline{\underbrace{(c_2 \cdot c_1)}_{x}/\underbrace{((c_2 \cdot c_3) \cdot c_2)}_{y}} \tag{5.47}$$

$$= (c_2 \cdot c_1)/\overline{((\underbrace{c_2}_{x} \cdot \underbrace{c_3}_{y}) \cdot \underbrace{c_2}_{x})} \tag{5.48}$$

$$=(c_2 \cdot c_1)/\overline{(\underbrace{c_2}_{x} \vee \underbrace{c_3}_{y})} \tag{5.49}$$

$$= \overline{(\underbrace{c_2}_{y} \cdot \underbrace{c_1}_{x})}/(c_2 \vee c_3) \tag{5.50}$$

$$=\overline{(\underbrace{c_1}_{x}/\underbrace{c_2}_{y})}/(c_2 \vee c_3) \tag{5.51}$$

$$= (c_1/c_2)/(c_2 \vee c_3) \tag{5.52}$$

$$\neq c_2 \tag{5.53}$$

The previous line is an instance of $x \neq x$. $\square$

Given the following propositions:

$$(x \cdot y) \cdot (y \cdot z) = y \tag{A1}$$

$$x/y = y \cdot x \tag{A2}$$

we want to prove:

$$x/((x/y)/z) = x/y. \tag{G}$$

The following lines are negations of goal:

$$c_1/((c_1/c_2)/c_3) \neq c_1/c_2. \tag{N1}$$

We can now proceed to prove the second axiom:

Proof.

$$\overline{\underbrace{(c_3 \cdot (c_2 \cdot c_1))}_{y} \cdot \underbrace{c_1}_{x}} = \overline{\underbrace{c_1}_{x}/\underbrace{(c_3 \cdot (c_2 \cdot c_1))}_{y}} \tag{5.54}$$

$$= c_1/\overline{(\underbrace{c_3}_{y} \cdot \underbrace{(c_2 \cdot c_1)}_{x})} \tag{5.55}$$

$$= c_1/\overline{(\underbrace{(c_2 \cdot c_1)}_{x}/\underbrace{c_3}_{y})} \tag{5.56}$$

$$= c_1/(\overline{(\underbrace{c_2}_{y} \cdot \underbrace{c_1}_{x})}/c_3) \tag{5.57}$$

$$= c_1/(\overline{(\underbrace{c_1}_{x}/\underbrace{c_2}_{y})}/c_3) \tag{5.58}$$

$$= c_1/((c_1/c_2)/c_3) \tag{5.59}$$

$$\neq c_1/c_2 \tag{5.60}$$

$$= \overline{\underbrace{c_1}_{x}/\underbrace{c_2}_{y}} \tag{5.61}$$

$$= \overline{\underbrace{c_2}_{y} \cdot \underbrace{c_1}_{x}} \tag{5.62}$$

$$(x \cdot (y \cdot z)) \cdot \overline{z} = (x \cdot (y \cdot z)) \cdot \overline{((\underbrace{y}_{x} \cdot z) \cdot (z \cdot w))} = (x \cdot (\underbrace{y \cdot z}_{y})) \cdot ((\underbrace{y \cdot z}_{y}) \cdot \varphi) = \underbrace{y \cdot z}_{y} \tag{5.63}$$

where $\varphi = z \cdot w$. Using the previous calculations we get

$$\overline{(\underbrace{c_3}_{x} \cdot (\underbrace{c_2}_{y} \cdot \underbrace{c_1}_{z})) \cdot \underbrace{c_1}_{z} = \underbrace{c_2}_{y} \cdot \underbrace{c_1}_{z}}$$

and also

$$\overline{(c_3 \cdot (c_2 \cdot c_1)) \cdot c_1 \neq c_2 \cdot c_1}$$

gives

$$F \tag{5.64}$$

□

□

Theorem 5.1.33. *Let* $(A, /, \vee)$ *be a* $GCG2$ *of type* $(2, 2)$. *In this conditions we can assure that* $(A, /)$ *is a Central Grupoid.*

Proof. Given the following propositions:

$$(x/y)/(y \vee z) = y \tag{A1}$$

$$x/((x/y)/z) = x/y \tag{A2}$$

we want to prove:

$$(x/y)/(y/z) = y. \tag{G}$$

The following lines are negations of our goal:

$$(c_1/c_2)/(c_2/c_3) \neq c_2. \tag{N1}$$

We can now proceed to prove the theorem:

Proof.

$$(x/y)/(\underbrace{\overline{y}}_{y}/z) = (x/y)/(\overline{((x/\underbrace{y}_{y})/(\underbrace{y}_{y} \vee \underbrace{u}_{z}))}/z) \tag{5.65}$$

$$= \underbrace{(x/y)}_{x}/(((\underbrace{x/y)}_{x}/\underbrace{(y \vee u)}_{y})/z) \tag{5.66}$$

$$= \underbrace{(x/y)}_{x}/\underbrace{(y \vee u)}_{y} \tag{5.67}$$

$$= \overline{(x/\underbrace{y}_{y})/(\underbrace{y}_{y} \vee \underbrace{u}_{z})} \tag{5.68}$$

$$= \underbrace{\overline{y}}_{y} \tag{5.69}$$

Using the previous calculations we get

$$\overline{(\underbrace{c_1}_{x}/\underbrace{c_2}_{y})/(\underbrace{c_2}_{y}/\underbrace{c_3}_{z}) = \underbrace{c_2}_{y}}$$

and also

$$\overline{(c_1/c_2)/(c_2/c_3) \neq c_2}$$

gives

$$F \tag{5.70}$$

□

□

We finish this chapter by proving three more properties regarding Central Groupoids.

5.1.9 More Properties of Central Groupoids

Theorem 5.1.34. *If A is a Central Grupoid, then for all $x, y \in A$ we have*

$$(xy = yx) \Rightarrow x = y.$$

Proof. Since A is a Central Grupoid, it follows by definition that

$$(xy)(yz) = y$$

for all $x, y, z \in A$. Replacing z by x we get that $(xy)(yx) = y$. Consequently,

$$y = (xy)(yx) = (yx)(xy) = x$$

as we wanted to prove. □

Theorem 5.1.35. *Central Groupoids are contained in Rectangular Groupoids.*

Proof. Consider $a, b, c, d \in A$ such that $ab = cd$ and let us suppose that the identity

$$(wy)(yz) = y$$

holds for all $w, y, z \in A$. Then, replacing y by xy and w by xx, we get that

$$((xx)(xy))((xy)z) = xy \Rightarrow x((xy)z) = xy. \tag{5.71}$$

Similarly, since $(xy)(yw) = y$, it follows that, substituting y by yz and w by zz,

$$(x(yz))((yz)(zz)) = yz \Rightarrow (x(yz))z = yz. \tag{5.72}$$

Moreover, we can see that for each $x \in A$,

$$(ab)(dx) = (cd)(dx) = d, \quad (xc)(ab) = (xc)(cd) = c.$$

Using equation 5.71 with $x = a, y = b, z = dx$ and equation 5.72 with $y = a, z = b, x = xc$, we conclude that

$$a((ab)(dx)) = ab \Rightarrow ad = ab, \quad ((xc)(ab))b = ab \Rightarrow cb = ab$$

as we wanted to prove.

□

Theorem 5.1.36 ([40], Theorem 5)**.** *Central Grupoids contain the class of algebras defined by the identity:*

$$[x((yz)w)] \cdot (zw) = z. \tag{5.73}$$

Proof. Suppose that for all $x, y, z, w \in A$ the equality

$$[x((yz)w)] \cdot (zw) = z.$$

holds. In these conditions, replacing the variable x by $x((w(yz))w)$, we get that

$$[(x((w(yz))w))((yz)w)] \cdot (zw) = z.$$

On the other hand, using equality 5.73 and replacing z by yz and also y by w, we get that

$$[x((w(yz))w)] \cdot ((yz)w) = yz.$$

Therefore, we conclude that

$$(yz)(zw) = [(x((w(yz))w))((yz)w)] \cdot (zw) = z$$

as we wanted to prove. □

5.2 Opening Opportunities and Concluding Remarks

The guiding goal of this thesis has been the following: *to produce material that allows to find/identify very general theories which are, at the same time, tractable.* Our strategy rested on three pillars: (i) to construct a *clean database* of algebraic classes that appear in the literature (hence mathematically relevant), (ii) to build the *poset* of inclusions between these classes (by signature and, when needed, with extended notions of inclusion), and (iii) to train and integrate human–machine workflows for deciding massive numbers of questions (proofs and refutations) with speed and auditability.

To complete this goal (general theories with deep results) it is now necessary to *extract theorems in bulk* (as we did for algebraic classes, an entire auto-formalization project in itself) and *test*, on a large scale, those results against the most general classes in our database to see which of them allow the generalization of the largest number of such results.

This project should build upon the present one, but should require a comparable amount of work: reusing the database of classes, the poset, and the automated theorem proving (ATP) pipelines, but shifting the focus to *meta-theorems* at the level of general classes.

Historically, groups served as a paradigm: enormous depth but limited generality. Semigroups, much more general, were for decades thought to admit only "universal algebra level" results (isomorphism theorems, homomorphism theorems, etc.). The work of Green [27] and Rees [61] changed this picture: *it is possible to prove deep theorems even in very general structures.* Our program aimed at finding classes even more general, but still with "anchor theorems". The preliminary evidence is promising, but more work is needed—particularly in the "slow" layer of the workflow and in the systematic engineering of special terms that unlock proofs. When the entire project will be completed, then it will be possible (hopefully) to emulate for theories more general than semigroups, the success story that the last 100 years saw in this class.

6

Using the Database to Produce Results with a Known Flavor (Conclusion)

As we saw, our work and methods provided hundreds/thousands of theorems. However, as we explain, several paths can be taken using our database in order to produce even more interesting results. For a future work, one could:

(i) Identities yielding solutions to the Yang–Baxter equation

It is known that certain classes (racks, quandles, braces) yield solutions to the Yang–Baxter equation (YBE) [18, 19]. The pipeline is:

1. Fix families of identities Σ that imply YBE (e.g., quandle axioms, or the additive/multiplicative mixture of left braces [63]).
2. Scan the database for classes that *imply* Σ (directly or via realization).
3. For each candidate class, export constructors for YBE solutions and validate with ATP/model finders.

(ii) Natural orders: from semigroups to other classes

There are many different natural orders in semigroups (or in some subclasses).
One example is the **Jones (natural) order** in *regular semigroups*:

$$a \leq b \quad \Longleftrightarrow \quad \exists\, e, f \in E(S) \text{ idempotents with } a = eb = bf.$$

The natural orders are very important in the theory of semigroups (in inverse, regular, etc), but they seem overlooked in other classes of algebras where the same definition might yield an order as well.
Therefore, we designed a scanning procedure as follows:

1. Formalize the order as a first-order predicate with idempotent parameters.

2. Search in classes with sufficient idempotents (or projectors) to replicate the construction.

3. Test reflexivity, antisymmetry, transitivity, and compatibility with ATP and small-model generators.

A particular illustration of this idea appears in Appendix B.

(iii) Mal'cev terms and congruences

A **Mal'cev term** is a ternary term $p(x, y, z)$ satisfying

$$p(x, x, y) = y, \quad p(x, y, y) = x.$$

A variety is *congruence permutable* if and only if it admits a Mal'cev term [44].
Examples:

- In groups, $p(x, y, z) = xy^{-1}z$ is Mal'cev.
- In modules and affine structures, $p(x, y, z) = x - y + z$ works.

As above, one could go through our entire database trying to find classes of algebras that are congruence permutable. Note that this project requires the extra difficulty of generating counteless candidates to be a Malcev term, and then to check if it is or it is not. To see more information regarding this subject, please verify Appendix D.

(iv) Creation of GAP packages using our library

Our project could dramatically extend the classical GAP libraries such as `SmallGroups`, `SmallSemi`, and `LOOPS`. The new packages can be constructed directly from our *library of important classes of algebras* and its associated poset of inclusions.
The workflow is as follows:

1. Start with a class of algebras extracted from the literature, defined in first-order logic and maximal in the global poset of inclusions, say class C.

2. Generate all small models of the class, typically up to some order bound (for instance, all models of size ≤ 8 or ≤ 10), using a finite model builder.

3. Package these models as a GAP library, complete with identifiers and metadata, in the same style as `SmallGroups` or `SmallSemi`.

4. Add commands that exploit the poset structure: for each subclass $\mathcal{D}$ that in the poset lies below C, the package includes functions to retrieve *all models of* $\mathcal{D}$.

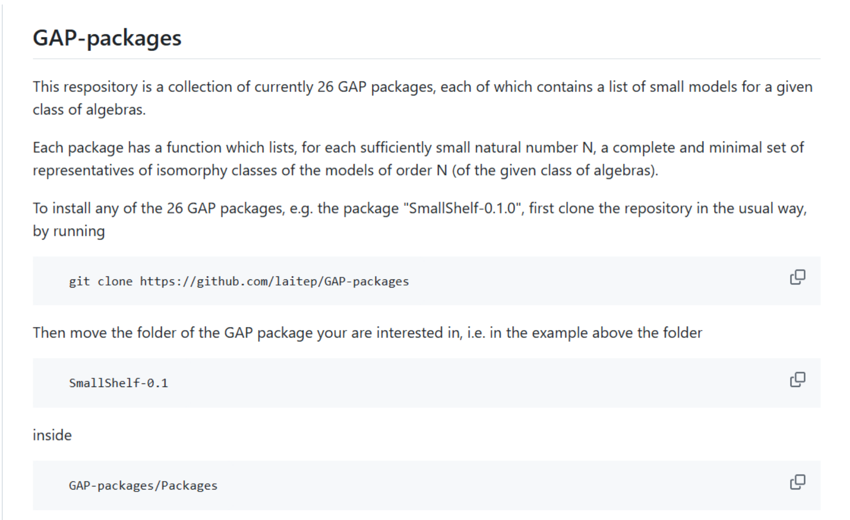
The result are *dozens of GAP packages*, each corresponding to a mathematically significant class of algebras, each with its own small-model library. This approach achieves a unique integration:

- The abstract database (axioms and inclusions) and the concrete small-model libraries are kept synchronized.

- The packages inherit the same API conventions, so that once a user knows how to query one package, they can query them all.

- The inclusion poset is built into the access commands: researchers can traverse upwards or downwards in the hierarchy to explore neighbouring classes automatically.

The following images show the instructions and one of those GAP packages that we created, specifically for the class of algebras Shelf defined by the identity $(x*y)*z = (x*z)*(y*z)$. The reader can also check these packages in the github link: https://github.com/laitep/GAP-packages.



GAP-packages

This respository is a collection of currently 26 GAP packages, each of which contains a list of small models for a given class of algebras.

Each package has a function which lists, for each sufficiently small natural number N, a complete and minimal set of representatives of isomorphy classes of the models of order N (of the given class of algebras).

To install any of the 26 GAP packages, e.g. the package "SmallShelf-0.1.0", first clone the repository in the usual way, by running

```
git clone https://github.com/laitep/GAP-packages
```

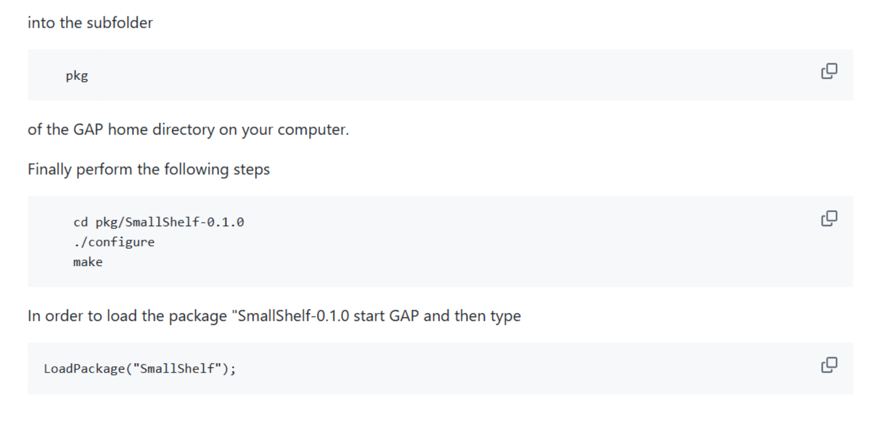
Then move the folder of the GAP package your are interested in, i.e. in the example above the folder

```
SmallShelf-0.1
```

inside

```
GAP-packages/Packages
```

Figure 6.1: Instructions to install the GAP packages (1)



into the subfolder

```
pkg
```

of the GAP home directory on your computer.

Finally perform the following steps

```
cd pkg/SmallShelf-0.1.0
./configure
make
```

In order to load the package "SmallShelf-0.1.0 start GAP and then type

```
LoadPackage("SmallShelf");
```

Figure 6.2: Instructions to install the GAP packages (2)

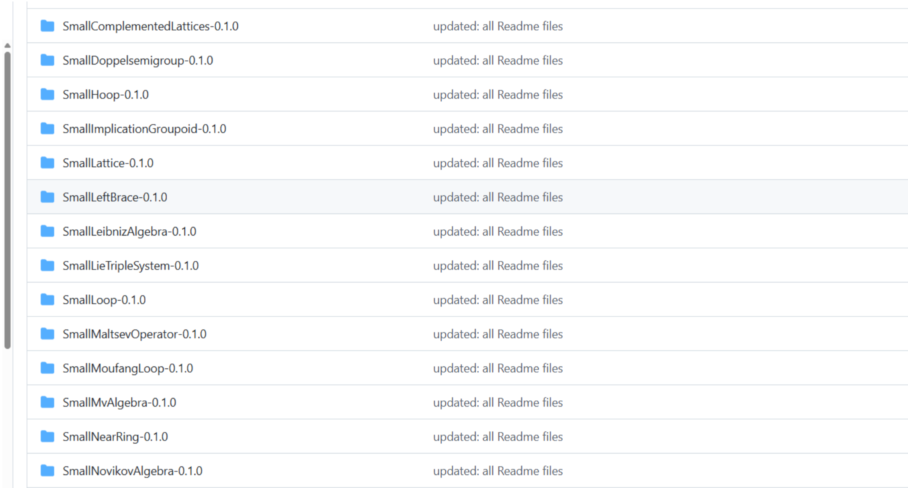


Figure 6.3: List of some of the GAP packages that were created

Figure 6.4: Loading the package SmallShelf

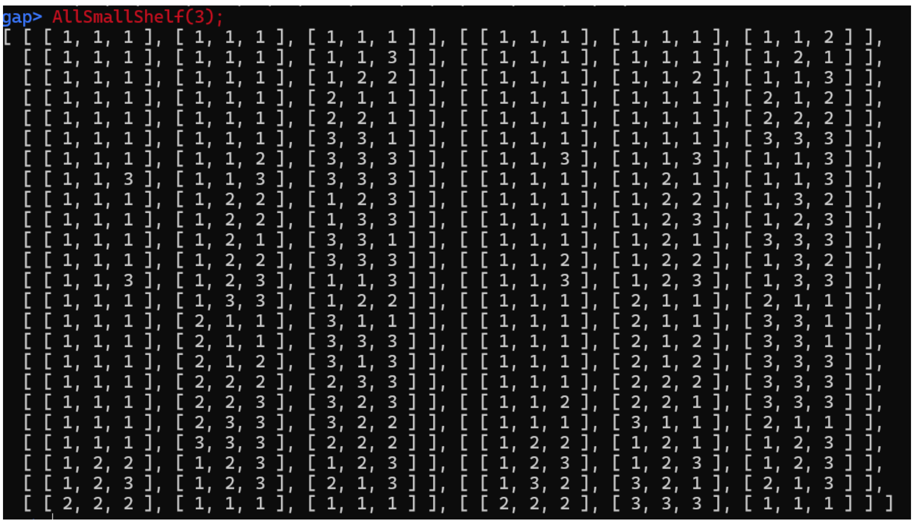


Figure 6.5: GAP package that shows all the non-isomorphic models for the class of algebras Shelf of order 3.

To see more information regarding this topic, see Appendix C.

Bibliography

[1] URL: https://www.proverx.dm.fct.unl.pt/login.php (cit. on pp. 6, 15).

[2] J. Araújo, M. Kinyon, and J. Konieczny. *Conjugacy in inverse semigroups*. Vol. 533. 2019, pp. 142–173 (cit. on p. 6).

[3] J. Araújo, M. Kinyon, and Y. Robert. "Varieties of regular semigroups with uniquely defined inversion". In: *Semigroup Forum* 76 (2024), pp. 205–228 (cit. on p. 17).

[4] *Archive of Formal Proofs (AFP)*. refereed repository for Isabelle/HOL, founded 2004 (cit. on p. 10).

[5] J. Avigad et al. *Theorem Proving in Lean 4*. online book (cit. on p. 10).

[6] M. Bancerek, C. Byliński, A. Grabowski, et al. "The role of the Mizar Mathematical Library for interactive proof development in Mizar". In: *J. Autom. Reason.* 61 (2018), pp. 9–32 (cit. on p. 9).

[7] K. Bansal et al. "HOList: an environment for machine learning of higher-order theorem proving". In: *ICML Proc.* Vol. 97. 2019, pp. 454–463 (cit. on p. 13).

[8] G. Birkhoff. *On the structure of abstract algebras*. Trans. Amer. Math. Soc. 57, 1935, pp. 228–255 (cit. on p. 4).

[9] S. Bonzio, A. Loi, and L. Peruzzi. *A duality for involutive bisemilattices*. 2017. arXiv: 1611.09560 [math.LO]. URL: https://arxiv.org/abs/1611.09560 (cit. on pp. 31, 32).

[10] N. G. de Bruijn. *Automath: a language for mathematics*. Tech. rep. TH-Report 68-WSK-05. Technische Hogeschool Eindhoven, 1968 (cit. on p. 9).

[11] K. Buzzard. *The Fermat's Last Theorem Project*. 2024. URL: https://leanprover-community.github.io/blog/posts/FLT-announcement/ (cit. on p. 12).

[12] I. R. Churchill, M. Elhamdadi, and N. Fernando. *The Cocycle structure of the Alexander f-quandles on finite fields*. 2017. arXiv: 1612.08968 [math.GT]. URL: https://arxiv.org/abs/1612.08968 (cit. on p. 31).

[13] T. Coquand and G. Huet. "The Calculus of Constructions". In: *Information and Computation* 76 (1988), pp. 95–120 (cit. on p. 10).

[14] J. M. Cornejo and H. P. Sankappanavar. *Semisimple Varieties of Implication Zroupoids*. 2015. arXiv: `1509.08502 [math.LO]`. URL: `https://arxiv.org/abs/1509.08502` (cit. on pp. 32, 33).

[15] J. M. Cornejo, A. Gallardo, and I. Viglizzo. "A categorial equivalence for semi-Nelson algebras". In: *Soft Computing* 25.22 (2021). ISSN: 1433-7479. DOI: `10.1007/s00500-021-06198-y`. URL: `http://dx.doi.org/10.1007/s00500-021-06198-y` (cit. on pp. 28, 29).

[16] N. N. Didurik and V. A. Shcherbacov. *Some properties of Neumann quasigroups*. 2018. URL: `https://arxiv.org/abs/1809.07095` (cit. on p. 28).

[17] B. Dinis and I. van den Berg. "On the quotient class of non-archimedean fields". In: *Indagationes Mathematicae* 28.4 (2017-08), pp. 784–795. ISSN: 0019-3577. DOI: `10.1016/j.indag.2017.05.001`. URL: `http://dx.doi.org/10.1016/j.indag.2017.05.001` (cit. on pp. 31, 33).

[18] V. Drinfeld. "On some unsolved problems in quantum group theory". In: *Quantum Groups*. Vol. 1510. Lecture Notes in Mathematics. Springer, 1992, pp. 1–8 (cit. on p. 44).

[19] P. Etingof, T. Schedler, and A. Soloviev. "Set-theoretical solutions to the quantum Yang–Baxter equation". In: *Duke Mathematical Journal* 100 (1999), pp. 169–209 (cit. on p. 44).

[20] *Formalized Mathematics*. journal site and archive, established 1990 (cit. on p. 10).

[21] *Formalizing Fermat's Last Theorem in Lean: A Landmark Mathematical Project*. 2024. URL: `https://lean-lang.org/use-cases/flt/` (cit. on p. 12).

[22] G. Frege. *Begriffsschrift: eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*. Halle, 1879 (cit. on p. 9).

[23] H. Geuvers and R. Nederpelt. "Characteristics of de Bruijn's early proof checker Automath". In: *Fundam. Inform.* 185 (2022), pp. 313–336 (cit. on p. 9).

[24] K. Gödel. "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I". In: *Monatsh. Math. Phys.* 38 (1931), pp. 173–198 (cit. on p. 9).

[25] G. Gonthier et al. "A machine-checked proof of the Odd Order Theorem". In: *ITP 2013*. Vol. 7998. LNCS. 2013 (cit. on p. 11).

[26] G. Gonthier. "Formal proof — the Four-Color Theorem". In: *Notices Amer. Math. Soc.* 55 (2008), pp. 1382–1393 (cit. on p. 10).

[27] J. A. Green. "On the structure of semigroups". In: *Annals of Mathematics (2)* 54 (1951), pp. 163–172 (cit. on pp. 1, 43).

[28] T. C. Hales et al. "A formal proof of the Kepler Conjecture". In: *Forum Math. Pi* (2017). arXiv:1501.02155 (2015) (cit. on p. 11).

[29] P. Hall. *Verbal and marginal subgroups*. 1940, pp. 156–157 (cit. on pp. 5, 6).

[30] P. M. Higgins. *Techniques of Semigroup Theory*. Oxford University Press, 1992 (cit. on p. 6).

[31] D. Hilbert and P. Bernays. *Grundlagen der Mathematik*. Vol. I. Berlin: Springer, 1934 (cit. on p. 9).

[32] J. Hirschorn. *Partial order embeddings with convex range*. 2007. arXiv: math/0701486 [math.RA]. URL: https://arxiv.org/abs/math/0701486 (cit. on p. 30).

[33] J. M. Howie. *Fundamentals of Semigroup Theory*. Oxford University Press, 1995 (cit. on p. 6).

[34] ImperialCollegeLondon. *FLT: Ongoing Lean formalisation of Fermat's Last Theorem*. 2024–. 2024. URL: https://github.com/ImperialCollegeLondon/FLT (cit. on p. 12).

[35] G. Irving et al. "DeepMath: deep sequence models for premise selection". In: *NeurIPS*. 2016, pp. 2235–2243 (cit. on p. 13).

[36] *Journal of Formalized Mathematics*. electronic portal and archive (cit. on p. 10).

[37] C. Kaliszyk and J. Urban. "HOL(y)Hammer: Online ATP service for HOL Light". In: *Math. Comput. Sci.* 9 (2015), pp. 5–22 (cit. on pp. 12, 13).

[38] C. Kaliszyk and J. Urban. "Learning-assisted automated reasoning with Flyspeck". In: *J. Autom. Reason.* 53 (2014), pp. 173–213 (cit. on pp. 12, 13).

[39] M. Kaufmann, P. Manolios, and J. S. Moore, eds. *Computer-Aided Reasoning: ACL2 Case Studies*. Kluwer, 2000 (cit. on p. 10).

[40] D. E. Knuth. *Notes on Central Groupoids*. 1970 (cit. on p. 42).

[41] A. G. Kurosh. *The Theory of Groups*. 1960 (cit. on p. 5).

[42] M. V. Lawson. *Inverse Semigroups: The Theory of Partial Symmetries*. World Scientific, 1998 (cit. on p. 6).

[43] W. Li et al. "IsarStep: a benchmark for high-level mathematical reasoning". In: *ICLR*. arXiv:2006.09265 (2020). 2021 (cit. on p. 13).

[44] A. I. Mal'cev. "On the general theory of algebraic systems". In: *Matematicheskii Sbornik* 35 (1954), pp. 3–20 (cit. on p. 45).

[45] M. E. Malandro. "Fast Fourier transforms for finite inverse semigroups". In: *Journal of Algebra* 324.2 (2010). ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2009.11.031. URL: http://dx.doi.org/10.1016/j.jalgebra.2009.11.031 (cit. on p. 29).

[46] W. McCune. *Solution of the Robbins problem*. 1997, pp. 263–276 (cit. on p. 4).

[47] N. Megill and D. W. Huffman. *Metamath: A Computer Language for Pure Mathematics*. 2020 edition (open access). 2020 (cit. on p. 10).

[48] G. Metcalfe, N. Olivetti, and D. Gabbay. *Sequent and Hypersequent Calculi for Abelian and Lukasiewicz Logics*. 2002. arXiv: cs/0211021 [cs.LO]. URL: https://arxiv.org/abs/cs/0211021 (cit. on p. 30).

[49] L. de Moura and S. Ullrich. "The Lean 4 theorem prover and programming language (system description)". In: (2021) (cit. on p. 10).

[50] H. Neumann. *Varieties of Groups*. 1967 (cit. on p. 5).

[51] T. Nipkow, L. C. Paulson, and M. Wenzel. *Isabelle/HOL: A Proof Assistant for Higher-Order Logic*. Vol. 2283. LNCS. Springer, 2002 (cit. on p. 10).

[52] U. Norell. "Towards a practical programming language based on dependent type theory". PhD thesis. Chalmers University of Technology, 2007 (cit. on p. 10).

[53] OpenAI. *Solving (some) formal math olympiad problems*. 2022. URL: https://openai.com/index/formal-math/ (cit. on p. 13).

[54] S. Owre, J. M. Rushby, and N. Shankar. "PVS: A Prototype Verification System". In: *CADE–11*. 1992 (cit. on p. 10).

[55] N. Papanikolaou. *Logic Column 13: Reasoning Formally about Quantum Systems: An Overview*. 2005. URL: https://arxiv.org/abs/cs/0508005 (cit. on pp. 29, 30).

[56] M. Pavicic and N. D. Megill. *Standard Logics Are Valuation-Nonmonotonic*. 2008. arXiv: 0812.2702 [cs.LO]. URL: https://arxiv.org/abs/0812.2702 (cit. on pp. 30, 33).

[57] M. Pedicini and F. Quaglia. *PELCR: Parallel Environment for Optimal Lambda-Calculus Reduction*. 2004. arXiv: cs/0407055 [cs.LO]. URL: https://arxiv.org/abs/cs/0407055 (cit. on p. 31).

[58] M. Petrich. *Inverse Semigroups*. Wiley, 1984 (cit. on p. 6).

[59] S. Polu and I. Sutskever. *Generative language modeling for automated theorem proving*. 2020 (cit. on p. 13).

[60] S. Polu et al. "Formal mathematics statement curriculum learning". In: *ICML*. 2022 (cit. on p. 13).

[61] D. Rees. "On semigroups". In: *Proceedings of the Cambridge Philosophical Society* 36 (1940), pp. 387–400 (cit. on pp. 1, 43).

[62] D. J. S. Robinson. *A Course in the Theory of Groups*. 1996 (cit. on p. 5).

[63] W. Rump. "Braces, radical rings, and the quantum Yang–Baxter equation". In: *Journal of Algebra* 307 (2007), pp. 153–170 (cit. on p. 44).

[64] C. Szegedy. *Short note on current prospects for automated mathematical reasoning*. 2025 (cit. on p. 14).

[65] T. Tao. "A pilot project in universal algebra to explore new ways to collaborate and use machine assistance". In: (2024). URL: https://terrytao.wordpress.com/2024/09/25/a-pilot-project-in-universal-algebra-to-explore-new-ways-to-collaborate-and-use-machine-assistance/ (cit. on p. 18).

[66] T. Tao. "The Equational Theories Project: A Brief Tour". In: (2024). URL: https://terrytao.wordpress.com/2024/10/12/the-equational-theories-project-a-brief-tour/ (cit. on p. 18).

[67] The Coq Development Team. *The Coq Proof Assistant Reference Manual*. various editions (cit. on p. 10).

[68] J. Urban. "MaLARea: a metasystem for automated reasoning in large theories". In: *CEUR Workshop Proc.* Vol. 257. 2007, pp. 17–20 (cit. on pp. 12, 13).

[69] J. Urban. "MPTP 0.2: Design, implementation, and initial experiments". In: *J. Autom. Reason.* 37 (2006), pp. 21–43 (cit. on pp. 12, 13).

[70] Q. Wang et al. *Exploration of neural machine translation in autoformalization of mathematics in Mizar*. 2019 (cit. on pp. 13, 14).

[71] A. N. Whitehead and B. Russell. *Principia Mathematica*. Vol. I. Cambridge: Cambridge University Press, 1910 (cit. on p. 9).

[72] K. Yang et al. "LeanDojo: theorem proving with retrieval-augmented language models". In: *NeurIPS Datasets & Benchmarks*. 2023 (cit. on p. 13).

[73] K. Zheng, J. M. Han, and S. Polu. "miniF2F: a cross-system benchmark for formal Olympiad-level mathematics". In: *ICLR*. arXiv:2109.00110 (2021). 2022 (cit. on p. 13).

A

Research Article on Marginal Subsemigroups and Commutators in Inverse Semigroups

Semigroup Forum
https://doi.org/10.1007/s00233-025-10548-9

RESEARCH ARTICLE

Marginal subsemigroups and commutators in inverse semigroups

Gonçalo Araújo[1] · João Araújo[1] · Michael Kinyon[2]

Received: 21 January 2025 / Accepted: 29 May 2025

Abstract
Marginal subgroups, introduced by P. Hall, are characteristic subgroups induced by group words. The goal of this paper is to extend the notion to inverse semigroups. Our first main result establishes that these marginal subsemigroups are full inverse subsemigroups. We then examine the special case in which the word is the commutator, showing that the induced marginal inverse subsemigroup coincides with the metacenter, which is a normal inverse subsemigroup. In the process we prove some results about commutators in inverse semigroups and in Clifford semigroups. The paper concludes with several open problems.

Keywords Marginal subgroups · Inverse semigroups · Clifford semigroups · Commutators

1 Introduction

Let $\omega(x_1, \ldots, x_n)$ be a group word. An element a of a group G is said to be *marginal for* ω if, for all $i \in \{1, \ldots, n\}$ and for all $x_i \in G$,

$$\omega(x_1, \ldots, ax_i, \ldots, x_n) = \omega(x_1, \ldots, x_i, \ldots, x_n) = \omega(x_1, \ldots, x_i a, \ldots, x_n)\ . \quad (1)$$

This paper is dedicated to Professor John Meakin

Communicated by Mikhail Volkov.

✉ João Araújo
jj.araujo@fct.unl.pt

Gonçalo Araújo
gg.araujo@campus.fct.unl.pt

Michael Kinyon
michael.kinyon@du.edu

1 Department of Mathematics, Faculdade de Ciências e Tecnologia, Universidade NOVA de Lisboa, 2829-516 Caparica, Portugal

2 Department of Mathematics, University of Denver, 2390 S York St, Denver, CO 80210, USA

Published online: 07 July 2025

The set $M_\omega(G)$ of all marginal elements of G is a characteristic, hence normal, subgroup of G, called the *marginal subgroup* for ω. Marginal subgroups were introduced by P. Hall in [3]. Following Hall (see, e.g., [13, p. 57]), marginal subgroups are sometimes defined asymmetrically by, say, by the set of the second equations in (1), however these definitions turn out to be equivalent.

An important example of a marginal subgroup of a group G is its *center* $Z(G) = \{a \in G : ax = xa, \forall x \in G\}$, which turns out to be the marginal subgroup for the *commutator* $[x, y] = x^{-1}y^{-1}xy$. One could, in fact, argue that the whole general theory of marginal subgroups is modeled on commutators and centers.

The only use of the notion of marginality in semigroup theory of which we are aware is the work of Moravec on what he called marginal completely regular semigroups [11]. What we do in this paper is in a different direction from Moravec's very interesting work.

In this paper we focus on inverse semigroups. Recall that a semigroup S is said to be an *inverse semigroup* if, for each $x \in S$, there exists a unique inverse $x^{-1} \in S$ satisfying $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. Standard references for inverse semigroup theory are [4, Chap. 5], [9] and [12]. Throughout this paper, we will use well known facts about inverse semigroups without explicit citation. Such facts include: the set $E(S)$ of all idempotents of S forms a commutative subsemigroup, and for all $x, y \in S$, $(x^{-1})^{-1} = x$ and $(xy)^{-1} = y^{-1}x^{-1}$.

Let S be an inverse semigroup and let $\omega(x_1, \ldots, x_n)$ be an inverse semigroup word. For $i \in \{1, \ldots, n\}$, we define an element $a \in S$ to be *i-th partial marginal for* ω if there exist idempotents $e, f \in E(S)$ such that, for all $x_i \in S$,

$$\omega(x_1, \ldots, ax_i, \ldots, x_n) = \omega(x_1, \ldots, ex_i, \ldots, x_n) \tag{2}$$

$$\omega(x_1, \ldots, x_i a, \ldots, x_n) = \omega(x_1, \ldots, x_i f, \ldots, x_n)\,. \tag{3}$$

Strictly speaking, we should indicate the dependence of e and f on the index i; however, we will show shortly (Lemma 4) that e and f depend only on a and they are equal to $aa^{-1} = a^{-1}a$. The set $M^i_\omega(S)$ of all such elements a is called *i-th partial margin* for ω. The set $M_\omega(S) = \bigcap_{i=1}^n M^i_\omega(S)$ is called the *margin* for ω and its elements are said to be marginal for ω. (In the group case, the refinement of the notion of marginal element into i-th partial marginal element is due to L.-C. Kappe [5].)

Recall that a subsemigroup of a semigroup is said to be *full* if it contains all idempotents. We can now state our first main result.

Theorem 1 *Let S be an inverse semigroup and let $\omega(x_1, \ldots, x_n)$ be an inverse semigroup word. Then:*

1. *For each $i \in \{1, \ldots, n\}$, $M^i_\omega(S)$ is a full inverse subsemigroup of S;*
2. *$M_\omega(S)$ is a full inverse subsemigroup of S.*

Note that we do not claim that the marginal inverse subsemigroup $M_\omega(S)$ is normal for arbitrary inverse semigroup words; see Problem 11 in §5.

In the special case of the commutator, we are able to characterize the marginal inverse subsemigroup. Recall that the *metacenter* of an inverse semigroup S is

$$Z(S) = \{a \in S : aa^{-1}xa = axa^{-1}a, \ \forall x \in S\}\,.$$

The metacenter is a normal inverse subsemigroup of S [6, 12].

Theorem 2 *Let S be an inverse semigroup and let $\omega(x, y) = [x, y]$ be the commutator. Then*

$$M_\omega(S) = Z(S)\,.$$

Consequently, $M_\omega(S)$ is a normal inverse subsemigroup of S.

In §2, we prove Theorem 1. In §3, we examine commutator identities in inverse semigroups, especially Clifford semigroups. In §4, we prove Theorem 2. Finally in §5, we pose a few problems.

2 Marginal Inverse Subsemigroups

Throughout this section, let S be an inverse semigroup and let $\omega(x_1, \ldots, x_n)$ be an inverse semigroup word.

We start with an easy observation which follows from $E(S)$ being a commutative subsemigroup of S.

Lemma 3 *For all $e_1, \ldots, e_n \in E(S)$,*

$$\omega(e_1, \ldots, e_n) = e_1 \cdots e_n\,.$$

An element a of an inverse semigroup S is *completely regular* if $aa^{-1} = a^{-1}a$, in which case we denote the idempotent $aa^{-1} = a^{-1}a$ by a^0. We now show that i-th partial marginal elements are completely regular.

Lemma 4 *Let S be an inverse semigroup, let $\omega(x_1, \ldots, x_n)$ be an inverse semigroup word, and let $a \in M^i_\omega(S)$. Then a is completely regular and for all $x_i \in S$,*

$$\omega(x_1, \ldots, ax_i, \ldots, x_n) = \omega(x_1, \ldots, a^0x_i, \ldots, x_n) \quad \textit{and} \tag{4}$$

$$\omega(x_1, \ldots, x_ia, \ldots, x_n) = \omega(x_1, \ldots, x_ia^0, \ldots, x_n) \tag{5}$$

Proof Let $e, f \in E(S)$ be as in (2), (3). We will first prove $e = f$, then $aa^{-1} = e$ and $a^{-1}a = e$. These together imply (4), (5).

First, we show

$$\omega(x_1, \ldots, e, \ldots, x_n) = \omega(x_1, \ldots, a, \ldots, x_n) = \omega(x_1, \ldots, f, \ldots, x_n)\,. \tag{6}$$

We compute

$$\begin{aligned}
\omega(x_1, \ldots, a, \ldots, x_n) &= \omega(x_1, \ldots, aa^{-1}a, \ldots, x_n) \\
&= \omega(x_1, \ldots, ea^{-1}a, \ldots, x_n) && \text{by (2)} \\
&= \omega(x_1, \ldots, ea^{-1}ae, \ldots, x_n) \\
&= \omega(x_1, \ldots, aa^{-1}ae, \ldots, x_n) && \text{by (2)}
\end{aligned}$$

$$\begin{aligned}
&= \omega(x_1, \dots, ae, \dots, x_n) \\
&= \omega(x_1, \dots, e^2, \dots, x_n) && \text{by (2)} \\
&= \omega(x_1, \dots, e, \dots, x_n)\,.
\end{aligned}$$

The second equality of (6) follows by a dual calculation.

Now in (6), set each $x_i = e$ and use Lemma 3 to get $e = ef$. Dually, we also have $f = ef$ and thus $e = f$. We will now use this freely when we invoke (3).

Next, starting with Lemma 3, we compute

$$\begin{aligned}
aa^{-1}e &= \omega(e, \dots, aa^{-1}e, \dots, e) \\
&= \omega(e, \dots, aa^{-1}a, \dots, e) && \text{by (3)} \\
&= \omega(e, \dots, a, \dots, e) \\
&= \omega(e, \dots, e, \dots, e) && \text{by (6)} \\
&= e && \text{by Lemma 3}
\end{aligned}$$

Together with a dual calculation, we have shown

$$ea^{-1}a = e = aa^{-1}e\,. \tag{7}$$

Now since $a^{-1}ea \in E(S)$,

$$\begin{aligned}
ea^{-1}ea &= \omega(e, \dots, a^{-1}ea, \dots, e) && \text{by Lemma 3} \\
&= \omega(e, \dots, a^{-1}e^2, \dots, e) && \text{by (3)} \\
&= \omega(e, \dots, a^{-1}e, \dots, e) \\
&= \omega(e, \dots, a^{-1}a, \dots, e) && \text{by (3)} \\
&= ea^{-1}a && \text{by Lemma 3} \\
&= e && \text{by (7)}
\end{aligned}$$

It follows that

$$ea^{-1} = ea^{-1}eaa^{-1} = ea^{-1}aa^{-1}e = ea^{-1}e\,. \tag{8}$$

Finally, we start again with Lemma 3 to get

$$\begin{aligned}
aa^{-1} &= \omega(aa^{-1}, \dots, aa^{-1}, \dots, aa^{-1}) && \text{by Lemma 3} \\
&= \omega(aa^{-1}, \dots, ea^{-1}, \dots, aa^{-1}) && \text{by (2)} \\
&= \omega(aa^{-1}, \dots, ea^{-1}e, \dots, aa^{-1}) && \text{by (8)} \\
&= \omega(aa^{-1}, \dots, aa^{-1}a, \dots, aa^{-1}) && \text{by (2) and (3)} \\
&= \omega(aa^{-1}, \dots, a, \dots, aa^{-1}) \\
&= \omega(aa^{-1}, \dots, e, \dots, aa^{-1}) && \text{by (6)}
\end{aligned}$$

$$
\begin{aligned}
&= aa^{-1}e && \text{by Lemma 3} \\
&= e\,, && \text{by (7)}
\end{aligned}
$$

A dual calculation shows $a^{-1}a = e$, and this completes the proof. □

Lemma 5 *If $a \in M^i_\omega(S)$ then $a^{-1} \in M^i_\omega(S)$. In particular, for all $x_i \in S$, $i \in \{1, \dots, n\}$,*

$$\omega(x_1, \dots, a^{-1}x_i, \dots, x_n) = \omega(x_1, \dots, a^0 x_i, \dots, x_n) \quad \textit{and} \tag{9}$$

$$\omega(x_1, \dots, x_i a^{-1}, \dots, x_n) = \omega(x_1, \dots, x_i a^0, \dots, x_n) \tag{10}$$

Proof This follows from using (4) twice in succession along with $aa^{-1} = a^{-1}a$:

$$
\begin{aligned}
\omega(x_1, \dots, ax_i, \dots, x_n) &= \omega(x_1, \dots, aa^{-1}x_i, \dots, x_n) \\
&= \omega(x_1, \dots, a^{-1}aa^{-1}x_i, \dots, x_n) \\
&= \omega(x_1, \dots, a^{-1}x_i, \dots, x_n)\,.
\end{aligned}
$$

□

Lemma 6 *If $a \in M^i_\omega(S)$, then a centralizes $E(S)$, that is, $ae = ea$ for all $e \in E(S)$.*

Proof For all $e \in E(S)$,

$$
\begin{aligned}
ea = ea^0a = a^0ea = aa^{-1}eea = a(ea)^{-1}ea& \\
&= a\,\omega(e, \dots, (ea)^{-1}(ea), \dots, e) && \text{by Lemma 3} \\
&= a\,\omega(e, \dots, a^{-1}ea, \dots, e) \\
&= a\,\omega(e, \dots, a^{-1}ea^0, \dots, e) && \text{by (5)} \\
&= a\,\omega(e, \dots, a^{-1}e, \dots, e) \\
&= a\,\omega(e, \dots, a^0e, \dots, e) && \text{by (9)} \\
&= aa^0e && \text{by Lemma 3} \\
&= ae\,,
\end{aligned}
$$

□

We can now prove our first main theorem.

Proof of Theorem 1 Let $a, b \in M^i_\omega(S)$. By Lemma 6, $a^0b = ba^0$. We use this as follows:

$$(ab)^0 = (ab)^{-1}(ab) = b^{-1}a^{-1}ab = b^{-1}ba^{-1}a = b^0a^0 = a^0b^0\,. \tag{11}$$

Now for all $x_i \in S$,

$$\omega(x_1, \dots, abx_i, \dots, x_n) = \omega(x_1, \dots, a^0bx_i, \dots, x_n) \qquad \text{by (4)}$$

$$
\begin{aligned}
&= \omega(x_1, \ldots, ba^0 x_i, \ldots, x_n) \\
&= \omega(x_1, \ldots, b^0 a^0 x_i, \ldots, x_n) \qquad \text{by (4)} \\
&= \omega(x_1, \ldots, (ab)^0 x_i, \ldots, x_n)\,,
\end{aligned}
$$

Thus (4) holds for ab, and the proof that (5) holds for ab is similar. Hence $ab \in M^i_\omega(S)$ and therefore $M^i_\omega(S)$ is a subsemigroup of S.

That $M^i_\omega(S)$ is closed under taking inverses is Lemma 5. Finally it is clear that every idempotent is i-th partial marginal, that is, $E(S) \subseteq M^i_\omega(S)$, and thus $M^i_\omega(S)$ is full. This proves part (1) of the theorem, and part (2) immediately follows. □

3 Commutator properties and identities

For the rest of this paper, we focus specifically on commutators $[x, y] = x^{-1}y^{-1}xy$ in inverse semigroups. In groups, commutators satisfy some well-known identities.

1. $[x, 1] = 1$;
2. $[x, y]^{-1} = [y, x]$;
3. $[z^{-1}xz, z^{-1}yz] = z^{-1}[x, y]z$;
4. $[x, yz] = [x, z][x, y]^z$;
5. $y^{-1}xy = x[x, y]$;
6. $[x, y^{-1}] = y[y, x]y^{-1}$;
7. $[[x, y^{-1}], z]^y[[y, z^{-1}], x]^z[[z, x^{-1}], y]^x = 1$;
8. $(xy)^2 = x^2y^2[y, x][[y, x], y]$.

Of course, we cannot expect many of these to hold in arbitrary inverse semigroups without suitable modifications. A couple of them are quite easy.

Lemma 7 *Let S be an inverse semigroup.*

1. *If $e \in E(S)$, then $[x, e] \in E(S)$ for all $x \in S$;*
2. *$[x, y]^{-1} = [y, x]$ for all $x, y \in S$;*

Proof 1. Since $x^{-1}ex \in E(S)$ for all $x \in S$, we have

$$[x, e]^2 = (x^{-1}ex)e(x^{-1}ex)e = (x^{-1}ex)^2e^2 = x^{-1}exe = [x, e]\,.$$

2. We have $[x, y]^{-1} = (x^{-1}y^{-1}xy)^{-1} = y^{-1}x^{-1}yx = [y, x]$. □

For the rest, we do not know if there are any suitable generalizations to arbitrary inverse semigroups; see Problem 14 in §5.

Recall that a *Clifford semigroup* is an inverse semigroup in which every element is completely regular, or equivalently, an inverse semigroup in which every idempotent is central. Clifford semigroups are characterized as semilattices of groups. More precisely, if S is a Clifford semigroup, then there exists a semilattice $(Y, \sqcap)$ and a family of disjoint subgroups S_α $(\alpha \in Y)$ such that $S = \bigcup_{\alpha\in Y} S_\alpha$ and $S_\alpha S_\beta \subseteq S_{\alpha\sqcap\beta}$ for all $\alpha, \beta \in Y$.

Not surprisingly, commutators in Clifford semigroups are much better behaved than in general inverse semigroups. They were first studied in detail by Kowol and Mitsch [7]. Among many other results, they showed that Clifford semigroups can be characterized using commutators.

Proposition 8 ([7], Cor. 3.2) *Let S be an inverse semigroup. The following are equivalent: (1) For all $x, y \in S$, $[x, y] \in E(S)$ implies $xy = yx$; (2) S is a Clifford semigroup.*

Our goal in this section is to show that group commutator identities hold in Clifford semigroups without change. We start with the following.

Lemma 9 *Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a Clifford semigroup. For $a, b \in S$, assume that for some $\alpha, \beta \in Y$, $a \in S_\alpha$, $b \in S_\beta$ and let $e = e_{\alpha \sqcap b}$ denote the identity element of $S_{\alpha \sqcap \beta}$. Then*

$$[ea, eb] = [a, b]\,.$$

Proof We have that $ea \in S_{(\alpha \sqcap \beta) \sqcap \alpha} = S_{\alpha \sqcap \beta}$ and $eb \in S_{(\alpha \sqcap \beta) \sqcap \beta} = S_{\alpha \sqcap \beta}$. Thus

$$\begin{aligned}
[ea, eb] &= (ea)^{-1_{S_{\alpha \sqcap \beta}}} (eb)^{-1_{S_{\alpha \sqcap \beta}}} (ea)(eb) \\
&= a^{-1_{S_\alpha}} e^{-1_{S_{\alpha \sqcap \beta}}} b^{-1_{S_\beta}} e^{-1_{S_{\alpha \sqcap \beta}}} eaeb \\
&= a^{-1_{S_\alpha}} eb^{-1_{S_\beta}} eeaeb \\
&= a^{-1_{S_\alpha}} b^{-1_{S_\beta}} ab \\
&= [a, b],
\end{aligned}$$

where the last but one equality holds because the element e is the identity in the group $S_{\alpha \sqcap \beta}$. □

Theorem 10 *The following hold in Clifford semigroups.*

1. $[z^{-1}xz, z^{-1}yz] = z^{-1}[x, y]z$;
2. $[xu, xvxw] = [xu, xw](xw)^{-1}[xu, xv]xw$;
3. $y^{-1}xy = x[x, y]$;
4. $[x, zy] = [x, y]y^{-1}[x, z]y$;
5. $[x, y^{-1}] = y[y, x]y^{-1}$;
6. $y^{-1}[[x, y^{-1}], z]yz^{-1}[[y, z^{-1}], x]zx^{-1}[[z, x^{-1}], y]x \in E(S)$;
7. $(xy)^2 = x^2y^2[y, x][[y, x], y]$.

Proof Taking into account Lemma 9 and the properties of commutators in groups, it follows that

1. $[z^{-1}xz, z^{-1}yz] = [ez^{-1}exze, ez^{-1}eyze] = ez^{-1}[ex, ey]ze = z^{-1}[x, y]z$;
2. We have

$$\begin{aligned}
[xu, xvxw] = [exu, exvexw] &= [exu, exw](exw)^{-1}[exu, exv]exw \\
&= [xu, xw](xw)^{-1}[xu, xv]xw;
\end{aligned}$$

3. $y^{-1}xy = (ey)^{-1}exey = ex[ex, ey] = x[x, y]$.
4. $[x, zy] = [ex, ezey] = [ex, ey](ey)^{-1}[ex, ez]ey = [x, y]y^{-1}[x, z]y$.
5. $[x, y^{-1}] = [ex, (ey)^{-1}] = ey[ey, ex](ey)^{-1} = y[y, x]y^{-1}$.
6. We compute

$$\begin{aligned}&y^{-1}[[x, y^{-1}], z]yz^{-1}[[y, z^{-1}], x]zx^{-1}[[z, x], y]x\\&= (ey)^{-1}[[ex, (ey)^{-1}], ez]ey(ez)^{-1}[[ey, (ez)^{-1}], ex]ez(ex)^{-1}[[ez, (ex)^{-1}], ey]ex\\&= e.\end{aligned}$$

It follows that $y^{-1}[[x, y^{-1}], z]yz^{-1}[[y, z^{-1}], x]zx^{-1}[[z, x^{-1}], y]x$ is an idempotent.

7. We compute

$$\begin{aligned}(xy)^2 = (exey)^2 &= (ex)^2(ey)^2[ey, ex][[ey, ex], ey]\\&= x^2y^2[y, x][[y, x], y].\end{aligned}$$

□

4 Marginal inverse subsemigroups for commutators

In this section, we prove Theorem 2. Let S be an inverse semigroup and let $\omega(x, y) = [x, y]$ be the commutator. We will prove that $M_\omega(S) = Z(S)$, the metacenter.

Assume first that $a \in Z(S)$ so that

$$axa^{-1}a = aa^{-1}xa \tag{12}$$

for all $x \in S$. Taking $x = a^{-1}a$, we obtain

$$a = a \cdot a^{-1}a \cdot a^{-1}a = aa^{-1} \cdot a^{-1}a \cdot a = a^{-1}a \cdot aa^{-1} \cdot a = a^{-1}aa,$$

using (12) in the second equality. Thus $aa^{-1} = a^{-1}a \cdot aa^{-1}$, that is, $aa^{-1} \le a^{-1}a$ in the natural order on $E(S)$. On the other hand, $a^{-1} \in Z(S)$ because the metacenter is an inverse subsemigroup of S [6, 12], so we also have $a^{-1}a \le aa^{-1}$. Thus $aa^{-1} = a^{-1}a$, that is, a is completely regular.

From (12), we obtain

$$a^0xa^0 = a^{-1}xa \tag{13}$$

for all $x \in S$. Thus

$$[ax, y] = (ax)^{-1}y^{-1}axy = x^{-1} \cdot a^{-1}y^{-1}a \cdot xy = x^{-1}a^0y^{-1}a^0xy = [a^0x, y]$$

and

$$[xa, y] = (xa)^{-1}y^{-1}xay = a^{-1}x^{-1}y^{-1}xa \cdot y = a^0x^{-1}y^{-1}xa^0y = [xa^0, y],$$

in both cases using (13) in the third equality. The other two cases, $[x, ay] = [x, a^0 y]$ and $[x, ya] = [x, ya^0]$, are immediately implied by Lemma 7(2). Therefore $a \in M_\omega(S)$.

Now assume $a \in M_\omega(S)$ so that $[ax, y] = [a^0 x, y]$ for all $x, y \in S$. Recall that a is completely regular (Lemma 4) and commutes with every idempotent (Lemma 6). We thus compute

$$\begin{aligned} a^0 xa &= a^0 xx^{-1}xa = a^0 xax^{-1}x = a(a^{-1}xax^{-1})x \\ &= a[a, x^{-1}]x = a[aa^0, x^{-1}]x = a[a^0, x^{-1}]xa \\ &= aa^0 xa^0 x^{-1}x = axx^{-1}xa^0 = axa^0 . \end{aligned}$$

Hence $a \in Z(S)$.

Therefore $M_\omega(S) = Z(S)$ as claimed. Finally, as already mentioned in §1, it is known that $Z(S)$ is normal [6, 12]. This completes the proof of Theorem 2.

5 Open problems

We have focused on marginal subsemigroups in this paper, but one could also approach the subject from a different direction. In the group case, a marginal subgroup is normal, hence defines a congruence. If S is a semigroup and ω is a semigroup word, define the *marginal equivalence relation* $\sim_\omega$ on S by $a \sim_\omega b$ if and only if

$$\omega(x_1, \dots, ax_i, \dots, x_n) = \omega(x_1, \dots, bx_i, \dots, x_n) \tag{14}$$

$$\omega(x_1, \dots, x_i a, \dots, x_n) = \omega(x_1, \dots, x_i b, \dots, x_n) \tag{15}$$

for all $x_i \in S$. Then define the *marginal congruence* $\equiv_\omega$ to be the smallest congruence on S containing $\sim_\omega$. In special classes of semigroups with additional operations, one would want to use appropriate notions of word and congruence.

In the case of inverse semigroups where ω is the commutator, it turns out the marginal equivalence relation and marginal congruence coincide, that is, $\sim_\omega$ is already a congruence. This can be shown directly, and in this case, the marginal congruence coincides with the central congruence of S [2, 6]. One can also reach the same conclusion starting with our result that the marginal inverse subsemigroup of the commutator coincides with the metacenter, hence is normal.

Problem 11 *Let S be an inverse semigroup and let $\omega(x_1, \dots, x_n)$ be an inverse semigroup word. Does the marginal congruence $\equiv_\omega$ coincide with the marginal equivalence relation $\sim_\omega$? Equivalently, is the marginal inverse subsemigroup $M_\omega(S)$ normal in S?*

Good candidates for investigation here are the power words $\omega(x) = x^n$.

In a more general direction, we suggest the following.

Problem 12 *Study marginal elements, subsemigroups, congruences, etc. in various classes of semigroups with the appropriate notion of what constitutes a word in that class.*

This might turn out to be particularly interesting in completely regular semigroups, even for commutators. In that case, there is more than one natural choice of commutator word, for example, $x^{-1}y^{-1}xy$ or $(yx)^{-1}xy$. Using PROVER9, we have been able to show that in cryptogroups (completely regular semigroups in which Green's $\mathcal{H}$-relation is a congruence), the marginal equivalence relation and marginal congruence for either choice of commutator coincide. However, we do not know what happens for general completely regular semigroups.

Problem 13 *Study marginal elements, subsemigroups, congruences, etc. for commutator words in completely regular semigroups.*

Finally, with reference to §3, we conclude with the following.

Problem 14 *Find appropriate generalizations of well-known group commutator identities to arbitrary inverse semigroups.*

Acknowledgements Our work was assisted by the system PROVERX [1, 8], by automated theorem prover PROVER9 and the finite model builder MACE4, both developed by McCune [10].

Funding Open access funding provided by FCT|FCCN (b-on).

References

1. Araújo, J., Kinyon, M., Robert, Y.: Varieties of regular semigroups with uniquely defined inversion. Port. Math. **76**(2), 205–228 (2019)
2. Freese, R., McKenzie, R.: Commutator Theory for Congruence Modular Varieties, London Mathematical Society Lecture Notes, vol. 125. Cambridge University Press, Cambridge (1987)
3. Hall, P.: Verbal and marginal subgroups. J. Reine Angew. Math. **182**, 156–157 (1940)
4. Howie, J. M.: Fundamentals of Semigroup Theory. London Mathematical Society Monographs. New Series, vol. 12. Oxford University Press, Oxford (1995)
5. Kappe, L.-C.: Engel margins in metabelian groups. Comm. Algebra **11**, 1965–1987 (1983)
6. Kinyon, M., Stanovský, D.: Abelianness and centrality in inverse semigroups, Proc. Edinburgh Math. Soc., to appear; Preprint arXiv:2209.13805v2, https://arxiv.org/abs/2209.13805 (2022)
7. Kowol, G., Mitsch, H.: Nilpotent inverse semigroups with central idempotents. Trans. Amer. Math. Soc. **271**, 437–449 (1982)
8. LaiTeP - Laboratory for Augmented Intelligence Theorem Proving: ProverX, https://www.proverx.dm.fct.unl.pt (2024)
9. Lawson, M.V.: Inverse Semigroups. The Theory of Partial Symmetries, World Scieintific, River Edge, NJ (1998)
10. McCune, W.: PROVER9 and MACE4, version LADR–2009–11A, http://www.cs.unm.edu/~mccune/prover9/ (2009)
11. Moravec, P.: Completely simple semigroups with nilpotent structure groups. Semigroup Forum **77**, 316–324 (2008)
12. Petrich, M.: Inverse Semigroups. John Wiley & Sons, New York (1984)

13. Robinson, D.: A Course in the Theory of Groups, Graduate Texts in Mathematics, vol. 80. Springer, New York (1996)

B

Research Article on Jones-Nambooripad Order in Universal Algebras

The Jones-Nambooripad Order in Universal Algebras

Gonçalo Araújo · Atle Hahn · Mikolas Janota · Ana-Catarina Monteiro · Josef Urban

Received: date / Accepted: date

Abstract It is well known that the following relation $a \leq b \Leftrightarrow (\exists e^2 = e, f^2 = f)\ a = eb = bf$ is a natural partial order on regular semigroups and that it plays a critical role in their theory, leading to very deep structural results. In this short note we prove that $\leq$, or its reflexive completion, is also a non-trivial natural partial order on other classes of non-regular semigroups or of non-associative algebras, including Directoids, λ-Lattices, and Schweizer Magmas. We hope that proving that these varieties admit this relation as a natural partial order may pave the way for similarly fruitful structural developments in their study.

1 Introduction

Let S be a semigroup. In S define the relation: for $a, b \in S$,

$$a \leq b \Leftrightarrow (\exists e^2 = e, f^2 = f)\ a = eb = bf. \tag{1.1}$$

It is well known that, when S is a semigroup of idempotents, $(S, \leq)$ is a partially ordered set, and $\leq$ is the *natural partial order* in bands (an order is said to be natural in some class of algebras when it is expressed in the language of the class). The same relation is the natural partial order for inverse semigroups, and its introduction had a great impact on the theory of inverse semigroups, namely, leading Schein and Meakin [8, 9, 10, 14], independently, to a structure theorem for the class. This was later generalized in very influential papers by Nambooripad [12, 13] (and independently by Hartwig [4]) who introduced a natural order for regular semigroups and used it to provide a structure theorem for that class. Subsequently, Jones [5, 6, 11] proved that the relation $\leq$ above coincides with Nambooripad's for regular semigroups. Therefore, we have a binary relation that serves as a natural partial order in some classes of semigroups and plays a critical role in key structure results. This raises the question: are there other varieties in which the relation $\leq$ is a non-trivial natural partial order?

This short note answers this question in the affirmative. We provide varieties of algebras T with operations $(op_i)_{i \leq n}$, $n \in \mathbf{N}$, containing a binary operation $* \in \{op_i \mid i \leq n\}$ such that for each model $(S, (op_i)_i)$ in T the relation on S defined by

$$a < b \Leftrightarrow (\exists e^2 = e, f^2 = f)\ a = eb = bf \tag{1.2}$$

G. Araújo
Center for Mathematics and Applications (CMA), Faculdade de Ciências e Tecnologia (FCT), Universidade Nova de Lisboa (UNL), 2829-516 Caparica, Portugal

A. Hahn (corresponding author)
CMA, FCT-UNL, 2829-516 Caparica, Portugal
E-mail: atle.hahn@gmail.com

M. Janota
Czech Institute of Informatics, Robotics and Cybernetics (CIIRC), Czech Technical University in Prague, Jugoslávských partyzánů 1580/3, 160 00 Praha 6, Czech Republic

A.-C. Monteiro
CMA, FCT-UNL, 2829-516 Caparica, Portugal

J. Urban
CIIRC, Czech Technical University in Prague, 160 00 Praha 6, Czech Republic

is antisymmetric and transitive. Clearly, if this is the case, then the reflexive relation $\leq$ on S, defined by

$$x \leq y \Leftrightarrow (x < y \text{ or } x = y) \tag{1.3}$$

will also be antisymmetric and transitive and hence a natural partial order relation in the variety T. The relation $\leq$ will be called the *reflexive completion* of $<$. Below, we list the found new equational theories where the relation defined by Eq. (1.2) is antisymmetric, transitive, and non-trivial (i.e., $x \leq y$ does not imply $x = y$).

2 Results

2.1 Pre-Directoids

The aim of this section is to establish the antisymmetry and transitivity of $<$ in important algebraic classes (that is, classes that have papers or books dedicated to them). But, in order to clarify the core arguments, we introduce an auxiliary class.

Definition 2.1 The variety of pre-directoids is the class of algebras of type (2) defined by the following identity:

(PD) $x \cdot ((x \cdot y) \cdot z) = (x \cdot y) \cdot z$.

Theorem 2.1 *(i) Let $(S, \cdot)$ be a pre-directoid, and let $<$ be the relation on S defined by*

$$x < y \quad \Leftrightarrow \quad \exists u, v \ (u^2 = u,\ v^2 = v,\ x = u \cdot y,\ x = y \cdot v). \tag{2.1}$$

Then $<$ is antisymmetric.

(ii) Assume that, additionally, $(S, \cdot)$ fulfills either of the following two identities

(I1) $x \cdot y = y \cdot x$. *(I2)* $(x \cdot y) \cdot x = x \cdot y$.

Then $<$ is also transitive and its reflexive completion $\leq$ is therefore a partial order relation on S.

Proof (i) Let $x, y \in S$ such that $x < y$ and $y < x$. We claim that $x = y$.

Since $x < y$, there exists $v \in S$ such that

$$x = y \cdot v. \tag{2.2}$$

Similarly, from $y < x$, there exists $u \in S$ such that

$$y = u \cdot x. \tag{2.3}$$

Thus $y \overset{(2.3)}{=} u \cdot x \overset{(2.2)}{=} u \cdot (y \cdot v) \overset{(2.3)}{=} u \cdot ((u \cdot x) \cdot v) \overset{(PD)}{=} (\underbracket{u \cdot x}) \cdot v \overset{(2.3)}{=} y \cdot v \overset{(2.2)}{=} x$.

(ii) Let $x, y, z \in S$ such that $x < y$ and $y < z$. We claim that $x < z$.

Since $x < y$, there exists $v_1 \in S$ such that

$$x = y \cdot v_1. \tag{2.4}$$

Similarly, $y < z$ implies that there exists $v_2 \in S$ such that

$$y = z \cdot v_2. \tag{2.5}$$

Observe that from this we get

$$z \cdot (y \cdot a) \overset{(2.5)}{=} z \cdot ((z \cdot v_2) \cdot a) \overset{(PD)}{=} (z \cdot v_2) \cdot a \overset{(2.5)}{=} y \cdot a. \tag{2.6}$$

and therefore

$$z \cdot x \overset{(2.4)}{=} z \cdot (y \cdot v_1) \overset{(2.6)}{=} (y \cdot v_1) \overset{(2.4)}{=} x \tag{2.7}$$

Next we claim that

$$x \cdot z = z \cdot x \overset{(2.7)}{=} x \tag{2.8}$$

If (I1) holds, this is obvious. If (I2) holds, this follows from $\underbracket{x} \cdot z \overset{(2.7)}{=} (z \cdot x) \cdot z \overset{(I2)}{=} z \cdot x \overset{(2.7)}{=} x$.

We conclude that $x \cdot \underbracket{x} \overset{(2.8)}{=} x \cdot (\underbracket{x \cdot z}) \overset{(2.8)}{=} x \cdot ((x \cdot z) \cdot z) \overset{(PD)}{=} (x \cdot z) \cdot z \overset{(2.8)}{=} x \cdot z \overset{(2.8)}{=} x$.

Thus x is idempotent and we have $x = z \cdot x = x \cdot z$. It is proved that $x < z$. The proof is complete.

2.1.1 Examples

We consider now important non-associative varieties in which the relation $<$ given by Eq. (2.1) is antisymmetric, transitive and non-trivial.

Definition 2.2 The variety of directoids [2] is the class of algebras of type (2) defined by the following identities:

(D1) $x \cdot x = x$.
(D2) $(x \cdot y) \cdot x = x \cdot y$.
(D3) $y \cdot (x \cdot y) = x \cdot y$.
(D4) $x \cdot ((x \cdot y) \cdot z) = (x \cdot y) \cdot z$.

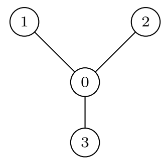
Clearly every directoid $(S, \cdot)$ is a pre-directoid where identity (I2) above is fulfilled. Therefore, by Theorem 2.1, the relation $<$ on S given by Eq. (2.1) is both antisymmetric and transitive and its reflexive completion $\leq$ is a partial order relation on S.

Here is an example (cf. Table 1 and Figure 1) of a non-associative order 4 directoid with non-trivial $\leq$.

Table 1 The multiplication table

$\cdot$	0	1	2	3
0	0	0	0	3
1	0	1	3	3
2	0	0	2	3
3	3	3	3	3

Fig. 1 Hasse diagram for $\leq$



Yet another class of non-associative algebras in which the relation $<$ given by Eq. (2.1) is antisymmetric, transitive and non-trivial, is the following.

Definition 2.3 The variety of λ-lattices [1] is the variety of type (2,2) defined by the following identities:

(LL1) $x \cdot y = y \cdot x$.
(LL2) $x \cdot ((x \cdot y) \cdot z) = (x \cdot y) \cdot z$.
(LL3) $x \cdot (x + y) = x$.
(LL4) $x + (x \cdot y) = x$.
(LL5) $x + y = y + x$.
(LL6) $x+((x+y)+z) = (x+y)+z$.

Let $(S, \cdot, +)$ be a λ-lattice. Clearly, both $(S, \cdot)$ and $(S, +)$ are commutative pre-directoids. Accordingly, by Theorem 2.1, the relation $<$ defined by Eq. (2.1), either on $\cdot$ or $+$, is both antisymmetric and transitive and the corresponding reflexive completions $\leq$ are partial order relations on S.

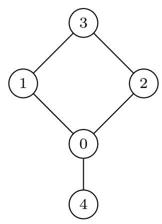
Here is an example (cf. Table 2 and Figure 2) of an order 5 λ-Lattice with non-associative $+$ and non-trivial $\leq$:

Table 2 Multiplication tables for $+$ and $\cdot$

$+$	0	1	2	3	4
0	0	0	0	0	4
1	0	1	4	1	4
2	0	4	2	2	4
3	0	1	2	3	4
4	4	4	4	4	4

$\cdot$	0	1	2	3	4
0	0	1	2	3	0
1	1	1	3	3	1
2	2	3	2	3	2
3	3	3	3	3	3
4	0	1	2	3	4

Fig. 2 Hasse diagram for $\leq$



2.2 Schweizer Magmas

This subsection presents another class of non-associative magmas generalizing Neumann quasigroups [15], where $<$ (Eq. (2.1)) is antisymmetric, transitive, and non-trivial.

Definition 2.4 Schweizer magmas [3] form a variety of type (2) defined by the following identity:

(SM) $(y \cdot z) \cdot (y \cdot x) = x \cdot z$.

Our goal is to prove the following theorem.

Theorem 2.2 *Let $(S,\cdot)$ be a Schweizer magma, and define the relation $<$ on S by*

$$x < y \quad \Leftrightarrow \quad \exists u, v \in S\ (u \cdot u = u,\ v \cdot v = v,\ x = u \cdot y,\ x = y \cdot v).$$

Then $<$ is antisymmetric and transitive and its reflexive completion $\leq$ is a partial order on S.

Before proving the theorem, we prove the following auxiliary result.

Lemma 1 *Let $(S,\cdot)$ be a Schweizer magma, and let $c, d, e \in S$ such that e is idempotent and*

$$c = d \cdot e. \tag{2.9}$$

Then:

(i) $d \cdot (d \cdot d) = c$ *(ii)* $c \cdot (d \cdot c) = c.$ *(iii)* $c \cdot (c \cdot c) = c.$

Proof (i) Firstly, we have that

$$\underbracket{c}\cdot(d \cdot a) \overset{(2.9)}{=} \underbracket{(d \cdot e)}\cdot(d \cdot a) \overset{(SM)}{=} a \cdot e. \tag{2.10}$$

In particular, we get $c \cdot (d \cdot e) = e \cdot e$ (replacing a with e in (2.10)) and $c \cdot (d \cdot d) = d \cdot e$ (replacing a with d in (2.10)). From these two equalities and the idempotency of e, we get

$$e = e \cdot e = c \cdot \underbracket{(d \cdot e)} = c \cdot \underbracket{(c \cdot (d \cdot d))}. \tag{2.11}$$

Thus $((z \cdot y) \cdot (z \cdot x)) \cdot ((z \cdot y) \cdot a) \overset{(SM)}{=} a \cdot (z \cdot x)$ and $\underbracket{((z \cdot y) \cdot (z \cdot x))}\cdot((z \cdot y) \cdot a) \overset{(SM)}{=} \underbracket{(x \cdot y)}\cdot((z \cdot y) \cdot a)$. The two previous equalities imply

$$a \cdot (z \cdot x) = (x \cdot y) \cdot ((z \cdot y) \cdot a). \tag{2.12}$$

Replacing x with $(z \cdot y)$ in the previous we get $a \cdot (z \cdot (z \cdot y)) \overset{(2.12)}{=} ((z \cdot y) \cdot y) \cdot ((z \cdot y) \cdot a) \overset{(SM)}{=} a \cdot y$. Therefore, we obtain the following identity:

$$a \cdot (b \cdot (b \cdot u)) = a \cdot u. \tag{2.13}$$

We can now conclude that $c \overset{(2.9)}{=} d \cdot e \overset{(2.11)}{=} \underbrace{d}_{a} \cdot (\underbrace{c}_{b} \cdot (\underbrace{c}_{b} \cdot \underbrace{(d \cdot d)}_{u})) \overset{(2.13)}{=} \underbrace{d}_{a} \cdot \underbrace{(d \cdot d)}_{u}$.

(ii) Just observe that $c \cdot (d \cdot c) \overset{(2.10)}{=} c \cdot e \overset{(2.11)}{=} \underbrace{c}_{a} \cdot (c \cdot (c \cdot \underbrace{(d \cdot d)}_{u})) \overset{(2.13)}{=} \underbrace{c}_{a} \cdot \underbrace{(d \cdot d)}_{u} \overset{(2.10)}{=} d \cdot e = c.$

(iii) Clearly, $c \cdot (c \cdot c) \overset{(ii)}{=} c \cdot (c \cdot (c \cdot (d \cdot c))) \overset{(2.13)}{=} c \cdot (d \cdot c) \overset{(ii)}{=} c.$

Now we can prove the main theorem.

Proof It is enough to prove that $<$ has the following property, which immediately implies both the antisymmetry and transitivity of $<$ and therefore also of its reflexive completion $\leq$:

$$(x < y \text{ and } y < z) \Rightarrow x = y. \tag{2.14}$$

Let $x, y, z \in S$ such that $x < y$ and $y < z$. We claim that $x = y$. Since $x < y$, there exists $v_1 \in S$ such that $x = y \cdot v_1$. From part (i) of Lemma 1 we get $y \cdot (y \cdot y) = x$. Similarly, since $y < z$, there exists $v_2 \in S$ such that $y = z \cdot v_2$; thus, from part (iii) of Lemma 1, $y \cdot (y \cdot y) = y$. Therefore, $y = y \cdot (y \cdot y) = x$.

From $x < y$ and $y < x$ we get $x = y$. From $x < y < z$ we get $x = y < z$. Theorem 2.2 is proved.

Here is an example (cf. Table 3 and Figure 3) of an order 4 non-associative Schweizer magma with non-trivial $\leq$:

To improve readability, we adopt the standard convention in non-associative magmas: juxtaposition takes precedence over $\cdot$, and brackets determine order. For example, $x(yv \cdot uz) := x \cdot ((y \cdot v) \cdot (u \cdot z))$.

Definition 2.5 Neumann magmas (cf. [3]) form a variety of type (2) defined by the following identity:

(NM) $x(yz \cdot yx) = z.$

Proposition 1 *Every Neumann magma is a Schweizer magma.*

Table 3 The multiplication table

$\cdot$	0	1	2	3
0	0	0	3	2
1	0	0	3	2
2	2	2	0	3
3	3	3	2	0

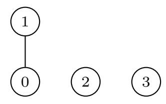


Fig. 3 Hasse diagram for $\leq$

In particular, the class of Schweizer magmas contains Neumann quasigroups (cf. [7] and p. 248 in [15]), which are equivalent to Schweizer quasigroups (cf. Theorem 21 in [3]).

Proof Observe first that (NM) implies $y = z(xy \cdot xz)$ and hence

$$u\,(\overline{y} \cdot zu) \overset{(NM)}{=} \underbrace{u}_{x}\left(\overline{\underbrace{z}_{y}\underbrace{(xy \cdot xz)}_{z}} \cdot \underbrace{z}_{y}\underbrace{u}_{x}\right) \overset{(NM)}{=} \underbrace{xy \cdot xz}_{z} = xy \cdot xz. \tag{2.15}$$

Next observe that $\underbrace{a}_{x}\underbrace{(bc)}_{y} \cdot \underbrace{a}_{x}\underbrace{b}_{z} \overset{(2.15)}{=} \underbrace{a}_{u}(\underbrace{bc}_{y} \cdot \underbrace{b}_{z}\underbrace{a}_{u}) \overset{(NM)}{=} c$, which leads us to the identity

$$x(yz) \cdot xy = z. \tag{2.16}$$

Further, we have $(\underbrace{a}_{x}\underbrace{d}_{y} \cdot \underbrace{a}_{x}\underbrace{c}_{z}) \cdot bd \overset{(2.15)}{=} \underbrace{b}_{u}(\underbrace{d}_{y} \cdot \underbrace{c}_{z}\underbrace{b}_{u}) \cdot \underbrace{b}_{x}\underbrace{d}_{y}$ and $\underbrace{b}_{x}(\underbrace{d}_{y} \cdot \underbrace{cb}_{z}) \cdot \underbrace{b}_{x}\underbrace{d}_{y} \overset{(2.16)}{=} \underbrace{cb}_{z}$, that is,

$$(xw \cdot xz) \cdot yw = zy. \tag{2.17}$$

Hence, $\underbrace{a}_{x}\underbrace{b}_{y} \cdot \underbrace{a}_{x}\underbrace{c}_{z} \overset{(2.15)}{=} \underbrace{(ac \cdot ac)}_{u}(\underbrace{b}_{y} \cdot \underbrace{c}_{z}\underbrace{(ac \cdot ac)}_{u}) \overset{(NM)}{=} (\underbrace{a}_{x}\underbrace{c}_{w} \cdot \underbrace{a}_{x}\underbrace{c}_{z})(\underbrace{b}_{y}\underbrace{\overline{c}}_{w}) \overset{(2.17)}{=} cb$, so that $xy \cdot xz = zy$.

Problem 1 *Identify other important varieties of algebras —meaning those for which at least one book is dedicated to their study— where $<$ is antisymmetric and transitive.*

Declarations

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

Competing interests

Not applicable.

Availability of data and materials

Not applicable.

Funding

G. Araújo, A. Hahn and A-C Monteiro: This work is funded by national funds through the FCT – Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 (https://doi.org/10.54499/UIDB/00297/2020) and UIDP/00297/2020 (https://doi.org/10.54499/UIDP/00297/2020) (Center for Mathematics and Applications)

Mikolas Janota and Josef Urban: This work was supported by the Ministry of Education, Youth and Sports within the dedicated program ERC CZ under the project *POSTMAN* no. LL1902 and co-funded by the European Union under the project *ROBOPROX* (reg. no. CZ.02.01.01/00/22_008/0004590).

Author contributions

This paper results from a comprehensive scan of well-known classes of algebras to identify those in which the natural partial order in regular semigroups also qualifies as a natural partial order. To this end, we downloaded the entire arXiv database, extracted the algebraic classes mentioned, and verified for each whether the result holds. When it did, we produced human proofs.

The extraction of relevant algebraic classes from the database was conducted by G. A., M. J., and J. U. during a six-month visit of the first author to Prague. Then the human verification of the identified classes was carried out by A. H., G. A., and C. M. The identification of the classes where the relation forms a partial order was performed by A. H., and the corresponding human proofs were provided by G. A., C. M., and A. H. All authors contributed to the final writing of the paper.

Acknowledgments

We wish to thank João Araújo for several very useful comments.

References

1. I. Chajda, H. Länger: Semimodular λ-lattices. Multiple-Valued Logic Soft Computing 39, 79-96 (2022). [https://arxiv.org/abs/1909.05186]
2. I. Chajda, H. Länger: Directoids, an algebraic approach to ordered sets, Heldermann Verlag (2011).
3. N.N. Didurik, V.A. Shcherbacov: Some properties of Neumann quasigroups. In: Conference on Applied and Industrial Mathematics: CAIM 2018, 20-22 septembrie 2018, Iași, România. Chișinău, Republica Moldova: Casa Editorial-Poligrafică "Bons Offices" SRL, 2018, Ediţia a 26-a, pp. 93-94. [https://arxiv.org/abs/1809.07095]
4. R.E. Hartwig, *How to partially order regular elements*, Math. Japonica **25**(1), 1–13 (1980).
5. P.M. Higgins, Techniques of Semigroup Theory, Oxford University Press (1992).
6. P. M. Higgins, The Mitsch order on a semigroup, Semigroup Forum, **49**, 261–266 (1994).
7. G. Higman and B. H. Neumann. Groups as groupoids with one law. Publ. Math. Debrecen, 2:215–221, 1952.
8. M. V. Lawson, Inverse Semigroups: The Theory of Partial Symmetries, World Scientific Publishing Co., Inc., River Edge, 1998.
9. S. Margolis, A tribute to John Meakin on the occasion of his 75th birthday, Semigroup Forum, **102**, 1–9 (2021).
10. J. Meakin, On the structure of inverse semigroups, Semigroup Forum, **12**(1), 6–14 (1976).
11. H. Mitsch, A natural partial order for semigroups, Proc. Amer. Math. Soc., **97**(3), 384–388 (1986).
12. K.S.S. Nambooripad, "Structure of Regular Semigroups". Memoirs of the American Mathematical Society, Volume 22, Number 224 (1979).
13. K.S.S. Nambooripad, "The natural partial order on a regular semigroup". Proceedings of the Edinburgh Mathematical Society. 23 (3): 249–260 (1980).
14. M. Petrich: Inverse Semigroups. Wiley, New York (1984).
15. Sh. K. Stein. On the foundations of quasigroups. Trans. Amer. Math. Soc., 85(1):228–256, 1957.

C

Research Article on Number of Non-Isomorphic Models of Common Algebras

Number of Non-isomorphic Models of Common Algebras

Gonçalo Araújo
Universidade Nova de Lisboa
Lisbon, Portugal

João Araújo
Universidade Nova de Lisboa
Lisbon, Portugal
https://docentes.fct.unl.pt/jj-araujo
jj.araujo@fct.unl.pt

Choiwah Chow
Universidade Aberta
Lisbon, Portugal
1702603@estudante.uab.pt

Mikoláš Janota
Czech Technical University in Prague
Czech Republic
http://people.ciirc.cvut.cz/~janotmik/
mikolas.janota@cvut.cz

João Ramires
Universidade Aberta
Lisbon, Portugal

Abstract

The number of models of order n, up to isomorphism, of an algebra is an important property of that algebra. To date, only a handful of such sequences of numbers are produced and available online in websites such as the oeis.org. We use the parallel model enumerator Mace4 to generate models of the algebras listed in the MarcieX database, count their numbers, and report them in this paper.

1 Introduction

The number of order n non-isomorphic models of a given class of algebras is a very important piece of information. It is also very difficult to find. For example, the number of non-isomorphic groupoids (or magmas) has been found [35]. Similar ideas can be applied to special classes of algebras. Closed formulas are known for some particular classes of groups [36], but finding them often requires very deep results, including the classification of finite simple groups. For groups in general, only bounds are known. The sequence of order n groups was chosen to be the first one in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [37].

But the results above, along with some of the same flavor, stand out as exceptions. The rule is that there is no closed formula for the order n non-isomorphic models of an important class of algebras (say, rings, semigroups, quasigroups, etc.). Therefore, the best we have now is some computation of the first terms of the sequence. For many classes of algebras not even the first values are computed for any meaningful n. The reason is that the task is far from trivial. We need to use some finite model enumerator to generate the models of order n according to the rules laid down by the axiomatic definition of that class of algebras. These enumerators use delicate symmetry breaking algorithms to reduce the number of redundant isomorphic models, but usually a large number of unnecessary models is generated. This has two adverse effects: time and memory are lost to generate them and afterwards a new algorithm and more time are needed to clean up the list. It often happens that the biggest bottleneck is precisely in this second part (get rid of the redundant generated models). For example, while there are only 1,627,672 semigroups of order 7, the finite model generator Mace4 [38] generates over 1 billion models of order 7; afterwards, its companion program Isofilter is not able to filter out all the isomorphic models in reasonable time (over 2 days) when the program had to be terminated.

In order to push the limits of the known sequences for general classes of algebras, we developed two new algorithms that apply to the two steps of the process. Our first symmetry breaking algorithm (to be published) dramatically reduces the number of redundant models generated; our second algorithm [39] reduces by orders of magnitude the time needed to discard the redundant models. For example, Mace4 generates 28.9 million models for the inverse semigroups of order 8. Without invariants, it takes Isofilter 150,703 seconds to filter out the isomorphic models to get 4,637 non-isomorphic models, but it takes less than 2% (2,873 seconds) of the original time for Isofilter to accomplish the same task with invariants (see Table 2 in [40]). The general idea is the following: to each model generated we attach a vector of parameters that are preserved under isomorphism. This induces a partition of he original list into many blocks (or parts) of non-isomorphic models (that is, isomorphic models belong to the same block). Therefore, rather than testing a given model against all others, we only test it against the models in the same block; in addition, as checking each block is independent from checking the others, the task can be parallelized.

With these two tools we were able to find the initial terms of the sequence of order n models of a given class of algebras for many classes whose sequence was unknown so far; we also were able to confirm and extend the sequences in the cases some terms were already known.

2 MarcieX

MarcieX [41] is a library of theories and theorems in the syntax of ProverX [42]. It has the advantage of being noiseless, unlike many other databases (such as TPTP [43], etc.) that computer scientists use to benchmark their research in Artificial Intelligence Theorem Proving (AITP). In addition, it provides some routines that allow the user to perform the following operations:

1. given a theorem, MarcieX scans its database to identify in which other classes of algebras of the same type, the analogue holds; for example, given the theorem *If in a group every element is the inverse of itself, then the group is commutative*, MarcieX will check all classes of algebras that have a binary operation and a unary operation, and check if the same result is true (eg, the same result holds for inverse semigroups);

2. given a class of algebras by the user, MarcieX will find the classes of algebras that contain it or are contained in it;

3. given some concept C that in some theory T has some property P (say, a binary relation that in semigroups is a congruence), MarcieX will find other classes of algebras not contained in T in which the concept C keeps property P;

4. given some model (say, a counter-example for some conjecture), MarcieX will identify pages of papers where a model isomorphic to that one appears.

In this paper we use MarcieX as a database of classes of algebras so that we can apply to them our finite model builder algorithms.

3 Counting Models

We use a 24-core Intel® Xeon®CPU E5-2630 v2 2.60GHz $\times 24$ computer, with 64 GB RAM and 600 GB disk space for generating models and for filtering out isomorphic models. Once isomorphic models for an order of an algebra are filtered out, we count the remaining non-isomorphic models and report the number.

We run finite model enumeration on algebras in the MarcieX database. We allocate at least 5 hours to the search, in parallel with all 24 cores of the computer, of models of each order of each algebras. The search will be aborted if the time limit is exceeded and if the search is deemed not likely to finish anytime soon. Another limitation of the hardware used is disk space. We stop the search of models of a particular order if the size of the file containing the models exceeds 500 GB. The search for an sequence of number of models starts from order 2, then proceed incrementally (by 1 each time) up to, but not including, the order in which the time limit or the disk space limit is reached.

We exclude in this paper algebras such as semigroups, quasigroups, and loops, etc., of which the number of models of small orders are well-known, or that they can easily be found in online resources such as the OEIS. We also exclude algebras that we are not able to find their numbers of models at least up to order 4.

We compare our results with the integer sequences in the OEIS. For OEIS sequences A000112, A001930, A084965, A087729, A181769, A226193, A346414, our results match theirs although we are not able to generate longer sequences than theirs, so, we do not report our results here. For sequences A178432 (see Section 5.22) and A305858 (see Section 6.32.4), we not only match, but also extend, their results. We found 100 new sequences that have not been reported to the OEIS that are reported in the following sections.

4 Classes of Semigroups

4.1 Type (2)

4.1.1 Cancellative monoids

A Cancellative monoid is a structure $\langle A, \cdot, 1\rangle$ such that

1. $(xy = xz \Rightarrow y = z)$;
2. $(xz = yz \Rightarrow x = y)$;
3. $x(yz) = (xy)z$;
4. $1x = x$;
5. $x1 = x$.

Cancellative monoids appear on [25] and the number of non-isomorphic models from order

4.1.2 Monoids

A Monoid is a structure $\langle A, \cdot, 1\rangle$ such that

1. $x(yz) = (xy)z$;
2. $1x = x$;
3. $x1 = x$.

Monoids appear on [27] and the number of non-isomorphic models from order

4.1.3 Commutative semigroups

A Commutative semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $xy = yx$.

Commutative semigroups appear on [28] and the number of non-isomorphic models from order

4.1.4 Archimedean commutative semigroups

An Archimedean commutative semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $xy = yx$;
3. $\forall x \forall y \exists z : \; xx = zy$.

Archimedean commutative semigroups appear on [28] and the number of non-isomorphic models from order

4.1.5 Nowhere commutative semigroups

A Nowhere commutative semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $xy = yx \Rightarrow x = y$.

Nowhere commutative semigroups appear on [28] and the number of non-isomorphic models from order

4.1.6 Left weakly commutative semigroups

A Left weakly commutative semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $\forall x \forall y \exists z : \; xyxy = yz$.

Left weakly commutative semigroups appear on [29] and the number of non-isomorphic models from order

4.1.7 Right weakly commutative semigroups

A Right weakly commutative semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $\forall x \forall y \exists z : \; xyxy = zx$.

Right weakly commutative semigroups appear on [29] and the number of non-isomorphic models from order

4.1.8 Weakly commutative semigroups

A Weakly commutative semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $\forall x \forall y \exists z : \; xyxy = yz$;
3. $\forall x \forall y \exists w : \; xyxy = wx$.

Weakly commutative semigroups appear on [29] and the number of non-isomorphic models from order

4.1.9 Conditionally commutative semigroups

A Conditionally commutative semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $xy = yx \Rightarrow xzy = yzx$.

Conditionally commutative semigroups appear on [29] and the number of non-isomorphic models from order

4.1.10 Quasi-commutative semigroups

A Quasi-commutative semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $xy = yxyx$.

Quasi-commutative semigroups appear on [29] and the number of non-isomorphic models from order

4.1.11 Right commutative semigroups

A Right commutative semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;

2. $xyz = xzy$.

Right commutative semigroups appear on [29] and the number of non-isomorphic models from order

4.1.12 Left commutative semigroups

A Left commutative semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;

2. $yzx = zyx$.

Left commutative semigroups appear on [29] and the number of non-isomorphic models from order

4.1.13 Externally commutative semigroups

An Externally commutative semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;

2. $xyz = zyx$.

Externally commutative semigroups appear on [29] and the number of non-isomorphic models from order

4.1.14 Medial semigroups

A Medial semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;

2. $xuvy = xvuy$.

Medial semigroups appear on [29] and the number of non-isomorphic models from order

4.1.15 E-2 semigroups

An E-2 semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;

2. $xyxy = xxyy$.

E-2 semigroups appear on [29] and the number of non-isomorphic models from order

4.1.16 Right cancellative semigroups

A Right cancellative semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $yx = zx \Rightarrow y = z$.

Right cancellative semigroups appear on [28] and the number of non-isomorphic models from order

4.1.17 Left cancellative semigroups

A Left cancellative semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $xy = xz \Rightarrow y = z$.

Left cancellative semigroups appear on [28] and the number of non-isomorphic models from order

4.1.18 Cancellative semigroups

A Cancellative semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $xy = xz \Rightarrow y = z$;
3. $yx = zx \Rightarrow y = z$.

Cancellative semigroups appear on [28] and the number of non-isomorphic models from order

4.1.19 Primitive semigroups

A Primitive semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $(y = xy \text{ and } y = yx) \Rightarrow x = y$.

Primitive semigroups appear on [28] and the number of non-isomorphic models from order

4.1.20 Null semigroups

A Null semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $xy = zw$.

Null semigroups appear on [28] and the number of non-isomorphic models from order

4.1.21 Semigroups with zero

A Semigroup with zero is a structure $\langle S, \cdot, 0 \rangle$ such that

1. $x(yz) = (xy)z$;
2. $0x = 0 = x0$.

Semigroups with zero appear in [100] and the number of non-isomorphic models from order 2 to order 6 is given in Table 1.

Table 1: Number of Semigroups with zero

Order:	2	3	4	5	6
#Models:	2	12	90	960	15,759

4.1.22 Left zero semigroups

A Left zero semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $xy = x$.

Left zero semigroups appear on [28] and the number of non-isomorphic models from order

4.1.23 Right zero semigroups

A Right zero semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $xy = y$.

Right zero semigroups appear on [28] and the number of non-isomorphic models from order

4.1.24 Unipotent semigroups

A Unipotent semigroup is a structure $\langle A, \cdot, 1\rangle$ such that

1. $x(yz) = (xy)z$;
2. $1 \cdot 1 = 1$;
3. $xx = x$ and $yy = y \Rightarrow x = y$.

Unipotent semigroups appear on [28] and the number of non-isomorphic models from order

4.1.25 Separative semigroups

A Separative semigroup is a structure $\langle A, \cdot\rangle$ such that

1. $x(yz) = (xy)z$;
2. $(xy = xx$ and $xx = yy) \Rightarrow x = y$.

Separative semigroups appear on [28] and the number of non-isomorphic models from order

4.1.26 2-Aperiodic semigroups

A 2-Aperiodic semigroup is a structure $\langle A, \cdot\rangle$ such that

1. $x(yz) = (xy)z$;
2. $xx = xxx$.

2-Aperiodic semigroups appear on [32] and the number of non-isomorphic models from order

4.1.27 2-Nilpotent semigroups

A 2-Nilpotent semigroup is a structure $\langle A, \cdot, 1\rangle$ such that

1. $x(yz) = (xy)z$;
2. $xx = 1$.

2-Nilpotent semigroups appear on [27] and the number of non-isomorphic models from order

4.1.28 Semigroups with Right Identity

A Semigroup with right identity is a structure $\langle A, \cdot, 1\rangle$ such that

1. $(xy)z = x(yz)$;
2. $x = x1$.

Semigroups with right identity appear on [1] and the number of non-isomorphic models from order

4.1.29 $\mathcal{J}$-Simple Semigroups

A $\mathcal{J}$-Simple semigroup is a structure $\langle A, \cdot \rangle$ such that

1. $(xy)z = x(yz)$;
2. $\forall x \forall y \exists z \exists w : \ x = zyw$.

$\mathcal{J}$-Simple semigroups appear on [1] and the number of non-isomorphic models from order

4.1.30 Right Groups

A Right group is a structure $\langle A, \cdot, 1 \rangle$ such that

1. $x(yz) = (xy)z$;
2. $1 \cdot 1 = 1$;
3. $\forall x \forall y \exists z : \ xz = y$.

Right groups appear on [1] and the number of non-isomorphic models from order

4.1.31 Id Semigroups

An Id semigroup is a structure $\langle A, \cdot, 1 \rangle$ such that

1. $x(yz) = (xy)z$;
2. $1 \cdot 1 = 1$.

Id semigroups appear on [1] and the number of non-isomorphic models from order

4.1.32 Left Groups

A Left group is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $\forall x \forall y \exists z : \ x = xzy$.

Left groups appear on [2] and the number of non-isomorphic models from order

4.2 Type (2,1)

4.2.1 Closable semilattice pseudo-complemented semigroups

A Closable semillatice pseudo-complemented semigroup is a structure $\langle A, \cdot, ', 0\rangle$ such that

1. $x(yz) = (xy)z$;
2. $x'x = 0$;
3. $x0' = x$;
4. $x'y' = y'x'$;
5. $x'(x'y)' = x'y'$;
6. $x''x = x$.

Closable semilattice pseudo-complemented semigroups appear in [110] and the number of non-isomorphic models from order 2 to order 8 is given in Table 2.

Table 2: Number of Closable semilattice pseudo-complemented semigroups

Order:	2	3	4	5	6	7	8
#Models:	1	4	24	191	2,046	30,693	1,661,964

4.2.2 Semilattice pseudo-complemented semigroups

A Semillatice pseudo-complemented semigroup is a structure $\langle A, \cdot, ', 0\rangle$ such that

1. $x(yz) = (xy)z$;
2. $x'x = 0$;
3. $x \cdot 0' = x$;
4. $x'y' = y'x'$;
5. $x'(x'y)' = x'y'$.

Semilattice pseudo-complemented semigroups appear on p. 84 of [110] and the number of non-isomorphic models from order 2 to order 8 is given in Table 3.

Table 3: Number of Semilattice pseudo-complemented semigroups

Order:	2	3	4	5	6	7	8
#Models:	2	8	54	540	7,252	124,712	3,680,984

4.2.3 Antidomain monoids

An Antidomain monoid is a structure $\langle A, \cdot, ', 1, 0\rangle$ such that

1. $\langle A, \cdot, 1\rangle$ is a monoid;
2. $x'x = 0$;
3. $x0 = 0$;
4. $x'y' = y'x'$;
5. $x''x = x$;
6. $x' = (xy)'(xy')'$;
7. $(xy)'x = (xy)'xy'$.

Antidomain monoids appear on p. 82 of [110] and the number of non-isomorphic models from order 2 to order 9 is given in Table 4.

Table 4: Number of Antidomain monoids

Order:	2	3	4	5	6	7	8	9
#Models:	1	2	8	40	261	2,444	33,055	1,682,176

4.2.4 Left closure semigroups

A Left closure semigroup is a structure $\langle A, \cdot, d, 1\rangle$ such that

1. $x(yz) = (xy)z$;
2. $d(x)x = x$;
3. $d(d(x)) = d(x)$;
4. $d(x)d(xy) = d(xy)$;
5. $d(x)d(y) = d(y)d(x)$.

Left closure semigroups appear on p. 76 of [110] and the number of non-isomorphic models from order 2 to order 7 is given in Table 5.

Table 5: Number of Left closure semigroups

Order:	2	3	4	5	6	7
#Models:	4	23	196	2,179	32,278	599,168

4.2.5 Left closure monoids

A Left closure monoid is a structure $\langle A, \cdot, d, 1\rangle$ such that

1. $\langle A, \cdot, d\rangle$ is a left closure semigroup;
2. $\langle A, \cdot, 1\rangle$ is a monoid.

Left closure monoids appear on p. 76 of [110] and the number of non-isomorphic models from order 2 to order 7 is given in Table 6.

Table 6: Number of Left closure monoids

Order:	2	3	4	5	6	7
#Models:	3	15	109	1,071	14,571	255,798

4.2.6 Quasi-MV-algebras

A Quasi-MV-algebra is a structure $\langle A, \oplus, ', 0, 1\rangle$ such that

1. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $x'' = x$;
3. $x \oplus 1 = 1$;
4. $(x' \oplus y)' \oplus y = (y' \oplus x)' \oplus x$;
5. $(x \oplus 0)' = x' \oplus 0$;
6. $(x \oplus 0) \oplus 0 = x \oplus 0$;
7. $0' = 1$.

Quasi-MV-algebras appear in [97] and the number of non-isomorphic models from order 2 to order 8 is given in Table 7.

Table 7: Number of Quasi-MV-algebras

Order:	2	3	4	5	6	7	8
#Models:	1	1	9	9	467	567	153,163

4.2.7 Groups

A Group is a structure $\langle A, \cdot, ^{-1}, 1\rangle$ such that

1. $x(yz) = (xy)z$;
2. $1x = x = x1$;
3. $x^{-1}x = xx^{-1} = 1$.

Groups appear on [2] and the number of non-isomorphic models from order

4.2.8 MV-algebras

An MV-algebra is a structure $\langle A, \oplus, \neg, 0\rangle$ such that

1. $0 \oplus u = u$;
2. $u \oplus v = v \oplus u$;
3. $(u \oplus v) \oplus w = u \oplus (v \oplus w)$;
4. $\neg\neg u = u$;
5. $u \oplus \neg 0 = \neg 0$;
6. $\neg(\neg u \oplus v) \oplus v = \neg(\neg v \oplus u) \oplus u$.

MV-algebras appear on [24] and the number of non-isomorphic models from order

4.2.9 Inverse monoids

An Inverse monoid is a structure $\langle A, \cdot, ^{\star}, 1\rangle$ such that

1. $x(yz) = (xy)z$;
2. $1x = x$;
3. $x1 = x$;
4. $(uv)^{\star} = v^{\star}u^{\star}$;

5. $(u^\star)^\star = u$;

6. $uu^\star u = u$;

7. $uu^\star vv^\star = vv^\star uu^\star$.

Inverse monoids appear on [26] and the number of non-isomorphic models from order

4.2.10 Regular semigroups

A Regular semigroup is a structure $\langle A, \cdot, ^{-1} \rangle$ such that

1. $x(yz) = (xy)z$;

2. $xx^{-1}x = x$.

Regular semigroups appear on [28] and the number of non-isomorphic models from order

4.2.11 Left regular semigroups

A Left regular semigroup is a structure $\langle A, \cdot, ^{-1} \rangle$ such that

1. $x(yz) = (xy)z$;

2. $x^{-1}xx = x$.

Left regular semigroups appear on [28] and the number of non-isomorphic models from order

4.2.12 Right regular semigroups

A Right regular semigroup is a structure $\langle A, \cdot, ^{-1} \rangle$ such that

1. $x(yz) = (xy)z$;

2. $xxx^{-1} = x$.

Right regular semigroups appear on [28] and the number of non-isomorphic models from order

4.2.13 2-Regular semigroups

A 2-Regular semigroup is a structure $\langle A, \cdot, ^{-1} \rangle$ such that

1. $x(yz) = (xy)z$;

2. $xxx^{-1}xx = xx$.

2-Regular semigroups appear on [31] and the number of non-isomorphic models from order

4.2.14 Inverse semigroups

An Inverse semigroup is a structure $\langle A, \cdot, ^{-1} \rangle$ such that

1. $x(yz) = (xy)z$;
2. $xx^{-1}x = x$;
3. $x^{-1}xx^{-1} = x^{-1}$;
4. $^{-1}$ is unique in the previous conditions.

Inverse semigroups appear on [28] and the number of non-isomorphic models from order

4.2.15 *-Semigroups

A *-Semigroup is a structure $\langle A, \cdot, ^{-1} \rangle$ such that

1. $(xy)z = x(yz)$;
2. $(x^{-1})^{-1} = x$;
3. $(xy)^{-1} = y^{-1}x^{-1}$.

*-Semigroups appear on [33] and the number of non-isomorphic models from order

4.2.16 U-Semigroups

A U-Semigroup is a structure $\langle A, \cdot, ^{-1} \rangle$ such that

1. $(xy)z = x(yz)$;
2. $(x^{-1})^{-1} = x$.

U-Semigroups appear on [33] and the number of non-isomorphic models from order

4.2.17 I-Semigroups

An I-Semigroup is a structure $\langle A, \cdot, ^{-1} \rangle$ such that

1. $(xy)z = x(yz)$;
2. $(x^{-1})^{-1} = x$;
3. $xx^{-1}x = x$.

I-Semigroups appear on [33] and the number of non-isomorphic models from order

4.2.18 Completely regular semigroups

A Completely regular semigroup is a structure $\langle A, \cdot, ^{-1} \rangle$ such that

1. $(xy)z = x(yz)$;
2. $xx^{-1}x = x$;
3. $xx^{-1} = x^{-1}x$.

Completely regular semigroups appear on [34] and the number of non-isomorphic models from order

4.2.19 Clifford semigroups

A Clifford semigroup is a structure $\langle A, \cdot, ^{-1} \rangle$ such that

1. $(xy)z = x(yz)$;
2. $xx^{-1}x = x$;
3. $xx^{-1} = x^{-1}x$;
4. $xx = x \Rightarrow xy = yx$.

Clifford semigroups appear on [34] and the number of non-isomorphic models from order

4.2.20 Cryptogroups

A Cryptogroup is a structure $\langle A, \cdot, ' \rangle$ such that

1. $(xy)z = x(yz)$;
2. $x = xx'x$;
3. $xx' = x'x$;
4. $x'' = x$;
5. $xx'yy'(xx'yy')' = xy(xy)'$.

Cryptogroups appear on [34] and the number of non-isomorphic models from order

4.2.21 Regular Cryptogroups

A Regular Cryptogroup is a structure $\langle A, \cdot, ' \rangle$ such that

1. $(xy)z = x(yz)$;
2. $x = xx'x$;
3. $xx' = x'x$;
4. $x'' = x$;
5. $xyzx(xyzx)' = xyxzx(xyxzx)'$.

Regular Cryptogroups appear on [34] and the number of non-isomorphic models from order

4.2.22 Left Regular Cryptogroups

A Left Regular Cryptogroup is a structure $\langle A, \cdot, ' \rangle$ such that

1. $(xy)z = x(yz)$;
2. $x = xx'x$;
3. $xx' = x'x$;
4. $x'' = x$;
5. $xy(xy)' = xx'yy'xx'$.

Left Regular Cryptogroups appear on [34] and the number of non-isomorphic models from order

4.2.23 Normal Cryptogroups

A Normal Cryptogroup is a structure $\langle A, \cdot, ' \rangle$ such that

1. $(xy)z = x(yz)$;
2. $x = xx'x$;
3. $xx' = x'x$;
4. $x'' = x$;
5. $xyzx(xyzx)' = xzyx(xzyx)'$.

Normal Cryptogroups appear on [34] and the number of non-isomorphic models from order

4.2.24 Left Normal Cryptogroups

A Left Normal Cryptogroup is a structure $\langle A, \cdot, ' \rangle$ such that

1. $(xy)z = x(yz)$;
2. $x = xx'x$;
3. $xx' = x'x$;
4. $x'' = x$;
5. $xyy'z = xzyy'$.

Left Normal Cryptogroups appear on [34] and the number of non-isomorphic models from order

4.2.25 Orthogroups

An Orthogroup is a structure $\langle A, \cdot, ' \rangle$ such that

1. $(xy)z = x(yz)$;
2. $x = xx'x$;
3. $xx' = x'x$;
4. $x'' = x$;
5. $xx'yy'(xx'yy')' = xx'yy'$.

Orthogroups appear on [34] and the number of non-isomorphic models from order

4.2.26 Regular Orthogroups

A Regular Orthogroup is a structure $\langle A, \cdot, ' \rangle$ such that

1. $(xy)z = x(yz)$;
2. $x = xx'x$;
3. $xx' = x'x$;
4. $x'' = x$;
5. $xyzx = xyxx'zx$.

Regular Orthogroups appear on [34] and the number of non-isomorphic models from order

4.2.27 Left Regular Orthogroups

A Left Regular Orthogroup is a structure $\langle A, \cdot, ' \rangle$ such that

1. $(xy)z = x(yz)$;
2. $x = xx'x$;
3. $xx' = x'x$;
4. $x'' = x$;
5. $xy = xyxx'$.

Left Regular Orthogroups appear on [34] and the number of non-isomorphic models from order

4.2.28 Normal Orthogroups

A Normal Orthogroup is a structure $\langle A, \cdot, ' \rangle$ such that

1. $(xy)z = x(yz)$;
2. $x = xx'x$;
3. $xx' = x'x$;
4. $x'' = x$;
5. $xyzz'x = xzz'yx$.

Normal Orthogroups appear on [34] and the number of non-isomorphic models from order

4.2.29 Semilattices of Groups

A Semilattice of groups is a structure $\langle A, \cdot, ' \rangle$ such that

1. $(xy)z = x(yz)$;
2. $x = xx'x$;
3. $\forall x \forall y \exists z : \ xy = yzx$.

Semilattices of groups appear on [1] and the number of non-isomorphic models from order

4.2.30 Completely Simple Semigroups

A Completely simple semigroup is a structure $\langle A, \cdot, ' \rangle$ such that

1. $(xy)z = x(yz)$;
2. $x = xx'x$;
3. $\forall x \forall y \exists z : \; x = xyxz$.

Completely simple semigroups appear on [1] and the number of non-isomorphic models from order

4.3 Type (2,1,1)

4.3.1 Bar inverse monoids

A Bar inverse monoid is a structure $\langle A, \cdot, ^\star, [], 1 \rangle$ such that

1. $x(yz) = (xy)z$;
2. $1x = x$;
3. $x1 = x$;
4. $(uv)^\star = v^\star u^\star$;
5. $(u^\star)^\star = u$;
6. $uu^\star u = u$;
7. $uu^\star vv^\star = vv^\star uu^\star$;
8. $[1] = 0$;
9. $[0] = 1$;
10. $u[v] = [uv]u$.

Bar inverse monoids appear on [26] and the number of non-isomorphic models from order

4.3.2 Intra-regular semigroups

An Intra-regular semigroup is a structure $\langle A, \cdot, ', ^{-1} \rangle$ such that

1. $x(yz) = (xy)z$;
2. $x'xxx^{-1} = x$.

Intra-regular semigroups appear on [28] and the number of non-isomorphic models from order

4.3.3 Domain range semigroups

A Domain range semigroup is a structure $\langle A, \cdot, d, r\rangle$ such that

1. $x(yz) = (xy)z$;
2. $d(x)x = x$;
3. $d(xy) = d(xd(y))$;
4. $d(d(x)y) = d(x)d(y)$;
5. $d(x)d(y) = d(y)d(x)$;
6. $d(r(x)) = r(x)$;
7. $xr(x) = x$;
8. $r(xy) = r(r(x)y)$;
9. $r(xr(y)) = r(x)r(y)$;
10. $r(x)r(y) = r(y)r(x)$;
11. $r(d(x)) = d(x)$.

Domain range semigroups appear on p. 81 of [110] and the number of non-isomorphic models from order 2 to order 4 is given in Table 8.

Table 8: Number of Domain range semigroups

Order:	2	3	4
#Models:	3	66	48,498

4.4 Classes of Bands

4.4.1 Bands

A Band is a structure $\langle A, \cdot\rangle$ such that

1. $x(yz) = (xy)z$;
2. $xx = x$.

Bands appear on [28] and the number of non-isomorphic models from order

4.4.2 Semilattices

A Semilattice is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $xx = x$;
3. $xy = yx$.

Semilattices appear on [28] and the number of non-isomorphic models from order

4.4.3 Regular bands

A Regular band is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $xx = x$;
3. $xyxzx = xyzx$.

Regular bands appear on [30] and the number of non-isomorphic models from order

4.4.4 Left regular bands

A Left regular band is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $xx = x$;
3. $xyx = xy$.

Left regular bands appear on [30] and the number of non-isomorphic models from order

4.4.5 Right regular bands

A Right regular band is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $xx = x$;
3. $xyx = yx$.

Right regular bands appear on [30] and the number of non-isomorphic models from order

4.4.6 Rectangular bands

A Rectangular band is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)z$;
2. $xx = x$;
3. $xyx = x$.

Rectangular bands appear on [30] and the number of non-isomorphic models from order

4.4.7 Left normal bands

A Left normal band is a structure $\langle A, \cdot \rangle$ such that

1. $(xy)z = x(yz)$;
2. $xx = x$;
3. $xyz = xzy$.

Left normal bands appear on [34] and the number of non-isomorphic models from order

4.4.8 Right normal bands

A Right normal band is a structure $\langle A, \cdot \rangle$ such that

1. $(xy)z = x(yz)$;
2. $xx = x$;
3. $yzx = zyx$.

Right normal bands appear on [34] and the number of non-isomorphic models from order

4.4.9 Normal bands

A Normal band is a band $\langle A, \cdot \rangle$ such that

1. $xyzx = xzyx$.

Normal bands appear in [87] and the number of non-isomorphic models from order 2 to order 9 is given in Table 9.

Table 9: Number of Normal bands

Order:	2	3	4	5	6	7	8	9
#Models:	3	8	30	114	536	2,698	15,441	96,868

5 Algebras of type (2)

5.1 Medial quasigroups

A Medial quasigroup is a structure $\langle A, \cdot \rangle$ such that

1. $\forall u \forall v \exists^1 w : \; wu = v$;
2. $\forall u \forall v \exists^1 w : \; uw = v$;
3. $(xy)(uv) = (xu)(yv)$.

Medial quasigroups appear on [16] and the number of non-isomorphic models from order

5.2 Left distributive quasigroups

A Left distributive quasigroup is a structure $\langle A, \cdot \rangle$ such that

1. $\forall u \forall v \exists^1 w : \; wu = v$;
2. $\forall u \forall v \exists^1 w : \; uw = v$;
3. $x(uv) = (xu)(xv)$.

Left distributive quasigroups appear on [16] and the number of non-isomorphic models from order

5.3 Right distributive quasigroups

A Right distributive quasigroup is a structure $\langle A, \cdot \rangle$ such that

1. $\forall u \forall v \exists^1 w : \; wu = v$;
2. $\forall u \forall v \exists^1 w : \; uw = v$;
3. $(xu)v = (xv)(uv)$.

Right distributive quasigroups appear on [16] and the number of non-isomorphic models from order

5.4 Distributive quasigroups

A Distributive quasigroup is a structure $\langle A, \cdot \rangle$ such that

1. $\forall u \forall v \exists^1 w: \; wu = v$;
2. $\forall u \forall v \exists^1 w: \; uw = v$;
3. $(xu)v = (xv)(uv)$;
4. $x(uv) = (xu)(xv)$.

Distributive quasigroups appear on [16] and the number of non-isomorphic models from order

5.5 Idempotent quasigroups

An Idempotent quasigroup is a structure $\langle A, \cdot \rangle$ such that

1. $\forall u \forall v \exists^1 w: \; wu = v$;
2. $\forall u \forall v \exists^1 w: \; uw = v$;
3. $xx = x$.

Idempotent quasigroups appear on [16] and the number of non-isomorphic models from order

5.6 Unipotent quasigroups

A Unipotent quasigroup is a structure $\langle A, \cdot, 1 \rangle$ such that

1. $\forall u \forall v \exists^1 w: \; wu = v$;
2. $\forall u \forall v \exists^1 w: \; uw = v$;
3. $xx = 1$.

Unipotent quasigroups appear on [16] and the number of non-isomorphic models from order

5.7 Left semi-symetric quasigroups

A Left semi-symetric quasigroup is a structure $\langle A, \cdot \rangle$ such that

1. $\forall u \forall v \exists^1 w: \; wu = v$;
2. $\forall u \forall v \exists^1 w: \; uw = v$;
3. $x(xy) = y$.

Left semi-symetric quasigroups appear on [16] and the number of non-isomorphic models from order

5.8 TS-quasigroups

A TS-quasigroup is a structure $\langle A, \cdot \rangle$ such that

1. $\forall u \forall v \exists^1 w : \ wu = v$;
2. $\forall u \forall v \exists^1 w : \ uw = v$;
3. $x(xy) = y$;
4. $xy = yx$.

TS-quasigroups appear on [16] and the number of non-isomorphic models from order

5.9 Right quasi-Jordan algebras

A Right quasi-Jordan algebra is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = x(zy)$;
2. $(y(xx))x = (yx)(xx)$.

Right quasi-Jordan algebras appear on [21] and the number of non-isomorphic models from order

5.10 LR-Grupoids

An LR-Grupoid is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = y(xz)$;
2. $(xy)z = (xz)y$.

LR-Grupoids appear on [18] and the number of non-isomorphic models from order

5.11 Jordan Grupoids

A Jordan Grupoid is a structure $\langle A, \cdot \rangle$ such that

1. $(xx)(ux) = ((xx)u)x$;
2. $xy = yx$.

Jordan Grupoids appear on [19] and the number of non-isomorphic models from order

5.12 BZ-algebras

A BZ-algebra is a structure $\langle A, \cdot, 0\rangle$ such that

1. $((xy)(zy))(xz) = 0$;
2. $xx = 0$;
3. $x0 = x$;
4. $((xy = 0)$ and $(yx = 0)) \Rightarrow x = y$.

BZ-algebras appear on [15] and the number of non-isomorphic models from order

5.13 BCC-algebras

A BCC-algebra is a structure $\langle A, \cdot, 0\rangle$ such that

1. $((xy)(zy))(xz) = 0$;
2. $xx = 0$;
3. $x0 = x$;
4. $((xy = 0)$ and $(yx = 0)) \Rightarrow x = y$;
5. $0x = 0$.

BCC-algebras appear on [15] and the number of non-isomorphic models from order

5.14 Phi-implicative BZ-algebras

A Phi-implicative BZ-algebra is a structure $\langle A, \cdot, 0\rangle$ such that

1. $((xy)(zy))(xz) = 0$;
2. $xx = 0$;
3. $x0 = x$;
4. $((xy = 0)$ and $(yx = 0)) \Rightarrow x = y$;
5. $xy = (xy)(y(0(0y)))$.

Phi-implicative BZ-algebras appear on [15] and the number of non-isomorphic models from order

5.15 Weakly positive implicative BZ-algebras

A Weakly positive implicative BZ-algebra is a structure $\langle A, \cdot, 0\rangle$ such that

1. $((xy)(zy))(xz) = 0$;
2. $xx = 0$;
3. $x0 = x$;
4. $((xy = 0) \text{ and } (yx = 0)) \Rightarrow x = y$;
5. $(xy)z = ((xz)z)(yz)$.

Weakly positive implicative BZ-algebras appear on [15] and the number of non-isomorphic models from order

5.16 Quasi-Directoids

A Quasi-Directoid is a structure $\langle A, \cdot\rangle$ such that

1. $xx = x$;
2. $x(xy) = xy$;
3. $y(xy) = xy$;
4. $x((xy)z) = (xy)z$.

Quasi-Directoids appear on [13] and the number of non-isomorphic models from order

5.17 Positive Implicative BCK-algebras

A Positive Implicative BCK-algebra is a structure $\langle A, \cdot, 0\rangle$ such that

1. $((xy)(xz))(zy) = 0$;
2. $x0 = x$;
3. $xy = 0 \text{ and } yx = 0 \Rightarrow y = x$;
4. $0x = 0$;
5. $(xy)z = (xz)(yz)$.

Positive Implicative BCK-algebras appear on [9] and the number of non-isomorphic models from order

5.18 RC-systems

A RC-system is a structure $\langle A, \cdot \rangle$ such that

1. $(xy)(xz) = (yx)(yz)$.

RC-systems appear on [7] and the number of non-isomorphic models from order

5.19 LD-systems

A LD-system is a structure $\langle A, \cdot \rangle$ such that

1. $x(yz) = (xy)(xz)$.

LD-systems appear on [5] and the number of non-isomorphic models from order

5.20 Left Bol loops

A Left Bol loop is a loop $\langle A, \cdot, 1 \rangle$ such that

1. $x(y(xz)) = (x(yx))z$.

Left Bol loops appear on p. 13 of [112] and the number of non-isomorphic models from order 2 to order 15 is given in Table 10.

Table 10: Number of Left Bol loops

Order:	2	3	4	5	6	7	8	9	10	11	12	13	14	15
#Models:	1	1	2	1	2	1	11	2	2	1	8	1	2	3

5.21 Right Bol loops

A Right Bol loop is a loop $\langle A, \cdot, 1 \rangle$ such that

1. $z((xy)x) = ((zx)y)x$.

Right Bol loops appear on p. 4 of [112] and the number of non-isomorphic models from order 2 to order 15 is given in Table 11.

Table 11: Number of Right Bol loops

Order:	2	3	4	5	6	7	8	9	10	11	12	13	14	15
#Models:	1	1	2	1	2	1	11	2	2	1	8	1	2	3

5.22 Involutory quandles

An Involutory quandle is a structure $\langle A, \cdot \rangle$ such that

1. $x(xy) = y$;
2. $x(yz) = (xy)(xz)$;
3. $xx = x$.

Involutory quandles appear on p. 2 of [111] and the number of non-isomorphic models from order 2 to order 11 is given in Table 12. The numbers of involutory quandles up to order 10 were previously reported in the OEIS sequence A178432. We have added the number of involutory quandles of order 11.

Table 12: Number of Involutory quandles

Order:	2	3	4	5	6	7	8	9	10	11
#Models:	1	3	5	13	41	142	665	4,288	36,455	436,672

5.23 Quasi-implication algebras

A Quasi-implication algebra is a structure $\langle A, \cdot \rangle$ such that

1. $(xy)x = x$;
2. $(xy)(xz) = (yx)(yz)$;
3. $((xy)(yx))x = ((yx)(xy))y$.

Quasi-implication algebras appear on p. 8 of [107] and the number of non-isomorphic models from order 2 to order 12 is given in Table 13.

Table 13: Number of Quasi-implication algebras

Order:	2	3	4	5	6	7	8	9	10	11	12
#Models:	1	1	2	3	5	8	13	20	33	53	88

5.24 Orthomodular implication algebras

An Orthomodular implication algebra is a structure $\langle A, \cdot, 1\rangle$ such that

1. $xx = 1$;
2. $x(yx) = 1$;
3. $(xy)x = x$;
4. $(xy)y = (yx)x$;
5. $(((xy)y)z)(xz) = 1$;
6. $(((((((((xy)y)z)z)z)x)x)z)x)x = (((xy)y)z)z$.

Orthomodular implication algebras appear on p. 4 of [107] and the number of non-isomorphic models from order 2 to order 12 is given in Table 14.

Table 14: Number of Orthomodular implication algebras

Order:	2	3	4	5	6	7	8	9	10	11	12
#Models:	1	1	2	2	4	6	11	17	33	63	137

5.25 Orthoimplication algebras

An Orthoimplication algebra is a structure $\langle A, \cdot\rangle$ such that

1. $(xy)x = x$;
2. $(xy)y = (yx)x$;
3. $x((yx)z) = xz$.

Orthoimplication algebras appear on p. 4 of [107] and the number of non-isomorphic models from order 2 to order 11 is given in Table 15.

Table 15: Number of Orthoimplication algebras

Order:	2	3	4	5	6	7	8	9	10	11
#Models:	1	1	2	2	4	6	11	17	33	63

5.26 Order algebras

An Order algebra is a structure $\langle A, \cdot \rangle$ such that

1. $xx = x$;
2. $(xy)x = yx$;
3. $(xy)y = xy$;
4. $x((xy)z) = x(yz)$;
5. $((xy)z)y = (xz)y$.

Order algebras appear in [90] and the number of non-isomorphic models from order 2 to order 7 is given in Table 16.

Table 16: Number of Order algebras

Order:	2	3	4	5	6	7
#Models:	2	7	36	251	2,306	26,568

5.27 Hilbert algebras

A Hilbert algebra is a structure $\langle A, \rightarrow, 1 \rangle$ such that

1. $x \rightarrow (y \rightarrow x) = 1$;
2. $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$;
3. $x \rightarrow y = 1$ and $y \rightarrow x = 1 \Rightarrow x = y$.

Hilbert algebras appear in [71] and the number of non-isomorphic models from order 2 to order 9 is given in Table 17.

Table 17: Number of Hilbert algebras

Order:	2	3	4	5	6	7	8	9
#Models:	1	2	6	21	95	550	4,036	37,603

5.28 Directoids

A Directoid is a structure $\langle A, \cdot \rangle$ such that

1. $xx = x$;

2. $(xy)x = xy$;

3. $y(xy) = xy$;

4. $x((xy)z) = (xy)z$.

Directoids appear in [67] and the number of non-isomorphic models from order 2 to order 7 is given in Table 18.

Table 18: Number of Directoids

Order:	2	3	4	5	6	7
#Models:	1	2	7	61	2,297	517,919

5.29 BCI-algebras

A BCI-algebra is a structure $\langle A, \cdot, 0 \rangle$ such that

1. $((xy)(xz))(zy) = 0$;

2. $(x(xy))y = 0$;

3. $xx = 0$;

4. $xy = yx = 0 \Rightarrow x = y$;

5. $x0 = 0 \Rightarrow x = 0$.

BCI-algebras appear in [44] and the number of non-isomorphic models from order 2 to order 8 is given in Table 19.

Table 19: Number of BCI-algebras

Order:	2	3	4	5	6	7	8
#Models:	2	5	22	118	974	10,834	162,555

5.30 BCK-algebras

A BCK-algebra is a structure $\langle A, \cdot, 0\rangle$ such that

1. $((xy)(xz))(zy) = 0$;
2. $x0 = x$;
3. $0x = 0$;
4. $xy = yx = 0 \Rightarrow x = y$;

BCK-algebras appear in [45] and the number of non-isomorphic models from order 2 to order 8 is given in Table 20.

Table 20: Number of BCK-algebras

Order:	2	3	4	5	6	7	8
#Models:	1	3	14	88	775	9,170	143,866

5.31 Commutative BCK-algebras

A Commutative BCK-algebra is a structure $\langle A, \cdot, 0\rangle$ such that

1. $((xy)(xz))(zy) = 0$;
2. $x0 = x$;
3. $0x = 0$;
4. $xy = yx = 0 \Rightarrow x = y$;
5. $x(xy) = y(yx)$.

Commutative BCK-algebras appear in [54] and the number of non-isomorphic models from order 2 to order 10 is given in Table 21.

Table 21: Number of Commutative BCK-algebras

Order:	2	3	4	5	6	7	8	9	10
#Models:	1	2	5	11	28	72	192	515	1,426

5.32 Quandles

A Quandle is a structure $\langle A, \lhd \rangle$ such that

1. $x \lhd x = x$;
2. $(x \lhd y) \lhd z = (x \lhd z) \lhd (y \lhd z)$;
3. $\forall x \forall y \exists^1 z: \; z \lhd x = y$.

Quandles appear on [23] and the number of non-isomorphic models from order

5.33 Algebras of type (2,1)

5.33.1 Left state BCK-algebras

A Left state BCK-algebra is a structure $\langle A, \cdot, \mu, 0 \rangle$ such that

1. $xy = 0 \Rightarrow \mu(x)\mu(y) = 0$;
2. $\mu(xy) = \mu(x)\mu(x(xy))$;
3. $\mu(\mu(x)\mu(y)) = \mu(x)\mu(y)$;
4. $((xy)(xz))(zy) = 0$;
5. $x0 = x$;
6. $xy = 0$ and $yx = 0 \Rightarrow y = x$;
7. $0x = 0$.

Left state BCK-algebras appear on [9] and the number of non-isomorphic models from order

5.33.2 Right state BKC-algebras

A Right state BKC-algebra is a structure $\langle A, \cdot, \mu, 0 \rangle$ such that

1. $xy = 0 \Rightarrow \mu(x)\mu(y) = 0$;
2. $\mu(xy) = \mu(x)\mu(y(yx))$;
3. $\mu(\mu(x)\mu(y)) = \mu(x)\mu(y)$;
4. $((xy)(xz))(zy) = 0$;
5. $x0 = x$;
6. $xy = 0$ and $yx = 0 \Rightarrow y = x$;
7. $0x = 0$.

Right state BKC-algebras appear on [9] and the number of non-isomorphic models from order

6 Algebras of type (2,2)

6.1 Wajsberg Hoops

A Wajsberg Hoop is a structure $\langle A, \cdot, +, 1\rangle$ such that

1. $x(yz) = (xy)z$;
2. $x1 = x$;
3. $xy = yx$;
4. $x + x = 1$;
5. $(x + y)x = (y + x)y$;
6. $x + (y + z) = (xy) + z$;
7. $(x + y) + y = (y + x) + x$.

Wajsberg Hoops appear on [20] and the number of non-isomorphic models from order

6.2 Bounded Hoops

A Bounded Hoop is a structure $\langle A, \cdot, +, 1, 0\rangle$ such that

1. $x(yz) = (xy)z$;
2. $x1 = x$;
3. $xy = yx$;
4. $x + x = 1$;
5. $(x + y)x = (y + x)y$;
6. $x + (y + z) = (xy) + z$;
7. $0 + x = 1$.

Bounded Hoops appear on [20] and the number of non-isomorphic models from order

6.3 Modular lattices

A Modular lattice is a structure $\langle A, \wedge, \vee \rangle$ such that

1. $(x \wedge z = x) \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z$;
2. $x \wedge y = y \wedge x$;
3. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$;
4. $x \wedge x = x$;
5. $x \vee (x \wedge y) = x$;
6. $x \wedge (x \vee y) = x$;
7. $x \vee y = y \vee x$;
8. $x \vee (y \vee z) = (x \vee y) \vee z$;
9. $x \vee x = x$.

Modular lattices appear on [22] and the number of non-isomorphic models from order

6.4 Unital Dialgebras

A Unital Dialgebra is a structure $\langle A, \cdot, +, 1 \rangle$ such that

1. $u(vw) = (uv)w$ and $(uv)w = u(v + w)$;
2. $(u + v) + w = u + (v + w)$ and $u + (v + w) = (uv) + w$;
3. $(u + v)w = u + (vw)$;
4. $1 + u = u$;
5. $u1 = u$.

Unital Dialgebras appear on [10] and the number of non-isomorphic models from order

6.5 Non Associative Dialgebras

A Non associative dialgebra is a structure $\langle A, \cdot, + \rangle$ such that

1. $(uv)w = u(v + w)$;
2. $u + (v + w) = (uv) + w$;
3. $(u + v)w = u + (vw)$.

Non associative dialgebras appear on [11] and the number of non-isomorphic models from order

6.6 Semidistributive lattices

A Semidistributive lattice is a structure $\langle A, \wedge, \vee \rangle$ such that

1. $x \vee y = x \vee z \Rightarrow x \vee (y \wedge z) = x \vee y$;
2. $x \wedge y = x \wedge z \Rightarrow x \wedge (y \vee z) = x \wedge y$;
3. $x \wedge y = y \wedge x$;
4. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$;
5. $x \wedge x = x$;
6. $x \vee (x \wedge y) = x$;
7. $x \wedge (x \vee y) = x$;
8. $x \vee y = y \vee x$;
9. $x \vee (y \vee z) = (x \vee y) \vee z$;
10. $x \vee x = x$.

Semidistributive lattices appear on [8] and the number of non-isomorphic models from order

6.7 RLC-systems

A RLC-system is a structure $\langle A, \cdot, + \rangle$ such that

1. $(xy)(xz) = (yx)(yz)$;
2. $(z + x) + (y + x) = (z + y) + (x + y)$;
3. $(yx) + (xy) = x$;
4. $x = (y + x)(x + y)$.

RLC-systems appear on [7] and the number of non-isomorphic models from order

6.8 ALD-systems

An ALD-system is a structure $\langle A, +, \cdot \rangle$ such that

1. $x(yz) = (xy)(xz)$;
2. $x(yz) = (x + y)z$;
3. $x(y + z) = (xy) + (xz)$.

ALD-systems appear on [5] and the number of non-isomorphic models from order

6.9 Left I-Semirings

A Left I-Semiring is a structure $\langle A, +, \cdot, 0, 1\rangle$ such that

1. $\langle A, +, 0\rangle$ is a commutative monoid;
2. $\langle A, \cdot, 1\rangle$ is a monoid;
3. $(x + y)z = (xz) + (yz)$;
4. $0x = 0$.

Left I-Semirings, also known as lazy semirings, appear in [118] and the number of non-isomorphic models from order 2 to order 8 is given in Table 22.

Table 22: Number of Left I-Semirings

Order:	2	3	4	5	6	7	8
#Models:	1	4	24	195	2,146	30,471	567,060

6.10 Doppelsemigroups

A Doppelsemigroup is a structure $\langle A, \vdash, \dashv\rangle$ such that

1. $(x \dashv y) \vdash z = x \dashv (y \vdash z)$;
2. $(x \vdash y) \dashv z = x \vdash (y \dashv z)$;
3. $(x \dashv y) \dashv z = x \dashv (y \dashv z)$;
4. $x \vdash (y \vdash z) = (x \vdash y) \vdash z$.

Doppelsemigroups appear on p. 11 of [119] and the number of non-isomorphic models from order 2 to order 5 is given in Table 23.

Table 23: Number of Doppelsemigroups

Order:	2	3	4	5
#Models:	8	77	1,217	68,177

6.11 Strong doppelsemigroups

A Strong doppelsemigroup is a structure $\langle A, \vdash, \dashv \rangle$ such that

1. $(x \dashv y) \vdash z = x \dashv (y \vdash z)$;
2. $(x \vdash y) \dashv z = x \vdash (y \dashv z)$;
3. $(x \dashv y) \dashv z = x \dashv (y \dashv z)$;
4. $x \vdash (y \vdash z) = (x \vdash y) \vdash z$;
5. $x \dashv (y \vdash z) = x \vdash (y \dashv z)$.

Strong doppelsemigroups appear on p. 12 of [119] and the number of non-isomorphic models from order 2 to order 5 is given in Table 24.

Table 24: Number of Strong doppelsemigroups

Order:	2	3	4	5
#Models:	8	65	841	56,428

6.12 Dimonoids

A Dimonoid is a structure $\langle A, \vdash, \dashv \rangle$ such that

1. $(x \dashv y) \dashv z = x \dashv (y \vdash z)$;
2. $(x \dashv y) \vdash z = x \vdash (y \vdash z)$;
3. $(x \dashv y) \dashv z = x \dashv (y \dashv z)$;
4. $x \vdash (y \vdash z) = (x \vdash y) \vdash z$.

Dimonoids appear on p. 12 of [119] and the number of non-isomorphic models from order 2 to order 5 is given in Table 25.

Table 25: Number of Dimonoids

Order:	2	3	4	5
#Models:	10	74	1,246	84,836

6.13 Commutative dimonoids

A Commutative dimonoid is a structure $\langle A, \vdash, \dashv \rangle$ such that

1. $(x \dashv y) \dashv z = x \dashv (y \vdash z)$;
2. $(x \dashv y) \vdash z = x \vdash (y \vdash z)$;
3. $(x \dashv y) \dashv z = x \dashv (y \dashv z)$;
4. $x \vdash (y \vdash z) = (x \vdash y) \vdash z$;
5. $x \vdash y = y \vdash$ and $x \dashv y = y \dashv x$.

Commutative dimonoids appear on p. 12 of [119] and the number of non-isomorphic models from order 2 to order 6 is given in Table 26.

Table 26: Number of Commutative dimonoids

Order:	2	3	4	5	6
#Models:	3	14	101	1,495	102,268

6.14 Diassociative semigroups

A Diassociative semigroup is a structure $\langle A, \leftarrow, \rightarrow \rangle$ such that

1. $(x \leftarrow y) \leftarrow z = x \leftarrow (y \leftarrow z) = x \leftarrow (y \rightarrow z)$;
2. $(x \rightarrow y) \leftarrow z = x \rightarrow (y \leftarrow z)$;
3. $(x \rightarrow y) \rightarrow z = (x \leftarrow y) \rightarrow z = x \rightarrow (y \rightarrow z)$.

Diassociative semigroups, also known as Associative dialgebras and Associative dimonoids, appear on p. 3 of [120] and the number of non-isomorphic models from order 2 to order 5 is given in Table 27.

Table 27: Number of Diassociative semigroups

Order:	2	3	4	5
#Models:	8	52	734	55,883

6.15 Duplicial semigroups

A Duplicial semigroup is a structure $\langle A, \leftarrow, \rightarrow \rangle$ such that

1. $(x \leftarrow y) \leftarrow z = x \leftarrow (y \leftarrow z)$;
2. $(x \rightarrow y) \leftarrow z = x \rightarrow (y \leftarrow z)$;
3. $(x \rightarrow y) \rightarrow z = x \rightarrow (y \rightarrow z)$.

Duplicial semigroups appear on p. 5 of [120] and the number of non-isomorphic models from order 2 to order 4 is given in Table 28.

Table 28: Number of Duplicial semigroups

Order:	2	3	4
#Models:	15	251	8,898

6.16 Left disemigroups

A Left disemigroup is a structure $\langle A, \vdash, \star \rangle$ such that

1. $\langle A, \vdash \rangle$ and $\langle A, \star \rangle$ are both semigroups;
2. $x \vdash (y \vdash z) = (x \star y) \vdash z$;
3. $x \vdash (y \star z) = (x \vdash y) \star z$.

Left disemigroups appear on p. 2 of [117] and the number of non-isomorphic models from order 2 to order 5 is given in Table 29.

Table 29: Number of Left disemigroups

Order:	2	3	4	5
#Models:	11	117	2,252	100,461

6.17 Right disemigroups

A Right disemigroup is a structure $\langle A, \star, \dashv \rangle$ such that

1. $\langle A, \dashv \rangle$ and $\langle A, \star \rangle$ are both semigroups;
2. $x \dashv (y \dashv z) = x \dashv (y \star z)$;
3. $(x \star y) \dashv z = x \star (y \dashv z)$.

Right disemigroups appear on p. 2 of [117] and the number of non-isomorphic models from order 2 to order 5 is given in Table 30.

Table 30: Number of Right disemigroups

Order:	2	3	4	5
#Models:	11	117	2,252	100,461

6.18 Disemigroups

A Disemigroup is a structure $\langle A, \vdash, \dashv \rangle$ such that

1. $\langle A, \vdash, \dashv \rangle$ is a left disemigroup;
2. $\langle A, \vdash, \dashv \rangle$ is a right disemigroup.

Disemigroups appear on p. 3 of [117] and the number of non-isomorphic models from order 2 to order 5 is given in Table 31.

Table 31: Number of Disemigroups

Order:	2	3	4	5
#Models:	6	41	513	51,391

6.19 Near-semirings

A Near-semiring is a structure $\langle S, +, \cdot \rangle$ such that

1. $x + (y + z) = (x + y) + z$;
2. $x(yz) = (xy)z$;
3. $x(y + z) = (xy) + (xz)$.

Near-semirings appear on p. 2 of [106] and the number of non-isomorphic models from order 2 to order 4 is given in Table 32.

Table 32: Number of Near-semirings

Order:	2	3	4
#Models:	22	531	25,364

6.20 Skew lattices

A Skew lattice is a structure $\langle A, \vee, \wedge \rangle$ such that

1. $\langle A, \vee \rangle$ is a band;
2. $\langle A, \wedge \rangle$ is a band;
3. $x \wedge (x \vee y) = x = x \vee (x \wedge y)$;
4. $(x \vee y) \wedge y = y = (x \wedge y) \vee y$.

Skew lattices appear in [104] and the number of non-isomorphic models from order 2 to order 10 is given in Table 33.

Table 33: Number of Skew lattices

Order:	2	3	4	5	6	7	8	9	10
#Models:	3	7	21	53	173	531	1,971	7,903	37,390

6.21 Semirings with zero

A Semiring with zero is a structure $\langle S, +, \cdot, 0 \rangle$ such that

1. $\langle S, +, 0 \rangle$ is a commutative monoid;
2. $\langle S, \cdot \rangle$ is a semigroup;
3. $x(y + z) = (xy) + (xz)$;
4. $(y + z)x = (yx) + (zx)$;
5. $0x = 0 = x0$.

Semirings with zero appear in [103] and the number of non-isomorphic models from order 2 to order 5 is given in Table 34.

Table 34: Number of Semirings with zero

Order:	2	3	4	5
#Models:	8	246	628	55,315

6.22 Semirings with identity and zero

A Semiring with identity and zero is a structure $\langle S, +, \cdot, 1, 0 \rangle$ such that

1. $\langle S, +, 0 \rangle$ is a commutative monoid;
2. $\langle S, \cdot, 1 \rangle$ is a monoid;
3. $x(y + z) = (xy) + (xz)$;
4. $(y + z)x = (yx) + (zx)$;
5. $0x = 0 = x0$.

Semirings with identity and zero appear in [102] and the number of non-isomorphic models from order 2 to order 5 is given in Table 35.

Table 35: Number of Semirings with identity and zero

Order:	2	3	4	5
#Models:	4	39	628	34,786

6.23 Semirings with identity

A Semiring with identity is a structure $\langle S, +, \cdot, 1 \rangle$ such that

1. $\langle S, + \rangle$ is a commutative semigroup;
2. $\langle S, \cdot, 1 \rangle$ is a monoid;
3. $x(y + z) = (xy) + (xz)$;
4. $(y + z)x = (yx) + (zx)$.

Semirings with identity appear in [101] and the number of non-isomorphic models from order 2 to order 5 is given in Table 36.

Table 36: Number of Semirings with identity

Order:	2	3	4	5
#Models:	4	46	663	35,451

6.24 Right hoops

A Right hoop is a structure $\langle A, \cdot, /, 1\rangle$ such that

1. $\langle A, \cdot, 1\rangle$ is a monoid;
2. $x/(yz) = (x/z)/y$;
3. $x/x = 1$;
4. $(x/y)y = (y/x)x$.

Right hoops appear in [99] and the number of non-isomorphic models from order 2 to order 11 is given in Table 37.

Table 37: Number of Right hoops

Order:	2	3	4	5	6	7	8	9	10	11
#Models:	1	2	5	10	23	49	111	244	545	1,203

6.25 Pocrims

A Pocrim is a structure $\langle A, \oplus, \ominus, 0\rangle$ such that

1. $((x \ominus y) \ominus (x \ominus z)) \ominus (z \ominus y) = 0$;
2. $x \ominus 0 = x$;
3. $0 \ominus x = 0$;
4. $(x \ominus y) \ominus z = x \ominus (z \oplus y)$;
5. $x \ominus y = y \ominus x = 0 \Rightarrow x = y$.

Pocrims appear in [96] and the number of non-isomorphic models from order 2 to order 8 is given in Table 38.

Table 38: Number of Pocrims

Order:	2	3	4	5	6	7	8
#Models:	2	4	14	45	217	1,159	7,457

6.26 Neardistributive lattices

A Neardistributive lattice is a structure $\langle A, \vee, \wedge \rangle$ such that

1. $\langle A, \vee, \wedge \rangle$ is a lattice;
2. $x \wedge (y \vee z) = x \wedge [y \vee (x \wedge [z \vee (x \wedge y)])]$;
3. $x \vee (y \wedge z) = x \vee [y \wedge (x \vee [z \wedge (x \vee y)])]$.

Neardistributive lattices appear in [86] and the number of non-isomorphic models from order 2 to order 11 is given in Table 39.

Table 39: Number of Neardistributive lattices

Order:	2	3	4	5	6	7	8	9	10	11
#Models:	1	1	2	4	9	22	60	174	532	1,700

6.27 Multiplicative semilattices

A Multiplicative semilattice is a structure $\langle A, \vee, \cdot \rangle$ such that

1. $\langle A, \vee \rangle$ is a semilattice;
2. $x(y \vee z) = xy \vee xz$;
3. $(x \vee y)z = xz \vee yz$.

Multiplicative semilattices appear in [83] and the number of non-isomorphic models from order 2 to order 4 is given in Table 40.

Table 40: Number of Multiplicative semilattices

Order:	2	3	4
#Models:	6	220	32,234

6.28 Hoops

A Hoop is a structure $\langle A, \cdot, \rightarrow, 1 \rangle$ such that

1. $\langle A, \cdot, 1 \rangle$ is a commutative monoid;
2. $x \rightarrow (y \rightarrow z) = (xy) \rightarrow z$;

3. $x \to x = 1$;

4. $(x \to y)x = (y \to x)y$.

Hoops appear in [72] and the number of non-isomorphic models from order 2 to order 13 is given in Table 41.

Table 41: Number of Hoops

Order:	2	3	4	5	6	7	8	9	10	11	12	13
#Models:	1	2	5	10	23	49	111	244	545	1,203	2,697	5,974

6.29 Complemented modular lattices

A Complemented modular lattice is a structure $\langle L, \vee, \wedge, 0, 1\rangle$ such that

1. $\langle L, \vee, \wedge, 0, 1\rangle$ is a complemented lattice;

2. $((x \wedge z) \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$.

Complemented modular lattices appear in [64] and the number of non-isomorphic models from order 2 to order 11 is given in Table 42.

Table 42: Number of Complemented modular lattices

Order:	2	3	4	5	6	7	8	9	10	11
#Models:	1	0	1	2	6	13	41	100	303	797

6.30 BCK-join-semilattice

A BCK-join-semilattice is a structure $\langle A, \vee, \to, 1\rangle$ such that

1. $(x \to y) \to ((y \to z) \to (x \to z)) = 1$;

2. $1 \to x = x$;

3. $x \to 1 = 1$;

4. $x \to (x \vee y) = 1$;

5. $x \vee ((x \to y) \to y) = ((x \to y) \to y)$;

6. $x \vee x = x$;

7. $x \vee y = y \vee x$;

8. $(x \vee y) \vee z = x \vee (y \vee z)$.

BCK-join-semilattices appear in [46] and the number of non-isomorphic models from order 2 to order 7 is given in Table 43.

Table 43: Number of BCK-join-semilattice

Order:	2	3	4	5	6	7
#Models:	1	3	14	87	745	8,282

6.31 BCK-meet-semilattices

A BCK-meet-semilattice is a structure $\langle A, \wedge, \rightarrow, 1\rangle$ such that

1. $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$;

2. $1 \rightarrow x = x$;

3. $x \rightarrow 1 = 1$;

4. $(x \wedge y) \rightarrow y = 1$;

5. $x \wedge ((x \rightarrow y) \rightarrow y) = x$;

6. $x \wedge x = x$;

7. $x \wedge y = y \wedge x$;

8. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.

BCK-meet-semilattices appear in [48] and the number of non-isomorphic models from order 2 to order 7 is given in Table 44.

Table 44: Number of BCK-meet-semilattices

Order:	2	3	4	5	6	7
#Models:	1	2	8	38	265	2,391

6.32 Algebras of type (2,2,1)

6.32.1 Digroups

A Digroup is a structure $\langle A, \vdash, \dashv, \dagger, 1\rangle$ such that

1. $\langle A, \vdash\ \rangle$ and $\langle A, \dashv\ \rangle$ are both semigroups;
2. $x \dashv (y \vdash z) = x \dashv (y \dashv z)$;
3. $(x \dashv y) \vdash z = (x \vdash y) \vdash z$;
4. $x \vdash (y \dashv z) = (x \vdash y) \dashv z$;
5. $1 \vdash x = x = x \dashv 1$;
6. $x \vdash x^\dagger = 1 = x^\dagger \dashv x$.

Digroups appear in [116] and the number of non-isomorphic models from order 2 to order 15 is given in Table 45.

Table 45: Number of Digroups

Order:	2	3	4	5	6	7	8	9	10	11	12	13	14	15
#Models:	2	2	4	2	6	2	10	4	7	2	17	2	8	5

6.32.2 Ortholattices

An Ortholattice is a structure $\langle L, \vee, \wedge, ', 0, 1\rangle$ such that

1. $\langle L, \vee, \wedge, 0, 1\rangle$ is a bounded lattice;
2. $x \vee x' = 1$;
3. $x \wedge x' = 0$;
4. $x'' = x$;
5. $(x \vee y)' = x' \wedge y'$;
6. $(x \wedge y)' = x' \vee y'$.

Ortholattices appear in [92] and the number of non-isomorphic models from order 2 to order 12 is given in Table 46.

Table 46: Number of Ortholattices

Order:	2	3	4	5	6	7	8	9	10	11	12
#Models:	1	0	1	3	13	46	226	1,047	5,432	29,002	163,783

6.32.3 Ockham algebras

A Ockham algebra is a structure $\langle A, \vee, \wedge, ', 0, 1\rangle$ such that

1. $\langle A, \vee, \wedge, 0, 1\rangle$; is a bounded distributive lattice;
2. $(x \wedge y)' = x' \vee y'$;
3. $(x \vee y)' = x' \wedge y'$;
4. $0' = 1$;
5. $1' = 0$.

Ockham algebras appear in [89] and the number of non-isomorphic models from order 2 to order 10 is given in Table 47.

Table 47: Number of Ockham algebras

Order:	2	3	4	5	6	7	8	9	10
#Models:	1	3	13	45	197	827	3,654	16,058	71,709

6.32.4 Near-rings

A Near-ring is a structure $\langle N, +, \cdot, -, 0\rangle$ such that

1. $\langle N, +, -, 0\rangle$ is a group;
2. $\langle N, \cdot\rangle$ is a semigroup;
3. $(x + y)z = xz + yz$.

Near-rings appear in [84] and the number of non-isomorphic models from order 2 to order 13 is given in Table 48. The numbers of near-rings up to order 7 were previously reported in the OEIS sequence A305858. We extend this sequence to order 13.

Table 48: Number of Near-rings

Order:	2	3	4	5	6	7	8	9	10	11	12	13
#Models:	3	5	35	10	99	24	3,856	486	535	139	54,694	454

6.32.5 Near-rings with identity

A Near ring with identity is a structure $\langle N, +, \cdot, -, 0, 1\rangle$ such that

1. $\langle N, +, -, 0, \cdot\rangle$ is a near-ring;
2. $x \cdot 1 = x = 1 \cdot x$.

Near-rings with identity appear in [85] and the number of non-isomorphic models from order 2 to order 19 is given in Table 49.

Table 49: Number of Near-rings with identity

Order:	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
#Models:	1	1	6	1	1	1	53	11	1	1	11	1	1	1	4,274	1	26	1

6.32.6 Involutive lattices

An Involutive lattice is a structure $\langle A, \vee, \wedge, \neg\rangle$ such that

1. $\langle A, \vee, \wedge\rangle$ is a lattice;
2. $\neg(x \wedge y) = \neg x \vee \neg y$;
3. $\neg\neg x = x$.

Involutive lattices appear in [73] and the number of non-isomorphic models from order 2 to order 13 is given in Table 50.

Table 50: Number of Involutive lattices

Order:	2	3	4	5	6	7	8	9	10	11	12	13
#Models:	1	1	3	4	12	20	61	122	389	906	3,047	8,028

6.32.7 Division rings

A Division ring is a structure $\langle R, +, \cdot, -, 0, 1\rangle$ such that

1. $\langle R, +, -, 0, \cdot, 1\rangle$ is a ring with identity;
2. $0 \neq 1$;
3. $x \neq 0 \Rightarrow \exists y\ (xy = 1)$.

Division rings, also known as Skew-fields, appear in [69] and the number of non-isomorphic models from order 2 to order 15 is given in Table 51.

Table 51: Number of Division rings

Order:	2	3	4	5	6	7	8	9	10	11	12	13	14	15
#Models:	2	3	3	5	0	7	4	6	0	11	0	13	0	0

6.32.8 Clifford algebras

A Clifford algebra is a structure $\langle A, +, \cdot, -, 0, 1\rangle$ such that

1. $x + y = y + x$;
2. $(x + y) + z = x + (y + z)$;
3. $x(yz) = (xy)z$;
4. $x(y + z) = xy + xz$;
5. $(y + z)x = yx + zx$;
6. $x + 0 = x$;
7. $1x = x$;
8. $x + (-x) = 0$.

Clifford algebras, also known as Geometric algebras, appear in [53] and the number of non-isomorphic models from order 2 to order 9 is given in Table 52.

Table 52: Number of Clifford algebras

Order:	2	3	4	5	6	7	8	9
#Models:	2	5	16	51	202	879	4,454	25,322

6.32.9 Commutative regular rings

A Commutative regular ring is a structure $\langle R, +, \cdot, -, 0, 1\rangle$ such that

1. $\langle R, +, -, 0, \cdot, 1\rangle$ is a ring with identity;
2. $\forall x \exists y \quad xyx = x$;
3. $xy = yx$.

Commutative regular rings appear in [61] and the number of non-isomorphic models from order 2 to order 13 is given in Table 53.

Table 53: Number of Commutative regular rings

Order:	2	3	4	5	6	7	8	9	10	11	12	13
#Models:	2	3	13	5	72	7	900	384	800	11	169,452	13

6.33 Algebras of type (2,2,1,1)

6.33.1 Left braces

A Left brace is a structure $\langle A, \cdot, +, ^{-1}, -, 0, 1\rangle$ such that

1. $(x(y+z)) + x = (xy) + (xz)$;
2. $x(yz) = (xy)z$;
3. $1x = x = x1$;
4. $xx^{-1} = 1 = x^{-1}x$;
5. $x + (y + z) = (x + y) + z$;
6. $0 + x = x = x + 0$;
7. $x + (-x) = 0 = (-x) + x$;
8. $x + y = y + x$.

Left braces appear on [6] and the number of non-isomorphic models from order

6.33.2 Skew braces

A Skew brace is a structure $\langle G, +, \cdot, ^{-1}, -, 1, 0\rangle$ such that

1. $\langle G, \cdot, ^{-1}, 1\rangle$ is a group;
2. $\langle G, +, -, 0\rangle$ is a group;
3. $x + (yz) = (x + y)x^{-1}(x + z)$;

Skew Braces appear on [3] and the number of non-isomorphic models from order

7 Algebras of type (2,2,2)

7.1 Quasigroups

A Quasigroup is a structure $\langle A, \cdot, \backslash, /\rangle$ such that

1. $x(x/y) = y$;
2. $(y\backslash x)x = y$;
3. $x/(xy) = y$;
4. $(yx)\backslash x = y$.

Quasigroups appear on [16] and the number of non-isomorphic models from order

7.2 Associative trioids

An Associative trioid is a structure $\langle A, \vdash, \dashv, \perp\rangle$ such that

1. $(x \dashv y) \dashv z = x \dashv (y \dashv z)$;
2. $(x \dashv y) \dashv z = x \dashv (y \vdash z)$;
3. $(x \vdash y) \dashv z = x \vdash (y \dashv z)$;
4. $(x \dashv y) \vdash z = x \vdash (y \vdash z)$;
5. $(x \vdash y) \vdash z = x \vdash (y \vdash z)$;
6. $(x \dashv y) \dashv z = x \dashv (y \perp z)$;
7. $(x \perp y) \dashv z = x \perp (y \dashv z)$;
8. $(x \dashv y) \perp z = x \perp (y \vdash z)$;

9. $(x \vdash y) \perp z = x \vdash (y \perp z)$;
10. $(x \perp y) \vdash z = x \vdash (y \vdash z)$;
11. $(x \perp y) \perp z = x \perp (y \perp z)$.

Associative trioids appear on p. 5 of [121] and the number of non-isomorphic models from order 2 to order 4 is given in Table 54.

Table 54: Number of Associative trioids

Order:	2	3	4
#Models:	15	185	5,914

7.3 RIF loops

A RIF loop is a loop $\langle A, \cdot, \backslash, /, 1\rangle$ such that

1. $x\backslash y = (x/1)y$;
2. $y/x = y(1/x)$;
3. $(xy)(z(xy)) = ((x(yz))x)y$.

RIF loops appear on p. 6 of [112] and the number of non-isomorphic models from order 2 to order 11 is given in Table 55.

Table 55: Number of RIF loops

Order:	2	3	4	5	6	7	8	9	10	11
#Models:	1	1	2	2	2	1	11	2	23	242

7.4 ARIF loops

An ARIF loop is a loop $\langle A, \cdot, \backslash, /, 1\rangle$ such that

1. $x(yx) = (xy)x$;
2. $(zx)((yx)y) = z((xy)x)y$.

ARIF loops appear on p. 6 of [112] and the number of non-isomorphic models from order 2 to order 11 is given in Table 56.

Table 56: Number of ARIF loops

Order:	2	3	4	5	6	7	8	9	10	11
#Models:	1	1	2	1	2	1	11	3	3	1

7.5 Extra loops

An Extra loop is a loop $\langle A, \cdot, \backslash, /, 1\rangle$ such that

1. $(x(yz))y = (xy)(zy)$.

Extra loops appear on p. 14 of [113] and the number of non-isomorphic models from order 2 to order 15 is given in Table 57.

Table 57: Number of Extra loops

Order:	2	3	4	5	6	7	8	9	10	11	12	13	14	15
#Models:	1	1	2	1	2	1	5	2	2	1	5	1	2	1

7.6 F-quasigroups

An F-quasigroup is a quasigroup $\langle A, \cdot, \backslash, /\rangle$ such that

1. $x(yz) = (xy)((x\backslash x)z)$;
2. $(zy)x = (z(x/x))(yx)$.

F-quasigroups appear on p. 8 of [112] and the number of non-isomorphic models from order 2 to order 12 is given in Table 58.

Table 58: Number of F-quasigroups

Order:	2	3	4	5	6	7	8	9	10	11	12
#Models:	1	5	13	19	6	41	88	116	20	109	155

7.7 Rectangular quasigroups

A Rectangular quasigroup is a structure $\langle A, \cdot, \backslash, /\rangle$ such that

1. $x\backslash(xx) = x$;
2. $(xx)/x = x$;

3. $x(x\backslash y) = x\backslash(xy)$;

4. $(x/y)y = (xy)/y$;

5. $(x\backslash y)\backslash((x\backslash y)(zu)) = (x\backslash(xz))u$;

6. $((xy)(z/u))/(z/u) = x((yu)/u)$.

Rectangular quasigroups appear on p. 71 of [114] and the number of non-isomorphic models from order 2 to order 5 is given in Table 59.

Table 59: Number of Rectangular quasigroups

Order:	2	3	4	5
#Models:	3	7	40	1,413

7.8 Rectangular loops

A Rectangular loop is a structure $\langle A, \cdot, \backslash, / \rangle$ such that

1. $x\backslash(xx) = x$;

2. $(xx)/x = x$;

3. $x(x\backslash y) = x\backslash(xy)$;

4. $(x/y)y = (xy)/y$;

5. $(x\backslash y)\backslash((x\backslash y)(zu)) = (x\backslash(xz))u$;

6. $((xy)(z/u))/(z/u) = x((yu)/u)$;

7. $(x\backslash x)(y\backslash y) = (x/x)(y/y)$.

Rectangular loops appear on p. 72 of [114] and the number of non-isomorphic models from order 2 to order 6 is given in Table 60.

Table 60: Number of Rectangular loops

Order:	2	3	4	5	6
#Models:	3	3	7	8	117

7.9 Left conjugacy closed loops

A Left conjugacy closed loop is a loop $\langle A, \cdot, \backslash, /, 1\rangle$ such that

1. $z(yx) = ((zy)/z)(zx)$.

Left conjugacy closed loops appear on p. 6 of [112] and the number of non-isomorphic models from order 2 to order 11 is given in Table 61.

Table 61: Number of Left conjugacy closed loops

Order:	2	3	4	5	6	7	8	9	10	11
#Models:	1	1	2	1	5	1	24	7	18	1

7.10 Right conjugacy closed loops

A Right conjugacy closed loop is a loop $\langle A, \cdot, \backslash, /, 1\rangle$ such that

1. $(xy)z = (xz)(z\backslash(yz))$.

Right conjugacy closed loops appear on p. 6 of [112] and the number of non-isomorphic models from order 2 to order 11 is given in Table 62.

Table 62: Number of Right conjugacy closed loops

Order:	2	3	4	5	6	7	8	9	10	11
#Models:	1	1	2	1	5	1	24	7	18	1

7.11 Conjugacy closed loops

A Conjugacy closed loop is a loop $\langle A, \cdot, \backslash, /, 1\rangle$ such that

1. $z(yx) = ((zy)/z)(zx)$;
2. $(xy)z = (xz)(z\backslash(yz))$.

Conjugacy closed loops appear on p. 14 of [113] and the number of non-isomorphic models from order 2 to order 11 is given in Table 63.

Table 63: Number of Conjugacy closed loops

Order:	2	3	4	5	6	7	8	9	10	11
#Models:	1	1	2	1	3	1	7	5	3	1

7.12 Multiplicative lattices

A Multiplicative lattice is a structure $\langle A, \vee, \wedge, \cdot \rangle$ such that

1. $\langle A, \vee, \wedge \rangle$ is a lattice;
2. $x(y \vee z) = xy \vee xz$;
3. $(x \vee y)z = xz \vee yz$.

Multiplicative lattices appear in [82] and the number of non-isomorphic models from order 2 to order 4 is given in Table 64.

Table 64: Number of Multiplicative lattices

Order:	2	3	4
#Models:	6	175	25,872

7.13 Moufang quasigroups

A Moufang quasigroup is a structure $\langle A, \cdot, \backslash, / \rangle$ such that

1. $\langle A, \cdot, \backslash, / \rangle$ is a quasigroup;
2. $ye = y \Rightarrow ((xy)z)x = x(y((ez)x))$.

Moufang quasigroups appear in [81] and the number of non-isomorphic models from order 2 to order 5 is given in Table 65.

Table 65: Number of Moufang quasigroups

Order:	2	3	4	5
#Models:	1	5	29	1,351

7.14 Lattice-ordered semigroups

A Lattice-ordered ring is a structure $\langle L, \vee, \wedge, \cdot \rangle$ such that

1. $\langle L, \vee, \wedge \rangle$ is a lattice;
2. $\langle L, \cdot \rangle$ is a semigroup;
3. $x(y \vee z) = xy \vee xz$;
4. $(x \vee y)z = xz \vee yz$.

Lattice-ordered semigroups appear in [75] and the number of non-isomorphic models from order 2 to order 6 is given in Table 66.

Table 66: Number of Lattice-ordered semigroups

Order:	2	3	4	5	6
#Models:	6	44	479	6,738	117,028

7.15 Generalized Boolean algebras

A Generalized Boolean algebra is a Brouwerian algebra $\langle A, \vee, \wedge, \rightarrow, 1 \rangle$ such that

1. $x \vee y = (x \rightarrow y) \rightarrow y$.

Generalized Boolean algebras appear in [70] and the number of non-isomorphic models from order 2 to order 9 is given in Table 67.

Table 67: Number of Generalized Boolean algebras

Order:	2	3	4	5	6	7	8	9
#Models:	4	0	12	0	60	0	1,171,417	0

7.16 Distributive lattice ordered semigroups

A Distributive lattice-ordered semigroup is a structure $\langle A, \vee, \wedge, \cdot \rangle$ such that

1. $\langle A, \vee, \wedge \rangle$ is a distributive lattice;
2. $\langle A, \cdot \rangle$ is a semigroup;
3. $x(y \vee z) = (xy) \vee (xz)$ and $(x \vee y)z = (xz) \vee (yz)$.

Distributive lattice ordered semigroups appear in [68] and the number of non-isomorphic models from order 2 to order 5 is given in Table 68.

Table 68: Number of Distributive lattice ordered semigroups

Order:	2	3	4	5
#Models:	6	44	479	5,492

7.17 BCK-lattices

A BCK-lattice is a structure $\langle A, \vee, \wedge, \to, 1\rangle$ such that

1. $(x \to y) \to ((y \to z) \to (x \to z)) = 1$;
2. $1 \to x = x$;
3. $x \to 1 = 1$;
4. $x \to (x \vee y) = 1$;
5. $x \vee ((x \to y) \to y) = ((x \to y) \to y)$;
6. $x \vee x = x$;
7. $x \vee y = y \vee x$;
8. $(x \vee y) \vee z = x \vee (y \vee z)$;
9. $(x \wedge y) \to y = 1$;
10. $x \wedge ((x \to y) \to y) = x$;
11. $x \wedge x = x$;
12. $x \wedge y = y \wedge x$;
13. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.

BCK-lattices appear in [47] and the number of non-isomorphic models from order 2 to order 8 is given in Table 69.

Table 69: Number of BCK-lattices

Order:	2	3	4	5	6	7	8
#Models:	1	2	8	38	265	2,391	27,485

7.18 Commutative lattice-ordered monoids

A Commutative lattice-ordered monoid is a structure $\langle A, \vee, \wedge, \cdot, 1\rangle$ such that

1. $\langle A, \vee, \wedge\rangle$ is a lattice;
2. $\langle A, \cdot, 1\rangle$ is a monoid;
3. $x(y \vee z) = xy \vee xz$;
4. $(x \vee y)z = xz \vee yz$;
5. $xy = yx$.

Commutative lattice-ordered monoids appear in [55] and the number of non-isomorphic models from order 2 to order 8 is given in Table 70.

Table 70: Number of Commutative lattice-ordered monoids

Order:	2	3	4	5	6	7	8
#Models:	2	6	31	199	1,598	15,515	181,538

7.19 Commutative lattice-ordered semigroups

A Commutative lattice-ordered semigroup is a structure $\langle A, \vee, \wedge, \cdot\rangle$ such that

1. $\langle A, \vee, \wedge\rangle$ is a lattice;
2. $\langle A, \cdot\rangle$ is a semigroup;
3. $x(y \vee z) = xy \vee xz$;
4. $(x \vee y)z = xz \vee yz$;
5. $xy = yx$.

Commutative lattice-ordered semigroups appear in [57] and the number of non-isomorphic models from order 2 to order 7 is given in Table 71.

Table 71: Number of Commutative lattice-ordered semigroups

Order:	2	3	4	5	6	7
#Models:	4	20	149	1,427	16,683	230,688

7.20 Algebras of type (2,2,2,1)

7.20.1 Trigroups

A Trigroup is a structure $\langle A, \vdash, \perp, \dashv, ^{-1}, 1\rangle$ such that

1. $\langle A, \vdash, \perp, \dashv \rangle$ is a trisemigroup;
2. $1 \vdash x = x = x \dashv 1$;
3. $x \vdash x^{-1} = 1 = x^{-1} \dashv x$ and $x \perp x^{-1} = 1 = x^{-1} \perp x$.

Trigroups appear in [115] and the number of non-isomorphic models from order 2 to order 4 is given in Table 72.

Table 72: Number of Trigroups

Order:	2	3	4
#Models:	3	46	43,973

7.20.2 Boolean monoids

A Boolean monoid is a structure $\langle A, \vee, \wedge, \cdot, \neg, 0, 1, e\rangle$ such that

1. $\langle A, \vee, 0, \wedge, 1, \neg\rangle$ is a Boolean algebra;
2. $\langle A, \cdot, e\rangle$ is a monoid;
3. $(x \vee y)z = (xz) \vee (yz)$;
4. $x(y \vee z) = (xy) \vee (xz)$;
5. $0x = 0 = x0$.

Boolean monoids appear in [51] and the number of non-isomorphic models from order 2 to order 16 is given in Table 73.

Table 73: Number of Boolean monoids

Order:	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
#Models:	1	0	5	0	0	0	83	0	0	0	0	0	0	0	242,547

8 Algebras of type (2,2,2,2)

8.1 Algebras of type (2,2,2,2,1)

8.1.1 Bilattices

A Bilattice is a structure $\langle L, \vee, \wedge, \oplus, \otimes, \neg \rangle$ such that

1. $\langle L, \vee, \wedge \rangle$ is a lattice;
2. $\langle L, \oplus, \otimes \rangle$ is a lattice;
3. $\neg(x \wedge y) = \neg x \vee \neg y$;
4. $\neg(x \vee y) = \neg x \wedge \neg y$;
5. $\neg\neg x = x$;
6. $\neg(x \oplus y) = \neg x \oplus \neg y$;
7. $\neg(x \otimes y) = \neg x \otimes \neg y$.

Bilattices appear in [49] and the number of non-isomorphic models from order 2 to order 9 is given in Table 74.

Table 74: Number of Bilattices

Order:	2	3	4	5	6	7	8	9
#Models:	0	0	1	3	32	284	3,839	62,735

9 Algebras of type (3)

9.1 Autodistributive Ternary Groupoids

An Autodistributive Ternary Groupoid is a structure $\langle A, a \rangle$ such that

1. $a(a(x, y, z), u, v) = a(a(x, u, v), a(y, u, v), a(z, u, v))$.

Autodistributive Ternary Groupoids appear on [17] and the number of non-isomorphic models from order

9.2 Cancellative Ternary Groupoids

A Cancellative Ternary Groupoid is a structure $\langle A, a\rangle$ such that

1. $a(u, v, x) = a(u, v, y) \Rightarrow x = y$;
2. $a(u, x, v) = a(u, y, v) \Rightarrow x = y$;
3. $a(x, u, v) = a(y, u, v) \Rightarrow x = y$.

Cancellative Ternary Groupoids appear on [14] and the number of non-isomorphic models from order

9.3 Left Cancellative Ternary Groupoids

A Left Cancellative Ternary Groupoid is a structure $\langle A, a\rangle$ such that

1. $a(u, v, x) = a(u, v, y) \Rightarrow x = y$.

Left Cancellative Ternary Groupoids appear on [14] and the number of non-isomorphic models from order

9.4 Middle Cancellative Ternary Groupoids

A Middle Cancellative Ternary Groupoid is a structure $\langle A, a\rangle$ such that

1. $a(u, x, v) = a(u, y, v) \Rightarrow x = y$.

Middle Cancellative Ternary Groupoids appear on [14] and the number of non-isomorphic models from order

9.5 Right Cancellative Ternary Groupoids

A Right Cancellative Ternary Groupoid is a structure $\langle A, a\rangle$ such that

1. $a(x, u, v) = a(y, u, v) \Rightarrow x = y$.

Right Cancellative Ternary Groupoids appear on [14] and the number of non-isomorphic models from order

9.6 Semicommutative Ternary Groupoids

A Semicommutative Ternary Groupoid is a structure $\langle A, a\rangle$ such that

1. $a(x, y, z) = a(z, y, x)$.

Semicommutative Ternary Groupoids appear on [14] and the number of non-isomorphic models from order

9.7 Commutative Ternary Groupoids

A Commutative Ternary Groupoid is a structure $\langle A, a \rangle$ such that

1. $a(x, y, z) = a(z, y, x)$;
2. $a(x, y, z) = a(z, x, y)$;
3. $a(x, y, z) = a(x, z, y)$;
4. $a(x, y, z) = a(y, x, z)$;
5. $a(x, y, z) = a(y, z, x)$.

Commutative Ternary Groupoids appear on [14] and the number of non-isomorphic models from order

9.8 Heaps

A Heap is a structure $\langle A, a \rangle$ such that

1. $a(v, w, a(x, y, z)) = a(a(v, w, x), y, z)$;
2. $y = a(x, x, y)$;
3. $y = a(y, x, x)$.

Heaps appear on [12] and the number of non-isomorphic models from order

9.9 Abelian Heaps

An Abelian heap is a structure $\langle A, a \rangle$ such that

1. $a(x, y, z) = a(z, y, x)$;
2. $a(v, w, a(x, y, z)) = a(a(v, w, x), y, z)$;
3. $y = a(x, x, y)$;
4. $y = a(y, x, x)$.

Abelian heaps appear on [12] and the number of non-isomorphic models from order

9.10 Median algebras

A Median algebra is a structure $\langle A, a \rangle$ such that

1. $a(u, u, v) = u$;
2. $a(u, v, w) = a(v, u, w)$ and $a(v, u, w) = a(v, w, u)$;
3. $a(a(u, v, w), x, y) = a(u, a(v, x, y), a(w, x, y))$.

Median algebras appear on [4] and the number of non-isomorphic models from order

10 Algebras of type (3,2)

10.1 Trusses

A Truss is a structure $\langle A, a, \cdot \rangle$ such that

1. $wa(x, y, z) = a(wx, wy, wz)$ and $a(x, y, z)w = a(xw, yw, zw)$;
2. $x(yz) = (xy)z$;
3. $a(x, y, z) = a(z, y, x)$;
4. $a(v, w, a(x, y, z)) = a(a(v, w, x), y, z)$;
5. $y = a(x, x, y)$;
6. $y = a(y, x, x)$.

Trusses appear on [12] and the number of non-isomorphic models from order

10.2 Unital Trusses

A Unital Truss is a structure $\langle A, a, \cdot, 1 \rangle$ such that

1. $wa(x, y, z) = a(wx, wy, wz)$ and $a(x, y, z)w = a(xw, yw, zw)$;
2. $x(yz) = (xy)z$;
3. $a(x, y, z) = a(z, y, x)$;
4. $a(v, w, a(x, y, z)) = a(a(v, w, x), y, z)$;
5. $y = a(x, x, y)$;
6. $y = a(y, x, x)$;

7. $x1 = x$;

8. $1x = x$.

Unital Trusses appear on [12] and the number of non-isomorphic models from order

11 Other Algebras

11.1 Ground mereotopologies

A Ground mereotopology is a structure $\langle A, P, C \rangle$ such that

1. P, C are relations defined on A;

2. $P(x, x)$;

3. $P(x, y)$ and $P(y, x) \Rightarrow x = y$;

4. $P(x, y)$ and $P(y, z) \Rightarrow P(x, z)$;

5. $C(x, x)$;

6. $C(x, y) \Rightarrow C(y, x)$;

7. $C(z, x)$ and $P(x, y) \Rightarrow C(y, z)$.

Ground mereotopologies appear on p. 5 of [109] and the number of non-isomorphic models from order 2 to order 7 is given in Table 75.

Table 75: Number of Ground mereotopologies

Order:	2	3	4	5	6	7
#Models:	3	11	63	518	6,789	141,499

11.2 Partially ordered semigroups

A Partially ordered semigroup is a structure $\langle A, \cdot, \leq \rangle$ such that

1. $\langle A, \cdot \rangle$ is a semigroup;

2. $\langle A, \leq \rangle$ is a partially ordered set;

3. $x \leq y \Rightarrow xz \leq yz$ and $zx \leq zy$.

Partially ordered semigroups appear in [95] and the number of non-isomorphic models from order 2 to order 5 is given in Table 76.

Table 76: Number of Partially ordered semigroups

Order:	2	3	4	5
#Models:	11	173	4,753	198,838

11.3 Partially ordered monoids

A Partially ordered monoid is a structure $\langle A, \cdot, 1, \leq \rangle$ such that

1. $\langle A, \cdot, 1 \rangle$ is a monoid;
2. $\langle A, \leq \rangle$ is a partially ordered set;
3. $x \leq y \Rightarrow wxz \leq wyz$.

Partially ordered monoids appear in [94] and the number of non-isomorphic models from order 2 to order 6 is given in Table 77.

Table 77: Number of Partially ordered monoids

Order:	2	3	4	5	6
#Models:	4	37	549	13,371	504,634

11.4 Ordered monoids with zero

An Ordered monoid with zero is a structure $\langle A, \cdot, 1, 0, \leq \rangle$ such that

1. $\langle A, \cdot, 1, \leq \rangle$ is an ordered monoid;
2. $x0 = 0 = 0x$.

Ordered monoids with zero appear in [91] and the number of non-isomorphic models from order 2 to order 9 is given in Table 78.

Table 78: Number of Ordered monoids with zero

Order:	2	3	4	5	6	7	8	9
#Models:	2	6	30	168	1,150	9,374	90,446	1,033,764

11.5 Dense linear orders

A Dense linear order is a structure $\langle A, \leq \rangle$ such that

1. $\langle A, \leq \rangle$ is a partially ordered set;
2. $x \leq y$ or $y \leq x$;
3. $x < y \Rightarrow \exists z \ (x < z, z < y)$.

Dense linear orders appear in [66] and the number of non-isomorphic models from order 2 to order 4 is given in Table 79.

Table 79: Number of Dense linear orders

Order:	2	3	4
#Models:	4	324	1,179,648

11.6 Commutative ordered monoids

A Commutative ordered monoid is a structure $\langle A, \cdot, 1, \leq \rangle$ such that

1. $\langle A, \cdot, 1 \rangle$ is a monoid;
2. $\langle A, \leq \rangle$ is a partially ordered set;
3. $x \leq y \Rightarrow wxz \leq wyz$;
4. $x \leq y$ or $y \leq x$.
5. $xy = yx$.

Commutative ordered monoids appear in [58] and the number of non-isomorphic models from order 2 to order 9 is given in Table 80.

Table 80: Number of Commutative ordered monoids

Order:	2	3	4	5	6	7	8	9
#Models:	2	6	22	92	426	2,146	11,634	67,472

11.7 Commutative partially ordered monoids

A Commutative partially ordered monoid is a structure $\langle A, \cdot, 1, \leq \rangle$ such that

1. $\langle A, \cdot, 1 \rangle$ is a monoid;
2. $\langle A, \leq \rangle$ is a partially ordered set;
3. $x \leq y \Rightarrow wxz \leq wyz$;
4. $xy = yx$.

Commutative partially ordered monoids appear in [59] and the number of non-isomorphic models from order 2 to order 7 is given in Table 81.

Table 81: Number of Commutative partially ordered monoids

Order:	2	3	4	5	6	7
#Models:	4	27	301	4,887	113,358	3,763,121

11.8 Commutative partially ordered semigroups

A Commutative partially ordered semigroup is a structure $\langle A, \cdot, \leq \rangle$ such that

1. $\langle A, \cdot \rangle$ is a semigroup;
2. $\langle A, \leq \rangle$ is a partially ordered set;
3. $x \leq y \Rightarrow xz \leq yz$ and $zx \leq zy$;
4. $xy = yx$.

Commutative partially ordered semigroups appear in [60] and the number of non-isomorphic models from order 2 to order 5 is given in Table 82.

Table 82: Number of Commutative partially ordered semigroups

Order:	2	3	4	5
#Models:	7	83	1,468	37,248

11.9 Complemented lattices

A Complemented lattice is a structure $\langle L, \vee, \wedge, 0, 1, \leq \rangle$ such that

1. $\langle L, \vee, \wedge \rangle$ is a lattice;
2. $0 \leq x$;
3. $x \leq 1$;
4. $\forall x \exists y \quad x \vee y = 1$ and $x \wedge y = 0$.

Complemented lattices appear in [63] and the number of non-isomorphic models from order 2 to order 8 is given in Table 83.

Table 83: Number of Complemented lattices

Order:	2	3	4	5	6	7	8
#Models:	1	0	1	4	41	600	13,941

11.10 Orthomodular lattices

An Orthomodular lattice is an ortholattice $\langle L, \vee, \wedge, ', 0, 1, \leq \rangle$ such that

1. $x \leq y \Rightarrow x \vee (x' \wedge y) = y$.

Orthomodular lattices appear in [93] and the number of non-isomorphic models from order 2 to order 14 is given in Table 84.

Table 84: Number of Orthomodular lattices

Order:	2	3	4	5	6	7	8	9	10	11	12	13	14
#Models:	1	0	1	1	2	2	5	4	9	9	17	17	33

11.11 Monadic algebras

A Monadic algebra is a structure $\langle A, \vee, \wedge, \neg, f, 0, 1 \rangle$ such that

1. $\langle A, \vee, \wedge, \neg, 0, 1 \rangle$ is a Boolean algebra;
2. $f(x \vee y) = f(x) \vee f(y)$;
3. $f(0) = 0$;

4. $x \leq f(x) = f(f(x))$;
5. $f(x) \wedge y = 0 \Leftrightarrow x \wedge f(y) = 0$.

Monadic algebras appear in [80] and the number of non-isomorphic models from order 2 to order 7 is given in Table 85.

Table 85: Number of Monadic algebras

Order:	2	3	4	5	6	7
#Models:	8	0	10,432	0	0	0

11.12 Concurrent semirings

A Concurrent semiring is a structure $\langle A, +, *, \cdot, 0, 1 \rangle$ such that

1. $\langle A, +, *, \cdot, 0, 1 \rangle$ is an idempotent bisemiring;
2. $\langle A, *, \cdot, 1, \leq \rangle$ is a concurrent monoid.

Concurrent semirings appear on p. 37 of [108] and the number of non-isomorphic models from order 2 to order 5 is given in Table 86.

Table 86: Number of Concurrent semirings

Order:	2	3	4	5
#Models:	1	4	43	14,029

11.13 Residuated partially ordered monoids

A Residuated partially ordered monoid is a structure $\langle A, \cdot, \backslash, /, 1, \leq \rangle$ such that

1. $\langle A, \leq \rangle$ is a partially ordered set;
2. $\langle A, \cdot, 1 \rangle$ is a monoid;
3. $xy \leq z \Leftrightarrow y \leq x \backslash z$;
4. $xy \leq z \Leftrightarrow x \leq z/y$.

Residuated partially ordered monoids appear in [98] and the number of non-isomorphic models from order 2 to order 7 is given in Table 87.

Table 87: Number of Residuated partially ordered monoids

Order:	2	3	4	5	6	7
#Models:	2	5	28	186	1,795	21,904

11.14 Linear Heyting algebras

A Linear Heyting algebra is a structure $\langle A, \vee, \wedge, \rightarrow, 0, 1, \leq \rangle$ such that

1. $\langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice;
2. $x \wedge y \leq z \Leftrightarrow y \leq x \rightarrow z$;
3. $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

Linear Heyting algebras, also known as Goedel algebras, appear in [77] and the number of non-isomorphic models from order 2 to order 4 is given in Table 88.

Table 88: Number of Linear Heyting algebras

Order:	2	3	4
#Models:	8	302	835,006

11.15 Brouwerian algebras

A Brouwerian algebra is a structure $\langle A, \vee, \wedge, \rightarrow, 1 \rangle$ such that

1. $\langle A, \vee, \wedge \rangle$ is a distributive lattice;
2. $x \leq 1$;
3. $x \wedge y \leq z \Leftrightarrow y \leq x \rightarrow z$.

Brouwerian algebras appear in [52] and the number of non-isomorphic models from order 2 to order 5 is given in Table 89.

Table 89: Number of Brouwerian algebras

Order:	2	3	4	5
#Models:	4	8	58	8,806

11.16 Commutative residuated partially ordered monoids

A Commutative residuated partially ordered monoid is a structure $\langle A, \cdot, \backslash, /, 1, \leq \rangle$ such that

1. $\langle A, \leq \rangle$ is a partially ordered set;
2. $\langle A, \cdot, 1 \rangle$ is a monoid;
3. $x \cdot y \leq z \Leftrightarrow y \leq x \backslash z$;
4. $x \cdot y \leq z \Leftrightarrow x \leq z/y$;
5. $xy = yx$.

Commutative residuated partially ordered monoids, also known as Lineales, appear in [62] and the number of non-isomorphic models from order 2 to order 7 is given in Table 90.

Table 90: Number of Commutative residuated partially ordered monoids

Order:	2	3	4	5	6	7
#Models:	2	5	24	131	1,001	9,248

11.17 Concurrent semiring with invariants

A Concurrent semiring with invariants is a structure $\langle A, +, *, \cdot, \varphi, 0, 1, \leq \rangle$ such that

1. $\langle A, +, *, \cdot, 0, 1 \rangle$ is a concurrent semiring;
2. $\varphi : A \to A$ is a closure operator;
3. $1 \leq \varphi(x)$;
4. $\varphi(x * y) \leq \varphi(x + y)$.

Concurrent semiring with invariants appear on p. 37 of [108] and the number of non-isomorphic models from order 2 to order 5 is given in Table 91.

Table 91: Number of Concurrent semiring with invariants

Order:	2	3	4	5
#Models:	1	8	199	10,260

11.18 Normal valued lattice-ordered groups

A Normal valued lattice-ordered group is a lattice-ordered group $\langle L, \vee, \wedge, \cdot, ^{-1}, e\rangle$ such that

1. $(x \vee x^{-1})(y \vee y^{-1}) \leq (y \vee y^{-1})^2(x \vee x^{-1})^2$.

Normal valued lattice-ordered groups appear in [88] and the number of non-isomorphic models from order 2 to order 7 is given in Table 92.

Table 92: Number of Normal valued lattice-ordered groups

Order:	2	3	4	5	6	7
#Models:	2	3	30	155	5,528	69,332

11.19 M-zeroids

An M-zeroid is a structure $\langle A, \vee, \wedge, +, -, 0, \leq\rangle$ such that

1. $\langle A, +\rangle$ is a commutative semigroup;
2. $\langle A, \wedge, \vee\rangle$ is a lattice;
3. $-x = x$;
4. $x + 0 = 0$;
5. $x + (-x) = 0$;
6. $x \leq y \Leftrightarrow 0 = -x + y$;
7. $x + (y \vee z) = (x + y) \vee (x + z)$.

M-zeroids appear in [78] and the number of non-isomorphic models from order 2 to order 9 is given in Table 93.

Table 93: Number of M-zeroids

Order:	2	3	4	5	6	7	8	9
#Models:	2	3	7	21	75	315	1,537	8,583

11.20 De Morgan monoids

A De Morgan monoid is a structure $\langle A, \vee, \wedge, \cdot, ', 1, \leq \rangle$ such that

1. $\langle A, \vee, \wedge \rangle$ is a distributive lattice;
2. $\langle A, \cdot, 1 \rangle$ is a commutative monoid;
3. $xy \leq z \Leftrightarrow y \leq (z'x)'$;
4. $x \leq xx$.

De Morgan monoids appear in [65] and the number of non-isomorphic models from order 2 to order 4 is given in Table 94.

Table 94: Number of De Morgan monoids

Order:	2	3	4
#Models:	18	882	174,062

11.21 BL-algebras

A BL-algebra is a structure $\langle A, \vee, \wedge, \cdot, \rightarrow, 0, 1 \rangle$ such that

1. $\langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded lattice;
2. $\langle A, \cdot, 1 \rangle$ is a commutative monoid;
3. $xy \leq z \Leftrightarrow y \leq x \rightarrow z$;
4. $(x \rightarrow y) \vee (y \rightarrow x) = 1$;
5. $x(x \rightarrow y) = x \wedge y$.

BL-algebras appear in [50] and the number of non-isomorphic models from order 2 to order 5 is given in Table 95.

Table 95: Number of BL-algebras

Order:	2	3	4	5
#Models:	6	34	606	854,822

11.22 Lattice-ordered rings

A Lattice-ordered ring is a structure $\langle L, \vee, \wedge, +, \cdot, -, 0\rangle$ such that

1. $\langle L, \vee, \wedge\rangle$ is a lattice;
2. $\langle L, +, \cdot, -, 0\rangle$ is a ring;
3. $x \leq y \Rightarrow x + z \leq y + z$;
4. $0 \leq x, y \Rightarrow 0 \leq xy$.

Lattice-ordered rings appear in [74] and the number of non-isomorphic models from order 2 to order 6 is given in Table 96.

Table 96: Number of Lattice-ordered rings

Order:	2	3	4	5	6
#Models:	14	51	2,238	4,655	492,120

11.23 Commutative lattice-ordered rings

A Commutative lattice-ordered ring is a structure $\langle L, \vee, \wedge, +, \cdot, -, 0\rangle$ such that

1. $\langle L, \vee, \wedge\rangle$ is a lattice;
2. $\langle L, +, \cdot, -, 0\rangle$ is a ring;
3. $x \leq y \Rightarrow x + z \leq y + z$;
4. $0 \leq x, y \Rightarrow 0 \leq x \cdot y$;
5. $xy = yx$.

Commutative lattice-ordered rings appear in [56] and the number of non-isomorphic models from order 2 to order 7 is given in Table 97.

Table 97: Number of Commutative lattice-ordered rings

Order:	2	3	4	5	6	7
#Models:	14	51	1,794	4,655	492,120	3,290,875

12 Conclusions

In this paper, we report new integer sequences for the number of models of order n (for small integer values of n) for 100 algebras. These sequences have not been reported to the OEIS. We also increased the length of the existing OEIS integer sequences A178432 and A305858. This is accomplished with improved model enumeration algorithms implemented in the parallel finite model enumerator Mace4, running on a 24-cores computer.

13 Acknowledgments

Gonçalo Araújo The results were supported by Fundação para a Ciência e a Tecnologia, through the project UI/BD/154405/2023.
João Araújo The results were supported by Fundação para a Ciência e a Tecnologia, through the projects UIDB/00297-/2020 (CMA), PTDC/MAT-PUR/31174/2017, UIDB/04621/2020 and UIDP/04621/2020.
Mikoláš Janota The results were supported by the Ministry of Education, Youth and Sports within the dedicated program ERC CZ under the project POSTMAN no. LL1902. This scientific article is part of the RICAIP project that has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 857306.

References

[1] Peter M. Higgins and Marcel Jackson, Algebras defined by equations, arXiv 1810.13012 (2020).

[2] Peter M. Higgins and Marcel Jackson, Equationally defined classes of semigroups, arXiv 2212.02699, (2023).

[3] Agata Smoktunowicz and Leandro Vendramin, On skew braces (with an appendix by N. Byott and L. Vendramin), European Mathematical Society - EMS - Publishing House GmbH.

[4] Indira Chatterji and Cornelia Drutu and Frederic Haglund, Kazhdan and Haagerup properties from the median viewpoint, arXiv 0704.3749.

[5] Patrick Dehornoy, The group of parenthesized braids, arXiv math/0407097.

[6] David Bachiller and Ferran Cedo and Eric Jespers, Solutions of the Yang-Baxter equation associated with a left brace, arXiv 1503.02814.

[7] Patrick Dehornoy, Set-theoretic solutions of the Yang-Baxter equation, RC-calculus, and Garside germs, arXiv 1403.3019.

[8] H. Thomas and N. Williams, Rowmotion in slow motion, Proceedings of the London Mathematical Society.

[9] R. A. Borzooei and A. Dvurečenskij and O. Zahiri, State BCK-algebras and State-Morphism BCK-algebras, arXiv 1304.6963.

[10] Liu Sheng and Yu Ping and Li Yan-yan and Du Yan-chun, Application of H infinite control to ship steering system, Journal of Marine Science and Application.

[11] Dong Liu and Naihong Hu, Leibniz Algebras Graded by Finite Root Systems, arXiv math/0510545.

[12] Tomasz Brzeziński, Trusses: Paragons, ideals and modules, arXiv 1901.07033.

[13] Stefano Bonzio and Ivan Chajda, Residuated Relational Systems, arXiv 1810.09335.

[14] Andrzej Borowiec and Wiesław A. Dudek and Steven Duplij, Bi-Element Representations of Ternary Groups, Informa UK Limited.

[15] Wieslaw A. Dudek and Janus Thomys, On some generalizations of BCC-algebras, Informa UK Limited.

[16] V. A. Shcherbacov, On the structure of left and right F-, SM- and E-quasigroups, arXiv 0811.1725.

[17] Andrzej Borowiec and Wieslaw A. Dudek and Steven Duplij, Basic concepts of ternary Hopf algebras, arXiv math/0306208.

[18] Dietrich Burde and Karel Dekimpe and Kim Vercammen, Complete LR-structures on solvable Lie algebras, arXiv 0906.1151.

[19] Jinsen Zhou and Guangzhe Fan, Generalized derivations of n-Hom Lie superalgebras, arXiv 1605.04555.

[20] Hector Freytes, Injectives in residuated algebras, arXiv 0806.4946.

[21] Murray R. Bremner, On the definition of quasi-Jordan algebra, arXiv 1008.2009.

[22] Daoud Clarke, Context-theoretic Semantics for Natural Language: an Algebraic Framework, arXiv 2009.10542.

[23] Zhiyun Cheng, The Chord Index, its Definitions, Applications, and Generalizations, arXiv 1606.01446.

[24] M. L. Dalla Chiara and R. Giuntini (2001), Quantum Logic, arXiv quant-ph/0101028.

[25] Tatiana Gateva-Ivanova and Shahn Majid (2007), Matched pairs approach to set-theoretic solutions of the Yang-Baxter equation, arXiv math/0507394.

[26] M. Pedicini and F. Quaglia (2004), PELCR: Parallel Environment for Optimal Lambda-Calculus Reduction, arXiv cs/0407055.

[27] P. A. Grillet (1995). Semigroups. CRC Press. ISBN 978-0-8247-9662-4.

[28] A. H. Clifford, G. B. Preston (1964). The Algebraic Theory of Semigroups Vol. I (Second Edition). American Mathematical Society. ISBN 978-0-8218-0272-4.

[29] Attila Nagy (2001). Special Classes of Semigroups. Springer. ISBN 978-0-7923-6890-8.

[30] Fennemore, Charles (1970), All varieties of bands, Semigroup Forum, 1 (1): 172-179, doi:10.1007/BF02573031.

[31] K. S. Harinath (1979), Some results on k-regular semigroups, Indian Journal of Pure and Applied Mathematics 10(11), 1422-1431.

[32] Mati Kilp, Ulrich Knauer, Alexander V. Mikhalev (2000), Monoids, Acts and Categories: with Applications to Wreath Products and Graphs, Expositions in Mathematics 29, Walter de Gruyter, Berlin, ISBN 978-3-11-015248-7.

[33] J. M. Howie (1995), Fundamentals of Semigroup Theory, Oxford University Press.

[34] Mario Petrich, Norman R. Reilly (1999), Completely Regular Semigroups, Volume 23 de Wiley-Interscience and Canadian Mathematics Series of Monographs and Texts.

[35] Harrison, M. A. The Number of Isomorphism Types of Finite Algebras. *Proceedings of the American Mathematical Society*, vol. 17, no. 3, 1966, pp. 731–37. JSTOR, available at https://doi.org/10.2307/2035402. Accessed 10 Jan. 2023.

[36] European Mathematical Society. *Enumeration of Finite Groups*, available at https://euro-math-soc.eu/review/enumeration-finite-groups, 2011

[37] N. J. A. Sloane et al., *The On-Line Encyclopedia of Integer Sequences*, available at https://oeis.org, 2022.

[38] McCune, W., Mace4 reference manual and guide, August 2003, available at https://www.cs.unm.edu/~mccune/prover9/mace4.pdf.

[39] Araújo, J., Chow, C. and Janota, M. Boosting isomorphic model filtering with invariants. *Constraints* 27, 360–379 (2022). https://doi.org/10.1007/s10601-022-09336-x

[40] Araújo, J., Chow, C. and Janota, M. Filtering Isomorphic Models by Invariants. *27th International Conference on Principles and Practice of Constraint Programming (CP 2021)* 4:1 – 4:9 (2021). https://drops.dagstuhl.de/opus/volltexte/2021/15295

[41] Araújo, J., Ramires, J., *MarcieX a model / theory / theorem database*, available at https://marciedb.pythonanywhere.com/, 2022.

[42] Sutcliffe, G., The TPTP Problem Library and Associated Infrastructure. From CNF to TH0, TPTP v6.4.0. *Journal of Automated Reasoning* 59 (2017), no. 4, pp. 483-502.

[43] Araújo, J., Ramires, J., *MarcieX a model / theory / theorem database*, available at https://marciedb.pythonanywhere.com/, 2022.

[44] Yoshinari Arai, Kiyoshi Iséki, Shôtarô Tanaka. (1966) Characterizations of BCI,BCK-algebras. Proceedings of the Japan Academy Vol. 42, Issue 2, (Jan 1966) , pgs 105-107.

[45] Yoshinari Arai, Kiyoshi Iséki, Shôtarô Tanaka. (1966) Characterizations of BCI,BCK-algebras. Proceedings of the Japan Academy Vol. 42, Issue 2, (Jan 1966) , pgs 105-107.

[46] Jansana, R., On the Deductive System of the Order of an Equationally Orderable Quasivariety, (2013).

[47] Shoar, S. K., Borzooei, R. A., Moradian, R., Radfar, A. (2018). PC-lattices: A Class of Bounded BCK-algebras. Bulletin of the Section of Logic, 47(1), 33–44. https://doi.org/10.18778/0138-0680.47.1.03

[48] Jansana, R., On the Deductive System of the Order of an Equationally Orderable Quasivariety, (2013).

[49] Bílková, M., Frittella, S., Majer, O., Nazari, S., How to reason with inconsistent probabilistic information? (2020), p. 5.

[50] Rui Paiva, Regivan Santiago, Benjamin Bedregal. (2019). On BL-Algebras and its Interval Counterpart. TEMA. 20. 241-256.

[51] Mark V. Lawson, Scott P, Characterizations of classes of countable Boolean inverse monoids, (2022).

[52] Bezhanishvili, G., Carai, L., Morandi, P., Heyting frames and Esakia duality, (2023), p. 12.

[53] Chisolm, E. (2012). Geometric Algebra. *arXiv*. https://doi.org/10.48550/arXiv.1205.5935

[54] Yoshinari Arai, Kiyoshi Iséki, Shôtarô Tanaka. (1966) Characterizations of BCI, BCK-algebras. Proc. Japan Acad. 42(2): 105-107

[55] Abbadini, M. (2021) Equivalence à la Mundici for commutative lattice-ordered monoids. Algebra Univers. 82, 45.

[56] D. G. Johnson. (1960) A structure theory for a class of lattice-ordered rings. Acta Math. 104(3-4): 163-215.

[57] P. J. McCarthy, Homomorphisms of certain commutative lattice ordered semigroups, (1966), p. 1.

[58] Samson Abramsky and Jouko Väänänen. (2009) From IF to BI: A Tale of Dependence and Separation. Synthese Vol. 167, No. 2, Knowledge, Rationality & Action (Mar., 2009), pp. 207-230

[59] W. P. R. Mitchell, H. Simmons, Monoid Based Semantics for Linear Formulas, (2001), pp. 1603-1604.

[60] Osamu Nakada. (1951) Partially ordered abelian semigroups. J. Fac. Sci. Hokkaido Univ. Ser. I Math. 11(4): 181-189.

[61] K. Smith, Commutative regular rings and Boolean-valued fields, (1984).

[62] Jenei, S., Ono, H., On involutive FLe-monoids, (2010), p. 721.

[63] Saki, A., Kiani, D., Complemented lattices of subracks, https://doi.org/10.48550/arXiv.1806.07929, (2018).

[64] Saki, A., Kiani, D., Complemented lattices of subracks, https://doi.org/10.48550/arXiv.1806.07929, (2018).

[65] John K. Slaney. (1989) On the structure of De Morgan monoids with corollaries on relevant logic and theories. Notre Dame J. Formal Logic 30(1): 117-129 (Winter 1989).

[66] A. P. Hazen. (1992) Interpretability of Robinson arithmetic in the ramified second-order theory of dense linear order. Notre Dame J. Formal Logic 33(1): 101-111 (Winter 1992).

[67] Ježek, J., Quackenbush, R., Directoids: algebraic models of up-directed sets, (1990), p. 50.

[68] Urquhart, A., Decision problems for distributive lattice-ordered semigroups, (1995), p. 400.

[69] Dauns, J., A concrete approach to division rings, (1982).

[70] J.C.E. Dekker. (1990) Isols and generalized Boolean algebras. Rocky Mountain J. Math. 20(1): 107-115 (Winter 1990).

[71] Hidegorô Nakano. (1950) Hilbert algebras. Tohoku Math. J. (2) 2(1): 4-23.

[72] Arthan, R., Oliva, P., Hoops, Coops and the Algebraic Semantics of Continuous Logic, (2012), p. 9.

[73] Olson, J., S., Involutive Residuated Lattices Based on Modular and Distributive Lattices, (2012), p. 374.

[74] Melvin Henriksen, J. R. Isbell. (1962) Lattice-ordered rings and function rings. Pacific J. Math. 12(2): 533-565.

[75] Kehayopulu, N., Lattice ordered semigroups and Γ-hypersemigroups, (2019), p. 1837.

[76] Jenei, S., Ono, H., On involutive FLe-monoids, (2010), p. 721.

[77] Spada, L., Fuzzy Logic and Algebra, (2006), p. 16

[78] Palmatier, J., B., Guzman, F., M-Zeroids: Structure and Categorical Equivalence, (2010).

[79] Bolesław Sobociński. (1976) A short equational axiomatization of modular ortholattices. Notre Dame J. Formal Logic 17(2): 311-316 (April 1976).

[80] Pinto, S., M., Martins, M., T., O., Pinto, M., C., Monadic Dynamic Algebras, (2005), p. 3.

[81] Dudek, W., A., On Belousov-Moufang quasigroups, (2020).

[82] Richard G. Burton. (1975) Fractional elements in multiplicative lattices. Pacific J. Math. 56(1): 35-49.

[83] P. H. Karvellas. (1974) A note on compact semirings which are multiplicative semilattices. Pacific J. Math. 55(1): 195-205.

[84] Tomasz Brzeziński, Stefano Mereta, Bernard Rybołowicz. (2022) From pre-trusses to skew braces. Publ. Mat. 66(2): 683-714.

[85] Krimmel, J., E., A Condition on Near-Rings with Identity, (1973), https://doi.org/10.1007/BF01300528.

[86] Pereira, R., Computing Congruences and Endomorphisms for Algebras of Type $(2^m, 1^n)$, (2022).

[87] Mario Petrich. (1973) Normal bands of commutative cancellative semigroups. Duke Math. J. 40(1): 17-32 (March 1973).

[88] Glass, A., M., W., Ordered permutation groups, (1981).

[89] Adams, M., E., Priestley, H., A., Equational bases for varieties of Ockham algebras, (1994), p. 370.

[90] Chajda, I., Halas, R., Order Algebras, (2002), p. 1.

[91] Antonio Cano Gómez, Jean-Eric Pin. Shuffle on positive varieties of languages. Theoretical Computer Science, 2004, 312, pp.433-461. ffhal-00112826.

[92] Ladislav Beran. (1976) Three identities for ortholattices. Notre Dame J. Formal Logic 17(2): 251-252 (April 1976).

[93] Bolesław Sobociński. (1976) A short equational axiomatization of orthomodular lattices. Notre Dame J. Formal Logic 17(2): 317-320 (April 1976).

[94] Bana Al Subaiei, James Renshaw. (2016) On Free Products and Amalgams of Pomonoids, Communications in Algebra, 44:6, 2455-2474.

[95] R. J. Koch. (1960) Ordered semigroups in partially ordered semigroups. Pacific J. Math. 10(4): 1333-1336.

[96] Arthan, R., Oliva, P., Hoops, Coops and the Algebraic Semantics of Continuous Logic, (2012), p. 8.

[97] Félix Bou, Francesco Paoli, Antonio Ledda, Matthew Spinks, Roberto Giuntini. (2010) The Logic of Quasi-MV Algebras. Journal of Logic and Computation, Volume 20, Issue 2, Pages 619–643 (April 2010).

[98] Franco Montagna, Constantine Tsinakis. (2007) Ordered groups with a modality. Journal of Pure and Applied Algebra, Volume 211, Issue 2, Pages 511-531 (November 2007)

[99] Jipsen, P., Kinyon, M. (2019) Nonassociative right hoops. Algebra Univers. 80, 47.

[100] Peter M. Higgins, Marcel Jackson. (2020) Algebras defined by equations. Journal of Algebra, Volume 555, Pages 131-156, ISSN 0021-8693.

[101] Zonnefeld, Caden G. (2021) Directed Graphs of the Finite Tropical Semiring. Rose-Hulman Undergraduate Mathematics Journal: Vol. 22 : Iss. 1 , Article 4.

[102] Mordeson, J., N., Malik, D., S., Fuzzy Automata and Languages, (2002), p. 455.

[103] Jens Zumbrägel. (2008) Classification of finite congruence-simple semirings with zero. Journal of Algebra and Its Applications Vol. 07, No. 03, pp. 363-377.

[104] Joao Pita Costa. (2012) On ideals of a skew lattice. Discussiones Mathematicae General Algebra and Applications 32, 5–6.

[105] Dauns, J., A concrete approach to division rings, (1982).

[106] Kumar, J., and Krishna, K. V. (2013). Affine Near-Semirings over Brandt Semigroups. *arXiv*. https://doi.org/10.1080/00927872.2013.833211

[107] Megill, N. D., and Pavicic, M. (2003). Quantum Implication Algebras. *arXiv.* https://doi.org/10.1023/B:IJTP.0000006007.58191.da

[108] Hoare, T., and Möller, B., and Struth, G., and Wehrman, I. (2011). Concurrent Kleene Algebra and its Foundations. *J. Log. Algebr. Program..* 80. 266-296. 10.1016/j.jlap.2011.04.005.

[109] Hahmann, Torsten and Grüninger, Michael. (2013). Complementation in Representable Theories of Region-Based Space. *Notre Dame J. of Formal Logic.* 54. 10.1215/00294527-1731344.

[110] Desharnais, J., Jipsen, P., Struth, G. (2009). Domain and Antidomain Semigroups. In: Berghammer, R., Jaoua, A.M., Möller, B. (eds) Relations and Kleene Algebra in Computer Science. RelMiCS 2009. *Lecture Notes in Computer Science,* vol 5827. Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-04639-1_6

[111] Lisitsa, A., and Vernitski, A. (2017). Automated Reasoning for Knot Semigroups and π-orbifold Groups of Knots. 10.1007/978-3-319-72453-9_1.

[112] Phillips, J. and Stanovský, David. (2010). Automated theorem proving in quasigroup and loop theory. *AI Commun..* 23. 267-283. 10.3233/AIC-2010-0460.

[113] Niebrzydowski, M. (2013). On some ternary operations in knot theory. *arXiv.* https://doi.org/10.48550/arXiv.1301.0391

[114] Krapež, Aleksandar. (2012). Varieties of rectangular quasigroups. *Quasigroups and Related Systems.* 20. 71-80.

[115] Biyogmam, G., and Tchamna, S., and Tcheka, C. (2020). From quotient trigroups to groups. *Quasigroups and Related Systems.* 28. 29-41.

[116] Zhang, G., and Chen, Y. (2018). A construction of free digroup. *arXiv.* https://doi.org/10.1007/s00233-021-10161-6

[117] Biyogmam, G. R., and Tcheka, C. (2019). From Trigroups To Leibniz 3-Algebras. *arXiv.* https://doi.org/10.48550/arXiv.1904.12030

[118] Höfner, P., Laws for Left I-Semiring, http://www.kleenealgebra.de/kleene_db/lemmas/single.php?algebra=Left%20I-Semiring, 2007. [Online; accessed 21-December-2022].

[119] Zhuchok, Anatolii. (2018). Relatively free doppelsemigroups, Potsdam University Press, 2018.

[120] Foissy, L., Manchon, D., and Zhang, Y. (2020). Families of algebraic structures. *arXiv.* https://doi.org/10.48550/arXiv.2005.05116

[121] Loday, J., and Ronco, M. O. (2002). Trialgebras and families of polytopes. *arXiv*. https://doi.org/10.48550/arXiv.math/0205043.

[122] Loday, J. (2001). Dialgebras. *arXiv*. https://doi.org/10.48550/arXiv.math/0102053

(Concerned with sequences A000112, A001930, A084965, A087729, A178432, A181769, A226193, A305858, and A346414.)

2020 *Mathematics Subject Classification*: Primary 08-08; Secondary 68W99.
Keywords: Mace4, finite model enumeration, algebra.

D

Research Article on Subtraction Algebras and Congruence-3-Permutable

Subtraction Algebras are Congruence-3-Permutable

Gonçalo Araújo*, Wolfram Bentz†‡, Atle Hahn*, Ana-Catarina Monteiro*

Dedicated to John Meakin.

Abstract

Bosbach showed that right complemented semigroups satisfy congruence-3-permutability. This result was subsequently extended to generalized right complemented semigroups by Hagemann and Mitschke. In this note, we show that congruence-3-permutability also holds in subtraction algebras, a variety of magmas inspired by the set-theoretic subtraction. To our knowledge, this result has not been known so far.

1 Introduction

An algebra $\mathbf{A}$ is congruence-3-permutable if for all congruences ϕ, θ of $\mathbf{A}$ the inclusion $\phi \circ \theta \circ \phi \subseteq \theta \circ \phi \circ \theta$ holds. In this case, $\phi \circ \theta \circ \phi = \theta \circ \phi \circ \theta = \theta \vee \phi$, and in particular $\phi \circ \theta \circ \phi$ is a congruence itself. A variety $\mathcal{V}$ is congruence-3-permutable is every algebra in $\mathcal{V}$ is congruence-3-permutable. This definition generalizes classical congruence-permutability (= congruence-2-permutability), that is, the congruence identity $\phi \circ \theta = \theta \circ \phi$. While a strictly weaker condition than congruence-permutability, 3-permutability is a strong enough property to have significant structural implications, and hence is of general interest. For example, congruence-3-permutable varieties are congruence-modular [8] and any compatible T_0-topology on one of their algebras is automatically Hausdorff [7]. Notably, neither of these results is true anymore for (correspondingly defined) congruence-4-permutable varieties (by [6], with a counter-example attributed to Day, and [4]).

Regarding semigroups, it was shown in [3] that right complemented semigroups are 3-permutable. Subsequently this result was extended to generalized right complemented semigroups in [5]. In both of these cases, the interaction of the main and the complementary operation play an important role. However, what has been missing so far is a determination if a type of complementary operation can force 3-permutability on its own.

The aim of this note is to prove that this is indeed the case for subtraction algebras, which use generalizations of set-theoretic subtraction (see Sec. 2 for the exact definition). To our knowledge, this result seems to not have been known. For example, in [10], several consequences of the axioms are listed, but the decisive identity for 3-permutability is missing.

2 The main result

Like other congruence lattice properties, 3-permutability in varieties is equivalent to the existence of term functions with certain identities.

Theorem 1 ([5]). *A variety $\mathcal{V}$ is congruence-3-permutable if and only if $\mathcal{V}$ has ternary term functions f, g such that the following identities are satisfied in all algebras in $\mathcal{V}$:*

(G1) $f(x, y, y) = x$;

(G2) $g(x, x, y) = y$;

(G3) $f(x, x, y) = g(x, y, y)$.

Subtraction algebras generalize the concept of set-theoretic substraction usually denoted by $\setminus$.

*Center for Mathematics and Applications (CMA), Faculdade de Ciências e Tecnologia (FCT), Universidade Nova de Lisboa (UNL), 2829–516, Caparica, Portugal

†Center for Mathematics and Applications (CMA), Faculdade de Ciências e Tecnologia (FCT), Universidade Nova de Lisboa (UNL), 2829–516, Caparica, Portugal

‡Universidade Aberta, R. Escola Politécnica, 147, 1269–001 Lisboa, Portugal; Wolfram.Bentz@uab.pt

Definition 1 ([2])**.** *A subtraction algebra* $\mathbf{S}$ *is a binary algebra* $(S, \cdot)$ *such that the following identities hold:*

(A1) $x \cdot (y \cdot x) = x$;

(A2) $x \cdot (x \cdot y) = y \cdot (y \cdot x)$;

(A3) $(x \cdot y) \cdot z = (x \cdot z) \cdot y$.

To reduce the amount of bracketing, we adopt the convention that the use of concatenation binds stronger than the explicit use of $\cdot$, so that $x \cdot yz$ is interpreted as $x \cdot (y \cdot z)$.
We remark that alternative definitions include an extra constant 0 and the axiom (A4): $0 \cdot 0 = 0$. As subtraction algebras can be shown to satisfy the identity $xx = yy$, this difference is not material for our results.

Proposition 1. *Suppose that the following identities hold*

(A1) $x \cdot yx = x$;

(A2) $x \cdot xy = y \cdot yx$;

(A3) $xy \cdot z = xz \cdot y$;

(A5) $f(x, y, z) = x \cdot yz$;

(A6) $g(x, y, z) = z \cdot yx$.

Then we have:

(G1) $f(x, y, y) = x$;

(G2) $g(x, x, y) = y$;

(G3) $f(x, x, y) = g(x, y, y)$.

Proof. We start by observing that

$$ab \cdot \underbrace{b}_{x} \overset{(A1)}{=} \underbrace{ab}_{x} \cdot (\underbrace{b}_{y} \cdot \underbrace{ab}_{x}) \overset{(A1)}{=} ab,$$

which implies the identity

$$xy \cdot y = xy. \tag{1}$$

We also have

$$a \underbrace{a}_{x} \overset{(A1)}{=} \underbrace{a}_{x}(\underbrace{a}_{x} \cdot \underbrace{ba}_{y}) \overset{(A2)}{=} ba \cdot (ba \cdot a) \overset{(1)}{=} ba \cdot ba,$$

$$a(ba \cdot c) \overset{(A3)}{=} a(bc \cdot a) \overset{(A1)}{=} a,$$

and

$$ab \cdot c \overset{(1)}{=} (ab \cdot b)c \overset{(A3)}{=} (ab \cdot c)b.$$

Hence we get the following identities

$$xx = yx \cdot yx, \tag{2}$$

$$x(yx \cdot z) = x, \tag{3}$$

and

$$xy \cdot z = (xy \cdot z)y, \tag{4}$$

respectively.
Further, notice that

$$a \underbrace{a}_{x} \overset{(3)}{=} \underbrace{a}_{x} \cdot \underbrace{a}_{x}(\underbrace{ba \cdot c}_{y}) \overset{(A2)}{=} (ba \cdot c) \cdot (ba \cdot c)a \overset{(4)}{=} (ba \cdot c)(ba \cdot c) \overset{(2)}{=} cc,$$

and so we get

$$xx = yy. \tag{5}$$

Therefore,

$$f(x, y, y) = x \cdot yy \overset{(5)}{=} x \cdot xx \overset{(A1)}{=} x,$$

which proves ($G1$).
Notice that now the proof of ($G2$) is immediate.
Finally,

$$f(x, x, y) \overset{(A5)}{=} x \cdot xy \overset{(A2)}{=} y \cdot yx \overset{(A6)}{=} g(x, y, y),$$

which proves ($G3$). □

Now, our main theorem is immediate.

Theorem 2. *The variety of subtraction algebras is congruence-3-permutable.*

Proof. The result follows immediately from Theorem 1 and Proposition 1. □

3 Acknowledgments

G. Araújo, W. Bentz, A. Hahn and A-C Monteiro: This work is funded by national funds through the FCT – Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 (https://doi.org/10.54499/UIDB/00297/2020) and UIDP/00297/2020 (https://doi.org/10.54499/UIDP/00297/2020) (Center for Mathematics and Applications).
In the search we were helped by the system ProverX [1, 9].

References

[1] J. Araújo, M. Kinyon, and Y. Robert: "Varieties of regular semigroups with uniquely defined inversion". Portugaliae Mathematica, Volume 76, No. 2 (2019)

[2] J. C. Abbott, Sets, Lattices and Boolean Algebras, Allyn and Bacon, Boston 1969.

[3] B. Bosbach: "Rechtskomplementäre Halbgruppen", Math. Z. 124 (1972) 273–288.

[4] J. Coleman, "Separation in topological Algebras", Algebra Universalis 35 (1996), 72-84.

[5] J.Hagemann and A.Mitschke, On n-permutable congruences, Algebra Universalis 3 (1973), 8–12.

[6] G. Grätzer, Universal Algebra, Second Edition, Springer-Verlag, New York, 1979.

[7] H.P.Gumm, "Topological implications in n-permutable varieties", Algebra Universalis 19 (1984), 319-321.

[8] B. Jónsson,"On the representation of lattices". Math. Scand. 1 (1953), 193–206.

[9] LaiTeP - Laboratory for Augmented Intelligence Theorem Proving: ProverX (2024) online access at https://www.proverx.dm.fct.unl.pt

[10] C. H. Park, "Neutrosophic ideal of Subtraction Algebras, Neutrosophic Sets and Systems", 24 (2019), 36–45.