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Stochastic Debt Sustainability Assessment – Different Approaches and Application to Portugal

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Abstract

Some of the literature on fiscal sustainability has developed approaches for stochastic simulations of the evolution of debt, quantifying the uncertainty inherent in the debt projections of standard debt sustainability assessments (DSA). I apply two different approaches to performing these stochastic debt sustainability assessments (SDSA) to Portugal, until 2027, approximating the methodologies used to allow for comparability of results. The simulated distributions are similar, suggesting a median decrease of the debt ratio, with significant risks to sustainability highlighted by relatively wide distributions. I argue that using both approaches is useful to corroborate the results obtained in the simulations.

Keywords: Fiscal sustainability, stochastic debt sustainability analysis, stochastic debt simulations, fan charts, vector auto-regression (VAR), Portugal

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1. Introduction

Fiscal sustainability is often referred to as the ability of a government to honor all its debt obligations, both now and in the future, given that government's adherence to some level of public consumption and investment. In other words, there is fiscal sustainability when the government obeys its intertemporal budget constraint: the present value of government outlays (including interest payments and debt repayments) should be equal to the present value of government revenues.

The stochastic debt projections which I will focus on in this work project are forward-looking, or prospective, approaches to assessing fiscal sustainability. On the other hand, the academic literature on fiscal sustainability has a significant body of work employing backward-looking, or retrospective, approaches to determining fiscal sustainability[1].

Hamilton and Flavin (1986), for example, employ unit root tests to the level of debt, alongside other econometric estimations, to understand whether, under certain assumptions, the U.S. government's borrowing constraint had been met.

Other research has focused on attempting to find cointegrating relationships necessary for fiscal sustainability. Hakkio & Rush (1991) and Ahmed and Rogers (1995) apply cointegration tests to government revenues and expenditures series.

Another part of the research has focused on estimating an econometric relationship between the primary balance (budget balance before interest expenses) and debt, the fiscal reaction function (FRF). Bohn (1998) estimates a linear regression given by (1).

$$b_t = \rho d_t + \alpha Z_t + \epsilon_t \tag{1}$$

Where b_t refers to the primary balance, d_t refers to the value of debt (start-of-period) and Z_t is a vector of controls at time t. Bohn (1998) argued that, given the stationarity of the other components of fiscal policy, a positive, linear response of the primary surplus to an increase

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[1] See Curtaşu (2011) for an overview of these approaches.

in the debt-to-GDP ratio is sufficient to ensure sustainability. When estimating fiscal response functions with U.S. fiscal data, the author found evidence in favor of a positive coefficient ρ. However, subsequent research by Ghosh et. al. (2013) has shown that this relationship is unlikely to be linear, establishing evidence of fiscal fatigue, that is, that the effect captured by ρ weakens for high debt-to-GDP ratios.

However, the fact that they assess past sustainability makes backward-looking approaches unable to meet the needs of international organizations, like the European Commission and the IMF, which are tasked with, and interested in, detecting, and addressing potential risks to future fiscal sustainability.

Consequently, besides these approaches, there are also forward-looking approaches to assess sustainability. International organizations, such as the European Commission, the IMF and the ECB perform regular debt sustainability assessments (DSA)[2]. The European Commission, for example, employs a DSA as part of its framework on monitoring compliance with the Stability and Growth Pact (SGP), allowing for timely identification of potential risks to fiscal sustainability. From these DSA exercises, a variety of sustainability indicators are gathered, allowing an assessment of the country's fiscal sustainability across different dimensions.

The conventional DSA's involve medium to long-term simulations of the debt-to-GDP ratio under certain macroeconomic forecasts and assumptions that, combined, yield a path of the evolution of debt over a given period. However, having only one path of the evolution of the debt ratio does not provide any information about the uncertainty surrounding such a prediction. One can try to qualify such uncertainty by comparing this initial debt path to others, created under less favorable macroeconomic assumptions, such as an increase to interest rates, a growth shock, or to the primary balance, or combinations of different shocks. These stress tests are useful in illustrating some of the sensitivity of the results of debt

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[2] See Alcidi and Gros (2018) for a review of the different frameworks and approaches for DSA's used by international organizations.

projections to adverse conditions. However, even if shocks are calibrated to match the observed variance of the underlying data, these shock scenarios do not account for the correlation between different types of shocks, nor do they attempt to predict the joint response of the variables relevant to debt accumulation, over time, to the shocks imposed. Furthermore, it is impossible to quantify how likely a particular stress scenario, when compared to any other set of conditions that could have been chosen.

Stochastic debt sustainability analysis (SDSA), therefore, attempt to combat the shortcomings of standard DSA stress tests and simulate an entire distribution of debt-to-GDP ratios over a certain period. Usually, a Monte Carlo type simulation is done to generate many debt-to-GDP ratios by simulating shocks to the variables involved in the debt accumulation equation, in a way that preserves the historical magnitude and correlation of the shocks. In the end, the probability of, under the assumptions of the simulation, the debt ratio increasing in 5 years' time, for example, can be explicitly quantified.

In this work project, I apply two different approaches to stochastic debt sustainability analysis that exist in the literature to the case of Portugal, one based on the estimation of Vector Auto Regressive (VAR) models, and another, simpler approach based on a variance-covariance matrix obtained from historical data, referred to as the simple approach.

Researching the literature, I had not found work applying these two different approaches and comparing their results. Usually, the methodologies used for the simulation vary considerably between the two approaches, meaning that contrasting the results is not very informative.

However, by uniformizing the methodologies regarding the rest of the simulation, as I do here, one can gain understanding into the potential uncertainty and risks surrounding the future evolution of public debt in Portugal, more particularly, and, by comparing the simulation results obtained, to gain understanding into the robustness of these results and approaches, more generally.

The remainder of this work project is structured as follows:

Section 2 contains a literature review of the two approaches. Section 3 details the methodology undertaken, and data used, for the simple approach. Section 4 does the same for the VAR model-based approach. Section 5 presents and discusses the results of the simulation. Finally, section 6 concludes.

2. Literature Review

VAR model-based approach

There are many contributions in the literature that perform SDSA simulations, based on fitting Vector Auto Regressive (VAR) models to historical data of macroeconomic variables used in the debt accumulation equation.

From the estimation of these VAR models, one can use the obtained residuals to generate new residuals with the same variances and covariances and, from these generated residuals, obtain many possible paths for the variables modelled in the VAR. These paths will correspond to many possible debt paths over the simulation period, which are often presented in the form of fan-charts.

Most contributions to the literature use higher frequency (usually quarterly) data in the VAR model[3], which then needs to be annualized to enter the debt accumulation equation, whose inputs are annual data.

Since Bohn (1998), the fiscal response function has become a common tool to model the dynamics of fiscal policy within the debt sustainability literature. In combination with VAR models, fiscal response functions estimated with historical data are commonly used in the literature to obtain forecasts of the primary balance.

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[3] Using annual data to estimate the VAR models would leave us with small time series which would compromise the quality of the VAR estimates. On the other hand, using quarterly or monthly data means that one must be careful in making sure that seasonality is properly accounted for. This is addressed later in this section.

The work by Ferruci and Penalver (2003) represents one of the earliest contributions to the literature on stochastic debt projections. In their paper, the authors fit a VAR to quarterly data on the real growth rate of GDP, the real exchange rate, and on the domestic and foreign interest rates of an emerging-market economy to obtain many future debt paths. Modelling the exchange rate in the VAR model is relevant to accurately model the effect of having a significant portion of debt denominated in foreign currency, an important feature of the debt dynamics in emerging-market economies.

Other early contributions include work by Garcia and Rigobon (2004) and by Celasun, Debrun and Ostry (2006).

Garcia and Rigobon (2004) fit monthly historical data on six macroeconomic variables (including the primary balance) of the Brazillian economy. This way, the model already forecasts future values for the primary balance, removing the need to estimate, separately, a fiscal response function.

Celasun, Debrun and Ostry use quarterly data on real GDP growth, real exchange rate depreciation and on the domestic and foreign interest rates from emerging-market economies are used to estimate the VAR model. Separately, a fiscal response function was estimated using a panel of annual data from emerging market economies. The estimated regression equation was used to forecast values for the primary balance, based on the debt-ratio, the output gap and a vector of other control variables. New residuals are also generated based on the estimated residuals from the fiscal response function, incorporating the uncertainty of the fiscal policy decision into the simulation.

After these contributions, more work developed extensions and modifications to this general methodology.

For example, after finding evidence of the non-normality of the shocks to the VAR variables, Frank and Ley (2009) also generate residuals to the VAR regressions with bootstrapping methods, instead of the joint-normality assumption. The authors experimented with structural breaks by fitting a Markov-Switching VAR to the historical data. This allowed them to obtain two sets of estimates and residuals depending on the state of the Markov-Switching VAR. The authors show that, across the two states, the residual's distributions differ considerably. Frank and Ley (2009) avoid the estimation of a FRF. Instead, the authors set the primary balance equal to the debt-stabilizing balance, the average primary balance that would ensure that debt would stay constant, at each period.

Other contributions to the literature also perform prospective sustainability analysis without the need to estimate a FRF. Tanner and Samake (2008)'s approach does not output a distribution of possible debt paths. Rather, they obtain a distribution of the debt-stabilizing primary balance.

On the other hand, other literature has tried to model non-linear fiscal decision dynamics, such as the work by Burger et. al. (2011), which uses Threshold Autoregressive (TAR) and State-Space modelling of the FRF. The use of a TAR model for the FRF allowed the authors to test for asymmetric responses of the primary balance to positive and negative output gaps, while State-Space modelling, which allows for the estimation of coefficients that change over time, was used to assess the evolution and stability of the coefficient of lagged debt in the FRF, ρ.

Kawakami and Romeu (2011) also focused on trying to include the uncertainty surrounding parameter estimates in the generated debt-paths. They also restrict the estimation of the VAR model, setting non-significant coefficients to zero. In an unrestricted VAR, the authors argue, the shocks applied to the variables may be amplified by imprecise coefficient estimates.

In the context of studying the effects of the implementation of a Eurobond program in the European Union, work by Tielens, Aarle and Hove (2014) estimate a VAR model with the short and long-term interest rates among the variables used. Information on the maturity structure of debt is used to combine the two interest rates outputted by the VAR model to obtain forecasts for the implicit interest rate on debt.

Bouabdallah et. al. (2017)'s work explained the methodological framework that the ECB uses to perform its debt sustainability analysis on Euro area sovereigns. The ECB used a VAR model to generate innovations to the non-fiscal determinants of debt but does not use a FRF to obtain predictions of the primary balance. The methodology used by the ECB uses short-term and long-term interest rates as part of the VAR model and calculates interest payments considering the maturity structure of debt.

This methodology for the ECB's SDSA has since been changed, with a Bayesian VAR model being estimated instead of the normal frequentist estimations of the model. (Bouabdallah and Cozmanca (2019), as cited in Braz and Campos (2019)).

Bańbura, Giannone and Reichlin (2008), for example, talk about the use of Bayesian estimation methods for VAR models as a solution to the problem of overfitting: As more variables and lags are added to a VAR model, the number of coefficients to be estimated increases rapidly, exhausting existing degrees of freedom. The higher number of estimated coefficients, although resulting in a better fit to the existing data, leads to the deterioration of forecast power for the model.

Simple approach

Another approach towards producing a stochastic DSA was introduced by di Giovanni and Gardner (2008), as a simpler alternative to the methodologies using VAR models and estimation of FRF's that had existed in the literature. This approach consists in generating shocks to the macroeconomic variables involved in the debt accumulation equation, based on their historical variance-covariance matrix, and applying these shocks to a baseline scenario of the variables, producing many possible scenarios of evolution of public debt.

Later, Beynet and Paviot (2012) apply this approach to Hungary, augmenting their analysis by accounting for possible exchange rate effects in Hungarian debt dynamics. Using quarterly data, shocks to the exchange rate were generated alongside shocks to the growth rate (which are used to generate shocks to the primary balance) and to the interest rates on debt. These shocks were applied to a baseline scenario based on OECD projections.

Berti (2013) applied this methodology to 24 EU countries. The baseline scenario around which shocks were applied is defined based on DG ECFIN's forecasts and macroeconomic assumptions. This baseline scenario is the same that is used for the deterministic debt projections in the European Commission's DSM (Debt Sustainability Monitor) model. In addition to allowing for exchange rate shocks (for countries not using the Euro), this work also accounted for separate shocks to the short and long-term interest rates since, depending on the maturity structure of public debt, these shocks will translate faster or slower into the implicit interest rate on debt. Following Berti's (2013) methodology, the European Commission has since used stochastic projections as part of its medium-term DSA.

After the introduction of the stochastic projections into the Commission's DSA, there have been some changes to the methodology behind generating the shocks. Namely, in the DSM, shocks to the primary balance are generated from the historical variance-covariance matrix, as opposed to Berti (2013), which derived the primary balance shocks from the shocks to the nominal growth rate. In addition, some baseline scenario assumptions have since been adjusted, namely regarding the paths of inflation and interest rates, to more accurately reflect market expectations.

3. Methodology and Data – Simple Approach

Methodology – Simple Approach

I follow the most recent approach used by the European Commission and presented in European Commission (2021a). The Commission's approach is, in turn, based on Berti (2013)'s methodology, with some changes.

The simulation takes place over five years, from 2023 to 2027. Five years is a common medium-term horizon for DSA's, and the most common simulation length used in most of the literature on SDSA that was reviewed. I use, as a starting point, DG ECFIN's Autumn 2022 Forecast's value of the debt ratio in 2022.

I simulate shocks to four macroeconomic variables involved in debt dynamics: the nominal growth rate, the short and long-term interest rates, and the primary balance. The European Commission approach does not generate exchange rate shocks for countries using the Euro. Since the amount of Portuguese debt denominated in foreign currency is, if not non-existent, negligible[4], I don't include exchange rate shocks in the simulations.

Quarterly data series are used. For each variable x, where x_q is the observation for quarter q, the time series are transformed into a series of quarterly shocks, δ_q^x, according to (2).

$$\delta_q^x = x_q - x_{q-1} \tag{2}$$

Quarterly shocks to the variables, ε_q^x, are generated from a joint-normal distribution with a mean of zero and a variance-covariance matrix equal to the one estimated from the series of δ_q^x. This variance-covariance matrix summarizes information on the variability of each variable and their co-movements. This way, the new shocks are created to match the historical shocks in their magnitude and in the direction and strength of their correlations.

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[4] ECB data on Portuguese government debt denominated in currencies other than the Euro indicates that it has been 0% of GDP since 2016.

Having generated enough quarterly shocks to cover for the projection period (2023-2027), they must be annualized. The annual shocks in year t to the nominal growth rate (ε_t^{ng}), the primary balance (ε_t^{b}) and the short-term interest rate, are calculated according to (3), (4) and (5).

$$\varepsilon_t^{ng} = \sum_{q=1}^{4} \varepsilon_q^{ng} \tag{3}$$

$$\varepsilon_t^{b} = \sum_{q=1}^{4} \varepsilon_q^{b} \tag{4}$$

$$\varepsilon_t^{i^S} = \sum_{q=1}^{4} \varepsilon_q^{i^S} \tag{5}$$

Equations (3) to (5) rule out the persistence of shocks to the nominal growth rate, the primary balance, and the interest rate beyond one year.

On the other hand, the annual long-term interest rate shocks, $\varepsilon_t^{i^L}$, are created to reflect the persistence of these shocks, according to (6). A shock to the long-term interest rate now will continue to be reflected in the interest paid on the long-term debt issued at these conditions, until that debt matures. Therefore, the shock in year t will be carried over to the following years in proportion to the share of long-term debt being rolled over in year t. This is done using the weighted average maturity of debt, T, which is around 7 years for Portugal[5].

$$\begin{cases} \varepsilon_t^{i^L} = \frac{1}{T}\sum_{q=1}^{4} \delta_q^{i^L}, if\ t = 2023 \\ \varepsilon_t^{i^L} = \frac{2}{T}\sum_{q=-4}^{4} \delta_q^{i^L}, if\ t = 2024 \\ \varepsilon_t^{i^L} = \frac{3}{T}\sum_{q=-8}^{4} \delta_q^{i^L}, if\ t = 2025 \\ \varepsilon_t^{i^L} = \frac{4}{T}\sum_{q=-12}^{4} \delta_q^{i^L}, if\ t = 2026 \\ \varepsilon_t^{i^L} = \frac{5}{T}\sum_{q=-16}^{4} \delta_q^{i^L}, if\ t = 2027 \end{cases} \tag{6}$$

Where $q = -4, -8, -12, -16$ refers to the first quarter of year $t-1$, $t-2$, $t-3$ and $t-4$, respectively.

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[5] This comes from ECB data on the average residual maturity of Portuguese debt.

Given the annual shocks to the interest rate on short and long-term debt, $\varepsilon_t^{i^S}$ and $\varepsilon_t^{i^L}$, and the shares of short and long-term debt, α^S and α^L, the shocks to the implicit interest, ε_t^i, are calculated as in (7).

$$\varepsilon_t^i = \alpha^S \varepsilon_t^{i^S} + \alpha^L \varepsilon_t^{i^L} \tag{7}$$

Afterwards, I apply the annual shocks to the baseline scenario path of the variables, over the projection horizon. The nominal growth rate, the primary balance, and the implicit interest rate, in year t, ng_t, b_t and i_t, respectively, are given by their baseline scenario value in that year, $\overline{ng}_t$, $\bar{b}_t$, $\bar{\imath}_t$ plus the annual shock, as in (8), (9) and (10).

$$g_t = \overline{ng}_t + \varepsilon_t^{ng} \tag{8}$$

$$b_t = \bar{b}_t + \varepsilon_t^b \tag{9}$$

$$i_t = \bar{\imath}_t + \varepsilon_t^i \tag{10}$$

The baseline path is defined by a set of macroeconomic assumptions. Importantly, the primary balance is forecast under the assumption of a constant structural primary balance after 2024. Despite having a constant structural component, the primary balance still changes cyclically, based on the output gap. Appendix I details how the baseline path is obtained, and the data used to calculate it.

After taking these steps, I am left with a series for each of the three variables over the projection horizon. Then, given the debt-to-GDP ratio in 2022, I feed the simulated values into (11), the debt accumulation equation, to calculate how the debt-to-GDP ratio would evolve from 2023 to 2027, for each draw of the simulation.

$$d_t = \frac{1 + i_t}{1 + ng_t} d_{t-1} - b_t + f_t \tag{11}$$

Where d_t is the debt-to-GDP ratio, ng_t is the nominal growth rate, i_t is the implicit interest rate, b_t is the primary balance as a percentage of GDP and f_t is the stock-flow adjustment as a percentage of GDP in year t.

The value of the stock-flow adjustment, f_t , in the first two years, 2023 and 2024, is given by the DG ECFIN's Autumn 2022 forecasts for this variable. This variable is set to zero for the remaining years.

For each round of the simulation, out of a total of 5000, I obtain a path of the debt-to-GDP obtaining 5000 different debt paths, which are shown in fan-charts. I also include a financial stress scenario, used in the European Commission's standard DSA, defined in Appendix I.

I compare the results with another case where the annual primary balance shocks, ε_t^b are not generated from the historical variance-covariance matrix but, instead, are generated from the nominal-growth rate shocks, as is done in Berti (2013), according to (12).

$$\varepsilon_t^b = \eta * \varepsilon_t^{ng} \tag{12}$$

Where η is the semi-elasticity of the budget balance with respect to the output gap[6].

Data

For the quarterly data used to generate the shocks, I obtain OECD data on quarterly short and long-term interest rates for Portugal; ESTAT data on quarterly Portuguese national accounts to calculate the quarterly nominal GDP growth rate series; ESTAT quarterly data on Portuguese government accounts is used to calculate the quarterly primary balance[7]. The data covers a common period of 1999-Q1 to 2022-Q2, for the first simulation, which requires quarterly primary balance data, and 1995-Q1 to 2022-Q2, for the second simulation, which does not.

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[6] This semi-elasticity of the budget balance for Portugal, which comes from the calculations of Mourre et al. (2019), is 0.538. This means that, given a 1% increase in the output gap, the primary balance should increase by 0.538% of GDP

[7] The primary balance is calculated by adding net interest payments to the government deficit/surplus.

The quarterly government and national accounts data used, in its raw format, presents significant seasonal behavior. I use seasonally adjusted versions of this data, made available by ESTAT. Given the methodology described above, failing to use seasonally adjusted data would result in generated annual shocks with a greater variability than observed historically.

If one were using unadjusted data, the volatility of δ_q^x will be affected by both the seasonal and non-seasonal fluctuations of the quarterly series.

Given that the generated shocks in each quarter are independent, the variance of the annual shocks to the primary balance, ε_t^b, created according to (4), is given by (13).

$$Var\left(\varepsilon_t^b\right) = 4 \cdot Var\left(\delta_q^b\right) \tag{13}$$

The variance of the generated annual shocks to the variable primary balance, ε_t^b, is four times the variance of the historical quarterly shocks to that same variable, δ_q^b. (13) also applies to the nominal growth rate. This means that part of the variability of δ_q^b and δ_q^{ng}, which comes from the seasonal behavior of these series, will be amplified. In other words, if the seasonal fluctuations of the historical quarterly series are more pronounced, this will lead to a higher variability of the generated annual shocks than the one observed in the historical annual time-series. This would, in turn, lead to unduly wide debt-to-GDP ratio distributions.

The interest rate series don't show any significant seasonal pattern.

The data used in defining the baseline scenario is detailed in Appendix I.

4. Data and Methodology - VAR model-based approach

Methodology

The approach chosen here follows Tielens, Aarle and Hove (2014), insofar as concerns the variables chosen to enter the VAR model, alongside the annualization of the variables. The remainder of the methodology is devised to allow for a fairer comparison between the results.

If all the assumptions had changed from one simulation to another, it would not be possible to meaningfully contrast the results.

I follow the European Commission's DSM model's approach to calculating the evolution of implicit interest rate for the baseline scenario, based on short and long-term interest rates. Likewise, I also follow the DSM model's assumptions regarding the primary balance under the baseline scenario. This means that I depart from some of the VAR approach literature, by not estimating a FRF.

Another reason precludes me from estimating a FRF. Namely, if one wants the results of the SDSA to be useful in identifying risks to sustainability to be addressed by fiscal policy, using an estimated FRF to model fiscal policy may prove confounding to this assessment, since the results would already account for some level of discretionary fiscal policy, as a response to the debt ratio changing, for example. The methodology I follow here, instead, results in a simulation under the assumption of no fiscal policy change.

A VAR model with four variables is estimated using quarterly data.

$$\begin{gathered} X_q = \beta_0 + \sum_{k=1}^{p} \beta_k X_{q-k} + \zeta_q \\ X_t \equiv \left(g_q, \pi_q, i_q^S, i_q^L\right),\ \zeta_q \sim N(0, \Omega) \end{gathered} \tag{14}$$

Where, g_q is the quarterly real GDP growth rate, π_q is the quarterly inflation rate, i_q^S is the short-term interest rate and i_q^L is the long-term interest rate, in quarter q. β_k is a vector of coefficients and ζ_q is a vector of error terms, well behaved and Gaussian with Ω as their variance-covariance matrix.

Before proceeding to the estimation, the data is tested for the existence of unit roots, with Augmented Dickey-Fuller and Philips-Perron unit root tests, detailed in Appendix II. From the tests, I find that the short and long-term interest rates seem to be I(1) variables, with the others being I(0). While some of the initial literature on this approach did not give much attention to the potential existence of unit roots in the data used to estimate the model, work by Eller and Urvová (2012), for example, tested for the existence of unit roots in the variables used, proceeding with the differencing of any I(1) variables before estimating the model.

Since differencing to avoid non-stationarity leads to a loss of some of the information contained in the variables in levels, I start by estimating the VAR model in levels and run the simulation using these estimates. Then, to understand whether the results are impacted by the unit-roots, I also reproduce the simulation for a VAR model estimated with only I(0) variables (I(1) variables differenced).

Besides this, in line with Kawakami and Romeu (2011) I compare the results of the simulation from the VAR in levels by estimating a restricted version of this model, which sets to zero any regressor coefficients not significant at the 90% confidence level, to understand to which extent uncertain coefficient estimates could be affecting the results obtained from the simulation.

All the information criteria point to an optimal model order of 1 for the VAR with the differenced interest rates and order 2 for the model in levels.

Having estimated the models, I generate the new residuals, $\widehat{\zeta_q}$, according to (15).

$$\widehat{\zeta_q} = W\widehat{v_q} \tag{15}$$

Where v_q are independent, random vectors drawn from a standard normal distribution, for each quarter q. W is a Cholesky factorization of the estimated residual's variance-covariance matrix, $\widehat{\Omega}$[8]. By using this procedure, one obtains vectors of quarterly shocks, $\widehat{\zeta_q}$, whose variance-covariance matrix is identical to that of the estimated residuals.

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[8] This method is equivalent to drawing random vectors from a multivariate normal distribution with mean zero and variance-covariance matrix $\widehat{\Omega}$.

Having generated enough residual vectors, $\widehat{\zeta_q}$, to span the simulation period, I feed these into the estimated VAR equations, alongside past variable values, recursively, to obtain a path for each quarterly variable, over the period.

Because the debt accumulation equation takes in annual data, the quarterly variables, x_q, are then annualized, according to (16), for g_q and π_q, and (17), for i_q^S and i_q^L.

$$x_t = \left(\prod_{q=1}^{4}(1 + x_q)\right)^{\frac{1}{4}} - 1 \tag{16}$$

$$x_t = \frac{\sum_{q=1}^{4} x_q}{4} \tag{17}$$

Given, g_t and π_t, I calculate the nominal growth rate, ng_t.

Using the same methodology for obtaining the implicit interest rate under the baseline path for the simple approach, (see Appendix I), I combine the simulated short and long-term interest rates through (18), to obtain the implicit interest rate, imp_t:

$$imp_t = \alpha_S i_t^S + \alpha_L imp_t^L \tag{18}$$

Where α_S is the share of short-term debt, α_L is the share of long-term debt[9] and imp_t^L is the implicit long-term interest rate, which is updated each year according to (19).

$$imp_t^L = \gamma_{t-1} i_t^L + (1 - \gamma_{t-1}) imp_{t-1}^L \tag{19}$$

Where γ_{t-1} is newly issued and matured long-term as a share of total long-term debt in $t - 1$, that is, the share of the long-term debt that was issued at the current long-term interest rate i_t^L. Each year, γ_{t-1} is calculated through (20).

$$\gamma_{t-1} = \frac{\alpha_L * \alpha_L^m * D_{t-2} + \alpha_L * \Delta D_{t-1}}{\alpha_L * (D_{t-2} + \Delta D_{t-1})} \tag{20}$$

Where α_L^m refers to the share of maturing long-term debt[10]. D_{t-2} refers to debt at the start of $t - 1$ and ΔD_{t-1} refers to the change in debt during year $t - 1$.

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[9] α_S comes from ESTAT data and is assumed to remain constant throughout the simulation period.

Equation (20), implicitly, assumes that when new debt is issued (or existing debt redeemed), the amount of long-term debt that is issued (or redeemed) is proportional to the existing share of long-term debt, α_L.

Like what is done when obtaining the baseline path under the simple approach, for the first year of the simulation, I obtain a value for imp_{t-1}^{L} such that (18) is verified, given the values of the implicit interest rate and the short-run interest rate[11].

The primary balance in each year, b_t, is calculated in the same way as what is done for the baseline path under the simple approach, through (21).

$$b_t = spb_t + \eta * ygap_t + c_t \tag{21}$$

Where spb_t refers to the structural primary balance before costs of ageing, $ygap_t$ is the output gap, in year t, η refers to semi-elasticity of the primary balance to the output gap, and c_t are the costs of ageing measured as a % of GDP.

The structural primary balance, spb_t, comes from European Commission's forecasts, for the first two years, and is assumed to remain constant for the remaining three years of the simulation. Even though the primary balance is not modelled in the VAR dynamics directly[12], and I do not estimate a FRF, it is still able to respond to shocks to the GDP growth rate, as can be seen in (21).

The output gap, $ygap_t$ is defined in (22).

$$ygap_t \equiv \frac{Y_t}{Y_t^{POT}} - 1 \tag{22}$$

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[10] α_L^m, in other words, is the percentage of long-term debt with residual maturity less than one year. This comes from ECB data on government debt securities. Namely, a value of 6.48% is used for α_L^m , which was obtained from the table at https://sdw.ecb.europa.eu/reports.do?node=1000003898.

[11] Namely, $imp_{t-1}^{L} = (imp_t - \alpha_S i_t^S)/\alpha_L$

[12] Celasun et. al. (2006) argue against modelling the primary balance within the quarterly VAR model since the quarterly primary balance series has small signal-to-noise ratio.

Where, Y_t and Y_t^{POT} refer to real GDP and potential real GDP, respectively. Starting from initial values, Y_{2021} and Y_{2021}^{POT}, each year of the simulation, I update the values of Y_t and Y_t^{POT} through (23) and (24).

$$Y_t = (1 + g_t)Y_{t-1} \tag{23}$$

$$Y_t^{POT} = (1 + g^{SS})\, Y_{t-1}^{POT} \tag{24}$$

Where g^{SS} refers to the steady-state annual real growth rate implied by the VAR model. g^{SS} is obtained from the equilibrium real growth rate given by the VAR model, as one iterates past values through the equations.

In the end, I calculate the debt in each year, according to (11).

Data

For the for quarterly data used in the VAR model, the short and long-term interest rate series are taken from OECD. The quarterly real growth rate and inflation rate (as measured by the GDP deflator) series are obtained from ESTAT data on national accounts. The data covers a common period from 1995-Q1 to 2022-Q2. Since values for 2022-Q3 and 2022-Q4 for X_q were not available, they are simulated from the VAR equations. This means that the simulation also shows some uncertainty for the debt-to-GDP ratio in 2022.

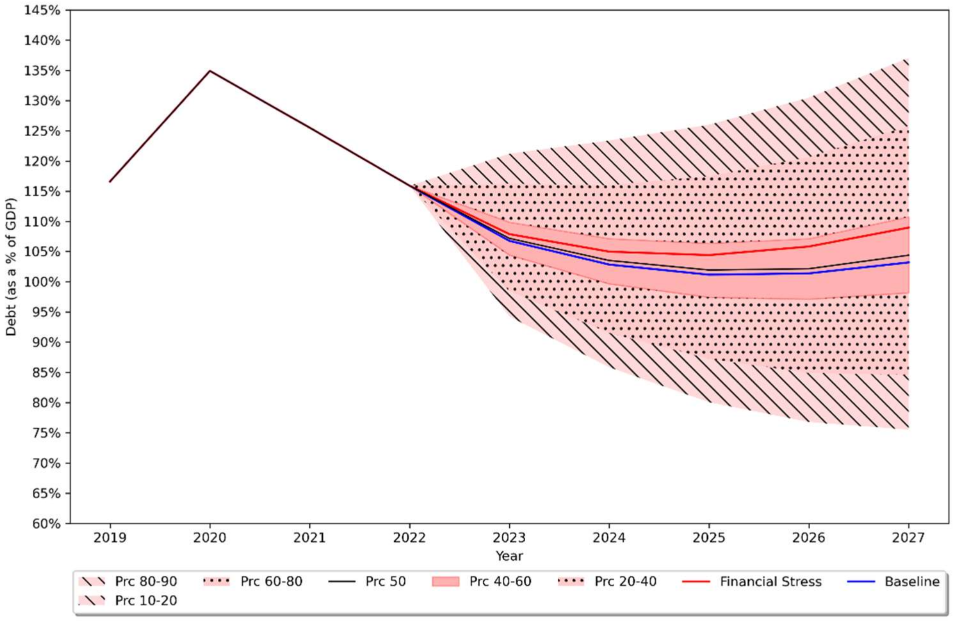
I use seasonally adjusted data provided by ESTAT to obtain series of the nominal growth rate and the inflation rate without seasonality. One could use unadjusted data if the VAR model was estimated with an order of at least $p = 4$. However, this would use up many degrees of freedom.

Regarding the annual data used in the simulation, spb_t, c_t and imp_{2021} come from AMECO. The stock-flow adjustments, f_t, are the same as in the simple approach. The values for c_t, come from European Commission (2021b). and Y_{2021}^{POT} and Y_{2021} come from Output Gaps Working Group (OGWG) data[13]. This data is also used in the defining the baseline scenario.

5. Results and discussion

The results of the simulations are shown in fan-charts that show the different percentiles of the debt-to-GDP ratio throughout the simulated years. The width of the debt cones shown corresponds to 90% of the simulated debt paths. The inner cones show 20% and 40% of all debt paths. I show the results in Figure 1, for the case where the primary balance shocks are generated through the variance-covariance matrix.

Figure 1 - Fan-chart for the simulation under the simple approach (primary balance shocks generated from historical variance-covariance matrix)



Under the baseline path, the Portuguese debt-to-GDP ratio is set to decrease over the five-year horizon (103.2% of GDP in 2027 compared to 115.9% of GDP in 2022). However, this decrease occurs mainly in the first three years, from 2023 to 2025, to 101.2% of GDP, with the debt level under the baseline increasing again after 2026. This first decrease is consistent

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[13] This data is freely available in the Output Gaps public group at https://circabc.europa.eu/ui/explore

with the relatively high structural primary balance forecasted by the European Commission for Portugal, and which is assumed to remain constant for the remaining years, assuming no discretionary fiscal adjustments during the period. On the other hand, increasing debt-servicing costs, brought about by the exit from the very low interest rate environment in developed economies, contribute to the upward trajectory taken by the baseline path at the end of the simulation period.

For the financial stress scenario, obtained by shocking interest rates up by 1pp., and including a risk-premium for high debt-to-GDP ratios, debt is still projected to decrease slightly over the simulation period, to 109% of GDP in 2027.

Overall, the results from the baseline scenario paint a scenario of favorable debt dynamics for Portugal over the next few years, conditions which are will slowly fade, as both interest rate rises, and GDP growth slows.

When looking at the results of the stochastic part of the simulation, according to the distribution of debt outcomes in 2027, there is an implied probability of 67.7% that Portuguese debt will not rise above the debt-to-GDP ratio in 2022. The median debt ratio in 2027 is very similar to the baseline, which is expected since the shocks are generated around this path.

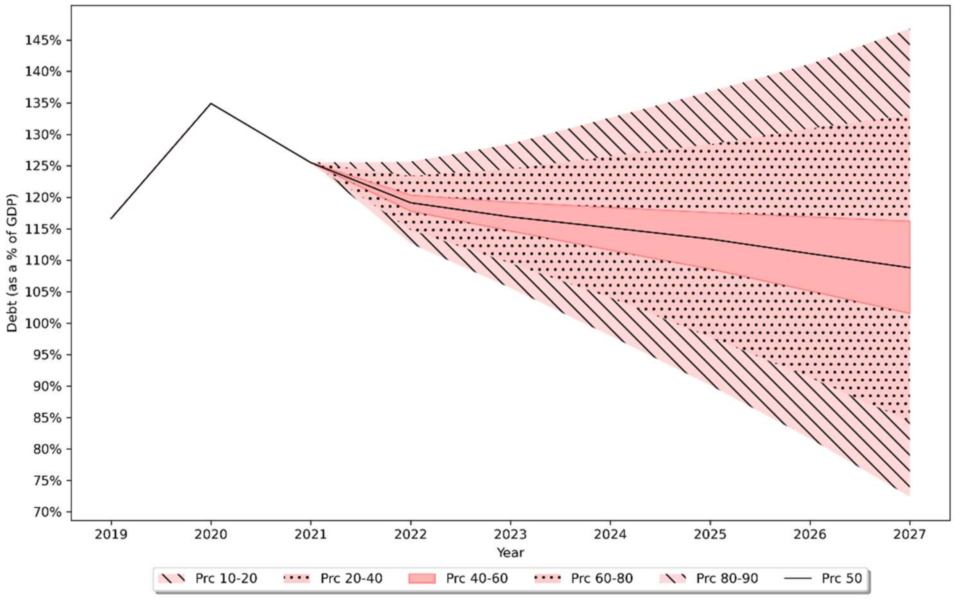
From these simulations, one can also gain an understanding of how extreme the financial stress test scenario is. This scenario is inside the inner cone, with around 40% of the simulated scenarios resulting in higher debt-to-GDP ratios. Therefore, when compared to the constellations of shocks generated in the simulation, the financial stress test is less extreme than most.

The width of the debt cone, representing 80% of all paths, is 37.5% of GDP, in 2024, and rises to of 61.5% of GDP, in 2027. This debt cone represents the uncertainty present in the baseline projections.

When comparing the width of the debt cones at T+5, shown in European Commission (2021a), for Portugal, with other countries, one finds that it is around 60% of GDP for Portugal, like what is obtained in my simulations, while it is smaller for most other Euro Area countries, indicating that Portugal suffers from higher uncertainty surrounding the evolution of debt.

The results of the simulation for the VAR model-based approach, for the model in levels, are shown in Figure 2. The fan-chart obtained from the model with I(0) variables is largely the same.

Figure 2 - Fan-chart showing the simulation under the VAR model-based approach (model in levels)

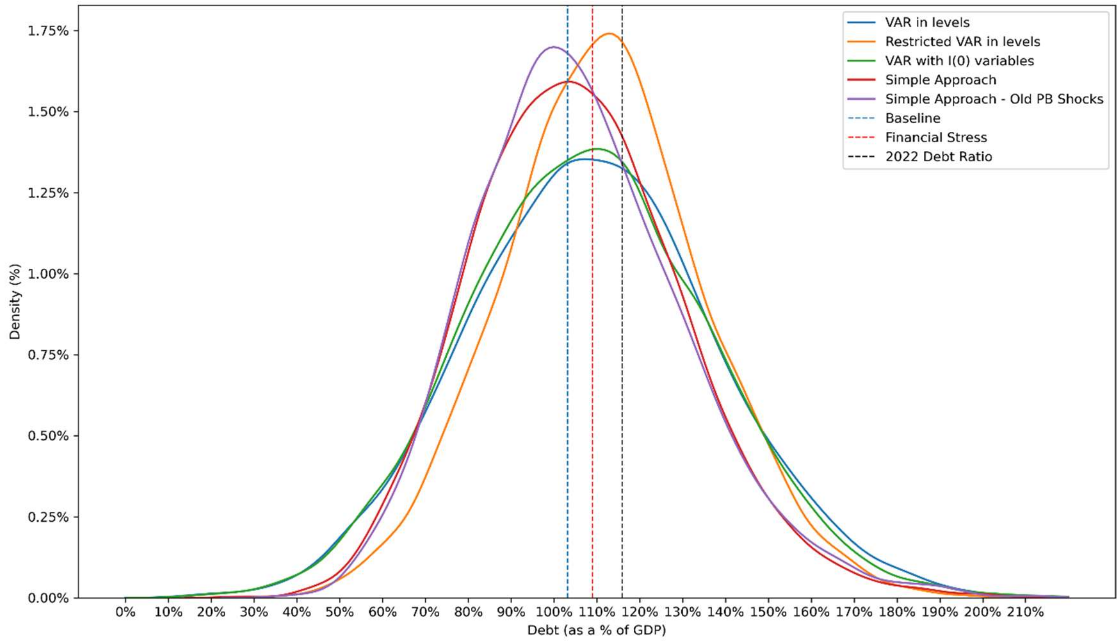


The results from the VAR model in levels also point to a slight decrease in the debt ratio over the simulation period, with a median of 115.1% of GDP in 2024 and 108.8% of GDP in 2027. The simulation points to an implied probability of 61.3% that the debt-to-GDP ratio will decrease, as compared to the value in 2022. This approach also leads to a slightly higher implied uncertainty for Portuguese debt dynamics, with the width of the outer cone rising to 74.4% of GDP in 2027.

The restricted VAR model, has similar median trajectories, but has a smaller implied uncertainty, more in line with what is obtained under the simple approach, with a difference between the 10th and 90th percentile in 2027 of 63.3% of GDP.

Figure 3 shows the density kernel for the distributions of the debt ratios in 2027 from all the simulations ran. In this case, the simulations generate slightly different distributions, but that have similar means and variances.

Figure 3 – Histogram density kernel for the simulated debt-to-GDP ratio distributions in 2027



Comparing the approaches, one finds that the results largely agree: Portuguese public debt is likely to decrease but the large dispersion of the distributions suggests a significant risk to debt rising to high levels due to unfavorable scenarios, which may require fiscal adjustment to ensure sustainability.

Because the simple approach can be applied to a baseline path, which can be readily changed to reflect current economic projections, it becomes a useful benchmark against which to compare the results of the VAR-based approach, serving as a sanity-check on the median path of these results. However, the simple approach simulates the shocks to the variables in a more rudimentary way, which could lead to constellations of shocks with unrealistic sizes and correlations. On the other hand, the VAR approach may be able to model the variables and their interactions more accurately, leading to more accurate combinations of generated shocks. Therefore, by following, as I do here, a methodology that allows for a meaningful comparison between these two approaches, which have different shortcomings, confidence in the results of the SDSA should increase if they are consistent. If they are not, further analysis should be done to understand and qualify any disparities between the results.

6. Conclusion

Having applied two different approaches for assessing debt sustainability under uncertainty to the case of Portugal, I obtain distributions for simulated debt outcomes in 2027. Overall, an average decrease in the debt ratio is expected over the next 5 years, with some significant risks to sustainability being highlighted by the width of the distributions obtained.

Encouragingly, the distributions obtained have similar moments, suggesting robustness of the results to the different approaches. I argue that employing a methodology, as I do here, that allows for comparing between the different approaches is helpful to validate these results.

However, there are still avenues for further work looking to explore and compare these methods. Specifically, instead of assuming that the shocks come from a normal distribution, one could explore bootstrapping methods to draw residuals directly from their empirical distributions. This could capture, for example, existing asymmetries in the shocks not captured by a normal distribution. As done by Frank and Ley (2009), methods for generating residuals with bootstrapping can also be used to draw clusters of consecutive residuals, instead of assuming their independence across periods.

The existence of asymmetries may lead to the results underestimating upward risks for debt and overestimating the possibility of downward paths. Furthermore, if adverse shocks are not temporally independent, and instead, sometimes end up clustered in periods of time, not capturing this effect would also underestimate the risks for debt increases.

Bayesian estimation methods for the VAR model, which are already employed in the ECB's new SDSA methodology, and which would, in theory, allow for better forecasting power, even as models become larger, should also be explored in future work.

In addition, more definitive analysis on the predictive power of these approaches could be done through rolling simulations, such that the results obtained across several years could be evaluated against the observed evolution of debt.

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