

Capital Games: Financial Risk Meets Cooperative Game Theory

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Several financial risk measures, such as VaR and CaR, have been proposed to be used by financial institutions. Due to diversification, the risk present in a portfolio obtained from the aggregation of individual portfolios is typically smaller than the sum of the risks of individual components, thus allowing for significant capital savings as portfolios are aggregated. In this paper we show that capital can be allocated in a financial institution, in an economically meaningful way, by means of a simple application of the Shapley Value for cooperative games. We also use the results obtained to show how to determine the associated returns to capital and management.

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1 Introduction

Risk management in financial institutions has registered, over the past few years, a significant improvement. As the shortcomings of traditional risk measures were evaluated, most of the times by adverse changes in the Profit and Loss accounts, demand for new risk management tools arose. Tremendous effort has been made towards identifying sources of risk, sophisticated instruments designed for risk management have been introduced in the market and new risk measures have been adopted by both regulators and financial institutions. This paper aims to show that using standard risk measures and simple concepts from game theory, the risk incurred by an institution can be allocated, in an economically meaningful way, to all its relevant components, thus improving on the *ad hoc* nature of the proportional method proposed by Bessis (1998). As a by product, a measure of management performance can be constructed.

One of today's industry standards for measuring market risk is Value at Risk, (VaR). Several models have been proposed for its evaluation (Jorion, 1997), and its use as a quantitative measure of market risk is now widespread. Regulators (Basle Committee, 1995a, b) set standards for VaR evaluation and industry participants measure it on a recurrent basis, using it as a day to day measure of risk as well as an instrument for capital allocation. Another measure of risk is Capital at Risk (CaR). Its value is related to market, credit and liquidity risks. It has also been gaining widespread attention, as more accurate measures of risk are demanded. See Bessis (1998).

VaR, and CaR, as well as other risk measures, can be computed for various

levels of aggregation. Typically, the aggregation of portfolios in a larger portfolio leads to a level of risk, in the larger portfolio, that is smaller than the simple sum of the measured risks present in the individual component portfolios, considered on a stand-alone basis. This phenomenon may be attributed to diversification/correlation effects, or simply to the adding up of offsetting risks. Thus portfolio aggregation shows that there are significant externalities involved, when it comes to measuring risks, as the risk of a larger portfolio, if properly managed, should not be greater than the sum of the risks of individual portfolios. Management will try to diversify, exploit correlation and incur in offsetting risks, thus approximating the risk return associated with her aggregated portfolio to the risk-return efficiency frontier.

Allocating capital to individual portfolios or, equivalently, allocating risks to individual portfolios, and managing those risks to obtain a return, is the manager's work. As we have seen spreading the risk among portfolios is a process that may involve significant externalities. Good and sound management will be able to internalize these externalities, managing an aggregated level of risk that will be smaller than the sum of individual risks in each sub-portfolio, thus enhancing the profit obtained from the resources under her control. In the end the manager will be able to improve the reward to the capital that has been allocated for her control. What is then the value of good management? Or, to put it in another way, from the total value created by capital usage, a process that involves capital and management skills, what is the value that should be allocated to capital and what is the value that is due to management skills?

If a firm can measure the value created by management, an equitable reward scheme to management skills can be implemented. Also, finding the value created by capital will allow the firm to make a correct allocation of the total capital available to each divisional level or business unit level. Thus efficiency will be promoted.

The aim of this paper is to show, by using some very simple concepts from cooperative game theory, that the total return on capital can be divided in two distinct components, one pertaining to capital and another pertaining to management, thus providing an answer to the questions posed above.

The remainder of the paper is as follows. The next section will review the Shapley value of a cooperative game. Then we will apply this solution to a capital-allocation game and show how to decompose the total return on capital into two components: one pertaining to capital and another pertaining to management. Short conclusions follow.

2 The Shapley value of a game

Consider a set of players, $\mathcal{N} = \{i : i = 1, \dots, N\}$ from which coalitions can be formed, a coalition of players C being any non-empty subset of \mathcal{N} . Let $L(\mathcal{N})$ be the set of all coalitions and v a function that has domain $L(\mathcal{N})$ and range in the reals. An N -person cooperative game is a pair (\mathcal{N}, v) with $v(S)$ being interpreted as the worth of coalition S . We assume that $v(\emptyset) = 0$. Shapley (1953) solved the problem of finding a mapping $\phi : \mathfrak{R}^{L(\mathcal{N})} \rightarrow \mathfrak{R}$ such that, when considering the cooperative game (\mathcal{N}, v) , the expected payoff to player i is $\phi_i(v)$, requiring the function ϕ to obey some sensible properties.

The first property concerns the role of dummy players, the ones that do not contribute in any way to any coalition. It is sensible to require that the allocated value to a dummy player be zero. If her contribution to any coalition is nil, including to the coalition formed by herself only, then its allocated value must be zero. The second property we should require is anonymity, encapsulating the idea that the value allocated to a player should not depend on its order in the set of players but only on his contribution. This idea is formalized by using a permutation device. A permutation is a complete reshuffling of the players. Making a permutation π in the set of players, and considering a permuted game, then permuting the allocated value in the permuted game, $\phi_{\pi(i)}(\pi v)$, one should get the original game's allocation value $\phi_i(v)$ for each player. Finally some form of "risk neutrality" is introduced, namely by the requirement that given any two games, u and v , the allocated value to any convex combination of these two games should be the (same) convex combination of the values allocated in the individual games. For a more complete and formal treatment of these three requirements see Myerson, 1991.

The three previous requirements are enough to determine the Shapley value of a game as being, (see Myerson , 1991, Shapley, 1953):

$$\varphi_i[v] = \sum_{\substack{T \subseteq N \\ i \in T}} \frac{(t-1)!(n-t)!}{n!} [v(T) - v(T - \{i\})]$$

$v(T) - v(T - \{i\})$ can be seen as the marginal contribution of player i to coalition T . The Shapley value of player i can be interpreted as the weighted

average of the contribution of player i to each coalition to which she belongs and is the only function that obeys the three previously referred requirements.

Notice that if a player is a dummy, meaning that it brings nothing to any coalition, then we have, for all $T \subseteq \mathcal{N}$

$$v(T) - v(T - \{i\}) = 0.$$

and, consequently,

$$\varphi_i[v] = 0.$$

If a player brings something to any coalition T she belongs to, then her Shapley value will be higher than her own value in the game v . In order to prove this result notice that if $\forall T \subset N$ we have

$$v(T) \geq v(T - \{i\}) + v(\{i\}),$$

then

$$v(T) - v(T - \{i\}) \geq v(\{i\}).$$

with the end result that

$$\varphi_i[v] \geq v(\{i\}),$$

since the Shapley value of player i is the weighted average of her contribution to all coalitions she belongs to. On the other hand, if a player is a burden, in the sense that there is no coalition that gains from her presence, meaning that $\forall T \subset N : i \in T$ we have

$$v(T) \leq v(T - \{i\}) + v(\{i\}).$$

then

$$v(T) - v(T - \{i\}) \leq v(\{i\}),$$

implying, given the definition of the Shapley value,

$$\varphi_i[v] \leq v(\{i\}).$$

We see that the Shapley value captures quite clearly the value of each player i , in terms of her contribution to any coalition, allocating a premium to “good” players and a discount to “bad” players.

3 Shapley meets VaR: Capital and management rewards

Suppose, for simplicity of exposition, that one organization has two separate portfolios, P_1 and P_2 ¹. Let P be the organization portfolio, resulting from the aggregation of the two individual portfolios. We assume that capital allocated to the organization and each of its portfolios is a measure of the risk involved. This is in accordance to the modern notions of VaR and CaR, that we interpret as measures, respectively, of market risk and overall risk. Let $K_i, i = 1, 2$ be the capital allocated to each portfolio and K be the total capital allocated to the organization. Suppose that $K \leq K_1 + K_2$, meaning that management diversifies its business, since the sum of individual portfolio risks is higher than the risk incurred in the organization. The fact that diversification allows for capital enhancement creates an externality,

¹It is straightforward to extend the argument to any finite number of portfolios. The assumptions is purely expositional.

expressed in the fact that K is smaller than the sum $K_1 + K_2$. The greater the overall externality, measured the difference $(K_1 + K_2) - K$, the more able the management is in diversifying. We measure the value of this externality, thus measuring management ability to use capital efficiently.

Measuring the value of the externality is, per-se, quite important. Capital is one of the scarcest resources, and the ablest managers will be the ones that will provide the greatest return for available capital and, at the same time, will be able to efficiently use available capital, thus enlarging the externality.

Let Π_i be profit generated by portfolio i , this profit being computed exclusive of capital costs. Given K_i higher values of Π_i translate into a greater efficiency in the use of capital, as for the same level of risk a greater return is being obtained. We can think of $r_i = \frac{\Pi_i}{K_i}$, $i = 1, 2$ as the return on capital for portfolio i , (RORAC)². Higher values for r_i , for a given K_i , translate into greater efficiency in the use of capital or, if we interpret K_i as a proxy for risk, the higher the value of r_i the closer we are to the risk return efficiency frontier.

Total return on capital allocated to the organization, K , can be measured as

$$r = \frac{r_1 K_1 + r_2 K_2}{K}$$

r can be interpreted as the RORAC of the organization. Now define a cooperative game, i.e., a capital-allocation game, played in the organization. The the players are the individual portfolios and the payoff to any coalition is the capital used by that coalition. Let K_1^S and K_2^S be the Shapley values for the

²See Bessis,1998.

capital allocation game played by the two separate portfolios. Additivity of the Shapley value means that $K = K_1^S + K_2^S$. We can interpret $K_i - K_i^S$ as the gain for portfolio i from being integrated in the organization. On a stand-alone basis, capital needed to cover risks in portfolio i is K_i . But as a member of the organization, and after considering all the benefits (in terms of diversification) that the portfolio brings to the organization, the amount of capital allocated to portfolio i is K_i^S , smaller than K_i as long as it contributes to the diversification effect.

Total return to the organization can then be expressed as

$$r = \frac{r_1 K_1 + r_2 K_2}{K} = \frac{r_1 K_1^S + r_2 K_2^S}{K} + \frac{r_1 (K_1 - K_1^S) + r_2 (K_2 - K_2^S)}{K}$$

The first ratio,

$$r_K = \frac{r_1 K_1^S + r_2 K_2^S}{K}$$

can be interpreted as the return on capital allocated to the organization, K .

The second term,

$$r_M = \frac{r_1 (K_1 - K_1^S) + r_2 (K_2 - K_2^S)}{K}$$

can be seen as excess return on K due to the efficient exploitation of the diversification/correlation effect, essentially due to management skills.

Note that higher individual portfolio returns (higher values of r_i) translate into a higher r_K , a weighted average of individual portfolio returns. As for r_M we have

$$\frac{\partial r_M}{\partial r_i} = \frac{(K_i - K_i^S)}{K}$$

When there is a positive externality effect ($K_i^S < K_i$) r_i and r_M move in the same direction, while a negative externality makes r_i and r_M move in opposite directions. Note also that, given the individual returns, r_i , the larger the positive individual externality effect, measured by the difference between $K_i - K_i^S$ the larger the value of the externality exploited in the managing process of portfolio i . Thus, management skills are revealed by both the values of r_i , that depend on the return on a given level of risk incurred by the portfolio, and on the ability to generate externalities (manage diversification/correlation) revealed by the difference $K_1 - K_1^S$ and $K_2 - K_2^S$.

When considering a large financial institution divided into several organizations, each incorporating several portfolios, the previous methodology allows a comparison of management ability, conditional on the prevailing organizational division. Different managers will generate different values of r_i and also different values for the capital externality. The r_M ratio, allowing comparisons independent from absolute capital allocation, permits comparison of management skills and thus the introduction of reward schemes directly linked to management ability.

4 VaR, Return to Capital and Management: Numerical Example.

Suppose that the business unit has 3 separate portfolios. VaR is computed for each separate portfolio, for each combination of portfolios and for the whole business unit. Hypothetical VaR values are given in the first column of Table 1.

Table 1.

Diversification effects are apparent. For example, VaR values for individual portfolios {1} and {2} are, respectively, 40 and 50. But the VaR value for the aggregate portfolio {1, 2} obtained by joining portfolios {1} and {2} is only 70, a reduction of 20 units *vis-a-vis* the sum of the VaR values of portfolios {1} and {2}. It should be clear from the previous table that VaR for the whole unit, being 100, is quite smaller than the sum of VaR values for the individual portfolios that constitute that unit, 135.

The Shapley value for the VaR game, presented in the third column, is computed in the Appendix, resulting in a capital allocation given by the vector (22.5, 42.5, 35.0). Notice that player 1 has a Shapley value much smaller than his individual value, translating his influence in risk reduction in the global portfolio and in all portfolios he belongs to.

From the Π values we get the return on risk adjusted capital of each portfolio as

$$\begin{aligned} r_1 &= \frac{8.5}{40} = 21.25\% \\ r_2 &= \frac{7.5}{50} = 15.00\% \\ r_3 &= \frac{6.0}{45} = 13.33\% \end{aligned}$$

The return on risk adjusted capital allocated to the organization, r , is

$$r = \frac{8.5 + 7.5 + 6.0}{100} = 22.00\%$$

This risk adjusted return can be decomposed in two components, the return to capital given by r_K where

$$r_K = \frac{.2125 * 22.5 + .15 * 42.5 + .1333 * 35}{100} = 15.822\%,$$

and the return to management, r_M ,

$$r_M = \frac{.2125 * (40 - 22.5) + .15 * (50 - 42.5) + .1333 * (45 - 35)}{100} = 6.178\%$$

Had we applied a simple proportional rule, as proposed by Bessis (1998), pp. 362, the resulting capital allocation would have been (29.63, 37.04, 33.33). Though simple to apply, the proportional capital allocation rule has a severe problem: dummy players get benefits that do not pertain to them. Considering the data in the following table

Portfolio	VaR	Shapley Value	Proportional Rule
{1}	40	40.0	38.519
{2}	50	47.5	48.148
{3}	45	42.5	43.333
{1, 2}	90	87.5	
{1, 3}	85	82.5	
{2, 3}	90	90.0	
{1, 2, 3}	130	130.0	

Table 2

Portfolio 1 is a dummy, as no coalition to which he belongs gains, in terms of risk, from his membership. Portfolios 2 and 3 are normal portfolios, in the

to an underestimation of risks present in dummy portfolios. As the simple proportional rule introduces a bias in risk allocation it should be discarded.

5 Conclusions

Several risk measures have been recently introduced, such as VaR and CAR. VaR, or Value at Risk, is now widely used for measuring market risk and as a tool in the capital allocation process. In this paper we have shown, using the Shapley value for cooperative games, how to derive capital allocations obeying some intuitive and economically meaningful properties. The capital allocations derived from the Shapley value can be used to decompose total return on capital into two components, one measuring direct return on capital and another measuring the value added by management. This last decomposition allows the determination of a risk adjusted measure of the value added by each factor, capital and management skills. We suggest that this last measure, the risk adjusted value added by management, as it depends directly from management skills in diversification and on the ability to obtain a return from a given capital, is a sound measure of management ability and thus providing an efficient way to implement compensation schemes .

In a large organization, with a large number of portfolios, computing the

Shapley value for each portfolio may be quite computer intensive and time consuming, since it would imply the computation of 2^n numbers. Extending the previous results to games with some coalition structure is a direction for future research.

6 Appendix: Computing Shapley Values

Using data from Table 1, the Shapley Value for the VaR game can be computed. Applying the results presented previously we get

	n	t	$\frac{(t-1)!(n-t)!}{n!} [v(T) - v(T - \{i\})]$	
$v(1)$	40	3	1	$\frac{40}{3}$
$v(1, 2) - v(2)$	20	3	2	$\frac{20}{6}$
$v(1, 3) - v(3)$	15	3	2	$\frac{15}{6}$
$v(1, 2, 3) - v(2, 3)$	10	3	3	$\frac{10}{3}$
			$\varphi_1[v] = 22.5$	

	n	t	$\frac{(t-1)!(n-t)!}{n!} [v(T) - v(T - \{i\})]$	
$v(2)$	50	3	1	$\frac{50}{3}$
$v(1, 2) - v(1)$	30	3	2	$\frac{30}{6}$
$v(2, 3) - v(3)$	45	3	2	$\frac{45}{6}$
$v(1, 2, 3) - v(1, 3)$	40	3	3	$\frac{40}{3}$
			$\varphi_2[v] = 42.5$	

	n	t	$\frac{(t-1)!(n-t)!}{n!} [v(T) - v(T - \{i\})]$	
$v(3)$	45	3	1	$\frac{45}{3}$
$v(1, 3) - v(1)$	20	3	2	$\frac{20}{6}$
$v(2, 3) - v(2)$	40	3	2	$\frac{40}{6}$
$v(1, 2, 3) - v(1, 2)$	30	3	3	$\frac{30}{3}$
			$\varphi_3[v] = 35.0$	

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