On a Theorem by Mas-Colell*

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Abstract

We consider anonymous games with a Lebesgue space of players in which either the action space or players' characteristics are denumerable. Our main result shows that the set of equilibrium distributions over actions coincides with the set of distributions induced by equilibrium strategies.

This result, together with Mas-Colell (1984)'s theorem, implies that any continuous, denumerable game has an equilibrium strategy. In particular, the theorems of Khan and Sun (1995) and Khan, Rath, and Sun (1997) can be obtained as corollaries of Mas-Colell's.

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1 Introduction

Based on the work of Hildenbrand (1974) and Hart, Hildenbrand, and Kohlberg (1974), Mas-Colell (1984) formulated the equilibrium notion of large anonymous games in terms of distributions. In this way, he departed from the usual formalization of an equilibrium as a strategy, i.e., as a measurable function from players into actions. This allowed him to obtain an existence theorem easily and under general conditions.

Since an equilibrium distribution is a probability distribution over characteristics and actions, it does not tell us what actions does each player takes in an equilibrium. Rather, and loosely, it tells us only the fraction of players with a certain characteristic that play a certain action. This is obviously in contrast with equilibrium strategies. Motivated by this, and despite the ease and generality that Mas-Colell's approach allows, researchers still investigate properties of equilibrium strategies (see the recent survey by Khan and Sun (2002)).

It is clear that the two approaches are related. In the concrete case of a game described by a measurable function from players into characteristics (i.e., payoff functions), this function together with an equilibrium strategy induces an equilibrium distribution of the game. In particular, this observation implies that the set of equilibrium distributions over actions — defined as the set of marginal distributions over the space of actions obtained from the equilibrium distributions — contains the set of distributions induced by equilibrium strategies. It also implies that each theorem establishing the existence of an equilibrium strategy also establishes the existence of an equilibrium distribution. This potential for extra generality accounts for the attractiveness of posing the existence problem in anonymous games with a continuum of players in terms of distributions.

In fact, Mas-Colell's theorem relies on strictly weaker assumptions than those of the known results on the existence of equilibrium strategies in games defined on the unit interval, which are stated in Schmeidler (1973), Rath (1992), Khan and Sun (1995) and Khan, Rath, and Sun (1997). Despite this extra generality, is it the case that these existence theorems can be obtained as corollaries of Mas-Colell's?

We show that, with the exception of Schmeidler's and Rath's theorems, Mas-Colell's implies all the others. We establish this relationship by showing that the set of equilibrium distributions over actions coincides with the set of distributions induced by equilibrium strategies in denumerable games. Denumerable games are games with either a denumerable action space or with denumerably many characteristics. Hence, since Khan and Sun's theorem applies only to games with denumerably many actions, it is then implied by Mas-Colell's together with our main result.

Furthermore, we show, as corollaries to Mas-Colell's theorem, that an equilibrium distribution exists for any game in the class considered by Khan, Rath, and Sun (1997).[1] When the action space is denumerable, our main result then implies Theorems 1 to 3 of Kahn, Rath and Sun. More generally, it shows that an equilibrium strategy exists for all denumerable games in this

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[1] These are games in which each player's payoff depends only on his choice and on the average choice of the others and the action space is a (non-necessarily denumerable) subset of either a separable Banach space with the weak topology, a separable Banach space with the norm topology, a dual space of a separable Banach space with the weak* topology or $\mathbb{R}^\infty$ (in this last case, only games with countably many actions will be considered).

class.

An important consequence of our results is that it yields a general existence theorem for equilibrium strategies in games with an uncountable action space. Thus, in contrast to the view expressed in Khan, Rath, and Sun (1997), it is possible to extend the equilibrium theory for anonymous games beyond the case of denumerable action space. The extension we present requires, however, that players' characteristics be denumerable. Hence, our results stress the importance of denumerability assumptions for the existence of equilibrium strategies in games with a continuum of players, highlighted in Khan, Rath, and Sun (1997), but they show that those assumptions can be imposed either on the action space or on players' characteristics.

In fact, the examples of Khan, Rath, and Sun (1997) show that our main result does not extent beyond denumerable games. In their examples, the set of distributions induced by equilibrium strategies is empty, while the set of equilibrium distributions over actions is not. It is this dependence on countability assumptions that prevents us from obtaining Schmeidler (1973)'s theorem on the existence of equilibrium strategies for games with a convex, compact subset of a euclidian space. It, therefore, also prevents us from obtaining Rath (1992)'s generalization of Schmeidler's theorem.

Besides extending Khan and Sun (1995) and Khan, Rath, and Sun (1997)'s results by allowing uncountable action spaces in games with denumerably many characteristics, we find our approach to be conceptually simpler. It amounts simply to asking when we can obtain an equilibrium strategy from an equilibrium distribution, whose existence is guaranteed by Mas-Colell's Theorem. That is, we divide the existence question into two related ones:

first, we ask when an equilibrium distribution might exist, the answer to which is given to us by Mas-Colell's theorem. Then, we ask if an equilibrium distribution over actions can be purified (or represented) by an equilibrium strategy. Our main result provides an answer to this question.

A recent and fruitful approach to the existence question in non-atomic games is the the one of Khan and Sun (1999), who consider games on a Loeb space of players. Although we also find Mas-Colell's distributional approach simpler than theirs, because it relies exclusively on standard methods, it seems that Khan and Sun's approach allows for additional generality. In fact, Khan and Sun's Theorem 1 implies Mas-Colell's existence theorem.

However, it is unknown whether or not Mas-Colell's theorem implies Khan and Sun's. As noted by Khan and Sun (1999, p. 474), Mas-Colell's theorem is equivalent to the existence of an equilibrium strategy in games on a hyperfinite Loeb counting space, but the argument that establishes this equivalence does not seem to extend to the case of a general Loeb space. Our approach can be useful in answering this question: If our main result holds in games on a Loeb space of players, then the equivalence between Mas-Colell's and Khan and Sun's theorems would be established.

Apart from the question of additional generality, Mas-Colell's theorem is also useful to relate several known existence results. In addition to the ones we have already mentioned, we have shown in Carmona (2005) that Mas-Colell's theorem is equivalent to the existence of an approximate equilibrium in all sufficiently large equicontinuous games (i.e., games in which players' characteristics form an equicontinuous family).[2] These relations are

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[2] There, we also show that both these results are equivalent to Khan and Sun (1999)'s

summarized in Figure 1.

We conclude this section with a remark on the proofs of our results. Although different, our approach in games with denumerably many actions is, nevertheless, quite close to the one used by Khan and Sun (1995). Here, as there, the main tool is the Bollobás and Varopoulos's Theorem. Thus, our approach relies on their insight into the importance of this theorem for the theory of large games. The results regarding games with denumerably many characteristics are new and rely on a simple modification of Skorokhod's Theorem (see Skorokhod (1956)).

2 Notation and Definitions

Let A be a non-empty, complete and separable metric space of *actions* endowed with its Borel $\sigma-$ algebra and $\mathcal{M}(A)$ be the set of Borel probability measures on A endowed with the weak convergence topology. By Parthasarathy (1967, Theorems II.6.2 and II.6.5), it follows that $\mathcal{M}(A)$ is a complete, separable metric space. Let $\mathcal{U}$ be a metric space of utility functions $u : A \times \mathcal{M}(A) \to \mathbb{R}$ endowed with its Borel $\sigma-$ algebra. The set $\mathcal{U}$ represents the space of *players' characteristics.*

A *game with a continuum of players* is characterized by a measurable function $U : [0,1] \to \mathcal{U}$, where the unit interval $[0,1]$ is endowed with the Lebesgue measure λ on the Lebesgue measurable sets and represents the set of *players.* We represent such game by $G = (([0,1],\lambda), U, A)$. A game G has *denumerably many characteristics* if $U([0,1])$ is a denumerable subset of $\mathcal{U}$. A

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asymptotic theorems.

game G has *denumerably many actions* if A is denumerable. Finally, a game is *denumerable* if it has either denumerably many actions or characteristics.

Let $\mathcal{U}_c$ denote the space of continuous utility functions $u : A \times \mathcal{M}(A) \to \mathbb{R}$ endowed with the supremum norm; it is a complete, separable metric space. A game G is *continuous* if A is compact and $\mathcal{U} = \mathcal{U}_c$.

Given a Borel probability measure τ on $\mathcal{U} \times A$, we denote by $\tau_{\mathcal{U}}$ and τ_A the marginals of τ on $\mathcal{U}$ and A respectively. The expression $u(a, \tau) \geq u(A, \tau)$ means $u(a, \tau) \geq u(a', \tau)$ for all $a' \in A$.

Given a game G with a continuum of players, a Borel probability measure τ on $\mathcal{U} \times A$ is an *equilibrium distribution for* G if

1. $\tau_{\mathcal{U}} = \lambda \circ U^{-1}$, and
2. $\tau(\{(u, a) \in \mathcal{U} \times A : u(a, \tau_A) \geq u(A, \tau_A)\}) = 1$.

We will use the following notation: $B_\tau = \{(u, a) \in \mathcal{U} \times A : u(a, \tau_A) \geq u(A, \tau_A)\}$. For a general game G with a continuum of players, an implicit requirement for τ to be an equilibrium distribution is that B_τ is measurable. Note that if G is continuous, then B_τ is closed, and so a Borel set; hence $\tau(B_\tau)$ is well defined for all distributions $\tau \in \mathcal{M}(\mathcal{U}_c \times A)$.

A *pure strategy* is a measurable function $f : [0, 1] \to A$. A pure strategy f is a *Nash equilibrium of* G if

$$U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(A, \lambda \circ f^{-1})$$

for almost all $t \in [0, 1]$.

Mas-Colell (1984)'s Theorem 1 asserts the existence of an equilibrium distribution in games with a continuum of players.

Theorem 1 (Mas-Colell) *An equilibrium distribution exists for all continuous games.*

In the following sections, we will show how Theorem 1 implies the existence of a Nash equilibrium in continuous, denumerable games.

3 Denumerable Games

In this section, we present our main result. Let G be a game with a continuum of players and let $\mathcal{D}(G)$ be the set of equilibrium distributions over actions. Formally,

$$\mathcal{D}(G) = \{\tau_A \in \mathcal{M}(A) : \tau \text{ is an equilibrium distribution of } G\}.$$

Also, let $D(G)$ be the set of distributions over actions induced by equilibrium strategies:

$$D(G) = \{\lambda \circ f^{-1} : f \text{ is a Nash equilibrium of } G\}.$$

It is clear that for any game G, denumerable or not, we have that $D(G) \subseteq \mathcal{D}(G)$. This follows simply from the fact that if f is a Nash equilibrium of G and τ denotes $\lambda \circ (U, f)^{-1}$, then τ_A belongs to $\mathcal{D}(G)$. In general, the inclusion is strict as it can be seen from the first example in Khan, Rath, and Sun (1997). However, if G is denumerable, then they coincide.

Theorem 2 *If G is a denumerable game, then $\mathcal{D}(G) = D(G)$.*

Theorem 2 follows from Theorems 3 and 4 below. We can interpret it as showing that for denumerable games there is no loss of generality by focusing on equilibrium strategies. That is, all equilibrium behavior is captured by equilibrium strategies.

Although interesting in its own right, Theorem 2 has an important consequence when applied to continuous games. If G is a continuous game, then, by Mas-Colell (1984, Theorem 1), G has an equilibrium distribution τ. Moreover, if G is also a denumerable game, then $\tau_A = \lambda \circ f^{-1}$ and f is a Nash equilibrium of G. Therefore, G has a Nash equilibrium. Thus, Theorem 2 implies the existence of Nash equilibria for continuous, denumerable games with a continuum of players.

Corollary 1 *A Nash equilibrium exists for all continuous, denumerable games.*

3.1 Games with Denumerably Many Actions

In this subsection we consider games with denumerable action spaces. The following purification result establishes part of Theorem 2.

Theorem 3 *Let G be a game with denumerably many actions. If τ is an equilibrium distribution of G, then there exists a Nash equilibrium $f : [0, 1] \to A$ such that $\tau_A = \lambda \circ f^{-1}$.*

The idea of the proof is to partition the set of players into as many sets as the number of actions available and then use this partition (which is in a one-to-one correspondence with the action space) to define the equilibrium strategy. This partition must satisfy two properties: the set of players playing action a must be a subset of those for whom a is a best-reply and its measure must equal $\tau_A(\{a\})$. The existence of such partition is guaranteed by the Bollobás and Varopoulos's theorem as stated in Khan and Sun (1995).

As a consequence of Theorem 3 and Mas-Colell's Theorem, we obtain Khan and Sun (1995)'s existence result (see their Theorem 10):

Corollary 2 (Khan and Sun) *A Nash equilibrium exists for all continuous games with denumerably many actions.*

3.2 Games with Denumerably Many Characteristics

Finally, we turn to games with denumerably many characteristics. For games in this class, we obtain a stronger representation theorem.

Theorem 4 *Let G be a game with denumerably many characteristics. If τ is an equilibrium distribution of G, then there exists a Nash equilibrium $f : [0,1] \to A$ such that $\tau = \lambda \circ (U, f)^{-1}$.*

Clearly, Theorem 4 implies the remaining part of Theorem 2, but it says more: for games with denumerably many characteristics, the set of equilibrium distributions coincides with the set of distributions induced by Nash equilibria and the function U describing the game.

The idea of its proof is simple: we define as many probability measures over the action space as the number of possible characteristics. Each such measure describes the distribution of choices made by players with a given characteristic. A modified version of Skorokhod (1956)'s theorem allows us to represent any such measure by a function defined on the set of players with the corresponding characteristic. Then, it is easy to see that these functions define the equilibrium strategy.

Theorem 4 implies part of Corollary 1. That is, it implies that any continuous game with denumerably many characteristics has a Nash equilibrium.

We emphasize that for this result, the action space can be uncountable. In this way, it is the first result on the existence of an equilibrium strategy when the action space is just a compact metric space.

4 Games based on the Average of Individual Choices

4.1 Bochner Integral with the Weak Topology

In this subsection, assume that A is a weakly compact subset of a separable Banach space X. Let $\overline{\text{co}}(A)$ denote the closed convex hull of A, which is also weakly compact (see Diestel and Uhl (1977, Theorem 11, p. 51)). It follows from Dunford and Schwartz (1957, Theorem V.6.3, p. 434) that both A and $\overline{\text{co}}(A)$ are metrizable. Note that the identity function $i : A \to A$ is μ – Bochner integrable and that $\int_A i \mathrm{d}\mu$ belongs to $\overline{\text{co}}(A)$ for every $\mu \in \mathcal{M}(A)$. Finally, let $\mathcal{U}_w$ be the space of weakly continuous real-valued functions on $A \times \overline{\text{co}}(A)$ endowed with the sup norm.

We define a class $\mathcal{G}_w$ of games with a continuum of players as follows: A game G in $\mathcal{G}_w$ is described by $(([0,1],\lambda),U,A)$ where $U : [0,1] \to \mathcal{U}_w$ is measurable.

An *equilibrium distribution for* $G \in \mathcal{G}_w$ is a Borel probability measure τ on $\mathcal{U}_w \times A$ satisfying

1. $\tau_{\mathcal{U}_w} = \lambda \circ U^{-1}$, and
2. $\tau(\{(u,a) \in \mathcal{U}_w \times A : u(a, \int_A i \mathrm{d}\tau_A) \geq u(A, \int_A i \mathrm{d}\tau_A)\}) = 1$,

where the integral $\int_A i\mathrm{d}\tau_A$ is a Bochner integral. A pure strategy f is a *Nash equilibrium of* G if

$$U(t)\left(f(t), \int_{[0,1]} f \mathrm{d}\lambda\right) \geq U(t)\left(A, \int_{[0,1]} f \mathrm{d}\lambda\right)$$

for almost all $t \in [0,1]$.

As a consequence of Mas-Colell's theorem and of Theorem 2, we obtain the following existence result.

Corollary 3 *An equilibrium distribution exists for all games* $G \in \mathcal{G}_w$. *Furthermore, if* G *is denumerable, then a Nash equilibrium exists.*

Note that when G is a game with denumerably many actions, Corollary 3 corresponds to Theorem 2 by Khan, Rath, and Sun (1997).

4.2 Bochner Integral with the Norm Topology

In this subsection, assume that A is a norm compact of a separable Banach space B. Then, $\overline{\mathrm{co}}(A)$ is also norm compact (see Diestel and Uhl (1977, Theorem 12, p. 51)). We let $\mathcal{U}_n$ be the space of norm continuous real-valued functions on $A \times \overline{\mathrm{co}}(A)$ endowed with the sup norm.

Finally, we let $\mathcal{G}_n$ be the class of games defined as in Subsection 4.1, except that now players' payoff functions belong to $\mathcal{U}_n$ (and A is norm compact).

As in Khan, Rath, and Sun (1997), Corollary 3 also applies to the case of norm compact action spaces. Note that A is weakly compact. Thus, since norm continuous functions on norm compact sets are weakly continuous, we obtain the following existence result.

Corollary 4 *An equilibrium distribution exists for all games $G \in \mathcal{G}_n$. Furthermore, if G is denumerable, then a Nash equilibrium exists.*

Similarly to what happened in the previous section, Corollary 4 corresponds to Corollary 1 by Khan, Rath, and Sun (1997) when G is a game with denumerably many actions.

4.3 Gel'fand Integral

In this subsection, let X^* be the dual of a separable Banach space X and assume that A is a weak* compact subset of X^*. Then, $\overline{\mathrm{co}}(A)$ is also weak* compact. It follows from Rudin (1971, Theorem 3.16, p. 70) that both A and $\overline{\mathrm{co}}(A)$ are metrizable. Note that the identity function $i : A \to A$ is $\mu-$ Gel'fand integrable and that $\int_A i \mathrm{d}\mu$ belongs to $\overline{\mathrm{co}}(A)$ for every $\mu \in \mathcal{M}(A)$. Finally, let $\mathcal{U}_g$ be the space of weak* continuous real-valued functions on $A \times \overline{\mathrm{co}}(A)$ endowed with the sup norm.

We define a class $\mathcal{G}_g$ of games with a continuum of players as follows: A game G in $\mathcal{G}_g$ is described by $(([0,1], \lambda), U, A)$ where $U : [0,1] \to \mathcal{U}_g$ is measurable.

An *equilibrium distribution for* $G \in \mathcal{G}_g$ is a Borel probability measure τ on $\mathcal{U}_g \times A$ satisfying

1. $\tau_{\mathcal{U}_g} = \lambda \circ U^{-1}$, and

2. $\tau(\{(u,a) \in \mathcal{U}_g \times A : u(a, \int_A i \mathrm{d}\tau_A) \geq u(A, \int_A i \mathrm{d}\tau_A)\}) = 1$,

where the integral $\int_A i \mathrm{d}\tau_A$ is a Gel'fand integral. A pure strategy f is a *Nash* *equilibrium of* G if

$$U(t)\left(f(t), \int_{[0,1]} f \mathrm{d}\lambda\right) \geq U(t)\left(A, \int_{[0,1]} f \mathrm{d}\lambda\right)$$

for almost all $t \in [0,1]$.

For games in this class, we also have the following existence result.

Corollary 5 *An equilibrium distribution exists for all games* $G \in \mathcal{G}_g$. *Furthermore, if* G *is denumerable, then a Nash equilibrium exists.*

Corollary 5 corresponds to Theorem 3 by Khan, Rath, and Sun (1997) when G is a game with denumerably many actions.

4.4 Games with Denumerably Many Actions in $\mathbb{R}^\infty$

In this subsection we consider the class of games defined in Section 5 of Khan, Rath, and Sun (1997). Let $\mathbb{R}^\infty$ be the space of all real sequences equipped with the product topology. The standard basis vectors are denoted by $\{e_i\}_{i=1}^\infty$. We let the space of actions be $A = \{e_i\}_{i=1}^\infty$. Again, $\overline{\mathrm{co}}(A)$ denotes the closed convex hull of A, which is equal to the set of all sequences of real numbers chosen from the closed unit interval. Finally, let $\mathcal{U}_\infty$ be the space of continuous real-valued functions on $A \times \overline{\mathrm{co}}(A)$ endowed with the sup norm.

We define a class $\mathcal{G}_\infty$ of games with a continuum of players as follows: A game G in $\mathcal{G}_\infty$ is described by $(([0,1],\lambda), U, A)$ where $U : [0,1] \to \mathcal{U}_\infty$ is measurable.

The following definitions are from Khan, Rath, and Sun (1997). If $(T, \mathcal{T}, \nu)$ is a probability space, then a function $f : T \to \mathbb{R}^\infty$ is measurable if $f^{-1}(\{e_i\}) \in$

$\mathcal{T}$. In this case,

$$\int_T f \mathrm{d}\nu = \sum_{i=1}^{\infty} e_i \nu(f^{-1}(\{e_i\})).$$

In particular, if $\mu \in \mathcal{M}(A)$, then $\int_A i\mathrm{d}\mu = \sum_{i=1}^{\infty} e_i \mu(\{e_i\})$ and so we trivially obtain the following change of variable formula:

$$\int_T f \mathrm{d}\nu = \int_A i \mathrm{d}\nu \circ f^{-1}.$$

Furthermore, if $j \in \mathbb{N}$ and $\pi_j : \mathbb{R}^{\infty} \to \mathbb{R}$ is the projection onto the j^{th} coordinate, then π_j is a simple function and its Lebesgue integral is $\int_A \pi_j \mathrm{d}\mu = \mu(\{e_j\})$.

An *equilibrium distribution for* $G \in \mathcal{G}_{\infty}$ is a Borel probability measure τ on $\mathcal{U}_{\infty} \times A$ satisfying

1. $\tau_{\mathcal{U}_{\infty}} = \lambda \circ U^{-1}$, and
2. $\tau(\{(u, a) \in \mathcal{U}_{\infty} \times A : u(a, \int_A i\mathrm{d}\tau_A) \geq u(A, \int_A i\mathrm{d}\tau_A)\}) = 1$.

A pure strategy f is a *Nash equilibrium of* G if

$$U(t)\left(f(t), \int_{[0,1]} f \mathrm{d}\lambda\right) \geq U(t)\left(A, \int_{[0,1]} f \mathrm{d}\lambda\right)$$

for almost all $t \in [0, 1]$.

Using both Corollary 1 and Mas-Colell's theorem, we can provide an alternative proof of Theorem 1 by Khan, Rath, and Sun (1997).

Corollary 6 (Kahn, Rath and Sun) *A Nash equilibrium exists for all games* $G \in \mathcal{G}_{\infty}$.

5 Concluding Remarks

In this paper we focus on three aspects of the theory of anonymous games with a continuum of players:

1. The relationship between the set of equilibrium distributions over actions and the set of distributions induced by Nash equilibria,
2. the existence of Nash equilibria in games with a compact action space and
3. the place of Mas-Colell's theorem in this theory.

Regarding the first issue, our main result (Theorem 2) shows that the two sets are equal in denumerable games. This implies that there is no gain in considering distributions for analyzing such games; in fact, equilibrium behavior is completely described by Nash equilibrium strategies.

Combining our main result with Mas-Colell's theorem, we obtain the existence of a Nash equilibrium strategy in continuous, denumerable games (Corollary 1). In particular, Corollary 1 yields, to the best of our knowledge, the first general existence theorem for large games with an arbitrarily compact metric space of actions.

We emphasize that our existence result, as well as those by Khan and Sun (1995) and Khan, Rath, and Sun (1997), follows directly from Mas-Colell's. This gives Mas-Colell's theorem a central place in the equilibrium theory of large games.

The importance of Mas-Colell's result is also highlighted by its unifying role. In fact, it allows us to relate many of the known results on the equilibrium theory of anonymous games, including both the cases of non-atomic and large finite games (see Figure 1).

We stress that two existence results are missing in Figure 1: those of Schmeidler (1973) and Rath (1992) for games defined on the unit interval. While it is clear that Rath's theorem implies Schmeidler's, it is unknown to us how each of them relates to the others listed in Figure 1, which is why they have been omitted the figure. It is also unknown whether Mas-Colell's theorem is equivalent to Kahn and Sun's on the existence of an equilibrium strategy in games on a general Loeb space. These are the main questions that this paper leaves open.

These questions can, however, be addressed as we did in this paper: we can ask whether the set of equilibrium distributions (denoted by $\mathcal{D}(G)$) is equal to the set of distributions induced by Nash equilibria (denoted by $D(G)$) in the class of games considered by Schmeidler (1973), Rath (1992) and Khan and Sun (1999). Actually, in the case of the first two, we can simply ask whether the set $\mathcal{I}(G) = \{\int_A i \mathrm{d}\mu : \mu \in \mathcal{D}(A)\}$ is equal to $I(G) = \{\int_A i \mathrm{d}\mu : \mu \in D(A)\}$. An affirmative answer to either one of these questions would imply that those results are also a corollary of Mas-Colell's.

A Appendix

A.1 Proof of Theorem 3

Let G be a game with denumerably many actions and τ be an equilibrium distribution of G. Note first that

$$\tau(C) = \tau(B_\tau \cap C) \tag{1}$$

for all measurable subsets C of $\mathcal{U} \times A$. This follows because $\tau(B_\tau) = 1$.

Let $A = \{a_i\}_{i=1}^{\infty}$. For all $i \in \mathbb{N}$ define

$$T_i = U^{-1}(\{u \in \mathcal{U} : (u, a_i) \in B_\tau\})$$

and

$$\tau_i = \tau_A(\{a_i\}).$$

Since the set $\{u \in \mathcal{U} : (u, a_i) \in B_\tau\}$ is measurable (see Aliprantis and Border (1999, Lemma 4.45, p. 148)), it follows that so is T_i for all $i \in \mathbb{N}$.

Let I be a finite subset of $\mathbb{N}$. Then,

$$\begin{aligned} \lambda(\cup_{i\in I} T_i) &= \lambda(U^{-1}(\cup_{i\in I}\{u \in \mathcal{U} : (u, a_i) \in B_\tau\})) \\ &= \tau_{\mathcal{U}}(\cup_{i\in I}\{u \in \mathcal{U} : (u, a_i) \in B_\tau\}) \\ &= \tau(\cup_{i\in I}(\{u \in \mathcal{U} : (u, a_i) \in B_\tau\} \times A)) \end{aligned}$$

Also,

$$\begin{aligned} \textstyle\sum_{i\in I} \tau_i &= \sum_{i\in I} \tau(\mathcal{U} \times \{a_i\}) \\ &= \tau(\cup_{i\in I}(\mathcal{U} \times \{a_i\})) \\ &= \tau(B_\tau \cap (\cup_{i\in I}(\mathcal{U} \times \{a_i\})). \end{aligned}$$

Since

$$\cup_{i\in I}(\{u \in \mathcal{U} : (u, a_i) \in B_\tau\} \times A) \subseteq B_\tau \cap (\cup_{i\in I}(\mathcal{U} \times \{a_i\}),$$

it follows that

$$\lambda(\cup_{i\in I} T_i) \geq \sum_{i\in I} \tau_i.$$

Hence, by the Theorem of Bollobás and Varopoulos (see Khan and Sun (1995, Theorem 4)), there exists a pairwise disjoint family $\{S_i\}_{i=1}^{\infty}$ where S_i is measurable, $S_i \subseteq T_i$ and $\lambda(S_i) = \tau_i$ for all $i \in \mathbb{N}$.

Let $t \in [0, 1]$. Define $f(t) = a_i$ if $t \in S_i$ and arbitrarily if $t \in [0, 1] \setminus (\cup_{i=1}^{\infty} S_i)$. Since $\lambda(\cup_{i=1}^{\infty} S_i) = \sum_{i=1}^{\infty} \tau_A(\{a_i\}) = 1$, then f is measurable.

Note that $\lambda \circ f^{-1}(\{a_i\}) = \lambda(S_i) = \tau_i = \tau_A(\{a_i\})$, which implies that $\lambda \circ f^{-1} = \tau_A$.

Finally, we show that f is a Nash equilibrium. Let $i \in \mathbb{N}$ and $t \in S_i$. Since $S_i \subseteq T_i = U^{-1}(\{u : (u, a_i) \in B_\tau\})$, then

$$U(t)(a_i, \tau_A) \geq U(t)(A, \tau_A).$$

We have that $f(t) = a_i$ and $\lambda \circ f^{-1} = \tau_A$; hence,

$$U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(A, \lambda \circ f^{-1}).$$

In conclusion, $U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(A, \lambda \circ f^{-1})$ for all $t \in \cup_{i=1}^{\infty} S_i$ and since $\lambda(\cup_{i=1}^{\infty} S_i) = 1$, it follows that f is a Nash equilibrium.

A.2 Proof of Theorem 4

Let G a be game with denumerably many characteristics and τ be an equilibrium distribution of G. We claim that there exists a measurable function $f : [0, 1] \to A$ such that $\tau = \lambda \circ (U, f)^{-1}$.

The proof of the above claim is as follows: Let $U([0,1]) = \{u_i\}_{i=1}^{\infty}$. For each $i \in \mathbb{N}$, let $T_i = U^{-1}(\{u_i\})$, $c_i = \lambda(T_i)$ and, if $c_i > 0$, $\lambda_i(E) = \lambda(E)/c_i$ for all measurable subsets E of T_i. Furthermore, again if $c_i > 0$, define $\tau_i(D) = \tau(\{u_i\} \times D)/c_i$ for all measurable subsets D of A. Then, there exists a measurable function $f_i : T_i \to A$ such that $\tau_i = \lambda_i \circ f_i^{-1}$.[3]

Define $f : [0,1] \to A$ by $f(t) = f_i(t)$ if $t \in T_i$ and $c_i > 0$, while if $t \in [0,1] \backslash (\cup_{i:c_i>0} T_i)$, we define $f(t)$ arbitrarily. Note that $\lambda([0,1] \backslash (\cup_{i:c_i>0} T_i)) = 0$ and so f is measurable.

Let B be a measurable subset of $\mathcal{U} \times A$ and define $B_i = \{a \in A : (u_i, a) \in B\}$ for all $i \in \mathbb{N}$. Note that $\{u_i\} \times B_i = (\{u_i\} \times A) \cap B$ and so, in particular, $\{u_i\} \times B_i$ is measurable. Also, note that $\tau(B) = \tau((U([0,1]) \times A) \cap B)$ and $\lambda \circ (U, f)^{-1}(B) = \lambda \circ (U, f)^{-1}((U([0,1]) \times A) \cap B)$. Then,

$$\begin{aligned}
\tau(B) &= \tau((U([0,1]) \times A) \cap B) \\
&= \sum_i \tau(\{u_i\} \times B_i) \\
&= \sum_i c_i \tau_i(B_i) \\
&= \sum_i c_i \lambda_i \circ f_i^{-1}(B_i).
\end{aligned} \tag{2}$$

---

[3] The existence of such f_i can be obtained by appropriately modifying the proof of Skorokhod (1956, Theorem 3.1.1, p. 281). We need only to redefine the sets $\Delta_{i_1,\ldots,i_k}$, which will be of the form T_i intersected with an interval. The continuity of the functions $c \mapsto \lambda(T_i \cap [a,c])/c_i$ and $c \mapsto \lambda(T_i \cap (a,c])/c_i$ for all $a \in [0,1]$, allows for $\lambda_i(\Delta_{i_1,\ldots,i_k}) = \tau_i(S_{i_1,\ldots,i_k})$, where $S_{i_1,\ldots,i_k}$ are as in Skorokhod's proof.

Also,

$$\begin{aligned}\lambda \circ (U,f)^{-1}(B) &= \lambda(\{t \in [0,1] : (U(t), f(t)) \in B\}) \\ &= \sum_i \lambda(\{t \in [0,1] : U(t) = u_i \text{ and } f(t) \in B_i\}) \\ &= \sum_i \lambda(\{t \in T_i : f_i(t) \in B_i\}) \\ &= \sum_i c_i \lambda_i(\{t \in T_i : f_i(t) \in B_i\}) \\ &= \sum_i c_i \lambda_i \circ f_i^{-1}(B_i).\end{aligned} \tag{3}$$

Thus, it follows that $\tau = \lambda \circ (U,f)^{-1}$.

To finish the argument, we claim that f is a Nash equilibrium of G. For notational convenience let $h = (U,f)$. We have

$$\begin{aligned}h^{-1}(B_\tau) &= \{t \in [0,1] : (U(t), f(t)) \in B_\tau\} \\ &= \{t \in [0,1] : U(t)(f(t), \tau_A) \geq U(t)(A, \tau_A)\} \\ &= \{t \in [0,1] : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(A, \lambda \circ f^{-1})\}.\end{aligned} \tag{4}$$

Since $\tau = \lambda \circ h^{-1}$ is an equilibrium distribution, then $\lambda(h^{-1}(B_\tau)) = 1$ and so $\lambda(\{t \in [0,1] : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(A, \lambda \circ f^{-1})\}) = 1$. Thus, f is a Nash equilibrium.

A.3 Lemmata for Corollary 3

In this subsection X denotes a separable Banach space and A a weakly compact subset of X. If a sequence $\{x_k\}_{k=1}^{\infty}$ converges to x in X with respect to the weak topology, we write $x_k \to_w x$. If $\mu \in \mathcal{M}(A)$ and $(T, \mathcal{T}, \nu)$ is a measure space, in order to avoid any confusion, we say that a function

$f : T \to X$ is strongly μ – measurable when f is μ – measurable as defined in Diestel and Uhl (1977).

Lemma 1 *Let A be a weakly compact subset of a separable Banach space and $\mu \in \mathcal{M}(A)$. Then, the identity function $i : A \to A$ is μ – Bochner integrable and $\int_A i\mathrm{d}\mu \in \overline{\mathrm{co}}(A)$.*

Proof. Let $k \in \mathbb{N}$. Since X is separable, there exists a countable set $\{x_j\}_{j=1}^{\infty} \subseteq X$ such that $X \subseteq \cup_{j=1}^{\infty} B_{1/k}(x_j)$ and so $A \subseteq \cup_{j=1}^{\infty} B_{1/k}(x_j)$. Define $B_1 = B_{1/k}(x_1)$ and $B_j = B_{1/k}(x_j) \setminus \left(\cup_{l=1}^{j-1} B_{1/k}(x_l)\right)$. Let $J_k \geq k$ be such that $\sum_{j=J_k+1}^{\infty} \mu(B_j) < 1/k$. Then, define i_k by $i_k(x) = x_j$ if $x \in B_j$ with $j \leq J_k$ and $i_k(x) = x_1$ otherwise. Clearly, $i_k : A \to A$ is a simple function for all k and $\lim_k ||i_k(x) - i(x)|| = 0$ for all $x \in A$. The last fact follows since if $x \in B_j$, then $||i_k(x) - i(x)|| \leq 1/k$ for all $k \geq j$. Thus, i is strongly μ – measurable.

Furthermore, since A is bounded, let M be such that $||x|| \leq M$ for all $x \in A$. Thus, $\int_A ||i||\mathrm{d}\mu \leq M < \infty$. This implies that i is Bochner integrable by Diestel and Uhl (1977, Theorem 2, p. 45).

Finally, $\int_A i\mathrm{d}\mu \in \overline{\mathrm{co}}(A)$ by Diestel and Uhl (1977, Corollary 8, p. 48). ■

Lemma 2 *Let A be a weakly compact subset of a separable Banach space and $\mu \in \mathcal{M}(A)$. If $\{\mu_k\}_k \subseteq \mathcal{M}(A)$ is such that $\mu_k \Rightarrow \mu$, then $\int_A i\mathrm{d}\mu_k \to_w \int_A i\mathrm{d}\mu$.*

Proof. Let $\Lambda \in X^*$. Then, $\Lambda : X \to \mathbb{R}$ is weakly continuous and so $\int_A \Lambda \mathrm{d}\mu_k \to \int_A \Lambda \mathrm{d}\mu$. Then, by Aliprantis and Border (1999, Lemma 11.44, p. 420) (which is a particular case of Hille's Theorem stated in Diestel and Uhl

(1977, Theorem 6, p. 47)), it follows that

$$\begin{aligned}&\Lambda\left(\int_A i\mathrm{d}\mu_k\right)=\int_A \Lambda\circ i\mathrm{d}\mu_k=\int_A \Lambda\mathrm{d}\mu_k\to\\&\int_A \Lambda\mathrm{d}\mu=\int_A \Lambda\circ i\mathrm{d}\mu=\Lambda\left(\int_A i\mathrm{d}\mu\right).\end{aligned}\tag{5}$$

Thus, $\int_A i\mathrm{d}\mu_k$ converges weakly to $\int_A i\mathrm{d}\mu$. ■

Lemma 3 *Let A be a weakly compact subset of a separable Banach space and $x\in\overline{\text{co}}(A)$. Then, there exists $\mu\in\mathcal{M}(A)$ such that $x=\int_A \mathrm{id}\mu$.*

Proof. Let $x\in\overline{\text{co}}(A)=\overline{\text{co}(A)}$ (by Dunford and Schwartz (1957, Lemma V.2.4 (ii), p. 415)) and let $\{x_k\}_{k=1}^{\infty}\subseteq \text{co}(A)$ be such that $x_k\to_w x$. Let $k\in\mathbb{N}$. Then, $x_k=\sum_{j=1}^m\lambda_j x_j$ for some $\{x_j\}\subseteq A$ and $\{\lambda_j\}_j$ with $\lambda_j\geq 0$ and $\sum_j\lambda_j=1$. Define μ_k by letting $\mu_k(\{x_j\})=\lambda_j$ for all $1\leq j\leq m$. It follows that $x_k=\int_A i\mathrm{d}\mu_k$.

Since $\mathcal{M}(A)$ is compact, we may assume that $\{\mu_k\}_k$ converges. Let $\mu=\lim_k\mu_k$. Then, $\int_A i\mathrm{d}\mu_k\to_w\int_A i\mathrm{d}\mu$. Since $x_k=\int_A i\mathrm{d}\mu_k$, then also $\int_A i\mathrm{d}\mu_k\to_w x$ and so $x=\int_A i\mathrm{d}\mu$. ■

Lemma 4 *Let $f:[0,1]\to A$ be measurable and $g:A\to A$ be $\lambda\circ f^{-1}$ – Bochner integrable. Then,*

$$\int_A g\mathrm{d}\lambda\circ f^{-1}=\int_{[0,1]} g\circ f\mathrm{d}\lambda.$$

Proof. Consider first the case in which g is simple and $g=\sum_{i=1}^n a_i\chi_{A_i}$. In this case, then $f\circ g$ is also simple with $g\circ f=\sum_{i=1}^n a_i\chi_{f^{-1}(A_i)}$. Then, it follows that

$$\int_A g\mathrm{d}\lambda\circ f^{-1}=\int_{[0,1]} g\circ f\mathrm{d}\lambda=\sum_{i=1}^n a_i\lambda(f^{-1}(A_i)).$$

We turn to the general case. Since g is $\lambda\circ f^{-1}$ – Bochner integrable, there exists a sequence $\{g_n\}$ of simple functions such that $||g_n - g|| \to 0$ almost everywhere, $\int_A ||g_n - g||\mathrm{d}\lambda\circ f^{-1} \to 0$ and

$$\int_A g\mathrm{d}\lambda\circ f^{-1} = \lim_n \int_A g_n\mathrm{d}\lambda\circ f^{-1}.$$

Let $Z \subseteq A$ be such that $\lambda\circ f^{-1}(Z) = 0$ and $||g_n(a) - g(a)|| \to 0$ for all $a \in Z^c$. Then, $W = f^{-1}(Z)$ is such that $\lambda(W) = 0$ and $||g_n\circ f(t) - g\circ f(t)|| \to 0$ for all $t \in W^c$. Thus, $||g_n\circ f - g\circ f|| \to 0$ almost everywhere, and since $g_n\circ f$ is simple, it follows that $g\circ f$ is strongly $\lambda\circ f^{-1}$ – measurable.

Furthermore, since A is bounded, then $||g_n\circ f - g\circ f||$ is also bounded, and it follows from the Lebesgue dominated convergence theorem that $\int_{[0,1]} ||g_n\circ f - g\circ f||\mathrm{d}\lambda \to 0$. Hence, $g\circ f$ is $\lambda\circ f^{-1}$ – Bochner integrable and

$$\int_{[0,1]} g\circ f\mathrm{d}\lambda = \lim_n \int_{[0,1]} g_n\circ f\mathrm{d}\lambda.$$

Furthermore,

$$\int_A g\mathrm{d}\lambda\circ f^{-1} = \lim_n \int_A g_n\mathrm{d}\lambda\circ f^{-1} = \lim_n \int_{[0,1]} g_n\circ f\mathrm{d}\lambda,$$

since $g_n\circ f$ is simple. This clearly implies that $\int_A g\mathrm{d}\lambda\circ f^{-1} = \int_{[0,1]} g\circ f\mathrm{d}\lambda$.
■

A.4 Proof of Corollary 3

Let $W : \mathcal{U}_w \to \mathcal{U}_c$ be defined by

$$W(u)(a,\mu) = u\left(a, \int_A i\mathrm{d}\mu\right) \tag{6}$$

for all $x \in A$ and $\mu \in \mathcal{M}(A)$. Since, by Lemma 1, i is μ – Bochner integrable and $\int_A i\mathrm{d}\mu \in \overline{\mathrm{co}}(A)$ for all μ, then W is well defined. Furthermore, by Lemma 2, $W(u) \in \mathcal{U}_c$.

Lemma 5 *The function W is injective and continuous.*

Proof. Let $u, v \in \mathcal{U}_w$ be such that $W(u) = W(v)$. Let $x \in A$, $z \in \overline{\text{co}}(A)$ and $\mu \in \mathcal{M}(A)$ be such that $z = \int_A \text{id} d\mu$. Since $W(u) = W(v)$, it follows that

$$u(x, z) = u\left(x, \int_A \text{id} d\mu\right) = W(u)(x, \mu) = W(v)(x, \mu) = v(x, z).$$

Thus, $u = v$. So, W is injective.

We next show that W is continuous. Let $u \in \mathcal{U}_w$ and $\{u_k\}_k \subseteq \mathcal{U}_w$ be such that $u_k \to u$. Then, for all $a \in A$ and $\mu \in \mathcal{M}(A)$,

$$|W(u_k)(a, \mu) - W(u)(a, \mu)| = \left|u_k\left(a, \int_A \text{id} d\mu\right) - u\left(a, \int_A \text{id} d\mu\right)\right| \leq ||u_k - u||, \tag{7}$$

and so $||W(u_k) - W(u)|| \leq ||u_k - u||$. This shows that $W(u_k) \to W(u)$ and so W is continuous. ■

Define $V : [0, 1] \to \mathcal{U}_c$ by $V = W \circ U$. Since W is continuous, then V is measurable. Thus, $(([0, 1], \lambda), V, A)$ defines a game G_V with a continuum of players. By Theorem 1, there exists an equilibrium distribution τ of G_V.

Since W is injective, define $g : W(\mathcal{U}_w) \to \mathcal{U}_w$ by $g = W^{-1}$.

Lemma 6 *The function g is continuous and $V^{-1}(g^{-1}(B)) = U^{-1}(B)$ for all subsets B of $\mathcal{U}_w$.*

Proof. Let $v \in \mathcal{U}$ and $\{v_k\}_k \subseteq \mathcal{U}$ be such that $v_k \to v$. Let $x \in A$ and $z \in \overline{\text{co}}(A)$ be given and let $\mu \in \mathcal{M}(A)$ be such that $z = \int_A \text{id} d\mu$. Then,

$$|g(v_k)(x, z) - g(v)(x, z)| = \left|W^{-1}(v_k)\left(x, \int_A \text{id} d\mu\right) - W^{-1}(v)\left(x, \int_A \text{id} d\mu\right)\right| \tag{8}$$
$$= |v_k(x, \mu) - v(x, \mu)| \leq ||v_k - v||,$$

and so $||g(v_k) - g(v)|| \leq ||v_k - v||$. This shows that $g(v_k) \to g(v)$ and so g is continuous.

Let $B \subseteq \mathcal{U}_w$. Since $V([0,1]) \subseteq W(\mathcal{U}_w)$ and $W^{-1} \circ W$ equals the identity in $\mathcal{U}_w$, then $g \circ V = W^{-1} \circ V = W^{-1} \circ W \circ U = U$. Hence, $V^{-1}(g^{-1}(B)) = (g \circ V)^{-1}(B) = U^{-1}(B)$. ■

We state the following lemma for later use.

Lemma 7 *For all Borel subsets B of A, then $\tau(\mathcal{U}_c \times B) = \tau(W(\mathcal{U}_w) \times B)$.*

Proof. Note first that

$$\begin{aligned}
\tau(W(\mathcal{U}_w) \times A) \quad &= \tau_{\mathcal{U}_c}(W(\mathcal{U}_w)) = \\
&= \lambda(V^{-1}(W(\mathcal{U}_w))) \\
&= \lambda(U^{-1}(\mathcal{U}_w)) \\
&= \lambda([0,1]) = 1.
\end{aligned}$$

This implies that $\tau((W(\mathcal{U}_w) \times A)^c) = 0$.

Hence,

$$\begin{aligned}
&\tau(\mathcal{U}_c \times B) = \\
&\tau((\mathcal{U}_c \times B) \cap (W(\mathcal{U}_w) \times A)) + \tau((\mathcal{U}_c \times B) \cap (W(\mathcal{U}_w) \times A)^c) = \\
&\tau(W(\mathcal{U}_w) \times B).
\end{aligned}$$

■

Define $h : W(\mathcal{U}_w) \times A \to \mathcal{U}_w \times A$ by $h = (g, i)$. Then, h is clearly measurable. Define $\mu \in \mathcal{M}(\mathcal{U}_w \times A)$ by

$$\mu(B) = \tau(h^{-1}(B)) \tag{9}$$

for all Borel subsets B of $\mathcal{U}_w \times A$.

Then, $\mu_{\mathcal{U}_w} = \lambda \circ U^{-1}$ since for all Borel measurable subsets $B \subseteq \mathcal{U}_w$ we have that

$$\begin{aligned} &\mu_{\mathcal{U}_w}(B) = \mu(B \times A) = \tau(h^{-1}(B \times A)) = \tau(f^{-1}(B) \times A) = \\ &\tau_{\mathcal{U}_c}(f^{-1}(B)) = \lambda \circ V^{-1}(f^{-1}(B)) = \lambda \circ U^{-1}(B). \end{aligned} \tag{10}$$

Furthermore, $\mu_A = \tau_A$ since for all Borel measurable subsets B of A we have that

$$\mu_A(B) = \mu(\mathcal{U}_w \times B) = \tau(W(\mathcal{U}_w) \times B) = \tau(\mathcal{U}_c \times B) = \tau_A(B). \tag{11}$$

Finally, we claim that

$$\mu\left(\left\{(u,a) \in \mathcal{U}_w \times A : u\left(a, \int_A \mathrm{id} \mathrm{d}\mu_A\right) \geq u\left(A, \int_A \mathrm{id} \mathrm{d}\mu_A\right)\right\}\right) = 1.$$

Let

$$B_\mu = \left\{(u,a) \in \mathcal{U}_w \times A : u\left(a, \int_A \mathrm{id} \mathrm{d}\mu_A\right) \geq u\left(A, \int_A \mathrm{id} \mathrm{d}\mu_A\right)\right\}.$$

Since $\mu(B_\mu) = \tau(h^{-1}(B_\mu))$, it is enough to show that $h^{-1}(B_\mu) = B_\tau$. This follows since

$$\begin{aligned} &h^{-1}(B_\mu) = \{(v,x) \in \mathcal{U}_c \times A : (f(v), x) \in B_\mu\} \\ &= \left\{(v,x) \in \mathcal{U}_c \times A : W^{-1}(v)\left(x, \int_A \mathrm{id} \mathrm{d}\mu_A\right) \geq W^{-1}(v)\left(A, \int_A \mathrm{id} \mathrm{d}\mu_A\right)\right\} \\ &= \{(v,x) \in \mathcal{U}_c \times A : v(x, \mu_A) \geq v(A, \mu_A)\} \\ &= \{(v,x) \in \mathcal{U}_c \times A : v(x, \tau_A) \geq v(A, \tau_A)\} = B_\tau. \end{aligned} \tag{12}$$

If, in addition, G is denumerable, then G_V is also denumerable. Hence, by Corollary 1, there exists a Nash equilibrium f.

Therefore,

$$V(t)(f(t), \lambda \circ f^{-1}) \geq V(t)(A, \lambda \circ f^{-1})$$

for almost all $t \in [0,1]$. Since $V(t) = W \circ U(t)$, $W(U(t))(a, \lambda \circ f^{-1}) = U(t)\left(a, \int_A i \mathrm{d}\lambda \circ f^{-1}\right)$ for all $a \in A$ and $t \in [0,1]$, and $\int_A i \mathrm{d}\lambda \circ f^{-1} = \int_{[0,1]} f \mathrm{d}\lambda$, it follows that

$$U(t)\left(f(t), \int_{[0,1]} f \mathrm{d}\lambda\right) \geq U(t)\left(A, \int_{[0,1]} f \mathrm{d}\lambda\right)$$

for almost all $t \in [0,1]$. Hence, f is a Nash equilibrium of G.

A.5 Lemmata for Corollary 5

In this subsection X denotes a separable Banach space, X^* its dual and A a weak* compact subset of X^*. If a sequence $\{x_k^*\}_{k=1}^{\infty}$ converges to x^* in X^* with respect to the weak* topology, we write $x_k^* \rightarrow_{w^*} x^*$.

Lemma 8 *Let* $\mu \in \mathcal{M}(A)$. *Then, the identity function* $i : A \rightarrow A$ *is* μ *– Gel'fand integrable and* $\int_A i \mathrm{d}\mu \in \overline{\mathrm{co}}(A)$.

Proof. Note first that i is weak* measurable. This follows because for all $x \in X$, the function $xi : A \rightarrow \mathbb{R}$ defined by $xi(a) = a(x)$ is measurable since it is weak* continuous. Furthermore, i is Gel'fand integrable by Aliprantis and Border (1999, 11.51, p.424): since xi is weak* continuous on a weak* compact set, then there exists $M \in \mathbb{R}$ such that $||xi(a)|| \leq B$ for all $a \in A$, which in turn implies that xi is integrable for all $x \in X$.

Finally, $\int_A i \mathrm{d}\mu \in \overline{\mathrm{co}}(A)$ by Aliprantis and Border (1999, Theorem 11.53, p. 424). ■

Lemma 9 *Let* $\mu \in \mathcal{M}(A)$*. If* $\{\mu_k\}_k \subseteq \mathcal{M}(A)$ *is such that* $\mu_k \Rightarrow \mu$*, then* $\int_A i \mathrm{d}\mu_k \to_{w^*} \int_A i \mathrm{d}\mu$*.*

Proof. For convenience, let x^* denote $\int_X i \mathrm{d}\mu$ and x_k^* denote $\int_A i \mathrm{d}\mu_k$. Let $x \in X$. By definition, $x^*(x) = \int_A xi \mathrm{d}\mu$, and similarly for x_k^*. Since $xi : A \to \mathbb{R}$ is weak* continuous, then $\int_A xi \mathrm{d}\mu_k \to \int_A xi \mathrm{d}\mu$. Since this holds for all $x \in X$, then $x_k^* \to_{w^*} x^*$. ■

Lemma 10 *Let* $x \in \overline{\mathrm{co}}(A)$*. Then, there exists* $\mu \in \mathcal{M}(A)$ *such that* $x = \int_A i \mathrm{d}\mu$*.*

Lemma 10 can be proven in the same way as Lemma 3.

Lemma 11 *Let* $f : [0,1] \to A$ *be measurable and* $g : A \to A$ *be continuous. Then,*

$$\int_A g \mathrm{d}\lambda \circ f^{-1} = \int_{[0,1]} g \circ f \mathrm{d}\lambda.$$

Proof. Note first that for all $x \in X$ we have that $(xg) \circ f = x(g \circ f)$. Since $xg : A \to \mathbb{R}$ is continuous and f is measurable, this implies that $x(g \circ f)$ is measurable for all $x \in X$. Hence, $g \circ f$ is weak* $-$ measurable.

Let $x_1^* = \int_A g \mathrm{d}\lambda \circ f^{-1}$ and $x_2^* = \int_{[0,1]} g \circ f \mathrm{d}\lambda$. Then, for all $x \in X$,

$$\begin{aligned} x_1^*(x) &= \int_A xg \mathrm{d}\lambda \circ f^{-1} \\ &= \int_{[0,1]} (xg) \circ f \mathrm{d}\lambda \\ &= \int_{[0,1]} x(g \circ f) \mathrm{d}\lambda \\ &= x_2^*(x) \end{aligned}$$

by the standard change of variable formula. Hence, $\int_A g \mathrm{d}\lambda \circ f^{-1} = \int_{[0,1]} g \circ f \mathrm{d}\lambda$. ■

A.6 Proof of Corollary 5

The proof of Corollary 5 follows the same scheme used in the proof of Corollary 3. Lemmata 8 – 11 imply that all the claims in the proof of Corollary 3 extend to the current setting.

A.7 Lemmata for Corollary 6

Lemma 12 *Let* $\mu \in \mathcal{M}(A)$*. If* $\{\mu_k\}_k \subseteq \mathcal{M}(A)$ *is such that* $\mu_k \Rightarrow \mu$*, then* $\int_A \mathrm{id} \mathrm{d}\mu_k \to \int_A \mathrm{id} \mathrm{d}\mu$*. Furthermore,* $\int_A \mathrm{id} \mathrm{d}\mu \in \overline{\mathrm{co}}(A)$*.*

Proof. Let $j \in \mathbb{N}$. Since π_j is continuous, then

$$\mu_k(\{e_j\}) = \int_A \pi_j \mathrm{d}\mu_k \to \int_A \pi_j \mathrm{d}\mu = \mu(\{e_j\}).$$

Since the j^{th} coordinate of $\int_A \mathrm{id} \mathrm{d}\mu_k$ equals $\mu_k(\{e_j\})$, while that of $\int_A \mathrm{id} \mathrm{d}\mu$ equals $\mu(\{e_j\})$, it follows that $\int_A \mathrm{id} \mathrm{d}\mu_k \to \int_A \mathrm{id} \mathrm{d}\mu$.

We finally show that $\int_A \mathrm{id} \mathrm{d}\mu \in \overline{\mathrm{co}}(A)$. Let $\{\mu_k\}_{k=1}^{\infty}$ be such that $\mu_k \Rightarrow \mu$ and μ_k has finite support for all k. Then,

$$\int_A \mathrm{id} \mathrm{d}\mu_k = \sum_{i: e_i \in \mathrm{supp}(\mu_k)} e_i \mu_k(\{e_i\}) \in \mathrm{co}(A)$$

and, since $\int_A \mathrm{id} \mathrm{d}\mu_k \to \int_A \mathrm{id} \mathrm{d}\mu$, it follows that $\int_A \mathrm{id} \mathrm{d}\mu \in \overline{\mathrm{co}}(A)$. ■

Lemma 13 *Let* $x \in \overline{\mathrm{co}}(A)$*. Then, there exists* $\mu \in \mathcal{M}(A)$ *such that* $x = \int_A \mathrm{id} \mathrm{d}\mu$*.*

Lemma 13 can be proven in the same way as Lemma 3.

A.8 Proof of Corollary 6

The proof of Corollary 6 follows the same scheme used in the proof of Corollary 3. Again we let $W : \mathcal{U}_\infty \to \mathcal{U}_c$ be defined by

$$W(u)(a, \mu) = u\left(a, \int_A \mathrm{id}\mu\right) \tag{13}$$

for all $x \in A$ and $\mu \in \mathcal{M}(A)$. By Lemma 12 $W(u) \in \mathcal{U}_c$. Furthermore, we can show that W is continuous.

One then defines $V : [0,1] \to \mathcal{U}_c$ by $V = W \circ U$ and a game with a continuum of players $G_V = (([0,1], \lambda), V, A)$. By Theorem 1, there exists an equilibrium distribution τ of G_V. Since A is countable, it follows by Theorem 3 that there exists a Nash equilibrium f.

Therefore,

$$V(t)(f(t), \lambda \circ f^{-1}) \geq V(t)(A, \lambda \circ f^{-1})$$

for almost all $t \in [0,1]$. Since $V(t) = W \circ U(t)$, $W(U(t))(a, \lambda \circ f^{-1}) = U(t)\left(a, \int_A i\mathrm{d}\lambda \circ f^{-1}\right)$ for all $a \in A$ and $t \in [0,1]$, and $\int_A i\mathrm{d}\lambda \circ f^{-1} = \int_{[0,1]} f\mathrm{d}\lambda$, it follows that

$$U(t)\left(f(t), \int_{[0,1]} f\mathrm{d}\lambda\right) \geq U(t)\left(A, \int_{[0,1]} f\mathrm{d}\lambda\right)$$

for almost all $t \in [0,1]$. Hence, f is a Nash equilibrium of G.

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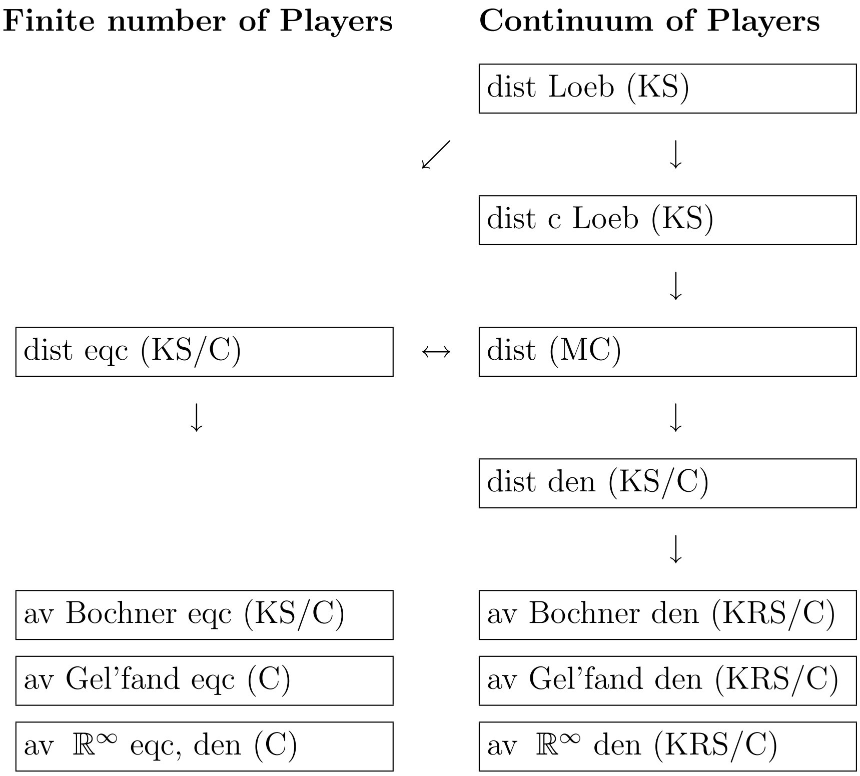

Figure 1: **Existence Results**

With the exception of Mas-Colell's (MC), all these results establish the existence of Nash equilibria. In all of them, action spaces are compact metric spaces and payoff functions are continuous. The symbol *dist* stands for games in which the externality is modeled as a distribution, while *av* is for those in which it is modeled as an average. The symbol *c* stands for counting, *den* for denumerable games and *eqc* for equicontinuous games. Finally, C stands for this paper and for Carmona (2005), KS for Khan and Sun (1995) and Khan and Sun (1999) and KRS for Khan, Rath, and Sun (1997).