Technical Annexes

This section provides additional and complementary elements to this work aiming at clarifying the author's mathematical and computational steps. Computations follow a consistent approach across all models. Many of this results were already exposed in the core of the document, but the reader can know more easily understand how it fits the overall processes in searching for a subgame perfect equilibria in quality choices.

<u>Model 1:</u> Homogeneous consumers and two qualities

1.1.Consumers Utility Function:

$$U(x) = v + s_1 - p_1 - t(x)$$

1.2.Indifferent consumer condition:

$$v + s_1 - p_1 - t(x) = v + s_2 - p_2 - t(1-x)$$

$$\Leftrightarrow x = \frac{s_1 - s_2 + p_2 - p_1 + t}{2t}$$

1.3.Demands for each firm:

$$D_1(p_i, p_j) = x = \frac{s_1 - s_2 + p_2 - p_1 + t}{2t}$$

$$D_2(p_i, p_j) = 1 - x = 1 - \left(\frac{s_1 - s_2 + p_2 - p_1 + t}{2t}\right) = \frac{s_2 - s_1 + p_1 - p_2 + t}{2t}$$

1.4.Firms' Profits are therefore:

$$\pi_1 = D_1.(p_1 - c_1) = \left[\left(\frac{s_1 - s_2 + p_2 - p_1 + t}{2t}\right)(p_1 - c_1)\right]$$

$$= \frac{p_1 s_1 - p_1 s_2 + p_1 p_2 - {p_1}^2 + p_1 t - c_1 s_1 + c_1 s_2 - c_1 p_2 + c_1 p_1 - c_1 t}{2t}$$

$$\pi_2 = D_2.(p_2 - c_2) = \left[\left(\frac{s_2 - s_1 + p_1 - p_2 + t}{2t}\right)(p_2 - c_2)\right]$$

$$= \frac{p_2 s_2 - p_2 s_1 + p_2 p_1 - {p_2}^2 + p_1 t - c_2 s_2 + c_2 s_1 - c_2 p_1 + c_2 p_2 - c_2 t}{2t}$$

1.5.Profits Maximization:

$$\frac{\partial \pi_1}{\partial p_1} = 0$$

$$\Leftrightarrow \frac{1}{2} + \frac{s_1 - s_2 + p_2 - 2p_1 + c_1}{2t} = 0$$

$$\Leftrightarrow p_1 = \frac{t}{2} + \frac{s_1 - s_2 + p_2 + c_1}{2}$$

$$\frac{\partial \pi_2}{\partial p_2} = 0$$

$$\Leftrightarrow \frac{1}{2} + \frac{s_2 - s_1 + p_1 - 2p_2 + c_2}{2t} = 0$$

$$\Leftrightarrow p_2 = \frac{t}{2} + \frac{s_2 - s_1 + p_1 + c_2}{2}$$

1.6.Finding equilibrium prices p_1^* and p_2^*:

$$\begin{cases} p_1 = \frac{t}{2} + \frac{s_1 - s_2 + p_2 + c_1}{2} \\ p_2 = \frac{t}{2} + \frac{s_2 - s_1 + p_1 + c_2}{2} \end{cases} = \begin{cases} p_1^* = \frac{s_1 - s_2 + 3t + 2c_1 + c_2}{3} \\ p_2^* = \frac{s_2 - s_1 + 3t + 2c_2 + c_1}{3} \end{cases}$$

1.7.Profits in equilibrium will therefore be:

$$\pi_1 = (D_1(p_1^*, p_2^*)).(p_1^* - c_1)$$

$$= \left[\left(\frac{s_1 - s_2 + p_2^* - p_1^* + t}{2t}\right)(p_1^* - c_1)\right]$$

$$= \left[\left(\frac{s_1 - s_2 + p_2^* - p_1^* + t}{2t}\right)\left(\frac{s_1 - s_2 + 3t + 2c_1 + c_2}{3} - c_1\right)\right]$$

$$= \left[\left(\frac{s_1 - s_2 +}{2t} + \frac{p_2^* - p_1^*}{2t}\right)\left(\frac{s_1 - s_2 + 3t + 2c_1 + c_2}{3} - c_1\right)\right]$$

$$= \left[\left(\frac{s_1 - s_2 +}{2t} + \frac{\frac{s_2 - s_1 + 3t + 2c_2 + c_1}{3} - \frac{s_1 - s_2 + 3t + 2c_1 + c_2}{3}}{2t}\right)\left(\frac{s_1 - s_2 + 3t + 2c_1 + c_2}{3} - c_1\right)\right]$$

$$= \frac{t}{2} + \frac{\Delta s_1 - \Delta c_1}{3} + \frac{\Delta s_1^2 - 2\Delta s_1 \Delta c_1 + \Delta c_1^2}{18t} \ with \ \Delta s_1 = s_1 - s_2 \ and \ \Delta c_1 = c_1 - c_2$$

$\pi_2 = (D_2(p_2^*, p_1^*)).(p_2^* - c_2)$[1]

$$= \frac{t}{2} + \frac{\Delta s_2 - \Delta c_2}{3} + \frac{\Delta s_2^2 - 2\Delta s_2 \Delta c_2 + \Delta c_2^2}{18t} \; with \; \Delta s_2 = s_2 - s_1 \; and \; \Delta c_2 = c_2 - c_1$$

<u>Model 2:</u> Heterogeneous consumers and two qualities

2.1.Indifferent consumer

$$v + \theta s_1 - p_1 - t(x) = v + \theta s_2 - p_2 - t(1-x)$$

$$\Leftrightarrow x = \frac{\theta(s_1 - s_2) + p_2 - p_1 + t}{2t}$$

$$\Leftrightarrow \theta = \frac{(p_2 - p_1) + t - 2tx}{s_1 - s_2}$$

2.2.Demands for each firm:

$$\int_{\underline{\theta}}^{\overline{\theta}} x(\theta) = \int_0^1 x(\theta) = D_1 = \frac{1}{2} + \frac{s_1 - s_2 + 2(p_2 - p_1)}{4t}$$

$$D_2 = 1 - D_1 = \frac{1}{2} + \frac{s_2 - s_1 + 2(p_1 - p_2)}{4t}$$

2.3.Firms' Profits are therefore:

$$\pi_1 = D_1.(p_1 - c_1) = \left[\left(\frac{1}{2} + \frac{s_1 - s_2 + 2(p_2 - p_1)}{4t}\right)(p_1 - c_1)\right] = \left[\left(\frac{1}{2} + \frac{\Delta s_1 + 2(p_2 - p_1)}{4t}\right)(p_1 - c_1)\right]$$

$$\pi_2 = D_2.(p_2 - c_2) = \left[\left(\frac{1}{2} + \frac{s_2 - s_1 + 2(p_1 - p_2)}{4t}\right)(p_2 - c_2)\right] = \left[\left(\frac{1}{2} + \frac{\Delta s_2 + 2(p_1 - p_2)}{4t}\right)(p_2 - c_2)\right]$$

$With \; \Delta s_1 = s_1 - s_2 \; and \; \Delta s_2 = s_2 - s_1$

2.5.Profits Maximization:

$$\frac{\partial \pi_1}{\partial p_1} = 0$$

[1] Recurring to the symmetric properties of the exercise

$$\Leftrightarrow \frac{1}{2} + \frac{2p_2 - 4p_1 + \Delta s_1 + c_1}{4t} = 0$$

$$\Leftrightarrow p_1 = \frac{t + p_2 + c_1}{2} + \frac{\Delta s_1}{4}$$

$$\frac{\partial \pi_2}{\partial p_2} = 0$$

$$\Leftrightarrow \frac{1}{2} + \frac{2p_1 - 4p_2 + \Delta s_2 + c_2}{4t} = 0$$

$$\Leftrightarrow p_2 = \frac{t + p_1 + c_2}{2} + \frac{\Delta s_2}{4}$$

2.6.Finding equilibrium prices p_1^* and p_2^*:

$$\begin{cases} p_1 = \frac{t + p_2 + c_1}{2} + \frac{\Delta s_1}{4} \\ p_2 = \frac{t + p_1 + c_2}{2} + \frac{\Delta s_2}{4} \end{cases} = \begin{cases} p_1^* = t + \frac{2c_1 + c_2}{3} + \frac{\Delta s_1}{6} \\ p_2^* = t + \frac{2c_2 + c_1}{2} + \frac{\Delta s_2}{4} \end{cases}$$

2.7.Profits in equilibrium will therefore be:

$$\pi_1 = (D_1(p_1^*, p_2^*)).(\, p_1^* - c_1)$$

$$= \left[\left(\frac{1}{2} + \frac{\Delta s_1 + \, 2(p_2^* - \, p_1^*)}{4t}\right)(\, p_1^* - c_1)\right]$$

$$= \left[\left(\frac{1}{2} + \frac{\Delta s_1 + \, 2(p_2^* - \, p_1^*)}{4t}\right)\left(t + \frac{\Delta s_1}{6} - \frac{\Delta c_1}{3}\right)\right]$$

$$= \left[\left(\frac{1}{2} + \frac{\Delta s_1}{4t} + \frac{(p_2^* - \, p_1^*)}{2t}\right)\left(t + \frac{\Delta s_1}{6} - \frac{\Delta c_1}{3}\right)\right]$$

$$= \left[\left(\frac{1}{2} + \frac{\Delta s_1}{4t} + \frac{\left(t + \frac{2c_2 + c_1}{2} + \frac{\Delta s_2}{4}\right) - \left(t + \frac{2c_1 + c_2}{3} + \frac{\Delta s_1}{6}\right)}{2t}\right)\left(t + \frac{\Delta s_1}{6} - \frac{\Delta c_1}{3}\right)\right]$$

$$= \frac{t}{2} + \frac{7\Delta s_1}{12} - \frac{\Delta c_1}{2} + \frac{\Delta s_1^2}{12t} + \frac{\Delta c_1^2}{9t} - \frac{4\Delta s_1 \Delta c_1}{18t} \; with \; \Delta s_1 = s_1 - s_2 \; and \; \Delta c_1 = c_1 - c_2$$

$$\pi_2 = (D_2(p_2^*, p_1^*)).(\, p_2^* - c_2)$$ [2]

$$= \frac{t}{2} + \frac{7\Delta s_2}{12} - \frac{\Delta c_2}{2} + \frac{\Delta s_2^2}{12t} + \frac{\Delta c_2^2}{9t} - \frac{4\Delta s_2 \Delta c_2}{18t} \; with \; \Delta s_2 = s_2 - s_1 \; and \; \Delta c_2 = c_2 - c_1$$

2.8.Effect of differentiation on the prices of the different products

[2] Recurring to the symmetric properties of the exercise

$$P_i^{H(H)} = t + c^H$$

$$P_i^{H(L)} = t + \frac{2c_i^H + c_j^L}{3} + \frac{\Delta s_1}{6}$$

$$P_i^{L(L)} = t + c^L$$

$$P_i^{L(H)} = t + \frac{2c_i^L + c_j^H}{3} + \frac{\Delta s_1}{6}$$

With $L = low\ quality\ fuel$; $H = high\ quality\ fuel$;

and $P_i^{K(Z)}$ *the price of the product* K (*offered by firm* i) *when the other firm offers quality* Z

<u>Model 3:</u> Heterogeneous non-continuous tastes and three qualities

In this model qualities are defined exogenously:

$s^H = 1$; $s^M = 0{,}5$; $s^L = 0$
(H = high quality fuel; M = medium quality fuel; L =low quality fuel)

Consumers are considered to have three different levels of preference for variety

1. θ^h - consumers that value quality the most
2. θ^m - consumers that have an intermediate preference for quality
3. θ^l- consumers with the lowest preference for quality

3.1.Indifference Relations

3.1.1.Between high and medium qualities for $\boldsymbol{\theta^h}$ consumers:

$$v + \theta^h - p^H - t(x) = v + \frac{\theta^h}{2} - p^M - t(1-x)$$

$$\Leftrightarrow x = \frac{p_2^M - P_1^H}{2t} + \frac{1}{2} + \frac{\theta^h}{4t}$$

3.1.2.Between high and medium qualities for $\boldsymbol{\theta^m}$ consumers:

$$v + \theta^m - p^H - t(x) = v + \frac{\theta^m}{2} - p^M - t(1-x)$$

$$\Leftrightarrow x = \frac{p_2^M - p_1^H}{2t} + \frac{1}{2} + \frac{\theta^m}{4t}$$

3.1.3.Between medium and low (offered by different firms) qualities for $\boldsymbol{\theta^m}$ consumers:

$$v + - p_1^L - t(x) = v + \frac{\theta^m}{2} - p^M - t(1-x)$$

$$\Leftrightarrow x = \frac{p_2^M - p_1^L}{2t} + \frac{1}{2} - \frac{\theta^m}{4t}$$

3.1.4.Between medium and low (offered by different firms) qualities for $\boldsymbol{\theta^l}$ consumers:

$$v + \ - p_1^L - t(x) = \ v + \ \frac{\theta^l}{2} - p^M \ - t(1-x)$$

$$\Leftrightarrow x = \frac{p_2^M - p_1^L}{2t} + \frac{1}{2} - \frac{\theta^l}{4t}$$

3.1.5.Between medium and low qualities for θ^l consumers:

$$v + \ - p_1^L - t(x) = \ v + \ \frac{\theta^l}{2} - p^M \ - t(1-x)$$

$$\Leftrightarrow x = \frac{p_2^L - p_1^L}{2t} + \frac{1}{2}$$

3.2.Demands for each quality and firm

$$D_1^H = \frac{1}{6} + \frac{p_2^M - p_1^L}{2t} \frac{\theta^h}{4t}$$

$$D_1^L = \frac{2}{6} + \frac{p_2^M + p_1^L - 2p_1^L}{2t} \frac{\theta^h}{4t}$$

$$D_2^M = \frac{2}{6} + \frac{p_1^H - p_1^L - 2p_2^M}{2t} + \frac{\theta^m - \theta^h}{4t}$$

$$D_2^L = \frac{1}{6} + \frac{p_1^L - p_2^L}{2t}$$

3.3.Profit Functions

$$\pi_1 = D_1^L.(\, p_1^L - c^L) + D_1^H.(\, p_1^H - c^H)$$

$$= \left[(p_1^L - c^L) \left(\frac{2}{6} + \frac{p_2^M + p_1^L - 2p_1^L}{2t} \frac{\theta^h}{4t} \right) \right] + \left[(p_1^H - c^H) \left(\frac{1}{6} + \frac{p_2^M - p_1^L}{2t} \frac{\theta^h}{4t} \right) \right]$$

$$\pi_2 = D_2^L.(\, p_2^L - c^L) + D_2^M.(\, p_2^M - c^M)$$

$$= \left[(p_2^L - c^L) \left(\frac{1}{6} + \frac{p_1^L - p_2^L}{2t} \right) \right] + \left[(p_2^M - c^M) \left(\frac{2}{6} + \frac{p_1^H - p_1^L - 2p_2^M}{2t} + \frac{\theta^m - \theta^h}{4t} \right) \right]$$

3.4.Profits maximization

$$\frac{\partial \pi_1}{\partial p_1^H} = 0$$

$$\Leftrightarrow p_1^H = \frac{t}{6} + \frac{p_2^M + c^H}{2} + \frac{\theta^h}{4}$$

$$\frac{\partial \pi_1}{\partial p_1^L} = 0$$

$$\Leftrightarrow p_1^L = \frac{t}{6} + \frac{p_2^M + p_2^L}{4} - \frac{\theta^m}{8} + \frac{c^L}{2}$$

$$\frac{\partial \pi_2}{\partial p_2^M} = 0$$

$$\Leftrightarrow p_2^M = \frac{t}{6} + \frac{p_1^H + p_1^L}{4} + \frac{\theta^m - \theta^h}{8} + \frac{c^M}{2}$$

$$\frac{\partial \pi_2}{\partial p_2^L} = 0$$

$$\Leftrightarrow p_2^L = \frac{t}{6} + \frac{p_1^L + c^L}{2}$$

3.5.Equilibrium prices, demands and resulting from solving the system with previous four prices, incorporating eq. prices in demand functions and rewriting profits functions results:

3.5.1.If both firms offer the same two qualities (medium and high):

$$P_1^{M^*} = c^M + \frac{t}{3}$$

$$P_2^{H^*} = c^H + \frac{t}{3}$$

$$\pi_i^* = \frac{3t}{18}$$

3.5.2.If both firms offer the same two qualities (medium and high):

$$P_1^{H^*} = c^H + \frac{t}{3}$$

$$P_2^{L^*} = c^L + \frac{t}{3}$$

$$\pi_i^* = \frac{3t}{18}$$

3.5.3.If alternatively, firms offer different qualities above the low-cost fuel level (medium and high):

$$p_i^H = \frac{t}{3} + \frac{5c^L + 14c^M + 26c^H}{45} + \frac{5\theta^m + 19\theta^h}{90}\,; \quad D_i^H \frac{1}{6} + \frac{5c^L + 14c^M - 19c^H}{90t} + \frac{5\theta^m + 19\theta^h}{180t}$$

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$$p_j^M = \frac{t}{3} + \frac{10c^L+28c^M+7c^H}{45} + \frac{10\theta^m-7\theta^h}{90}; \quad D_j^M = \frac{1}{3} + \frac{20c^L-34c^M+14c^H}{90t} + \frac{20\theta^m-13\theta^h}{180t}$$

$$p_i^L = \frac{t}{3} + \frac{2c^H+8c^M+35c^L}{45} - \frac{10\theta^m+\theta^h}{90}; \qquad D_i^L = \frac{1}{3} + \frac{4c^H+57c^M-20c^L}{90t} - \frac{40\theta^m+11\theta^h}{360t}$$

$$p_j^L = \frac{t}{3} + \frac{c^H+4c^M+40c^L}{45} - \frac{10\theta^m+\theta^h}{180}; \qquad D_j^L \frac{1}{6} + \frac{c^H+4c^M-5c^L}{90t}$$

$$\pi_1^L = \left[\left(\frac{t}{3} + \frac{2c^H+8c^M-10c^L}{45} - \frac{10\theta^m+\theta^h}{90}\right)\left(\frac{1}{3} + \frac{4c^H+57c^M-20c^L}{90t} - \frac{40\theta^m+11\theta^h}{360t}\right)\right]$$

$$\pi_1^H = \left[\left(\frac{t}{3} + \frac{5c^L+14c^M-19c^H}{45} + \frac{5\theta^m+19\theta^h}{90}\right)\left(\frac{1}{6} + \frac{5c^L+14c^M-19c^H}{90t} + \frac{5\theta^m+19\theta^h}{180t}\right)\right]$$

$$\pi_2^L = \left[\left(\frac{t}{3} + \frac{c^H+4c^M-5c^L}{45} - \frac{10\theta^m+\theta^h}{180}\right)\left(\frac{1}{6} + \frac{c^H+4c^M-5c^L}{90t}\right)\right]$$

$$\pi_2^M = \left[\left(\frac{t}{3} + \frac{10c^L-17c^M+7c^H}{45} + \frac{10\theta^m-7\theta^h}{90}\right)\left(\frac{1}{3} + \frac{20c^L-34c^M+14c^H}{90t} + \frac{20\theta^m-13\theta^h}{180t}\right)\right]$$