

Minimum rank and path cover number for generalized and double generalized cycle star graphs

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Abstract

For a given connected (undirected) graph $G = (V, E)$, with $V = \{1, \dots, n\}$, the minimum rank of G is defined to be the smallest possible rank over all symmetric matrices $A = [a_{ij}]$ such that for $i \neq j$, $a_{ij} = 0$ if, and only if, $\{i, j\} \notin E$. The path cover number of G is the minimum number of vertex-disjoint paths occurring as induced subgraphs of G that cover all the vertices of G . When G is a tree, the values of the minimum rank and of the path cover number are known as well the relationship between them. We study these values and their relationship for all graphs that have at most two vertices of degree greater than two: generalized cycle stars and double generalized cycle stars.

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1 Introduction

Let $G = (V, E)$ be a simple, connected and undirected graph with vertex set $V = \{1, \dots, n\}$ and edge set E .

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We say that a real $n \times n$ symmetric matrix $A = [a_{ij}]$ is associated with G if for $i \neq j$ ($i, j \in \{1, \dots, n\}$), $a_{ij} = 0$ if, and only if, $\{i, j\} \notin E$. The set of all symmetric matrices associated with G is denoted by $\mathcal{S}(G)$.

The **maximum multiplicity** of G , $M(G)$, is the maximum multiplicity for a single eigenvalue among matrices in $\mathcal{S}(G)$. The **minimum rank** of G , $mr(G)$, is defined to be the smallest possible rank over matrices in $\mathcal{S}(G)$. A **path cover** of G is a set of induced paths of G that do not intersect, but do cover all vertices of G . The **path cover number** of G , $P(G)$, is the minimum number of induced paths of a path cover of G . The number $\Delta(G)$ is the maximum of $p - q$ such that there exist q vertices of G whose deletion leaves p paths. Isolated vertices count as (degenerate) paths. The **maximum rank deficiency** for G , is $m(G) = n - mr(G)$, where n is the order of G .

In [5] Johnson and Leal Duarte proved that, for each tree T ,

$$M(T) = \Delta(T) = P(T) = m(T).$$

We aim to prove similar results for generalized and double generalized cycle stars.

2 Generalized cycle star and path cover number

All graphs discussed in this paper are simple, connected and undirected.

Let G be a graph and let v be a vertex of G . We denote by $G - v$ the subgraph of G obtained by deleting v and all edges incident with v . If S is a set of vertices, we let $G - S$ denote the graph obtained by removing each vertex of S and all associated incident edges. If G is a tree, the components of $G - v$ are called **branches** of G at v .

A tree T in which there is at most one vertex of degree greater than two is a **generalized star**. In a generalized star, a vertex v is a **central vertex** if its neighbors are pendant vertices (vertices of degree one) of their branches, and each branch is a path. Note that a path is a degenerate generalized star, in which every vertex is a central vertex. When referring to a path as a generalized star, one vertex has been fixed as the central vertex.

The **arms** of a generalized star T are the branches of T at v , where v is the central vertex of T .

Definition 2.1. A **generalized cycle star** is a connected graph H with at most one vertex with degree greater than two. If there exists a vertex v of degree greater than two, then v is the **central vertex** of H , otherwise one vertex v is fixed as the central vertex.

A generalized cycle star H where every non central vertex has degree two is a **cycle star**.

Observation 2.2.

1. If H is a generalized cycle star there is a generalized star T such that $H = T$ or H can be obtained from T by joining by an edge some of the pendant vertices of T to its central vertex. The central vertex of H is the central vertex of T .
2. Let H be a generalized cycle star with central vertex v . All the connected components of $H - v$ are paths. If W is the set of vertices of a connected component of $H - v$ then the subgraph of H induced by $W \cup \{v\}$ is a path or is a cycle.

Definition 2.3. Let H be a generalized cycle star with central vertex v . The connected components of $H - v$ are the **arms** of H . If B is an arm of H and W is the set of vertices of B , the subgraph H' of H induced by $W \cup \{v\}$ is called a **petal** if H' is a cycle and is called a **ray** if H' is a path.

If H is a generalized cycle star with central vertex v , c petals and r rays we say that H is a (v, c, r) -**generalized cycle star**.

Example 2.4. 1. The graph in Figure 2 is a generalized cycle star with 3 petals, 2 rays and central vertex v . This graph can be obtained from the generalized star T , represented in Figure 1, by joining by an edge some of the pendant vertices of T to its central vertex.

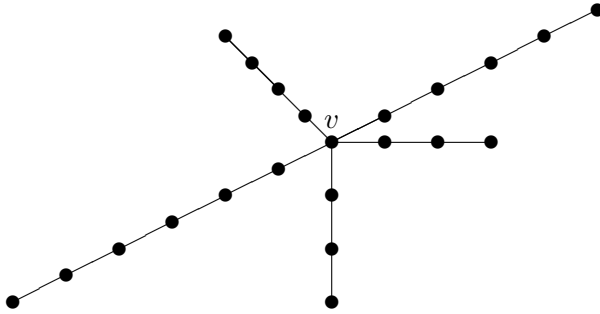


Figure 1: Generalized star

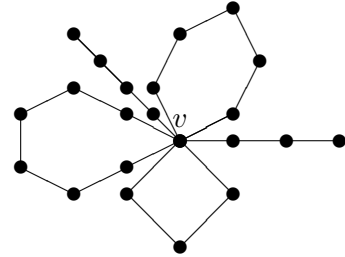


Figure 2: $(v, 3, 2)$ -generalized cycle star

2. The graph in Figure 4 is a cycle star. It can be obtained from the generalized star T in Figure 3 by joining by an edge each of the pendant vertices of T to the central vertex v .

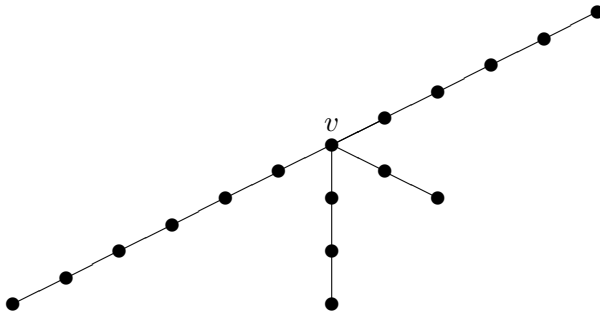


Figure 3: Generalized star

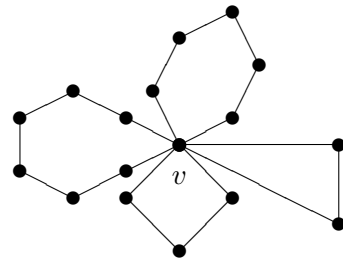


Figure 4: Cycle star graph

Observation 2.5. Let H be a (v, c, r) -generalized cycle star. Then:

1. The degree of the central vertex v is $2c + r$.
2. If $c = 0$, the graph is a generalized star.
3. If $r = 0$, the graph is a cycle star.

Proposition 2.6. *Let H be a (v, c, r) -generalized cycle star. Then*

$$\Delta(H) = \begin{cases} 1, & \text{if } r \leq 1 \text{ and } c = 0; \\ c + r - 1, & \text{if } r > 1 \text{ or } c \neq 0. \end{cases}$$

Proof

If $r \leq 1$ and $c = 0$, H is a path and $\Delta(H) = 1$. For the remaining cases the conclusion follows easily since if we delete the central vertex of H we obtain $c + r$ paths and the removal of any other vertex increases the number of paths by a maximum of one. \square

Proposition 2.7. *Let H be a (v, c, r) -generalized cycle star. Then*

$$P(H) = \begin{cases} c + 1, & \text{if } r \leq 1; \\ c + r - 1, & \text{if } r > 1. \end{cases}$$

Proof

Let $\{S_1, \dots, S_{P(H)}\}$ be a minimal path cover for H .

Let $F = H - v$. Then $P(F) = c + r$ since F is the union of $c + r$ disjoint paths.

Case 1: $r \leq 1$

Consider the path Q induced by the central vertex v and the remaining vertices of the ray when $r = 1$; if $r = 0$ let Q be the graph containing only the single vertex v .

Removing from the graph H the path Q we obtain c disjoint paths; these paths and Q are induced paths that do not intersect and that cover all vertices of H . Thus $P(H) \leq c + 1$.

To verify that $c + 1 \leq P(H)$ we can suppose, without loss of generality, that $v \in S_{P(H)}$. Then each of the remaining paths S_i has vertices from only one arm of H .

Let $W_{P(H)}$ be the set of vertices of $S_{P(H)} - v$. If $W_{P(H)} = \emptyset$, then $\{S_1, \dots, S_{P(H)-1}\}$ is a path cover for F . Thus, in this case, $P(F) < P(H)$. Since $P(F) = c$ if $r = 0$ and $P(F) = c + 1$ if $r = 1$, we conclude that $c + 1 \leq P(H)$. Suppose now that $W_{P(H)} \neq \emptyset$. We have the following cases:

1. $W_{P(H)}$ has vertices of two distinct petals, C and C' , of H .

By the definition of $P(H)$ there must exist paths S_k and S_l such that the vertices of S_k are all the vertices of C that do not belong to $S_{P(H)}$ and the vertices of S_l are all the vertices of C' that do not belong to $S_{P(H)}$.

Let W_k be the set of vertices of C that belong to $W_{P(H)}$ and let W_l be the set of vertices of C' that belong to $W_{P(H)}$. Let S'_k be the subgraph induced by the union of the set of vertices of S_k with W_k and let S'_l be the subgraph induced by the union of the set of vertices of S_l with W_l . Note that $S'_k = C - v$ and $S'_l = C' - v$. Let $S'_i = S_i$, for $i \in \{1, \dots, P(H) - 1\} \setminus \{k, l\}$. Then $\{S'_1, \dots, S'_{P(H)-1}\}$ is a path cover for F . Thus $P(F) < P(H)$ and $c + 1 \leq P(H)$.

2. $W_{P(H)}$ has vertices of a petal C and vertices of a ray R of H .

In this case $r = 1$. As $v \in S_{P(H)}$, it is easy to conclude that all vertices of $R - v$ are in $W_{P(H)}$ otherwise there would be a path cover for H with number of paths less than $P(H)$. Also, there must exist a path S_k such that the vertices of S_k are all the vertices of C that do not belong to $S_{P(H)}$. Let W_k be the set of vertices of C that belong to

$W_{P(H)}$. Let S'_k be the subgraph induced by the union of the set of vertices of S_k with W_k .

Then $\{S_1, \dots, S_{k-1}, S'_k, S_{k+1}, \dots, S_{P(H)-1}, S_{P(H)} - (W_k \cup \{v\})\}$ is a path cover for F . Thus $P(F) \leq P(H)$. Since, in this case, $P(F) = c+1$, we conclude that $c+1 \leq P(H)$.

3. $W_{P(H)}$ is contained in the set of the vertices of a ray of H .

Then $\{S_1, \dots, S_{P(H)} - v\}$ is a path cover for F . Thus $c+1 = P(F) \leq P(H)$.

4. $W_{P(H)}$ is contained in the set of the vertices of a petal C of H .

By the definition of $P(H)$ there must exist a path S_k such that the vertices of S_k are all the vertices of C that do not belong to $S_{P(H)}$.

Let $S'_k = C - v$. Then $\{S_1, \dots, S_{k-1}, S'_k, S_{k+1}, \dots, S_{P(H)-1}\}$ is a path cover for F . Thus $P(F) \leq P(H) - 1$ and $c+1 \leq P(H)$.

Case 2: $r > 1$

Let R_1, \dots, R_r be the rays of H and let Q be the concatenation of the rays R_1 and R_2 . Removing from the graph the path Q we obtain $c+r-2$ disjoint paths; these paths and Q are paths that do not intersect and that cover all vertices of H . Thus $P(H) \leq c+r-2+1 = c+r-1$.

Since $v \in S_{P(H)}$ and $S_{P(H)}$ is a path, then $S_{P(H)} - v$ is a path (if v is a pendent vertex of $S_{P(H)}$) or is the union of two paths. Thus, if we remove the vertex v from $S_1, \dots, S_{P(H)}$ we obtain γ paths that do not intersect and that cover all vertices of F , which $\gamma \leq P(H) + 1$.

Thus $P(F) \leq P(H) + 1 \leq c+r-1+1 = c+r = P(F)$. Therefore $P(H) = c+r-1$. \square

3 Maximum multiplicity and minimum rank of a generalized cycle star

Let G be a graph and let v be a vertex of G . The **rank-spread** of G at vertex v is defined to be $r_v(G) = mr(G) - mr(G - v)$.

The vertex v is a **cut-vertex** of a connected graph G if, and only if, $G - v$ is disconnected. If H is a (v, c, r) -generalized cycle star with $c+r > 1$ then the central vertex v is a cut-vertex of H .

Proposition 3.1. [1] *Let v be a cut-vertex of a graph G . For $i = 1, \dots, s$, let W_i be the set of vertices of the i th component of $G - v$ and let G_i be the subgraph induced by $\{v\} \cup W_i$. Then*

$$r_v(G) = \min \left\{ \sum_{i=1}^s r_v(G_i), 2 \right\}$$

and

$$mr(G) = \sum_{i=1}^s mr(G_i - v) + \min \left\{ \sum_{i=1}^s r_v(G_i), 2 \right\}.$$

Observation 3.2. The following results are well known and straightforward:

1. If P_n is a path with n vertices, then $mr(P_n) = n - 1$.

2. If C_n is a cycle with n vertices, then $mr(C_n) = n - 2$.

Proposition 3.3. *Let H be a (v, c, r) -generalized cycle star with n vertices. Then*

$$mr(H) = \begin{cases} n - c - 1, & \text{if } r \leq 1; \\ n - c - r + 1, & \text{if } r > 1. \end{cases}$$

Proof

If H is a cycle or a path the result follows. Suppose that H is neither a cycle nor a path. As the central vertex v is a cut-vertex of H , we can use Proposition 3.1. For $i = 1, \dots, c+r$, let W_i be the set of vertices of the i th component of $H - v$, let $n_i = |W_i|$ and let H_i be the subgraph induced by $\{v\} \cup W_i$.

Note that for $i = 1, \dots, c+r$, H_i is a path or a cycle but $H_i - v$ is always a path. By the Observation 3.2 we have

$$mr(H_i) = \begin{cases} (n_i + 1) - 1, & \text{if } H_i \text{ is a path;} \\ (n_i + 1) - 2, & \text{if } H_i \text{ is a cycle} \end{cases}$$

and

$$mr(H_i - v) = n_i - 1.$$

It follows from the definition of rank-spread that

$$r_v(H_i) = mr(H_i) - mr(H_i - v) = \begin{cases} 1, & \text{if } H_i \text{ is a path;} \\ 0, & \text{if } H_i \text{ is a cycle.} \end{cases}$$

By Proposition 3.1

$$r_v(H) = \min \left\{ \sum_{i=1}^{c+r} r_v(H_i), 2 \right\} = \begin{cases} r, & \text{if } r \leq 1; \\ 2, & \text{if } r > 1 \end{cases}$$

and thus

$$mr(H) = \sum_{i=1}^{c+r} mr(H_i - v) + \min \left\{ \sum_{i=1}^{c+r} r_v(H_i), 2 \right\} = \begin{cases} n - 1 - c, & \text{if } r \leq 1; \\ n - (c + r) + 1, & \text{if } r > 1. \end{cases}$$

□

Theorem 3.4. *Let H be a generalized cycle star. Then*

$$P(H) = M(H) = m(H).$$

Proof

Suppose that H is a (v, c, r) -generalized cycle star with n vertices. By definition, $m(H) = n - mr(H)$. Using Proposition 3.3 we have:

$$m(H) = \begin{cases} c + 1, & \text{if } r \leq 1 \\ c + r - 1, & \text{if } r > 1 \end{cases}.$$

By Proposition 2.7 we conclude that $P(H) = m(H)$. By [4] (pp. 34-5), $m(H) = M(H)$, for each graph H . Thus

$$P(H) = M(H) = m(H).$$

□

From Proposition 2.6, Proposition 2.7 and Theorem 3.4 we can easily conclude the next result.

Corollary 3.5. *Let H be a generalized cycle star with at least two rays. Then*

$$P(H) = M(H) = m(H) = \Delta(H).$$

4 Connected graphs with at most two vertices with degree greater than two

Let u and v be vertices of a graph G . Two paths from u to v are **internally disjoint** if, apart from u and v , no vertex of G lies on both paths.

Definition 4.1. Let H_1 and H_2 be disjoint generalized cycle stars with central vertices v_1 and v_2 , respectively. A graph G which can be obtained from H_1 and H_2 by joining a nonempty collection of pairwise internally disjoint paths from v_1 to v_2 is a **double generalized cycle star with central vertices v_1 and v_2** . If t is the number of pairwise internally disjoint paths from v_1 to v_2 and d is the number of these paths with length greater than 1 we say that G is a (H_1, H_2, t, d) -double generalized cycle star.

Example 4.2. 1. The graph in Figure 5 is a $(H_1, H_2, 3, 2)$ -double generalized cycle star with central vertices v_1 and v_2 , where H_1 is a $(v_1, 3, 1)$ -generalized cycle star and H_2 is a $(v_2, 2, 3)$ -generalized cycle star.

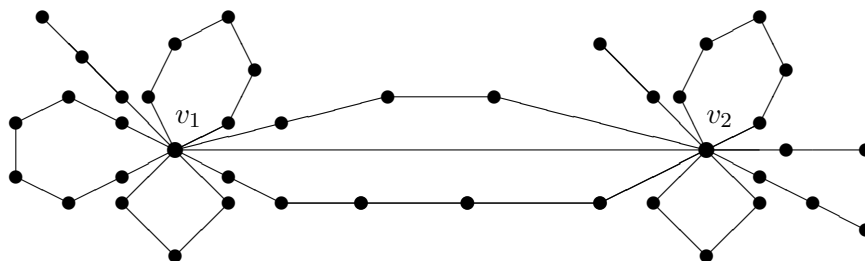


Figure 5: Double generalized cycle star

2. The graph in Figure 6 is a $(H_1, H_2, 3, 2)$ -double generalized cycle star with central vertices v_1 and v_2 , where H_1 is a $(v_1, 0, 0)$ -generalized cycle star and H_2 is a $(v_2, 2, 3)$ -generalized cycle star.

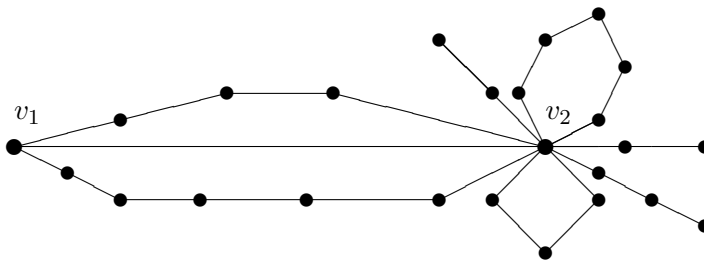


Figure 6: $(H_1, H_2, 3, 2)$ -double generalized cycle star

Definition 4.3. A graph that is a nonempty collection of t internally disjoint paths from a vertex v_1 to a vertex v_2 , with $d > 1$ paths of length greater than 1 is called a (v_1, v_2) -**rugby ball**.

Observation 4.4. A (v_1, v_2) -rugby ball is a (H_1, H_2, t, d) -double generalized cycle star with $d > 1$, where H_1 is a $(v_1, 0, 0)$ -generalized cycle star and H_2 is a $(v_2, 0, 0)$ -generalized cycle star.

Example 4.5. The graph in Figure 8 is a (v_1, v_2) -rugby ball with $d = 4$.

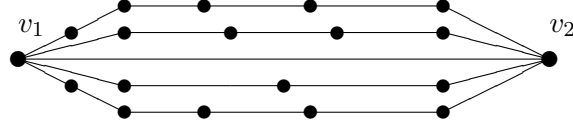


Figure 7: Rugby ball

Observation 4.6.

1. Every connected graph with exactly two vertices with degree higher than two is a double generalized cycle star.
2. Every double generalized cycle star is a graph with at most two vertices with degree greater than two.

4.1 Double generalized cycle star and path cover number

Lemma 4.7. *Let G be a (H_1, H_2, t, d) -double generalized cycle star where H_1 is a (v_1, c_1, r_1) -generalized cycle star and H_2 is a (v_2, c_2, r_2) -generalized cycle star. Then*

$$P(G) \geq c_1 + c_2 + r_1 + r_2 + d - 2.$$

Proof

Note that if $H = G - \{v_1, v_2\}$, then $P(H) = c_1 + c_2 + r_1 + r_2 + d$ since if we remove from the graph G the vertices v_1 and v_2 we obtain $c_1 + c_2 + r_1 + r_2 + d$ disjoint paths.

Let $\{S_1, \dots, S_{P(G)}\}$ be a minimal path cover for G . If v_1 and v_2 are in the same path S_i , then $S_i - \{v_1, v_2\}$ is a path or a union of two or three paths. If $v_1 \in S_i$ and $v_2 \in S_j$ with $i \neq j$, then $S_i - v_1$ is a path or a union of two paths and $S_j - v_2$ is also a path or a union of two paths. Thus, if we remove the vertices v_1 and v_2 from the paths $\{S_1, \dots, S_{P(G)}\}$ we obtain at most $P(G) + 2$ paths that do not intersect and that cover all vertices of H .

Thus $c_1 + c_2 + r_1 + r_2 + d = P(H) \leq P(G) + 2$. Therefore $P(G) \geq c_1 + c_2 + r_1 + r_2 + d - 2$. \square

Proposition 4.8. *Let G be a (H_1, H_2, t, d) -double generalized cycle star where H_1 is a (v_1, c_1, r_1) -generalized cycle star and H_2 is a (v_2, c_2, r_2) -generalized cycle star. We have*

$$P(G) = c_1 + c_2 + r_1 + r_2 + d - 2,$$

for each of the following cases:

1. $d > 3$;
2. $r_1 > 1$ and $r_2 > 1$;
3. $d = 3$ and $r_1 + r_2 \geq 1$;
4. $d = 2$ and $r_1 + r_2 > 1$;
5. $d = 1$ and $\min\{r_1, r_2\} = 1$ and $r_1 \neq r_2$;
6. $d = 1 = t$ and $r_1 = r_2 = 1$.

Proof

Let P_1, P_2, \dots, P_d be the family of internally disjoint paths from v_1 to v_2 with length greater than 1. Let R_1, R_2, \dots, R_{r_1} and $R'_1, R'_2, \dots, R'_{r_2}$ be the rays of H_1 and H_2 , respectively. For each of the following cases consider the paths Q_1 and Q_2 defined as indicated:

1. For $d > 3$, let Q_1 be the concatenation of paths $P_1 - v_2$ and $P_2 - v_2$ and let Q_2 be the concatenation of paths $P_3 - v_1$ and $P_4 - v_1$.
2. If $r_1 > 1$ and $r_2 > 1$, let Q_1 be the concatenation of paths R_1 and R_2 and let Q_2 be the concatenation of paths R'_1 and R'_2 .
3. For $d = 3$ and $r_1 + r_2 \geq 1$ suppose, wlog, that $r_1 \geq 1$. Let Q_1 be the concatenation of paths $P_1 - v_1$ and $P_2 - v_1$ and let Q_2 be the concatenation of paths $P_3 - v_2$ and R_1 .
4. For $d = 2$ and $r_1 + r_2 > 1$ consider the following sub-cases:
 - If $r_1 = 0$ then $r_2 > 1$. Let Q_1 be the concatenation of paths $P_1 - v_2$ and $P_2 - v_2$ and let Q_2 be the concatenation of paths R'_1 and R'_2 .
 - If $r_1 = 1$ then $r_2 \geq 1$. Let Q_1 be the concatenation of paths R_1 and $P_1 - v_2$ and Q_2 be the concatenation of paths $P_2 - v_1$ and R'_1 .
 - If $r_1 > 1$, let Q_1 be the concatenation of paths R_1 and R_2 and let Q_2 be the concatenation of paths $P_1 - v_1$ and $P_2 - v_1$.
5. If $d = 1$ and $\min\{r_1, r_2\} = 1$ and $r_1 \neq r_2$, suppose, wlog, that, $r_2 = 1$ and $r_1 > 1$. Let Q_1 be the concatenation of paths R_1 and R_2 and let Q_2 be the concatenation of paths R'_1 and $P_1 - v_1$.

In each of the previous cases, removing the paths Q_1 and Q_2 from the graph G we obtain $c_1 + c_2 + r_1 + r_2 + d - 4$ disjoint paths; these paths along with Q_1 and Q_2 are paths that do not intersect and that cover all vertices of G . Thus $P(G) \leq c_1 + c_2 + r_1 + r_2 + d - 2$.

6. If $d = 1 = t$ and $r_1 = r_2 = 1$, let Q_1 be the concatenation of paths R_1 , P_1 and R'_1 . In this case, removing from the graph G the path Q_1 we obtain $c_1 + c_2$ disjoint paths; these paths and Q_1 are paths that do not intersect and that cover all vertices of G . Thus $P(G) \leq c_1 + c_2 + 1 = c_1 + c_2 + r_1 + r_2 + d - 2$.

Since by Lemma 4.7 we have that $P(G) \geq c_1 + c_2 + r_1 + r_2 + d - 2$, we can conclude that $P(G) = c_1 + c_2 + r_1 + r_2 + d - 2$.

□

Observation 4.9. Let G be a (H_1, H_2, t, d) -double generalized cycle star where H_1 is a (v_1, c_1, r_1) -generalized cycle star and H_2 is a (v_2, c_2, r_2) -generalized cycle star.

1. The values of $P(G)$ for the cases that are not considered in the previous proposition are summarized in the following table and can easily be deduced.

Cases	$\mathbf{P(G)}$
$r_1 = r_2 = 0 \wedge d < t \wedge d \leq 1$	$c_1 + c_2 + r_1 + r_2 + d + 1$
$r_1 = r_2 = 0 \wedge d < t \wedge d = 2$ $r_1 = r_2 = 0 \wedge d = t \wedge d \leq 2$ $r_1 + r_2 = 1 \wedge d < t \wedge d \leq 1$ $((r_1 = 0 \wedge r_2 > 1) \vee (r_2 = 0 \wedge r_1 > 1)) \wedge d = 0$	$c_1 + c_2 + r_1 + r_2 + d$
$r_1 = r_2 = 0 \wedge d = 3$ $r_1 + r_2 = 1 \wedge d = t = 1$ $r_1 + r_2 = 1 \wedge d = 2$ $r_1 = r_2 = 1 \wedge d < t \wedge d \leq 1$ $((r_1 = 0 \wedge r_2 > 1) \vee (r_2 = 0 \wedge r_1 > 1)) \wedge d = 1$ $((r_1 = 1 \wedge r_2 > 1) \vee (r_2 = 1 \wedge r_1 > 1)) \wedge d = 0 \wedge t = 1$	$c_1 + c_2 + r_1 + r_2 + d - 1$

2. From the previous table and Proposition 4.8 we have

$$c_1 + c_2 + r_1 + r_2 + d - 2 \leq P(G) \leq c_1 + c_2 + r_1 + r_2 + d + 1.$$

Proposition 4.10. *Let G be a (H_1, H_2, t, d) -double generalized cycle star where H_1 is a (v_1, c_1, r_1) -generalized cycle star and H_2 is a (v_2, c_2, r_2) -generalized cycle star. If $d > 1$ and $r_1 + r_2 > 1$, then*

$$\Delta(G) = c_1 + c_2 + r_1 + r_2 + d - 2.$$

Proof

If we remove the central vertices v_1 and v_2 from G we obtain $c_1 + c_2 + r_1 + r_2 + d$ paths so, $\Delta(G) \geq c_1 + c_2 + r_1 + r_2 + d - 2$. If we remove a vertex different from the central vertices we obtain at most one more path. Then $\Delta(G) = c_1 + c_2 + r_1 + r_2 + d - 2$. □

4.2 Minimum rank of a rugby ball

Theorem 4.11. *Let $G = (V, E)$ be a (v_1, v_2) -rugby ball with n vertices and $d > 1$ internally disjoint paths with length greater than 1. Then*

$$mr(G) = n - d.$$

Proof

Note that $G - v_1$ is a generalized star with $n - 1$ vertices, $d > 1$ rays and central vertex v_2 . Thus, by Proposition 3.3, $mr(G - v_1) = n - 1 - d + 1 = n - d$. By [6] we have $mr(G) \geq mr(G - v_1)$. Therefore $mr(G) \geq n - d$.

In the following lemma we prove that there exists a matrix $B \in S(G)$ such that $\text{rank}(B) \leq n - d$ which enable us to conclude the equality. □

$$B'_{a,b} = \left[\begin{array}{cccc|cccc|cccc|cccc|cc}
0 & 1 & 0 & 0 & & & & & & & & & & & & & & & 1 & 0 \\
1 & 0 & 0 & 0 & & & & & & & & & & & & & & & & 0 & 0 \\
0 & 0 & 1 & 0 & & 0 & & & & 0 & & & & & & & & & & -1 & 0 \\
0 & 0 & 0 & 0 & & & & & & & & & & & & & & & & 1 & 1 \\
\hline
& & & & 0 & 1 & 0 & 0 & 0 & & & & & & & & & & & 1 & 0 \\
& & & & & 1 & 0 & 0 & 0 & 0 & & & & & & & & & & 0 & 0 \\
& & & & 0 & 0 & 0 & 1 & 0 & & & & & & & & & & & -1 & 0 \\
& & & & & 0 & 0 & 1 & 0 & 0 & & & & & & & & & & 0 & 0 \\
& & & & & 0 & 0 & 0 & 0 & 0 & & & & & & & & & & 1 & 1 \\
\hline
& & & & & & & & & & 0 & 1 & 0 & 0 & 0 & 0 & & & & 1 & 0 \\
& & & & & & & & & & & 1 & 0 & 0 & 0 & 0 & 0 & & & 0 & 0 \\
& & & & & & & & & & & 0 & 0 & 0 & 1 & 0 & 0 & & & -1 & 0 \\
& & & & & & & & & & & 0 & 0 & 1 & 0 & 0 & 0 & & & 0 & 0 \\
& & & & & & & & & & & 0 & 0 & 0 & 0 & -1 & 0 & & & 1 & 0 \\
& & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & & & 1 & 0 \\
& & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & & & 1 & 1 \\
\hline
1 & 0 & -1 & 1 & 1 & 0 & -1 & 0 & 1 & 1 & 1 & 0 & -1 & 0 & 1 & 1 & 1 & & & a & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & 1 & b
\end{array} \right]$$

Note that

$$B'_{a,b} = SB_{a,b}Q,$$

where

$$S = S_1 \oplus S_2 \oplus S_3 \oplus S_4 \oplus \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } Q = Q_1 \oplus Q_2 \oplus Q_3 \oplus Q_4 \oplus \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ with}$$

$$S_1 = E_{4,3}^{(-1)} E_{3,1}^{(-1)} \in \mathcal{M}_4$$

$$Q_1 = C_{3,1}^{(-1)} C_{4,3}^{(-1)} \in \mathcal{M}_4$$

$$S_2 = E_{5,3}^{(-1)} E_{3,1}^{(-1)} \in \mathcal{M}_5$$

$$Q_2 = C_{3,1}^{(-1)} C_{5,3}^{(-1)} \in \mathcal{M}_5$$

$$S_3 = E_{6,5}^{(+1)} E_{5,3}^{(-1)} E_{3,1}^{(-1)} \in \mathcal{M}_6$$

$$Q_3 = C_{3,1}^{(-1)} C_{5,3}^{(-1)} C_{6,5}^{(+1)} \in \mathcal{M}_6$$

$$S_4 = E_{7,6}^{(+1)} E_{6,5}^{(+1)} E_{5,3}^{(-1)} E_{3,1}^{(-1)} \in \mathcal{M}_7$$

$$Q_4 = C_{3,1}^{(-1)} C_{5,3}^{(-1)} C_{6,5}^{(+1)} C_{7,6}^{(+1)} \in \mathcal{M}_7$$

Now, considering $B'_{a,b}$ and successively performing the elementary row and column operations:

- $R_{23} - R_2 \rightarrow R_{23}; R_{23} + R_3 \rightarrow R_{23}; C_{23} - C_2 \rightarrow C_{23}; C_{23} + C_3 \rightarrow C_{23};$
- $R_{23} - R_6 \rightarrow R_{23}; R_{23} + R_8 \rightarrow R_{23}; C_{23} - C_6 \rightarrow C_{23}; C_{23} + C_8 \rightarrow C_{23};$
- $R_{23} - R_{11} \rightarrow R_{23}; R_{23} + R_{13} \rightarrow R_{23}; R_{23} + R_{14} \rightarrow R_{23};$
 $C_{23} - C_{11} \rightarrow C_{23}; C_{23} + C_{13} \rightarrow C_{23}; C_{23} + C_{14} \rightarrow C_{23};$
- $R_{23} - R_{17} \rightarrow R_{23}; R_{23} + R_{19} \rightarrow R_{23}; R_{23} + R_{20} \rightarrow R_{23}; R_{23} + R_{21} \rightarrow R_{23};$
 $C_{23} - C_{17} \rightarrow C_{23}; C_{23} + C_{19} \rightarrow C_{23}; C_{23} + C_{20} \rightarrow C_{23}; C_{23} + C_{21} \rightarrow C_{23},$

we obtain

$$D_{a,b} = \left[\begin{array}{cccc|cccc|cccc|cccc|cc}
0 & 1 & 0 & 0 & & & & & & & & & & & & & 0 & 0 \\
1 & 0 & 0 & 0 & & & & & & & & & & & & & 0 & 0 \\
0 & 0 & 1 & 0 & & 0 & & & & 0 & & & & & & & 0 & 0 \\
0 & 0 & 0 & 0 & & & & & & & & & & & & & 1 & 1 \\
\hline
& & & & 0 & 1 & 0 & 0 & 0 & & & & & & & & 0 & 0 \\
& & & & 0 & 1 & 0 & 0 & 0 & & & & & & & & 0 & 0 \\
& & & & 0 & 0 & 0 & 1 & 0 & & & & & & & & 0 & 0 \\
& & & & 0 & 0 & 1 & 0 & 0 & & & & & & & & 0 & 0 \\
& & & & 0 & 0 & 0 & 0 & 0 & & & & & & & & 1 & 1 \\
\hline
& & & & 0 & & & & & 0 & 1 & 0 & 0 & 0 & 0 & & & 0 & 0 \\
& & & & 0 & & & & & 1 & 0 & 0 & 0 & 0 & 0 & & & 0 & 0 \\
& & & & 0 & & & & & 0 & 0 & 0 & 1 & 0 & 0 & & & 0 & 0 \\
& & & & 0 & & & & & 0 & 0 & 1 & 0 & 0 & 0 & & & 0 & 0 \\
& & & & 0 & & & & & 0 & 0 & 0 & 0 & -1 & 0 & & & 0 & 0 \\
& & & & 0 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & & & 1 & 1 \\
\hline
& & & & 0 & & & & & 0 & 1 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 \\
& & & & 0 & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 \\
& & & & 0 & & & & & 0 & 0 & 0 & 1 & 0 & 0 & 0 & & 0 & 0 \\
& & & & 0 & & & & & 0 & 0 & 1 & 0 & 0 & 0 & 0 & & 0 & 0 \\
& & & & 0 & & & & & 0 & 0 & 0 & 0 & -1 & 0 & 0 & & 0 & 0 \\
& & & & 0 & & & & & 0 & 0 & 0 & 0 & 0 & -1 & 0 & & 0 & 0 \\
& & & & 0 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 1 & 1 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & a+2 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & b
\end{array} \right]$$

Note that $D_{a,b} = S'_4 S'_3 S'_2 S'_1 B'_{a,b} Q'_1 Q'_2 Q'_3 Q'_4$, where

$$\begin{aligned} S'_1 &= E_{23,3}^{(+1)} E_{23,2}^{(-1)} \in \mathcal{M}_{24} & Q'_1 &= C_{23,2}^{(-1)} C_{23,3}^{(+1)} \in \mathcal{M}_{24} \\ S'_2 &= E_{23,8}^{(+1)} E_{23,6}^{(-1)} \in \mathcal{M}_{24} & Q'_2 &= C_{23,6}^{(-1)} C_{23,8}^{(+1)} \in \mathcal{M}_{24} \\ S'_3 &= E_{23,14}^{(+1)} E_{23,13}^{(+1)} E_{23,11}^{(-1)} \in \mathcal{M}_{24} & Q'_3 &= C_{23,11}^{(-1)} C_{23,13}^{(+1)} C_{23,14}^{(+1)} \in \mathcal{M}_{24} \\ S'_4 &= E_{23,21}^{(+1)} E_{23,20}^{(+1)} E_{23,19}^{(+1)} E_{23,17}^{(-1)} \in \mathcal{M}_{24} & Q'_4 &= C_{23,17}^{(-1)} C_{23,19}^{(+1)} C_{23,20}^{(+1)} C_{23,21}^{(+1)} \in \mathcal{M}_{24} \end{aligned}$$

As the 4th, the 9th, the 15th and the 22th rows are equal we can use the 22th row to transform the 4th, the 9th and the 15th rows into null rows. Similar arguments apply to the columns. For this purpose we successively perform the following elementary row and column operations,

- $R_4 - R_{22} \rightarrow R_4; R_9 - R_{22} \rightarrow R_9; R_{15} - R_{22} \rightarrow R_{15};$
- $C_4 - C_{22} \rightarrow C_4; C_9 - C_{22} \rightarrow C_9; C_{15} - C_{22} \rightarrow C_{15},$

obtaining

2. $\mathbf{k_i} \equiv 1(\mathbf{mod4})$

(a) For $k_i = 1$ let $S_i = Q_i = I_{k_i}$.

(b) For $k_i = 5$ let

$$S_i = E_{5,3}^{(-1)} E_{3,1}^{(-1)} \quad \text{and} \quad Q_i = C_{3,1}^{(-1)} C_{5,3}^{(-1)}$$

(c) For $k_i > 5$ let

$$S_i = E_{k_i, k_i-2}^{(-1)} E_{k_i-2, k_i-4}^{(-1)} E_{k_i-4, k_i-6}^{(-1)} \cdots E_{5,3}^{(-1)} E_{3,1}^{(-1)}$$

$$Q_i = C_{3,1}^{(-1)} C_{5,3}^{(-1)} \cdots C_{k_i-4, k_i-6}^{(-1)} C_{k_i-2, k_i-4}^{(-1)} C_{k_i, k_i-2}^{(-1)}$$

3. $\mathbf{k_i} \equiv 2(\mathbf{mod4})$

(a) For $k_i = 2$ let

$$S_i = E_{2,1}^{(+1)} \quad \text{and} \quad Q_i = C_{2,1}^{(+1)}$$

(b) For $k_i = 6$ let

$$S_i = E_{6,5}^{(+1)} E_{5,3}^{(-1)} E_{3,1}^{(-1)} \quad \text{and} \quad Q_i = C_{3,1}^{(-1)} C_{5,3}^{(-1)} C_{6,5}^{(+1)}$$

(c) For $k_i > 6$ let

$$S_i = E_{k_i, k_i-1}^{(+1)} E_{k_i-1, k_i-3}^{(-1)} E_{k_i-3, k_i-5}^{(-1)} \cdots E_{5,3}^{(-1)} E_{3,1}^{(-1)}$$

$$Q_i = C_{3,1}^{(-1)} C_{5,3}^{(-1)} \cdots C_{k_i-3, k_i-5}^{(-1)} C_{k_i-1, k_i-3}^{(-1)} C_{k_i, k_i-1}^{(+1)}$$

4. $\mathbf{k_i} \equiv 3(\mathbf{mod4})$

(a) For $k_i = 3$ let

$$S_i = E_{3,2}^{(+1)} E_{2,1}^{(+1)} \quad \text{and} \quad Q_i = C_{2,1}^{(+1)} C_{3,2}^{(+1)}$$

(b) For $k_i > 3$ let

$$S_i = E_{k_i, k_i-1}^{(+1)} E_{k_i-1, k_i-2}^{(+1)} E_{k_i-2, k_i-4}^{(-1)} E_{k_i-4, k_i-6}^{(-1)} \cdots E_{3,1}^{(-1)}$$

$$Q_i = C_{3,1}^{(-1)} C_{5,3}^{(-1)} \cdots C_{k_i-2, k_i-4}^{(-1)} C_{k_i-1, k_i-2}^{(+1)} C_{k_i, k_i-1}^{(+1)}$$

For $i \in \{1, \dots, d\}$, let $D_i = S_i B_i Q_i$. It is easy to check that the last row and the last column of D_i are null, the remaining rows and columns of D_i have exactly one non null element and D_i has the form

$$D_i = \left[\begin{array}{ccc|c} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{array} \right]. \quad (1)$$

Let $S = S_1 \oplus S_2 \oplus \cdots \oplus S_d \oplus \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_d \oplus \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and consider

In what follows, $k_0 = 0$ and for $i \in \{1, \dots, d\}$, $\theta_i = k_0 + k_1 + \dots + k_{i-1}$.

1. $\mathbf{k}_i \equiv \mathbf{0}(\mathbf{mod}4)$

(a) For $k_i = 4$ let

$$S'_i = E_{n-1, \theta_i+3}^{(+1)} E_{n-1, \theta_i+2}^{(-1)} \quad \text{and} \quad Q'_i = C_{n-1, \theta_i+2}^{(-1)} C_{n-1, \theta_i+3}^{(+1)}$$

(b) For $k_i > 4$ let

$$S'_i = E_{n-1, \theta_i+k_i-1}^{(+1)} E_{n-1, \theta_i+k_i-2}^{(-1) \frac{k_i-2}{2}} E_{n-1, \theta_i+k_i-4}^{(-1) \frac{k_i-4}{2}} \cdots E_{n-1, \theta_i+4}^{(-1) \frac{4}{2}} E_{n-1, \theta_i+2}^{(-1) \frac{2}{2}}$$

$$Q'_i = C_{n-1, \theta_i+2}^{(-1) \frac{2}{2}} C_{n-1, \theta_i+4}^{(-1) \frac{4}{2}} \cdots C_{n-1, \theta_i+k_i-4}^{(-1) \frac{k_i-4}{2}} C_{n-1, \theta_i+k_i-2}^{(-1) \frac{k_i-2}{2}} C_{n-1, \theta_i+k_i-1}^{(+1)}$$

2. $\mathbf{k}_i \equiv \mathbf{1}(\mathbf{mod}4)$

(a) For $k_i = 1$ let $S'_i = Q'_i = I_n$.

(b) For $k_i = 5$ let

$$S'_i = E_{n-1, \theta_i+4}^{(+1)} E_{n-1, \theta_i+2}^{(-1)} \quad \text{and} \quad Q'_i = C_{n-1, \theta_i+2}^{(-1)} C_{n-1, \theta_i+4}^{(+1)}$$

(c) For $k_i > 5$ let

$$R'_i = E_{n-1, \theta_i+k_i-1}^{(-1) \frac{k_i-1}{2}} E_{n-1, \theta_i+k_i-3}^{(-1) \frac{k_i-3}{2}} E_{n-1, \theta_i+k_i-5}^{(-1) \frac{k_i-5}{2}} \cdots E_{n-1, \theta_i+4}^{(-1) \frac{4}{2}} E_{n-1, \theta_i+2}^{(-1) \frac{2}{2}}$$

$$Q'_i = C_{n-1, \theta_i+2}^{(-1) \frac{2}{2}} C_{n-1, \theta_i+4}^{(-1) \frac{4}{2}} \cdots C_{n-1, \theta_i+k_i-5}^{(-1) \frac{k_i-5}{2}} C_{n-1, \theta_i+k_i-3}^{(-1) \frac{k_i-3}{2}} C_{n-1, \theta_i+k_i-1}^{(-1) \frac{k_i-1}{2}}$$

3. $\mathbf{k}_i \equiv \mathbf{2}(\mathbf{mod}4)$

(a) For $k_i = 2$ let

$$S'_i = E_{n-1, \theta_i+1}^{(+1)} \quad \text{and} \quad Q'_i = C_{n-1, \theta_i+1}^{(+1)}$$

(b) For $k_i = 6$ let

$$S'_i = E_{n-1, \theta_i+5}^{(+1)} E_{n-1, \theta_i+4}^{(+1)} E_{n-1, \theta_i+2}^{(-1)} \quad \text{and} \quad Q'_i = C_{n-1, \theta_i+2}^{(-1)} C_{n-1, \theta_i+4}^{(+1)} C_{n-1, \theta_i+5}^{(+1)}$$

(c) For $k_i > 6$ let

$$R'_i = E_{n-1, \theta_i+k_i-1}^{(+1)} E_{n-1, \theta_i+k_i-2}^{(-1) \frac{k_i-2}{2}} E_{n-1, \theta_i+k_i-4}^{(-1) \frac{k_i-4}{2}} E_{n-1, \theta_i+k_i-6}^{(-1) \frac{k_i-6}{2}} \cdots E_{n-1, \theta_i+4}^{(-1) \frac{4}{2}} E_{n-1, \theta_i+2}^{(-1) \frac{2}{2}}$$

$$Q'_i = C_{n-1, \theta_i+2}^{(-1) \frac{2}{2}} C_{n-1, \theta_i+4}^{(-1) \frac{4}{2}} \cdots C_{n-1, \theta_i+k_i-6}^{(-1) \frac{k_i-6}{2}} C_{n-1, \theta_i+k_i-4}^{(-1) \frac{k_i-4}{2}} C_{n-1, \theta_i+k_i-2}^{(-1) \frac{k_i-2}{2}} C_{n-1, \theta_i+k_i-1}^{(+1)}$$

4. $\mathbf{k}_i \equiv \mathbf{3}(\mathbf{mod}4)$

(a) For $k_i = 3$ let

$$S'_i = E_{n-1, \theta_i+2}^{(+1)} E_{n-1, \theta_i+1}^{(+1)} \quad \text{and} \quad Q'_i = C_{n-1, \theta_i+1}^{(+1)} C_{n-1, \theta_i+2}^{(+1)}$$

(b) For $k_i > 3$ let

$$S'_i = E_{n-1, \theta_i+k_i-1}^{(+1)} E_{n-1, \theta_i+k_i-2}^{(+1)} E_{n-1, \theta_i+k_i-3}^{(-1) \frac{k_i-3}{2}} E_{n-1, \theta_i+k_i-5}^{(-1) \frac{k_i-5}{2}} \cdots E_{n-1, \theta_i+4}^{(-1) \frac{4}{2}} E_{n-1, \theta_i+2}^{(-1) \frac{2}{2}}$$

$$Q'_i = C_{n-1, \theta_i+2}^{(-1) \frac{2}{2}} C_{n-1, \theta_i+4}^{(-1) \frac{4}{2}} \cdots C_{n-1, \theta_i+k_i-5}^{(-1) \frac{k_i-5}{2}} C_{n-1, \theta_i+k_i-3}^{(-1) \frac{k_i-3}{2}} C_{n-1, \theta_i+k_i-2}^{(+1)} C_{n-1, \theta_i+k_i-1}^{(+1)}$$

Note that the matrix $D_{a,b}$ has d equal rows that correspond to the last row of each block. Using the row $n-2$ (the last of these equal rows), we can transform the remaining equal

rows into null rows without changing the other rows of the blocks D_1, \dots, D_d and the zero blocks. Similar arguments apply to the columns.

To this end, consider

$$S'' = E_{\theta_1+k_1, n-2}^{(-1)} \cdots E_{\theta_{d-1}+k_{d-1}, n-2}^{(-1)}$$

$$Q'' = C_{\theta_1+k_1, n-2}^{(-1)} \cdots C_{\theta_{d-1}+k_{d-1}, n-2}^{(-1)}.$$

The matrix $A_{a,b} = S'' D_{a,b} Q''$ has the form:

$$A_{a,b} = \left[\begin{array}{c|c|c|c|c} & & & & 0 \ 0 \\ & & & & \vdots \\ & & & & \vdots \\ D_1 & 0 & \cdots & 0 & 0 \ 0 \\ & & & & 0 \ 0 \\ \hline & & & & 0 \ 0 \\ & & & & \vdots \\ & & & & \vdots \\ 0 & D_2 & \cdots & 0 & 0 \ 0 \\ & & & & 0 \ 0 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline & & & & 0 \ 0 \\ & & & & \vdots \\ & & & & \vdots \\ 0 & 0 & \cdots & D_d & 0 \ 0 \\ & & & & 1 \ 1 \\ \hline 0 \ \cdots \ 0 \ 0 & 0 \ \cdots \ 0 \ 0 & \cdots & 0 \ \cdots \ 0 \ 1 & \\ 0 \ \cdots \ 0 \ 0 & 0 \ \cdots \ 0 \ 0 & \cdots & 0 \ \cdots \ 0 \ 1 & M'_{a,b} \end{array} \right]$$

Note that $A_{a,b} = A_{a,b}(\{n-2, n-1, n\}) \oplus N_{a,b}$, where $N_{a,b}$ is defined by

$$N_{a,b} = \begin{cases} \begin{bmatrix} 0 & 1 & 1 \\ 1 & a + \delta & 0 \\ 1 & 0 & b \end{bmatrix} & \text{if } \{v_1, v_2\} \notin E; \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & a + \delta & 1 \\ 1 & 1 & b \end{bmatrix} & \text{if } \{v_1, v_2\} \in E. \end{cases}$$

Thus $\text{rank}(A_{a,b}) = \text{rank}(A_{a,b}(\{n-2, n-1, n\})) + \text{rank}(N_{a,b})$. Since $A_{a,b}(\{n-2, n-1, n\})$ has $d-1$ null rows, $\text{rank}(A_{a,b}(\{n-2, n-1, n\})) \leq n-3-(d-1)$.

To study the rank of the matrix $N_{a,b}$ we have to consider two cases:

Case 1 : If $\{v_1, v_2\} \notin E$ then $\text{rank}(N_{a,b}) = \text{rank} \left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & a + \delta & 0 \\ 1 & 0 & b \end{bmatrix} \right) = \text{rank} \left(\begin{bmatrix} 1 & a + \delta & 0 \\ 0 & 1 & 1 \\ 0 & 0 & b + a + \delta \end{bmatrix} \right)$

Case 2 : If $\{v_1, v_2\} \in E$, $\text{rank}(N_{a,b}) = \text{rank} \left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & a + \delta & 1 \\ 1 & 1 & b \end{bmatrix} \right) = \text{rank} \left(\begin{bmatrix} 1 & a + \delta & 1 \\ 0 & 1 & 1 \\ 0 & 0 & b + a + \delta - 2 \end{bmatrix} \right)$

In each case, there exist values of a and b such that $\text{rank}(N_{a,b}) = 2$ and consequently,

$$\text{rank}(A_{a,b}) \leq n - 3 - (d - 1) + 2 = n - d,$$

which allow us to conclude that $\text{rank}(B_{a,b}) \leq n - d$, and the proof is complete. \square

4.3 Minimum rank of a double generalized cycle star

Lemma 4.15. *Let G be a (H_1, H_2, t, d) -double generalized cycle star with n vertices where H_1 is a $(v_1, 0, 0)$ -generalized cycle star and H_2 is a (v_2, c_2, r_2) -generalized cycle star.*

If $d > 1$ then

$$mr(G) = n - c_2 - r_2 - d + \min\{r_2, 2\}.$$

Proof

If $c_2 + r_2 = 0$, then G is a rugby ball and the result follows from Theorem 4.11. Assume that $c_2 + r_2 \neq 0$. Then the vertex v_2 is a cut-vertex of the graph G . Let k be the number of connected components of $G - v_2$. For $i = 1, \dots, k$, let W_i be the set of vertices of the i th component of $G - v_2$, let G_i be the subgraph induced by $W_i \cup \{v_2\}$ and assume that $v_1 \in W_k$. Observe that $G_i - v_2$ is a path, for $i = 1, \dots, k - 1$, and $G_k - v_2$ is a generalized star. Let $n_i = |W_i|$, for $i = 1, \dots, k$. By Proposition 3.1 and by Proposition 3.3

$$\begin{aligned} mr(G) &= (n_1 - 1) + \dots + (n_{k-1} - 1) + (n_k - d + 1) + r_{v_2}(G) \\ &= \sum_{i=1}^k n_i - (k - 1) - d + 1 + r_{v_2}(G) = n - k - d + 1 + r_{v_2}(G). \end{aligned}$$

We know that $r_{v_2}(G) = \min\left\{\sum_{i=1}^k r_{v_2}(G_i), 2\right\}$ where $r_{v_2}(G_i) = mr(G_i) - mr(G_i - v_2)$.

For $i \in \{1, \dots, k - 1\}$, G_i is a path or a cycle.

If G_i is a path then $r_{v_2}(G_i) = ((n_i + 1) - 1) - (n_i - 1) = 1$ and if G_i is a cycle then $r_{v_2}(G_i) = ((n_i + 1) - 2) - (n_i - 1) = 0$. On the other hand, G_k is a rugby ball, thus $r_{v_2}(G_k) = (n_k + 1 - d) - (n_k - d + 1) = 0$.

Therefore, $r_{v_2}(G) = \min\{r_2, 2\}$ and $mr(G) = n - k - d + 1 + \min\{r_2, 2\}$.

Since $k = c_2 + r_2 + 1$, then $mr(G) = n - c_2 - r_2 - d + \min\{r_2, 2\}$. \square

Proposition 4.16. *Let G be a (H_1, H_2, t, d) -double generalized cycle star with n vertices where H_1 is a (v_1, c_1, r_1) -generalized cycle star and H_2 is a (v_2, c_2, r_2) -generalized cycle star.*

If $d > 1$, then

$$mr(G) = n - c_1 - c_2 - r_1 - r_2 - d + \min\{r_1 + r_2, 2\}.$$

Proof

If $c_1 + r_1 = 0$, the result follows from Lemma 4.15. Assume that $c_1 + r_1 \neq 0$. Then the vertex v_1 is a cut-vertex of the graph G . Let s be the number of connected components of $G - v_1$. For $j = 1, \dots, s$, let Y_j be the set of vertices of the j th component of $G - v_1$, let G_j be the subgraph induced by $Y_j \cup \{v_1\}$ and assume that $v_2 \in Y_s$. Observe that $G_j - v_1$ is a path,

for $j = 1, \dots, s-1$, and $G_s - v_1$ is a generalized cycle star. Let $n_j = |Y_j|$, for $j = 1, \dots, s$. By Proposition 3.1, $r_{v_1}(G) = \min \left\{ \sum_{j=1}^s r_{v_1}(G_j), 2 \right\}$ where $r_{v_1}(G_j) = mr(G_j) - mr(G_j - v_1)$.

For $j \in \{1, \dots, s-1\}$, G_j is a path or a cycle. If G_j is a path then $r_{v_1}(G_j) = 1$, if G_j is a cycle then $r_{v_1}(G_j) = 0$. On the other hand, G_s satisfies the conditions of Lemma 4.15 thus

$$r_{v_1}(G_s) = (n_s + 1 - c_2 - r_2 - d + \min\{r_2, 2\}) - (n_s - c_2 - r_2 - d + 1) = \min\{r_2, 2\}.$$

Therefore,

$$r_{v_1}(G) = \min \left\{ \sum_{j=1}^s r_{v_1}(G_j), 2 \right\} = \min \{r_1 + \min\{r_2, 2\}, 2\} = \min \{r_1 + r_2, 2\}.$$

Since,

$$\begin{aligned} \sum_{j=1}^s mr(G_j - v_1) &= \sum_{j=1}^{s-1} mr(G_j - v_1) + mr(G_s - v_1) \\ &= (n_1 - 1) + \dots + (n_{s-1} - 1) + (n_s - c_2 - r_2 - d + 1) \\ &= (n - 1) - (s - 1) - c_2 - r_2 - d + 1 \\ &= n - (c_1 + r_1 + 1) - c_2 - r_2 - d + 1 \\ &= n - c_1 - c_2 - r_1 - r_2 - d \end{aligned}$$

we conclude that

$$m_r(G) = \sum_{j=1}^s mr(G_j - v_1) + r_{v_1}(G) = n - c_1 - c_2 - r_1 - r_2 - d + \min \{r_1 + \min\{r_2, 2\}, 2\}.$$

□

Observation 4.17. Let G be a (H_1, H_2, t, d) -double generalized cycle star with n vertices where H_1 is a (v_1, c_1, r_1) -generalized cycle star and H_2 is a (v_2, c_2, r_2) -generalized cycle star.

1. If $d > 1$ and $r_1 + r_2 > 1$, by Proposition 4.16, $mr(G) = n - c_1 - c_2 - r_1 - r_2 - d + 2$.
2. We can now proceed analogously to the proof of Proposition 4.16 to deduce the values of $m_r(G)$ for $d \leq 1$, which are summarized in the following table.

Cases	$mr(\mathbf{G})$
$r_1 = r_2 = 0 \wedge d < t \wedge d \leq 1$	$n - c_1 - c_2 - r_1 - r_2 - d - 1$
$r_1 = r_2 = 0 \wedge d = t = 1$ $r_1 + r_2 = 1 \wedge d < t \wedge d \leq 1$ $((r_1 = 0 \wedge r_2 > 1) \vee (r_2 = 0 \wedge r_1 > 1)) \wedge d = 0$	$n - c_1 - c_2 - r_1 - r_2 - d$
$r_1 = r_2 = 1 \wedge d < t \wedge d \leq 1$ $r_1 + r_2 = 1 \wedge d = t = 1$ $((r_1 = 0 \wedge r_2 > 1) \vee (r_2 = 0 \wedge r_1 > 1)) \wedge d = 1$ $((r_1 = 1 \wedge r_2 > 1) \vee (r_2 = 1 \wedge r_1 > 1)) \wedge d = 0$	$n - c_1 - c_2 - r_1 - r_2 - d + 1$
$r_1 \geq 1 \wedge r_2 \geq 1 \wedge d = t = 1$ $r_1 \geq 1 \wedge r_2 \geq 1 \wedge r_1 + r_2 > 2 \wedge 1 = d < t$ $r_2 > 1 \wedge r_1 > 1 \wedge d = 0 < t$	$n - c_1 - c_2 - r_1 - r_2 - d + 2$

3. From the previous table and the Proposition 4.16 we have

$$n - c_1 - c_2 - r_1 - r_2 - d - 1 \leq mr(G) \leq n - c_1 - c_2 - r_1 - r_2 - d + 2.$$

From the Propositions 4.8, 4.10 and 4.16 we obtain the following result:

Corollary 4.18. *Let G be a (H_1, H_2, t, d) -double generalized cycle star with n vertices where H_1 is a (v_1, c_1, r_1) -generalized cycle star and H_2 is a (v_2, c_2, r_2) -generalized cycle star.*

If $d > 1$ and $r_1 + r_2 > 1$, we have

$$P(G) = M(G) = m(G) = \Delta(G).$$

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