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DEPARTMENT
OF MATHEMATICS

OLEG TURCAN
Bachelor of Science in Mathematics

SINGULAR INTEGRAL OPERATORS ON REARRANGEMENT-INVARIANT BANACH FUNCTION SPACES

MASTER IN MATHEMATICS AND APPLICATIONS

NOVA University Lisbon
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Adviser: Oleksiy Karlovych
Associate Professor, NOVA University Lisbon

Co-adviser: Cláudio António Rainha Aires Fernandes
Associate Professor, NOVA University Lisbon

Examination Committee:

Chair: Magda Stela de Jesus Rebelo
Assistant Professor, NOVA University Lisbon

Rapporteur: Helena Maria Narciso Mascarenhas
Assistant Professor, Instituto Superior Técnico, Universidade de Lisboa

Adviser: Oleksiy Karlovych
Associate Professor, NOVA University Lisbon

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Singular integral operators on rearrangement-invariant Banach function spaces

Dedicated to Paulo
and
to my brothers.

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"A mathematician is a person who can find analogies
between theorems; a better mathematician is one who
can see analogies between proofs and the best
mathematician can notice analogies between theories."
(Stefan Banach)

Abstract

Given the Cauchy singular integral operator S acting on a reflexive rearrangement-invariant Banach function space, our goal is to study the Fredholmness of the operator $aP+Q$ where $P=\frac{1}{2}(I+S)$, $Q=\frac{1}{2}(I-S)$ and a is a semi-almost periodic function.

We start by equipping the space of the measurable functions finite μ-a.e., defined on a σ-finite measure space, with a metric. Then, we define the Banach function spaces and we proceed to a detailed study of their properties. Next up, we define the rearrangement invariant spaces and study the boundedness of the Hilbert transform H acting on these spaces. The operators S and H share the same behaviour, since $S=iH$, where i represents the imaginary unit. Further, we develop the necessary theory of compact and Fredholm operators. Finally, we prove our main result saying that, if a is a semi-almost periodic function and the operator $aP+Q$ is Fredholm, then a_lP+Q and a_rP+Q are invertible on the reflexive rearrangement-invariant space, where a_l and a_r are the left and right almost periodic representatives of a, respectively, provided by the Sarason theorem. If a is a purely almost periodic function, then $a=a_l=a_r$ and the above result implies that the invertibility and the Fredholmness of this operator are equivalent.

Keywords: Banach function space, rearrangement-invariant space, Fredholm operator, compact operator, Cauchy singular integral operator, Boyd indices, semi-almost periodic function.

Resumo

Dado o operador integral singular de Cauchy S sobre um espaço funcional de Banach invariante após rearranjo que seja reflexivo, o nosso objetivo é estudar o caso em que o operador $aP + Q$ é de Fredholm, onde $P = \frac{1}{2}(I + S)$, $Q = \frac{1}{2}(I - S)$ e a é uma função semi-quase periódica.

Começamos por munir o espaço de todas as funções mensuráveis que são finitas μ-a.e. definidas num espaço de medida σ-finita, com uma métrica. Depois, definimos os espaços funcionais de Banach e procedemos ao estudo detalhado das suas propriedades. A seguir, desenvolvemos a teoria necessária de operadores compactos e de Fredholm. Finalmente, provamos o resultado principal que diz que, se a é uma função semi-quase periódica e se $aP + Q$ é de Fredholm, então $a_l P + Q$ e $a_r P + Q$ são invertíveis no espaço reflexivo invariante após rearranjo, onde a_l e a_r são representantes quase periódicos esquerdo e direito de a, respectivamente, fornecidos pelo teorema de Sarason. Se a é puramente quase periódica, como $a = a_l = a_r$, o resultado acima diz que a noção de invertibilidade e de Fredholm deste operador são equivalentes.

Palavras-chave: Espaço funcional de Banach, espaço invariante após rearranjo, operador de Fredholm, operador compacto, operador singular integral de Cauchy, índices de Boyd, função semi-quase periódica.

Contents

Chapter 1

Introduction

Let $\mathcal{B}(X, Y)$ denote the Banach space of all bounded linear operators acting from a Banach space X to a Banach space Y. An operator $A \in \mathcal{B}(X, Y)$ is said to be Fredholm if its image $\operatorname{im} A$ is closed in Y and

$$\dim \ker A < \infty, \quad \dim Y / \overline{\operatorname{im} A} < \infty.$$

Invertibility and Fredholmness are important concepts in the study of differential equations, and they are used to determine the well-posedness and uniqueness of solutions to boundary value problems. They are also important in the study of integral equations

$$Kf = g,$$

where g is a given real-valued function and K is a linear operator defined by

$$(Kf)(x) := \int_a^b k(x, t) f(t) dt.$$

The function k is said to be the kernel of the integral operator K. This equation is also called: the Fredholm equation of the first kind. One of the simplest examples is that when $k \in C([a, b] \times [a, b])$, because it turns K to be compact. In the case that

$$k(x, t) = \frac{1}{\pi(x - t)},$$

the integral has a singularity, hence it does not exist in the proper way, neither in the improper way. Thus, this thesis aims to study the Hilbert transform H given by

$$f(x) \mapsto \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)}{x - t} dt$$

defined on reflexive rearrangement-invariant Banach function spaces, where the integral is understood in the Cauchy principal value sense, i.e.,

$$\text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)}{x - t} dt := \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{|x - t| > \varepsilon} \frac{f(t)}{x - t} dt.$$

More specifically, we will study necessary conditions for the Fredholmness of the operator $aP + Q$ where a is a semi-almost periodic function, $P = \frac{1}{2}(I + S)$ and $Q = \frac{1}{2}(I - S)$, where S is the Cauchy singular integral operator which is a multiple of the Hilbert transform by the imaginary unit.

A Banach function space is a generalization of the Lebesgue spaces L^p, which were first studied by Frigyes Riesz in December of 1910. They are Banach spaces and are defined over $\mathcal{M}_0$, the set of all measurable functions finite almost everywhere, in which the norm is related to the underlying measure in an appropriate way. Some other examples of Banach function spaces are: Orlicz spaces, variable exponent Lebesgue spaces, Lorentz spaces, Morrey spaces etc. Throughout this writing, we consider all measure spaces to be σ-finite.

In Chapter 2, we aim to expose, constructively, a result saying that $\mathcal{M}_0$ is a metrizable space of functions. Although the result is simpler on finite measure spaces, when we pass to a more general case, we need to appeal to some topological notions, namely one of them being the property of the product topology, saying that the convergence in the product space is equivalent to pointwise convergence. We then claim that a countable product of complete metric spaces is metrizable and its metric is also complete. Furthermore, this metric satisfies the property of the product topology referred previously. This will allows us to equip the space $\mathcal{M}_0$, where the underlying measure is σ-finite, with a metric that is isometrically isomorphic to the product of $\mathcal{M}_0$ with finite underlying measures.

Chapter 3 begins with the axioms of Banach function spaces X and the study their elementary properties. Then, we introduce the notion of the associate space X' which can be treated as a "subspace" of the dual space X^* of the Banach function space X. The space X' has close connections with X as it completely characterizes the norm of the elements of the original space X. Moreover, the second associate space X'' of X is understood to be the associate space of X' and it is further proven that its elements are exactly the elements of X and that the norm of $X'' = (X')'$ coincides with the norm of X. In the following section, the notion of absolutely continuous norm of a function is introduced. So X_a stands for the set of elements of X that have absolutely continuous norm and we say that X has absolutely continuous norm if X_a coincides with the whole space X. This notion correlates deeply with the dominated convergence theorem, in the sense of Theorem 3.27. In particular, the dominated convergence theorem implies that $L^p = (L^p)_a$, for $1 \leq p < \infty$. We then set X_b to be the closure of the set of simple functions. It is worthwhile to mention that this set contains the set of all functions that have absolutely continuous norm, hence, in order to show that $X = X_a$, it is sufficient to check that the simple functions have absolutely continuous norm. Yet, it is crucial to take a closer look to the set X_a, as the duality and separability of X depend on that subspace. In fact, one of the most interesting property is the fact that $X = X_a$ is equivalent to $X^* = X'$. Moreover, if $X = X_a$ and $X' = (X')_a$ then X is reflexive. In the same section, we extend the notion of the adjoint operator of Hilbert spaces to Banach spaces with absolutely continuous norm. The last section stands for the separability. Thus, a Banach function space X is separable if and only if the underlying measure is separable and $X = X_a$. If X^* is separable, then X is reflexive.

In Chapter 4, we aim to expose a result about the necessary and sufficient conditions for the boundedness of the Hilbert transform on rearrangement-invariant spaces. These are Banach function spaces with the property that every pair $f, g \in \mathcal{M}_0$ agrees in norm whenever f and g are equimeasurable, that is, when the measures of their strict superlevel sets are equal for each level. For every function $f \in \mathcal{M}_0$, one particular function of crucial importance arises and it is the decreasing rearrangement, which is defined by

$$f^*(t) = \inf\{\lambda \geq 0 : \ \mu_f(\lambda) \leq t\},$$

where μ_f is the measure of the strict superlevel set of f defined by t, i.e.,

$$\mu_f(\lambda) = \mu\{x \in R : \ |f(x)| > \lambda\}.$$

The decreasing rearrangement f^* rearranges the values of f in a decreasing order. Further, it is proven that f and f^* are equimeasurable, hence, in a rearrangement-invariant space, they agree in norm. From the Hardy-Littlewood inequality we can derive that, for every $f, g, \tilde{g} \in \mathcal{M}_0$, where $\tilde{g}$ is equimeasurable to g, we have

$$\int_R |f\tilde{g}| \, d\mu \leq \int_0^\infty f^*(s) g^*(s) ds.$$

The measure space (R, μ) is said to be resonant if we achieve the equality by taking the supremum over all such $\tilde{g}$, for any $f, g \in \mathcal{M}_0$. So, if (R, μ) is a resonant measure space and X is a rearrangement-invariant Banach function space, then the Luxemburg representation theorem allows us to construct a new rearrangement-invariant Banach function space $\overline{X}$ over the Lebesgue measure space $(\mathbb{R}^+, m)$ with the property that, for every $f \in \mathcal{M}_0$, yields

$$\|f\|_X = \|f^*\|_{\overline{X}}.$$

Next, the Boyd indices are defined based on the dilation operator

$$(E_t f)(s) = f(st), \quad (0 < s, t < \infty),$$

which acts on $\mathcal{M}_0(\mathbb{R}^+, m)$. They can be written as follows:

$$\underline{\alpha}_X = \lim_{t \downarrow 0} \frac{\log h_X(t)}{\log t}, \quad \overline{\alpha}_X = \lim_{t \to \infty} \frac{\log h_X(t)}{\log t},$$

where

$$h_X(t) = \left\| E_{1/t} \right\|_{\mathcal{B}(\overline{X})}, \quad (0 < t < \infty).$$

We will show that the Boyd indices of the Lebesgue spaces L^p are $\underline{\alpha}_{L^p} = \overline{\alpha}_{L^p} = 1/p$. Generally, we have the inequality $0 \leq \underline{\alpha}_X \leq \overline{\alpha}_X \leq 1$, for every space X. The last result of this chapter is that the Hilbert transform is bounded on a rearrangement-invariant Banach function space X if and only if we have

$$0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1.$$

In the last chapter, we begin with the definition of an algebra, C^*-algebra, almost periodic and semi-almost periodic functions. Then we state Sarason's theorem which will be useful to solve the main problem. Next, we recall some properties of complementary spaces, compact operators and the Riesz-Schauder theory in order to introduce the notion of a Fredholm operator. A bounded linear operator $A \in \mathcal{B}(X, Y)$ acting on Banach spaces is Fredholm if and only if there is $F \in \mathcal{B}(Y, X)$ such that $AF - I_Y$ and $FA - I_X$ are compact operators. Let $\mathcal{B}(X) := \mathcal{B}(X, X)$ and let $\mathcal{K}(X)$ be the set of all compact operators on X. Then $\mathcal{K}(X)$ is a closed two-sided ideal in the Banach algebra $\mathcal{B}(X)$. This enables us to define the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$. Hence, a bounded linear operator is Fredholm if and only if it is invertible modulo the ideal of compact operators. We lately proceed to derive from a cited result that, if the limit operator of any compact operator is zero and if $A \in \mathcal{B}(X)$ is a Fredholm operator, then its limit operator, if it exists, is invertible. This will be useful in the proof of the main result which says that, if a is semi-almost periodic function and $aP + Q$ is Fredholm on a reflexive rearrangement-invariant space satisfying $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1$, then $a_l P + Q$ and $a_r P + Q$ are invertible operators, where a_l and a_r are the left and right almost-periodic representatives of a, respectively, given by the Sarason theorem. In the case that a is a purely almost periodic function, since $a = a_l = a_r$ by Sarason's theorem, we get that, $aP + Q$ is invertible if and only if $aP + Q$ is Fredholm. This result is a partial generalization of Corollary 18.11 from [4].

Chapter 2

Metric in the space of measurable functions

In this chapter, we will show that the linear space $\mathcal{M}_0$ of all measurable almost everywhere finite functions is metrizable. The proof is constructive. In order to prove this result in full generality, we will need a result on metrizability of a countable product of metric spaces. Further we introduce the metric on a finite measure space (see Theorem 2.12 below). Finally, we extend the construction of the metric to the case of σ-finite measure spaces. Moreover, a consequence of this result is that the Lebesgue space $L^p(\mathbb{R}, m)$, $(1 \leq p \leq \infty)$, where m represents the Lebesgue measure, is continuously embedded in $\mathcal{M}_0$.

2.1 Countable product of metric spaces

Let us start introducing the notion of product topology which is very useful, since the convergence in a product topology is equivalent to componentwise convergence.

Definition 2.1. Let $(X_i)_{i\in I}$ be a family of topological spaces. The cartesian product of topological spaces $\prod_{i\in I} X_i$ is called *product space*. For each $j \in I$, consider the set

$$\mathcal{S}_j = \left\{\pi_j^{-1}(U_j) : \, U_j \text{ is an open set of } X_j\right\},$$

where $\pi_j \colon \prod_{i\in I} X_i \to X_j$ is the projection map from the product space onto X_j. Then $\mathcal{S} = \bigcup_{j\in I} \mathcal{S}_j$ defines, as a subbasis, a topology for $\prod_{i\in I} X_i$, called the *product topology*.

Remark. The open sets of the product topology are unions of sets of the form $\prod_{i\in I} U_i$ where U_i is an open set of X_i and U_i is not X_i for finitely many i. If the topology of each X_i is generated by a basis, then the product topology is generated by the subbasis $\mathcal{S} = \bigcup_{j\in I} \mathcal{S}_j$ where

$$\mathcal{S}_j = \left\{ \pi_j^{-1}(B_j) : \, B_j \text{ is a basis element of } X_j \right\}.$$

Lemma 2.2 (see [19, Exercise 6, subchapter 19]). *Let $(x^{(k)})_{k\in\mathbb{N}}$ be a sequence of elements of the product space $X = \prod_{i\in I} X_i$ and let $x \in X$. Then $x^{(k)} \to x$ (in the product topology) if and only if $\pi_j(x^{(k)}) \to \pi_j(x)$ for each $j \in I$.*

Proof. Suppose that $x^{(k)} \to x$. For $j \in I$ take U_j to be an open set of X_j containing $\pi_j(x)$. Then $x \in \pi_j^{-1}(U_j)$. By definition, there is $p \in \mathbb{N}$ such that

$$\forall k \geq p \quad x^{(k)} \in \pi_j^{-1}(U_j)$$

which is equivalent to

$$\forall k \geq p \quad \pi_j(x^{(k)}) = x_j^{(k)} \in U_j.$$

Now, suppose that $\pi_j(x^{(k)}) \to \pi_j(x)$ for each $j \in I$. Let B be an element basis of the product topology containing x. Then, $B = \prod_{i\in I} U_i$ where U_i is an open set of X_i and, $U_i \neq X_i$ for finitely many i, say, for $i \in J \subset I$. Then, for $j \in I$, $\pi_j(x) = x_j \in \pi_j(B) = U_j$. By hypothesis, there is $p_j \in \mathbb{N}$ satisfying:

$$\forall k \geq p_j \quad x_j^{(k)} \in U_j.$$

Since J is finite, we can take $p = \max_{j\in J} p_j$ and claim that, for any $j \in I$,

$$\forall k \geq p \quad x_j^{(k)} \in U_j.$$

Indeed, it is true, since for $j \notin J$, $U_j = X_j$. Thus

$$\forall k \geq p \quad x^{(k)} \in \prod_{i\in I} U_i = B,$$

as desired. □

The following result is at the core of our original problem.

Theorem 2.3 (see [7, Theorem 4.2.2] and [7, Theorem 4.3.12]). *For each $n \in \mathbb{N}$, let (X_n, ρ_n) be a metric space, with ρ_n bounded by 1 and, on $X = \prod_{n\in\mathbb{N}} X_n$, let us define the metric*

$$\rho(x, y) = \sum_{n\in\mathbb{N}} \frac{\rho_n(x_n, y_n)}{2^n},$$

for every $x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in X$.

(1) Then the product topology coincides with the topology induced by the metric ρ.

(2) If (X_n, ρ_n) is complete, for each $n \in \mathbb{N}$, then (X, ρ) is complete.

Proof. (1) To show that an open set of the product topology is a ρ-open, we fix $B = \prod_{n\in\mathbb{N}} \mathcal{B}_{\varepsilon_n}(y_n, X_n)$ where $\mathcal{B}_{\varepsilon_n}(y_n, X_n)$ is the open ball of X_n with radius $\varepsilon_n > 0$, finite for finitely many n, centered at y_n. Without loss of generality, we suppose that $\varepsilon_1, ..., \varepsilon_t$ are finite. Our goal is to find a finite $\varepsilon > 0$, such that $\mathcal{B}_\varepsilon(y, X) \subset B$, where $\mathcal{B}_\varepsilon(y, X)$ is the open ball of X centered at $y = (y_n)_{n\in\mathbb{N}}$ with radius ε. Firstly, note that, for each $n \in \mathbb{N}$, and for any $x \in \mathcal{B}_\varepsilon(y, X)$,

$$\frac{\rho_n(x_n, y_n)}{2^n} \leq \rho(x, y) < \varepsilon.$$

So, fix $\varepsilon = 2^{-t}\min\{\varepsilon_1, ..., \varepsilon_t\}$ and choose an $x \in \mathcal{B}_\varepsilon(y, X)$. Obviously, the choice of ε does not depend on x. Then, for each $j \in \mathbb{N}$,

$$2^{-j}\rho_j(x_j, y_j) < \varepsilon.$$

In particular, for $j = 1, ..., t$,

$$\rho_j(x_j, y_j) < \underbrace{2^{j-t}}_{\leq 1}\min\{\varepsilon_1, ..., \varepsilon_t\} \leq \varepsilon_j.$$

Thus, for each $y = (y_n)_{n\in\mathbb{N}} \in X$ and for each $B = \prod_{n\in\mathbb{N}} \mathcal{B}_{\varepsilon_n}(y_n, X_n)$ a basis element of the product topology, there is $\varepsilon > 0$ satisfying $\mathcal{B}_\varepsilon(y, X) \subset B$.

Of course, we would like to have $\mathcal{B}_{\delta_x}(x, X) \subset B$ for each $x \in B$. Let $x \in B$, and choose $\delta = \min\{\varepsilon_1 - \rho_1(x_1, y_1), ..., \varepsilon_t - \rho_t(x_t, y_t)\}$. Then for $j = 1, ..., t$ and for each $z \in C = \prod_{n=1}^{t} \mathcal{B}_\delta(x_n, X_n) \times \prod_{n\geq t+1} X_n$,

$$\rho_j(z_j, y_j) \leq \rho_j(z_j, x_j) + \rho_j(x_j, y_j) < \delta + \rho_j(x_j, y_j) \leq \varepsilon_j.$$

Hence $C \subset B$. If we apply the last proved result to C, we obtain $\mathcal{B}_r(x, X) \subset C$, for some $r > 0$. Therefore $\mathcal{B}_r(x, X) \subset B$ which means that every basis element of the product topology is ρ-open.

For the other part, pick $\mathcal{B}_\varepsilon(x, X)$ for some $x \in X$ and $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that

$$\frac{1}{2^N} < \frac{\varepsilon}{2}.$$

Then, if we choose $B = \prod_{n=1}^{N} \mathcal{B}_{\frac{\varepsilon}{2N}}(x_n, X_n) \times \prod_{n\geq N+1} X_n$, we have $B \subset \mathcal{B}_\varepsilon(x, X)$. Indeed, if $y \in B$, then

$$\begin{aligned}
\rho(x, y) &= \sum_{n=1}^{N} \frac{\rho_n(x_n, y_n)}{2^n} + \sum_{n\geq N+1} \frac{\rho_n(x_n, y_n)}{2^n} \\
&\leq \sum_{n=1}^{N} \rho_n(x_n, y_n) + \sum_{n\geq N+1} \frac{1}{2^n} \\
&< \sum_{n=1}^{N} \frac{\varepsilon}{2N} + \frac{1}{2^N} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

We would like to have $B_y \subset \mathcal{B}_\varepsilon(x, X)$ where B_y is a basis element of the product topology "centered" at y, for an arbitrary $y \in X$. But, since $\mathcal{B}_\varepsilon(x, X)$ is ρ-open itself, for each $y \in \mathcal{B}_\varepsilon(x, X)$ there is $\delta_y > 0$ satisfying $\mathcal{B}_{\delta_y}(y, X) \subset \mathcal{B}_\varepsilon(x, X)$. Applying the last result, we can guarantee the existence of a basis element of the product topology B_y such that $B_y \subset \mathcal{B}_{\delta_y}(y, X)$. Consequently $B_y \subset \mathcal{B}_\varepsilon(x, X)$, as desired.

(2) Suppose that $\left(x^{(k)}\right)_{k\in\mathbb{N}}$ is a Cauchy sequence of X. Then a similar reasoning as the proof of Lemma 2.2 can lead us to the conclusion that $\left(x_n^{(k)}\right)_{k\in\mathbb{N}}$ is a Cauchy sequence in X_n, for every $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$, there is x_n in X_n satisfying $x_n^{(k)} \to x_n$ as $k \to \infty$. Applying again Lemma 2.2, we conclude that $x^{(k)} \to x$ in the product topology thus, in ρ, where $x = (x_n)_{n\in\mathbb{N}} \in X$. □

2.2 Borel-Cantelli lemma

Let us now introduce some notations and well-known results of the measure theory that will allow us to construct a metric on the space of measurable functions that are finite a.e. (almost everywhere).

Definition 2.4. Let (R, Σ, μ) be a measure space and $E \in \Sigma$.

- We say that E is of *finite measure* if $\mu(E) < \infty$.
- We say that E is *σ-finite* if there exists a sequence of μ-measurable sets $(E_n)_{n\in\mathbb{N}}$ of finite measure verifying $E = \bigcup_{n\in\mathbb{N}} E_n$.

Of course, if R is of finite measure or σ-finite measure, the given measure space is said to be of finite measure or σ-finite measure, respectively.

Definition 2.5. Let (R, Σ, μ) be a measure space and $B \in \Sigma$ a non-empty set. We define the restriction measure space (B, Σ_B, μ_B), calling it *measure subspace* of (R, Σ, μ), where the collection of measurable sets is

$$\Sigma_B := \{A \cap B : \ A \in \Sigma\}$$

and μ_B is μ restricted to the domain Σ_B.

In real analysis $\limsup$ of a sequence of real numbers is defined as being the biggest sublimit of the sequence. The set of sublimits is always a non-empty set. In fact, if a sequence is bounded then, by the Bolzano-Weierstrass theorem, there is a convergent subsequence. If a sequence is not bounded, for instance, if it is not upper bounded, then clearly there is a subsequence converging to $+\infty$.

Now, consider an arbitrary sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ and for each $n \in \mathbb{N}$,

$$A_n = \{a_k : \ k \geq n\} \subset \mathbb{R}.$$

Clearly, $(A_n)_{n\in\mathbb{N}}$ is a descending chain of sets, that is

$$\forall n \in \mathbb{N} \quad A_{n+1} \subset A_n.$$

Thus, by real analysis once again, the real-valued sequence defined by $b_n = \sup A_n$ is decreasing. Consequently $(b_n)_{n\in\mathbb{N}}$ admits a limit (finite or not) and it is equal to $\inf_{n\in\mathbb{N}} b_n$. In fact,

$$\limsup_{n\to\infty} a_n = \inf_{n\in\mathbb{N}} \sup_{k\geq n} a_k.$$

Analogously,

$$\liminf_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} \inf_{k\geq n} a_k.$$

Now, consider a sequence of sets $(A_n)_{n\in\mathbb{N}}$. Then,

$$B_n = \bigcup_{k\geq n} A_k, \quad (n \in \mathbb{N}), \tag{2.1}$$

is a descending chain. If we think about infimum as intersection and supremum as union in terms of sets, then it motivates the following definition.

Definition 2.6. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets. Then the *upper limit* of the sequence is defined by

$$\limsup_{n\to\infty} A_n := \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k.$$

Similarly, the *lower limit* of the sequence $(A_n)_{n\in\mathbb{N}}$ is defined by

$$\liminf_{n\to\infty} A_n := \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k.$$

Lemma 2.7 (Borel-Cantelli, see [8, Corollary 28.4.3])**.** *Suppose that (R, Σ, μ) is a measure space and consider a sequence of measurable sets $(A_n)_{\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \mu(A_n) < \infty$. Then*

$$\mu\left(\limsup_{n\to\infty} A_n\right) = 0.$$

Proof. Let $s_n = \sum_{j=1}^{n} \mu(A_j)$ for $n \in \mathbb{N}$. By the hypotesis, $s_n \to s = \sum_{n\in\mathbb{N}} \mu(A_n)$. Clearly, the sequence $r_n = s - s_n = \sum_{j\geq n+1} \mu(A_j)$ tends to zero. Set $(B_n)_{n\in\mathbb{N}}$ to be defined as in (2.1). Since $(B_n)_{n\in\mathbb{N}}$ is a descending chain and

$$\mu(B_1) = \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu(A_n) < \infty,$$

we therefore have

$$\mu\left(\limsup_{n\to\infty} A_n\right) = \mu\left(\bigcap_{n\in\mathbb{N}} B_n\right) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu\left(\bigcup_{k\geq n} A_k\right) \leq \lim_{n\to\infty} \underbrace{\sum_{k\geq n} \mu(A_k)}_{r_{n-1}} = 0,$$

hence the conclusion. □

2.3 Convergence in measure

Let (R, Σ, μ) be a measure space. We denote by $\mathcal{M}(R, \Sigma, \mu)$ the set of all μ-measurable functions, or simply, the set of all measurable functions (if there is no danger of ambiguity), with the domain R and codomain $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ or $\dot{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. The set of all μ-measurable functions with domain R and codomain $[0, \infty]$ will be denoted by $\mathcal{M}^+(R, \Sigma, \mu)$, and $\mathcal{M}_0(R, \Sigma, \mu)$ will be the set of all μ-measurable functions that are finite μ-a.e. As usual, two functions coinciding μ-a.e. will be identified. Instead, if there is no danger of ambiguity, we can write just $\mathcal{M}$, $\mathcal{M}^+$ or $\mathcal{M}_0$.

Definition 2.8. Let (R, Σ, μ) be a measure space, $(f_n)_{n\in\mathbb{N}}$ a sequence of μ-measurable functions and $f \in \mathcal{M}$.

- We say that $(f_n)_{n\in\mathbb{N}}$ *converges to f a.e.* (almost everywhere) and write $f_n \to f$ a.e., if
$$\lim_{n\to\infty} f_n(x) = f(x),$$
for almost every $x \in R$.

- We say that $(f_n)_{n\in\mathbb{N}}$ *converges in measure* to f, and write $f_n \to f$ in measure, if for every $\varepsilon > 0$, we have $\mu(|f_n - f| > \varepsilon) \to 0$, which denotes the following limit
$$\lim_{n\to\infty} \mu\{x \in R : |f_n(x) - f(x)| > \varepsilon\} = 0.$$

- The sequence $(f_n)_{n\in\mathbb{N}}$ is said to *converge a.u.* (almost uniformly) to f, and write $f_n \to f$ a.u., if, for every fixed $\varepsilon > 0$ we can guarantee the existence of a measurable set A satisfying $\mu(A) < \varepsilon$ and $f_n \to f$ uniformly in $R \setminus A$, i.e.,
$$\lim_{n\to\infty} \sup_{x\in R\setminus A} |f_n(x) - f(x)| = 0.$$

Remark. Notice that the definition of $(f_n)_{n\in\mathbb{N}}$ converging uniformly almost everywhere to f is $\|f_n - f\|_\infty \to 0$, i.e.,
$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{x\in R} |f_n(x) - f(x)| = 0.$$

So, if a sequence of functions converges uniformly almost everywhere then it converges almost uniformly.

Theorem 2.9 (Egorov's Theorem, see [8, Theorem 28.6.2])**.** *Suppose that (R, Σ, μ) is a finite measure space, (f_n) a sequence of measurable functions and $f \in \mathcal{M}$. Then $f_n \to f$ a.e. if and only if $f_n \to f$ a.u.*

Proof. Suppose that $f_n \to f$ a.e. and consider, for each $n, k \in \mathbb{N}$, the following sets:

- $B_{n,k} = \{x \in R : \, |f_n(x) - f(x)| > 1/k\}$;
- $A_{n,k} = \bigcup_{m \geq n} B_{m,k}$;
- $N = \{x \in R : \, f_n(x) \nrightarrow f(x)\}$.

For the beggining, fix $k \in \mathbb{N}$. Then we have

$$\bigcap_{n \in \mathbb{N}} A_{n,k} = \limsup_{n \to \infty} B_{n,k} \subset N.$$

Indeed, if $x \notin N$ then, for some $n_k \in \mathbb{N}$,

$$\forall n \geq n_k \quad |f_n(x) - f(x)| \leq 1/k.$$

This is

$$x \notin \bigcup_{n \geq n_k} B_{n,k} = A_{n_k,k}$$

which proves the above inclusion. Hence $\bigcap_{n \in \mathbb{N}} A_{n,k}$ is a null set. Since the measure space is finite and $(A_{n,k})_{n \in \mathbb{N}}$ is a descending chain, we can guarantee that $\mu(A_{n,k}) \to 0$ as $n \to \infty$. So, for an arbitrary $\varepsilon > 0$, there is n_k satisfying

$$n \geq n_k \implies \mu(A_{n,k}) < \varepsilon/2^k.$$

Consequently, if we take $A = \bigcup_{k \in \mathbb{N}} A_{n_k,k}$, then

$$\mu(A) \leq \sum_{k \in \mathbb{N}} \mu(A_{n_k,k}) < \varepsilon \underbrace{\sum_{k \in \mathbb{N}} \frac{1}{2^k}}_{=1}.$$

To conclude the proof, we claim that $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f in $R \setminus A$. In fact, it is equivalent to, for any $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that

$$\forall x \in R \setminus A \quad n \geq n_k \implies |f_n(x) - f(x)| \leq 1/k.$$

In terms of sets, this is

$$R \setminus A \subset \{x \in R : \, |f_n(x) - f(x)| \leq 1/k \; (\forall n \geq n_k)\} = \bigcap_{n \geq n_k} \{x \in R : \, |f_n(x) - f(x)| \leq 1/k\}$$

which is equivalent to, for any $k \in \mathbb{N}$,

$$A \supset \bigcup_{n \geq n_k} \underbrace{\{x \in R : \, |f_n(x) - f(x)| > 1/k\}}_{B_{n,k}} = A_{n_k,k},$$

which is true, by construction.

Conversely, suppose that $f_n \to f$ a.u. By definition, for each $k \in \mathbb{N}$ we can guarantee the existence of $A_k \in \Sigma$ satisfying $\mu(A_k) < 1/k$ and $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f in $R \setminus A_k$. Set $A = \bigcap_{k \in \mathbb{N}} A_k$. Then $\mu(A) \leq \inf_{k \in \mathbb{N}} \mu(A_k) = 0$ and $(f_n)_{n \in \mathbb{N}}$ converges pointwise to f in $R \setminus A$. □

Remark. The above statement is not true in the case of infinite measure spaces. As a counterexample, consider the Lebesgue measure on $\mathbb{R}$ and $(f_n)_{n\in\mathbb{N}}$ to be $f_n = \chi_{[n,n+1]}$, where χ_E is the indicator function of $E \subset \mathbb{R}$. For $x \in \mathbb{R}$ there is $p \in \mathbb{N}$ greater than x. Since $\chi_{[n,n+1]}(x) = 0$ for any $n \geq p$, it follows that $f_n(x) \to 0$ as $n \to \infty$. Therefore $f_n \to 0$ a.e.

In fact $(f_n)_{n\in\mathbb{N}}$ does not converge almost uniformly to 0. Suppose the contrary. Then, there is a finite measure set A satisfying

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}\setminus A} \chi_{[n,n+1]}(x) = 0,$$

which means that

$$\forall x \in \mathbb{R} \setminus A \quad \chi_{[n,n+1]}(x) = 0,$$

for sufficiently big n, say, for $n \geq p$, for some $p \in \mathbb{N}$. It implies that

$$\forall n \geq p \quad \mathbb{R} \setminus A \cap [n, n+1] = \emptyset$$

or alternatively

$$\mathbb{R} \setminus A \cap [p, +\infty[= \emptyset.$$

It follows that $[p, +\infty[\subset A$, which contradicts the fact that $m(A) < \infty$.

Corollary 2.10 (see [8, Proposition 29.4.1])**.** *Suppose that (R, Σ, μ) is a finite measure space. If a sequence of measurable functions $(f_n)_{n\in\mathbb{N}}$ converges a.e. to a measurable function f, then $f_n \to f$ in measure.*

Proof. By the previous theorem, $f_n \to f$ a.u. Fix $\varepsilon > 0$. Then there is $A \in \Sigma$ such that $\mu(A) < \varepsilon$ and $f_n \to f$ uniformly in $R \setminus A$. Thus, for each $\delta > 0$ there is $p \in \mathbb{N}$ such that

$$\forall n \geq p \quad \sup_{x\in R\setminus A} |f_n(x) - f(x)| \leq \delta$$

which, in terms of sets, means that

$$R \setminus A \subset \bigcap_{n\geq p} \{x \in R : \, |f_n(x) - f(x)| \leq \delta\}.$$

Hence, for each $n \geq p$, $\mu(|f_n - f| > \delta) \leq \mu(A) < \varepsilon$ and the conclusion follows. □

Theorem 2.11 (see [8, Theorem 29.4.2])**.** *Let (R, Σ, μ) be a finite measure space and let (f_n) be a sequence of elements of $\mathcal{M}_0$ such that $f_n \to f$ in measure. Then there exists $(f_{n_k})_{k\in\mathbb{N}}$ a subsequence of (f_n) converging to f a.e.*

Proof. For each $k \in \mathbb{N}$ there is $n_k \in \mathbb{N}$ satisfying

$$n \geq n_k \implies \mu\{x \in R : \, |f_n(x) - f(x)| > 1/k\} < 1/2^k.$$

Assume, without loss of generality, that the natural valued sequence $(n_k)_{k\in\mathbb{N}}$ is strictly increasing. Let $B_k = \{x \in R : \ |f_{n_k}(x) - f(x)| > 1/k\}$ for every $k \in \mathbb{N}$. Since

$$\sum_{k\in\mathbb{N}} \mu(B_k) \le \sum_{k\in\mathbb{N}} 1/2^k = 1,$$

the Borel-Cantelli lemma provides us the equality $\mu(\limsup B_k) = 0$. So, by definition, if $x \notin \limsup B_k$, then there is $k \in \mathbb{N}$ satisfying

$$m \ge k \implies |f_{n_m}(x) - f(x)| \le 1/m.$$

Whence, $f_{n_k}(x) \to f(x)$ as $k \to \infty$. □

2.4 Metric associated to convergence in measure

A metric can characterize the convergence in measure. Take the function $\varphi_0\colon [0, +\infty[\to \mathbb{R}$ defined by

$$\varphi_0(t) = \begin{cases} t & \text{if } t \le 1, \\ 1 & \text{otherwise.} \end{cases} \tag{2.2}$$

Theorem 2.12 (see [8, Theorem 29.4.3]). *Let (R, Σ, μ) be a finite measure space. Consider the mapping $d_0\colon \mathcal{M}_0 \times \mathcal{M}_0 \to [0, +\infty[$ defined by*

$$d_0(f, g) = \int_R \varphi_0(|f - g|) d\mu.$$

Then d_0 is a metric. Furthermore, for any sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_0$ and for some $f \in \mathcal{M}_0$, $d_0(f_n, f) \to 0$ if and only if $f_n \to f$ in measure.

Proof. Fix $f, g, h \in \mathcal{M}_0$. Since φ_0 is increasing and subadditive,

$$\varphi_0(|f - h|) \le \varphi_0(|f - g| + |g - h|) \le \varphi_0(|f - g|) + \varphi_0(|g - h|).$$

Hence the triangle inequality of d_0 is satisfied, by monotonicity of integral. Also $d_0(f, g) = 0$ is equivalent to $\varphi_0(|f - g|) = 0$ a.e., which happens if and only if $\mu(f \ne g) = 0$. Thus $(\mathcal{M}_0, d_0)$ is a metric space.

Now suppose that $f_n \to f$ in measure and fix $\varepsilon > 0$. Then, there is $p \in \mathbb{N}$ such that,

$$\forall n \ge p \quad \mu\left(|f_n - f| > \frac{\varepsilon}{2\mu(R)}\right) < \varepsilon/2.$$

So, for any $n \ge p$,

$$\begin{aligned} d_0(f_n, f) &= \int_{|f_n - f| > \frac{\varepsilon}{2\mu(R)}} \varphi_0(|f_n - f|) d\mu + \int_{|f_n - f| \le \frac{\varepsilon}{2\mu(R)}} \varphi_0(|f_n - f|) d\mu \\ &\le \int_{|f_n - f| > \frac{\varepsilon}{2\mu(R)}} d\mu + \int_{|f_n - f| \le \frac{\varepsilon}{2\mu(R)}} \frac{\varepsilon}{2\mu(R)} d\mu \\ &\le \mu\left(|f_n - f| > \frac{\varepsilon}{2\mu(R)}\right) + \frac{\varepsilon}{2\mu(R)} \mu(R) \\ &< \varepsilon. \end{aligned}$$

Conversely, suppose $d_0(f_n, f) \to 0$. Note that for $c, t \geq 0$,

$$t > c \implies \varphi_0(t) \geq \varphi_0(c) = c,$$

whenever $c \leq 1$. Then, if we fix $c \in \,]0, 1]$, by monotonicity of μ and by Markov's inequality, we get

$$\mu(|f_n - f| > c) \leq \mu\left(\varphi_0(|f_n - f|) \geq c\right) \leq \frac{1}{c} \int_R \varphi_0(|f_n - f|) d\mu = d_0(f_n, f)/c \to 0,$$

as $n \to \infty$. □

Theorem 2.13 (see [8, Theorem 29.4.6]). *Let (R, Σ, μ) be a finite measure space. Then the metric space $(\mathcal{M}_0, d_0)$ is complete, where d_0 is the metric from Theorem 2.12.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a d_0-Cauchy sequence of $\mathcal{M}_0$. By the previous theorem, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. So, for each $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ satisfying

$$m, n \geq n_k \implies \mu\left(|f_n - f_m| > 1/2^k\right) < 1/2^k.$$

Without loss of generality, suppose that $(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence. Put $B_k = \left(\left|f_{n_k} - f_{n_{k+1}}\right| > 1/2^k\right)$ and let $C = \limsup_{n\to\infty} B_k$. Since $\sum_{k\in\mathbb{N}} \mu(B_k) \leq 1$, by the Borel-Cantelli lemma, $\mu(C) = 0$. For $x \notin C$ there is $k_0 \in \mathbb{N}$ such that

$$\forall k \geq k_0 \quad \left|f_{n_k}(x) - f_{n_{k+1}}(x)\right| \leq 1/2^k.$$

In particular, if $s > r \geq k$, for $k \geq k_0$,

$$|f_{n_r}(x) - f_{n_s}(x)| \leq \sum_{l=r}^{s-1} \left|f_{n_l}(x) - f_{n_{l+1}}(x)\right| \leq \sum_{l\geq k} 1/2^l = 2/2^k.$$

Hence $(f_{n_k}(x))_{k\in\mathbb{N}}$ is a $\mathbb{K}$-Cauchy sequence. Thus, $f_{n_k}(x) \to f(x)$ as $k \to \infty$. Since it happens for $x \notin C$, we can assume that $f|_C = 0$, and deduce that $f_{n_k} \to f$ a.e. for some $f \in \mathcal{M}_0$ as k increases, by nullity of C. By Corollary 2.10, $f_{n_k} \to f$ in measure. Applying Theorem 2.12, $d_0(f_{n_k}, f) \to 0$. Therefore $d_0(f_n, f) \to 0$. □

In order to extend the previous results to σ-finite measure spaces, we will study the convergence in $\mathcal{M}_0(R, \Sigma, \mu)$ indirectly.

Consider a σ-finite measure space (R, Σ, μ). Suppose that $(R_m)_{m\in\mathbb{N}}$ is a pairwise disjoint sequence of measurable sets such that $0 < \mu(R_m) < \infty$ for any $m \in \mathbb{N}$, and $R = \bigcup_{m\in\mathbb{N}} R_m$. For $A \in \Sigma$ such that $\mu(A) < \infty$ set $\pi_A \colon \mathcal{M}_0(R, \Sigma, \mu) \to \mathcal{M}_0(A, \Sigma_A, \mu_A)$ to be the restriction mapping with respect to A, that is, $\pi_A(f) = f|_A$, for $f \in \mathcal{M}_0(R, \Sigma, \mu)$. Denote (R_m, Σ_m, μ_m) instead of $(R_m, \Sigma_{R_m}, \mu_{R_m})$. Consider on $\mathcal{M}_0(R_m, \Sigma_m, \mu_m)$, the metric

$$d_m(f, g) = \int_{R_m} \varphi_0(|f - g|) d\mu_m,$$

with the function φ_0 defined in (2.2). One can easily check that the metric spaces $(\prod_{m\in\mathbb{N}}\mathcal{M}_0(R_m,\Sigma_m,\mu_m),\varrho)$ and $(\mathcal{M}_0(R,\Sigma,\mu),d)$ are isometric, where

$$\varrho((f_m)_{m\in\mathbb{N}},(g_m)_{m\in\mathbb{N}})=\sum_{m\in\mathbb{N}}\frac{1}{2^m}\frac{d_m(f_m,g_m)}{\mu(R_m)},$$

for $(f_m)_{m\in\mathbb{N}},(g_m)_{m\in\mathbb{N}}\in\prod_{m\in\mathbb{N}}\mathcal{M}_0(R_m,\Sigma_m,\mu_m)$, and

$$d(f,g)=\sum_{m\in\mathbb{N}}\frac{d_m(\pi_{R_m}(f),\pi_{R_m}(g))}{2^m\mu(R_m)},$$

where $f,g\in\mathcal{M}_0(R,\Sigma,\mu)$. For $f\in\mathcal{M}_0(R,\Sigma,\mu)$, the correspondent element in the product space is $(\pi_{R_m}(f))_{m\in\mathbb{N}}$ and, for $(f_m)_{m\in\mathbb{N}}$, the correspondent element in $\mathcal{M}_0(R,\Sigma,\mu)$ is f such that $f(x)=f_m(x)$, if $x\in R_m$. Since each of $\mathcal{M}_0(R_m,\Sigma_m,\mu_m)$ has a complete metric

$$\varrho_m(f_m,g_m)=\frac{d_m(f_m,g_m)}{\mu(R_m)}\leq 1,\quad (m=1,2,...),$$

by Theorem 2.3, ϱ is a complete metric which is equivalent to product topology (and so d is). Furthermore, Lemma 2.2 says that convergence in the total space is equivalent to convergence componentwise, that is, a sequence of elements of $\mathcal{M}_0(R,\Sigma,\mu)$ converges to $f\in\mathcal{M}_0(R,\Sigma,\mu)$ if and only if $f_n\chi_{R_m}$ converges to $f\chi_{R_m}$, for each $m\in\mathbb{N}$.

Consider now the following lemma to finalize this topic.

Lemma 2.14. *Let (R,Σ,μ) be a σ-finite measure space and let $(R_m)_{m\in\mathbb{N}}$ be a sequence of pairwise disjoint elements of Σ of finite measure that covers all the R. Then, the following statements are equivalent:*

(1) for each $m\in\mathbb{N}$, $f_n\to f$ in measure on R_m;

(2) $f_n\to f$ in measure, on finite measure sets.

Proof. (2) $\implies$ (1) is immediate. Suppose then (1), take A to be a finite measure set and put $S_m=A\cap R_m$, for any $m\in\mathbb{N}$. Notice that $\mu(A)=\sum_{m\in\mathbb{N}}\mu(S_m)<\infty$. So if we fix $\delta>0$, there must be $p\in\mathbb{N}$ such that

$$\sum_{m\geq p+1}\mu(S_m)<\frac{\delta}{2}.$$

By hypotesis, there is $N_m\in\mathbb{N}$ such that

$$\forall n\geq N_m\quad \mu\left(R_m\cap\{|f_n-f|>\varepsilon\}\right)<\frac{\delta}{2p},$$

for each $m \in \mathbb{N}$ and $\varepsilon > 0$. Set $N = \max\{N_1, ..., N_p\}$. Then, for $n \geq N$ we have

$$\begin{aligned} \mu\left(A \cap \{|f_n - f| > \varepsilon\}\right) &= \sum_{m=1}^{p} \mu\left(S_m \cap \{|f_n - f| > \varepsilon\}\right) + \sum_{m \geq p+1} \mu\left(S_m \cap \{|f_n - f| > \varepsilon\}\right) \\ &\leq \sum_{m=1}^{p} \mu\left(R_m \cap \{|f_n - f| > \varepsilon\}\right) + \sum_{m \geq p+1} \mu(S_m) \\ &< \sum_{m=1}^{p} \frac{\delta}{2p} + \frac{\delta}{2} = \delta. \end{aligned}$$

□

The following result will summarise what was discussed in this chapter.

Theorem 2.15. *Let (R, Σ, μ) be a σ-finite measure space, (f_n) a sequence of $\mathcal{M}_0(R, \Sigma, \mu)$ and $f \in \mathcal{M}_0(R, \Sigma, \mu)$, with usual notations. The following statements are equivalent:*

(I) $f_n \to f$ *in the metric space* $\mathcal{M}_0(R, \Sigma, \mu)$*;*

(II) $(\pi_{R_m}(f_n))_{m\in\mathbb{N}} \to (\pi_{R_m}(f))_{m\in\mathbb{N}}$ *in the metric space* $\prod_{m\in\mathbb{N}} \mathcal{M}_0(R_m, \Sigma_m, \mu_m)$*;*

(III) $\pi_{R_m}(f_n) \to \pi_{R_m}(f)$ *in* $\mathcal{M}_0(R_m, \Sigma_m, \mu_m)$*, for any* $m \in \mathbb{N}$*;*

(IV) $\pi_{R_m}(f_n) \to \pi_{R_m}(f)$ *in measure, for any* $m \in \mathbb{N}$*;*

(V) $f_n\chi_{R_m} \to f\chi_{R_m}$ *in measure, for any* $m \in \mathbb{N}$*;*

(VI) $f_n \to f$ *in measure on finite measure sets.*

Furthermore, $\mathcal{M}_0(R, \Sigma, \mu)$ *is a complete metric space.*

Proof. $(I) \Leftrightarrow (II)$ holds true, by isometry, $(II) \Leftrightarrow (III)$ holds true, by Lemma 2.2, $(III) \Leftrightarrow (IV)$ holds true by Theorem 2.12, $(IV) \Leftrightarrow (V)$ holds true by definition and $(V) \Leftrightarrow (VI)$ holds true by Lemma 2.14.

Finally, since each $\mathcal{M}_0(R_m, \Sigma_m, \mu_m)$ is complete, by Theorem 2.13, the completeness of $\mathcal{M}_0(R, \Sigma, \mu)$ follows by Theorem 2.3 (2). □

With that in mind, we conclude that $\mathcal{M}_0(R, \Sigma, \mu)$ is a complete metric space, where (R, Σ, μ) is a σ-finite measure space, and we can study the convergence in measure on finite measure sets instead of the convergence in metric on $\mathcal{M}_0(R, \Sigma, \mu)$.

Chapter 3

Banach function spaces

3.1 Structure and properties

From this chapter on, we shall assume once for all that (R, Σ, μ) is a σ-finite measure space and that there is a sequence of pairwise disjoint sets $(R_m)_{m\in\mathbb{N}}$ such that $0 < \mu(R_m) < \infty$ for any $m \in \mathbb{N}$ and $R = \bigcup_{m\in\mathbb{N}} R_m$. From now on, for brevity, we write (R, μ) instead of (R, Σ, μ).

Definition 3.1. Let (R, μ) be a measure space. A mapping $\rho\colon \mathcal{M}^+ \to [0, \infty]$ is called a *Banach function norm* (or simply, *function norm*) if, for any $f, g, f_n \in \mathcal{M}^+$ $(n = 1, 2, ...)$ and $a > 0$, it satisfies the following proprieties:

(P1) $\rho(f) = 0 \iff f = 0$ a.e.;

(P2) $\rho(af) = a\rho(f)$;

(P3) $\rho(f + g) \le \rho(f) + \rho(g)$ [triangle inequality];

(P4) $g \le f$ a.e. $\implies \rho(g) \le \rho(f)$ [lattice propriety];

(P5) $f_n \uparrow f$ a.e. $\implies \rho(f_n) \uparrow \rho(f)$ [Fatou's propriety];

(P6) $\mu(E) < \infty \implies \rho(\chi_E) < \infty$;

(P7) $\mu(E) < \infty \implies \int_E f d\mu \le C_E \rho(f)$,
for some positive real number C_E independent of f.

Theorem 3.2 (see [2, Chapter 1, Theorem 1.2])**.** *The Lebesgue functional* $\rho_p\colon \mathcal{M}^+ \to [0, \infty]$ *defined as*

$$\rho_p(f) := \begin{cases} \left(\int_R f^p d\mu\right)^{1/p} & \textit{if } 1 \le p < \infty, \\ \operatorname{ess\,sup}_R f & \textit{if } p = \infty, \end{cases}$$

is a function norm.

Proof. The proprieties (P1), (P2), (P4) and (P6) are immediate. (P3) and (P5) hold by Minkowski's inequality and monotone convergence, respectively. For (P7) choose a set of finite measure E and $f \in \mathcal{M}^+$. If $1 < p < \infty$, by Holder's inequality with q such that $1/p + 1/q = 1$, we get

$$\int_E f d\mu = \int_R f \chi_E d\mu \le \underbrace{\left(\int_R f^p d\mu\right)^{1/p}}_{\rho_p(f)} \underbrace{\left(\int_R \chi_E^q d\mu\right)^{1/q}}_{C_E}.$$

For $p = \infty$ we have:

$$\int_E f d\mu \le \int_E \operatorname*{ess\,sup}_R f d\mu = \rho_p(f) \underbrace{\int_R \chi_E d\mu}_{C_E}.$$

The case when $p = 1$ is obvious. ☐

Definition 3.3. Let ρ be a Banach function norm. The collection $X = X(\rho)$ of all functions f in $\mathcal{M}$ for which $\rho(|f|) < \infty$ is called a *Banach function space.* For each $f \in X$ we define

$$\|f\|_X := \rho(|f|).$$

Definition 3.4. Let X, Y be two metric spaces such that $X \subset Y$. Then we say that the space X is *embedded* into the space Y if the inclusion operator $i \colon X \to Y$, $x \mapsto x$ is continuous. To refer to that we will write $X \hookrightarrow Y$.

Recall that a simple function is a linear combination of indicator functions, that is, f is a simple function if

$$f = \sum_{k=1}^{n} a_k \chi_{E_k},$$

for $n \in \mathbb{N}$, $a_1, ..., a_n$ scalars and $E_1, ..., E_n$ measurable sets.

Theorem 3.5 (see [2, Chapter 1, Theorem 1.4])**.** *Let (R, μ) be a measure space and $\rho \colon \mathcal{M}^+ \to [0, \infty]$ a Banach function norm. Then $(X, \|\cdot\|_X)$ is a normed space, where $X = X(\rho)$ and, for each $f \in X$,*

$$\|f\|_X := \rho(|f|).$$

Additionally the inclusions

$$S \subset X \hookrightarrow \mathcal{M}_0$$

hold, where S is the set of simple functions with domain on R. In particular, if $f_n \to f$ in X, then $f_n \to f$ in measure on finite measure sets, and hence some subsequence converges to f a.e.

Proof. Of course $\|\cdot\|_X$ is a norm on X, by (P1), (P2) and (P3). A simple function f with domain on R is, by definition, $f = \sum_{i=1}^k a_i\chi_{E_i}$ for $k \in \mathbb{N}$, $a_i \in \mathbb{K}$ and finite measure sets E_i. Using the proprieties of modulus and Banach function norm, we get

$$\rho(|f|) = \rho\left(\left|\sum_{i=1}^k a_i\chi_{E_i}\right|\right) \le \rho\left(\sum_{i=1}^k |a_i|\,|\chi_{E_i}|\right) \le \sum_{i=1}^k |a_i|\,\rho\left(|\chi_{E_i}|\right).$$

Thus $S \subset X$. If there is $f \in X$ with $E = |f|^{-1}[\{\infty\}]$ whose measure is different from 0 then $\int_E |f|\,d\mu$ is not finite, which contradicts (P7). So it must be that $X \subset \mathcal{M}_0$ remaining for us to prove that $i\colon X \to \mathcal{M}_0$ is continuous. Choose $f_n \to f$ on X, a set E of finite measure and $\varepsilon > 0$. Then, applying Markov's inequality and (P7) we get

$$\mu\{x \in E : |f_n(x) - f(x)| > \varepsilon\} \le \frac{1}{\varepsilon}\int_E |f_n - f|\,d\mu \le \frac{1}{\varepsilon}C_E\underbrace{\rho(|f_n - f|)}_{\|f_n - f\|_X}.$$

Hence $f_n \to f$ in measure on E. Since E is an arbitrary finite measure set, by Theorem 2.15, $f_n \to f$ in the metric space $\mathcal{M}_0$. Thus i is sequentially continuous. Since X and $\mathcal{M}_0$ are metric spaces, we conclude that $X \hookrightarrow \mathcal{M}_0$.

Since $f_n \to f$ in measure on finite measure sets, $f_n\chi_{R_m} \to f\chi_{R_m}$ in measure, for every $m \in \mathbb{N}$. Using Theorem 2.15 with Theorem 2.11, for each $m \in \mathbb{N}$, we guarantee $\left(\alpha_n^{(m)}\right)$ an increasing sequence such that $f_{\alpha_n^{(m)}}\chi_{R_m} \to f\chi_{R_m}$ a.e. Since every subsequence $\left(f_{\alpha_n^{(m)}}\chi_{R_m}\right)_{n\in\mathbb{N}}$ of $(f_n\chi_{R_m})_{n\in\mathbb{N}}$ $(m = 1, 2, ...)$ converges in measure to $f\chi_{R_m}$, without a loss of generality we can suppose that if $m_1 \le m_2$, then $\left(\alpha_n^{(m_2)}\right)$ is a subsequence of $\left(\alpha_n^{(m_1)}\right)$. Then $\left(\alpha_n^{(n)}\right)$ is a sequence such that $f_{\alpha_n^{(n)}}\chi_{R_m} \to f\chi_{R_m}$ a.e., for all the $m \in \mathbb{N}$. In other words, $f_{\alpha_n^{(n)}} \to f$ a.e. ☐

Lemma 3.6 (see [2, Chapter 1, Lemma 1.5])**.** *Let $X = X(\rho)$ be a Banach function space and suppose that (f_n) is a sequence of elements of X.*

(i) If $0 \le f_n \uparrow f$ a.e., then either $f \notin X$ and $\|f_n\|_X \uparrow \infty$ or $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$;

(ii) (Fatou's lemma) If $f_n \to f$ a.e. and $\liminf_{n\to\infty} \|f_n\|_X < \infty$, then $f \in X$ and

$$\|f\|_X \le \liminf_{n\to\infty} \|f_n\|_X < \infty.$$

Proof. The first assertion is immediate, by (P5) from the definition of Banach function norm. For the second part, we cannot use (P5) yet. Instead, we set $h_n(x) = \inf_{m\ge n} |f_n(x)|$. By construction, $h_n \le h_{n+1}$ a.e., for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} h_n(x) = \sup_{n\in\mathbb{N}} h_n(x) = \liminf_{n\to\infty} |f_n(x)| = |f(x)|$$

for almost every $x \in R$. Hence $h_n \uparrow |f|$ a.e. Now, since by definition, $h_n \le |f_m|$ a.e. for each $m \ge n$, applying (P2), we can see that

$$\forall m \ge n \quad \rho(h_n) \le \|f_m\|_X.$$

Hence

$$\rho(h_n) \leq \inf_{m \geq n} \|f_m\|_X \,.$$

Using this and using the fact that $h_n \uparrow |f|$ a.e. with (P5) we have

$$\|f\|_X = \sup_{n \in \mathbb{N}} \rho(h_n) \leq \sup_{n \in \mathbb{N}} \inf_{m \geq n} \|f_m\|_X = \liminf_{n \to \infty} \|f_n\|_X \,.$$

Consequently $f \in X$, since the right-hand side expression is finite and f is the pointwise limit of a sequence of measurable functions. □

Theorem 3.7 (Riesz-Fischer property; see [2, Chapter 1, Theorem 1.6])**.** *Let X be a Banach function space. Suppose that (f_n) is a sequence of elements of X such that*

$$\sum_{n \in \mathbb{N}} \|f_n\|_X < \infty.$$

Then $\sum_{n \in \mathbb{N}} f_n$ converges in X and

$$\left\| \sum_{n \in \mathbb{N}} f_n \right\|_X \leq \sum_{n \in \mathbb{N}} \|f_n\|_X \,.$$

In particular, X is complete.

Proof. Let $t_N = \sum_{n \leq N} |f_n|$ so that $0 \leq t_N \uparrow t := \sum_{n \in \mathbb{N}} |f_n|$. Notice that

$$\|t_N\|_X \leq \sum_{n \leq N} \|f_n\|_X \leq \sum_{n \in \mathbb{N}} \|f_n\|_X < \infty.$$

Applying (i) from the previous lemma, we get $t \in X$. Since $X \subset \mathcal{M}_0$, for almost every $x \in R$, $\sum_{n \in \mathbb{N}} |f_n(x)|$ converges and so does $\sum_{n \in \mathbb{N}} f_n(x)$, by completness of $\mathbb{K}$. Thus if $s_N = \sum_{n \leq N} f_n$ then $s_N \to f := \sum_{n \in \mathbb{N}} f_n$ a.e. and, for fixed $M \in \mathbb{N}$, $s_N - s_M \to f - s_M$ a.e., as well. Furthermore,

$$\liminf_{N \to \infty} \|s_N - s_M\|_X \leq \liminf_{N \to \infty} \sum_{n=M+1}^{N} \|f_n\|_X = \sum_{n \geq M+1} \|f_n\|_X \to 0,$$

as $M \to \infty$, since the right-hand side sum is the rest sum of a convergent series. By Fatou's lemma, it follows that $f - s_M \in X$ (hence $f \in X$) and $s_M \to f$ in X. But then, for every $M \in \mathbb{N}$,

$$\|f\|_X \leq \|f - s_M\|_X + \|s_M\|_X \leq \|f - s_M\|_X + \sum_{n \leq M} \|f_n\|_X \,,$$

hence, the desired inequality follows, when we take the limit as $M \to \infty$. Since every absolutely convergent series is convergent, the completeness follows by a well-known result of the introductory functional analysis (see, e.g., [25, Chapter IV, Theorem 1.4]). □

The next theorem summarizes for future reference the basic proprieties of Banach function spaces established so far.

Theorem 3.8 (see [2, Chapter 1, Theorem 1.7]). *Suppose that ρ is a function norm and let*

$$X = \{f \in \mathcal{M} : \ \rho(|f|) < \infty\}.$$

For each $f \in X$ let $\|f\|_X = \rho(|f|)$. Then $(X, \|\cdot\|_X)$ is a Banach space and the following proprieties hold for all $f, g, f_1, f_2, \ldots \in \mathcal{M}$ and all measurable sets E of R:

(i) *If $|g| \le |f|$ a.e. and $f \in X$, then $g \in X$ and $\|g\|_X \le \|f\|_X$. In particular, $f \in X$ if and only if $|f| \in X$ and in that case f and $|f|$ have the same norm in X.*

(ii) *(The Fatou property) Suppose that $f_n \in X$, for any $n \in \mathbb{N}$ and $0 \le f_n \uparrow f$ a.e. If $f \in X$, then $\|f_n\|_X \uparrow \|f\|_X$, whereas if $f \notin X$, then $\|f_n\|_X \uparrow \infty$.*

(iii) *(Fatou's lemma) If $f_n \in X$ for any $n \in \mathbb{N}$, $f_n \to f$ a.e. and $\liminf_{n\to\infty} \|f_n\|_X < \infty$, then $f \in X$ and*

$$\|f\|_X \le \liminf_{n\to\infty} \|f_n\|_X .$$

(iv) *Every simple function belongs to X.*

(v) *To each set E of finite measure there corresponds a constant $C_E < \infty$ satisfying*

$$\int_E |f|\, d\mu \le C_E \|f\|_X$$

for all $f \in X$.

(vi) *If $f_n \to f$ in X, then $f_n \to f$ in measure on every set of finite measure. In particular, some subsequence of (f_n) converges to f a.e.*

Theorem 3.9 (see [2, Chapter 1, Theorem 1.8]). *Let X and Y be Banach function spaces over the same measure space. If $X \subset Y$, then $X \hookrightarrow Y$.*

Proof. Suppose that the linear operator $i \colon X \to Y$ is not bounded. Then there exists a sequence (f_n) in X such that $\|f_n\|_X = 1$ and $\|f_n\|_Y > n^3$ for each $n \in \mathbb{N}$. By replacing f_n with its absolute value, the norm does not change. So we may suppose, without loss of generality, that $f_n \ge 0$ for any $n \in \mathbb{N}$. By the Riesz-Fischer property, it follows that $\sum_{n\in\mathbb{N}} n^{-2} f_n$ converges, say, to $f \in X$. Hence $f \in Y$ which is impossible, since

$$0 \le n^{-2} f_n \le \sum_{n\in\mathbb{N}} n^{-2} f_n = f$$

and

$$\|f\|_Y \ge n^{-2} \|f_n\|_Y > n,$$

for any $n \in \mathbb{N}$. Therefore, i must be a bounded linear operator. □

Corollary 3.10 (see [2, Chapter 1, Corollary 1.9]). *If two Banach function spaces consist of the same set of functions, then their norms are equivalent.*

Proof. By the previous theorem, $X \hookrightarrow X$, that is, the inclusion map $i\colon X \to X$ is a bounded linear operator. Since i is the identity map on the Banach space X, the Banach isomorphism theorem implies that i must be invertible. $\square$

3.2 The associate space

Definition 3.11. If ρ is a function norm, its *associate norm* ρ' is defined on $\mathcal{M}^+$ by

$$\rho'(g) := \sup\left\{ \int_R fg d\mu :\ f \in \mathcal{M}^+,\ \rho(f) \le 1 \right\}.$$

Theorem 3.12 (see [2, Chapter 1, Theorem 2.2]). *The associate norm ρ' of ρ is itself a function norm.*

Proof. If $\rho(f) \le 1$, then by Theorem 3.5 f is finite a.e. Hence, if $g = 0$ a.e., then $\int_R fg d\mu = 0$, for all $f \in \mathcal{M}^+$ and so $\rho'(g) = 0$. Conversely, if $\rho'(g) = 0$, then by definition, $\int_R fg d\mu = 0$ for any $f \in \mathcal{M}^+$ satisfying $\rho(f) \le 1$. If E is a set such that $0 < \mu(E) < \infty$, then (P1) and (P6) from the definition of the function norm gives us the inequality $0 < \rho(\chi_E) < \infty$. Taking $f = \frac{\chi_E}{\rho(\chi_E)}$ so that $\rho(f) = 1$, we obtain

$$\int_R fg d\mu = (\rho(\chi_E))^{-1} \int_E g d\mu = 0,$$

which implies that $g = 0$ a.e. on E. Since we are working on σ-finite measure space, it follows that $g = 0$ a.e. The properties (P2)-(P4) are immediate.

Now suppose that (g_n) is a sequence in $\mathcal{M}^+$ such that $g_n \uparrow g$ a.e. for some $g \in \mathcal{M}^+$. So, by the lattice property $\rho'(g_n) \le \rho'(g)$ for any $n \in \mathbb{N}$. If $\rho(g_n) = 0$ for some $n \in \mathbb{N}$, then there is nothing to prove. We therefore suppose that $\rho'(g) < \infty$ for every $n \in \mathbb{N}$. Let ξ be a real number such that $\rho'(g) > \xi$. By definition of supremum, there is $f \in \mathcal{M}^+$ such that $\rho(f) \le 1$ and $\int_R fg d\mu > \xi$. Now $fg_n \uparrow fg$ implies that $\int_R fg_n d\mu \uparrow \int_R fg d\mu$ by the monotone convergence theorem. Hence there is $N \in \mathbb{N}$ such that $\int_R fg_n d\mu > \xi$ for $n \ge N$. Therefore $\rho'(g_n) \uparrow \rho'(g)$.

If E is a set satisfying $\mu(E) < \infty$, then for any $f \in \mathcal{M}^+$ with $\rho(f) \le 1$,

$$\int_R fg d\mu = \int_E f d\mu \le C_E \rho(f) \le C_E.$$

Hence $\rho'(\chi_E) \le C_E < \infty$.

For the remaining property, we suppose that $\mu(E) > 0$ since $\mu(E) = 0$ is a trivial case. Set $f = \frac{\chi_E}{\rho(\chi_E)}$ so that $\rho(f) = 1$. By definition,

$$\rho'(g) \ge \int_R fg d\mu = (\rho(\chi_E))^{-1} \int_E g d\mu.$$

Therefore we conclude that ρ' satisfies the property (P7) by setting $C'_E = \rho(\chi_E)$. $\square$

Definition 3.13. Let ρ be a function norm and let $X = X(\rho)$ be the Banach function space. Let ρ' be the associate norm of ρ. The Banach function space $X(\rho')$ determined by ρ' is called the *associate space* of X and is denoted by X'. Hence its norm is defined by

$$\|g\|_{X'} := \left\{ \int_R |fg|\, d\mu : \ f \in X, \ \|f\|_X \leq 1 \right\}.$$

Theorem 3.14 (Hölder's inequality; see [2, Chapter 1, Theorem 2.4])**.** *Let X be a Banach function space and X' be its associate space. If $f \in X$ and $g \in X'$ then fg is integrable. Furthermore,*

$$\int_R |fg|\, d\mu \leq \|f\|_X \, \|g\|_{X'} .$$

Proof. If $\|f\|_X = 0$ then $f = 0$ a.e. and there is nothing left to prove. Otherwise, we choose $\tilde{f} = \frac{f}{\|f\|_X}$ so that $\left\|\tilde{f}\right\|_X = 1$ and

$$\|g\|_{X'} \geq \int_R \left|\tilde{f}g\right| d\mu = (\|f\|_X)^{-1} \int_R |fg|\, d\mu,$$

which implies the required inequality. □

Theorem 3.15 (see [6, Theorem 3.39])**.** *If $p, p' \in [1, +\infty]$ are such that $1/p + 1/p' = 1$ then $L^{p'}$ is the associate space of L^p.*

Proof. The classical Hölder's inequality says that we have

$$\int_R |fg|\, d\mu \leq \|f\|_p \, \|g\|_{p'} ,$$

for every $f \in L^p$ and $g \in L^{p'}$. So, it remains to prove that

$$\sup \left\{ \int_R |fg|\, d\mu : \ \|f\|_p \leq 1 \right\} \geq \|g\|_{p'} ,$$

for every $g \in L^{p'}$.

For the case when $1 < p < \infty$, if we take $g \in L^{p'}$, such that $g \neq 0$, and $f = c\,|g|^{p'-1}$ for some constant c, then $|fg| = c\,|g|^{p'}$. Thus

$$\|fg\|_1 = c\,\|g\|_{p'}^{p'} .$$

So, if we take $c = \|g\|_{p'}^{1-p'}$ then

$$\|fg\|_1 = \|g\|_{p'} ,$$

lasting us to prove that $\|f\|_p \leq 1$. Indeed

$$\|f\|_p = \|g\|_{p'}^{1-p'} \left(\int_R |g|^{p(p'-1)}\, d\mu \right)^{1/p} = \|g\|_{p'}^{1-p'} \, \|g\|_{p'}^{\frac{p'}{p}} = \|g\|_{p'}^{1+p'\left(\frac{1}{p}-1\right)} = 1.$$

For $p = \infty$, if $g \in L^1$, then, by setting $f = \operatorname{sgn}(g)$, we get

$$\int_R |fg|\, d\mu = \int_R |g|\, d\mu = \|g\|_1 ,$$

and since $\|f\|_\infty = 1$, the conclusion follows.

Finally, for $p = 1$, we will instead show that

$$\sup\left\{ \frac{\int_R |fg|\, d\mu}{\|f\|_1} \,:\, f \neq 0 \right\} \geq \|g\|_\infty .$$

For each $\varepsilon > 0$ such that $\varepsilon < \|g\|_\infty$, consider

$$E = \{x \in R :\, |g(x)| > \|g\|_\infty - \varepsilon/2\} .$$

Since R is σ-finite, there is a finite measure set $F \subset E$ such that $\mu(F) > 0$. If we put $f = \chi_F$, we therefore obtain

$$\int_R |fg|\, d\mu = \int_F |g|\, d\mu \geq (\|g\|_\infty - \varepsilon/2) \int_F d\mu > (\|g\|_\infty - \varepsilon)\, \|f\|_1 ,$$

which is

$$\frac{\int_R |fg|\, d\mu}{\|f\|_1} > \|g\|_\infty - \varepsilon.$$

By the definition of supremum, we achieve the desired inequality. □

From the Hölder inequality we know that $g \in X'$ implies that $fg \in L^1$, for all $f \in X$. We shall now show that, in fact, the converse also holds true.

Theorem 3.16 (Landau's resonance theorem; see [2, Chapter 1, Lemma 2.6]). *Let X be Banach function space and let X' be its associate space. Then $g \in X'$ if and only if $fg \in L^1$ for every $f \in X$.*

Proof. If $g \in X'$, then by Hölder's inequality $fg \in L^1$ for every $f \in X$.

Now, suppose that $fg \in L^1$, for any $f \in X$ and suppose that $\|g\|_{X'} = \infty$. Then, by definition of the norm function, there are $f_n \in X$ such that $\|f_n\| = 1$ and $\int_R |f_n g|\, d\mu > n^3$ for all the $n \in \mathbb{N}$. But then $f = \sum_{n \in \mathbb{N}} n^{-2} f_n \in X$ by the Riesz-Fischer property. Without loss of generality, suppose that $f_n \geq 0$. Consequently

$$\int_R |fg|\, d\mu \geq n^{-2} \int_R |f_n g|\, d\mu > n,$$

which is a contradiction. □

Definition 3.17. Let X be a Banach function space and let X' be its associate space. Then the associate space $(X')'$ of X is called *second associate space* and it is denoted by X''.

Theorem 3.18 (Lorentz-Luxemburg theorem; see [2, Chapter 1, Theorem 2.7]). *Let X be a Banach function space. Then $X = X''$ (in sense of set theory) and, for any $f \in \mathcal{M}^+$,*

$$\|f\|_X = \|f\|_{X''} .$$

Proof. If $f \in X$, by the Hölder's inequality $fg \in L^1$ for every $g \in X'$. Hence, by the Landau resonance theorem $f \in X''$, and

$$\|f\|_{X''} = \sup \left\{ \int_R |fg| \, d\mu : \; g \in X', \, \|g\|_{X'} \leq 1 \right\} \leq \|f\|_X \, .$$

Conversely, let $f \in X''$. For every $n \in \mathbb{N}$, we set

$$f_n(x) = \min \{ |f(x)| \, , n \} \, \chi_{R_n}(x),$$

where the sequence (R_n) is defined in the convention made on the beggining of this chapter. Now

$$\forall n \in \mathbb{N} \quad 0 \leq f_n \leq n \chi_{R_n}.$$

Applying Theorem 3.8 (i) and (iv), it follows that $f_n \in X$. Furthermore, by Fatou's property,

$$0 \leq f_n \uparrow |f|$$

in both X and X''. So, to verify $\|f\|_X \leq \|f\|_{X''}$ it is sufficient to prove that

$$\forall n \in \mathbb{N} \quad \|f_n\|_X \leq \|f_n\|_{X''} \, .$$

We shall now suppose that f and $n \in \mathbb{N}$ are fixed and assume, without loss of generality, that $\|f_n\|_X > 0$. Consider the space

$$M_n = \{ g \in \mathcal{M} : \; \text{supp } g \subset R_n \} \, ,$$

equipped with the norm

$$\|g\|_{M_n} := \int_{R_n} |g_n| \, d\mu.$$

Since L^1 is complete, $\left(M_n, \|\cdot\|_{M_n}\right)$ is a Banach space. Let S_X be the closed unit ball on X. Then the set

$$U = S_X \cap M_n$$

is a convex subset of M_n. If $(h_k)_{k \in \mathbb{N}} \subset U$ is such that $h_k \to h$ in M_n, then $h_k \to h$ in measure on finie measure sets, hence some subsequence converges a.e. to h, say $(h_{\alpha_k})_{k \in \mathbb{N}}$. Since $h_{\alpha_k} \in S_X$, by Fatou's lemma $h \in S_X$. But supp $h_{\alpha_k} \subset R_n$ for all the $k \in \mathbb{N}$, whence supp $h \subset R_n$, thus $h \in U$. Therefore U is a closed subset of M_n.

Let now $\lambda > 1$. Then

$$g = \lambda \frac{f_n}{\|f_n\|_X} \in M_n \setminus U.$$

By the second geometric version of Hahn-Banach theorem (see [23, Theorem 3.4]), there is a closed hyperplane that separates strictly $\{g\}$ and U. In other words, by the Riesz Representation theorem (see [6, Theorem 3.43]), there is some non-zero function $\varphi \in L^\infty$ such that supp $\varphi \subset M_n$ and

$$\Re \left(\int_{R_n} h \varphi d\mu \right) < \gamma < \Re \left(\int_{R_n} g \varphi d\mu \right),$$

for some real number γ and for every $h \in U$. By writing φ in the polar form, $\varphi = |\varphi|\,\psi$ and observing that $\psi\,|h| \in U$ if and only if h does, we obtain

$$\sup_{h\in U}\int_{R_n} |\varphi h|\, d\mu \le \gamma < \Re\left(\int_{R_n} \varphi g d\mu\right) \le \int_{R_n} |\varphi g|\, d\mu.$$

For every $h \in S_X$, we have

$$h(x) = \lim_{k\to\infty} h_k(x) = \lim_{k\to\infty} \min\{h(x), k\}\, \chi_{R_n}(x),$$

for $x \in R_n$. Obviously, $h_k \in M_n$, so $h_k \in U$, since $|h_k| \le |h|$ in R, whence $\|h_k\|_X \le \|h\|_X \le 1$. A function $h \in S_X$ thus can be approximated by the sequence of functions (h_k) in U. Since $U \subset S_X$, by the monotone convergence theorem,

$$\sup_{h\in U}\int_{R_n} |\varphi h|\, d\mu = \sup_{h\in S_X}\int_{R_n} |\varphi h|\, d\mu.$$

Hence

$$\|\varphi\|_{X'} = \sup_{h\in S_X}\int_{R_n} |\varphi h|\, d\mu \le \gamma < \int_{R_n} |\varphi g|\, d\mu = \frac{\lambda}{\|f_n\|_X}\int_{R_n} |\varphi f_n|\, d\mu,$$

or, equivalently

$$\|f_n\|_X < \lambda \int_{R_n} \left|\frac{\varphi}{\|\varphi\|_{X'}} f_n\right| d\mu \le \lambda \sup_{\phi\in S_{X'}}\int_{R_n} |\phi f_n|\, d\mu = \lambda\, \|f_n\|_{X''}\,,$$

where $S_{X'}$ is the unit ball in X'. Letting $\lambda \downarrow 1$, we get the desired inequality. □

Lemma 3.19 (see [2, Chapter 1, Lemma 2.8]). *The norm of a function g in the associate space X' can be given by*

$$\|g\|_{X'} = \sup\left\{\left|\int_R fgd\mu\right| :\ f \in X,\ \|f\|_X \le 1\right\}.$$

Proof. Since $\left|\int fg\right| \le \int |fg|$, clearly the left-hand-side integral does not exceed the value of $|g\|_{X'}$. In order to prove the reverse inequality, denote the unit ball of X as S_X. Let

$$E = \{x \in R:\ g(x) \ne 0\}\,.$$

The polar form of g is given by, $g = |g|\,\psi$ with $|\psi| = 1$, and so $|g| = g\overline{\psi}$ on E. Thus, for any $f \in S_X$,

$$\int_R |fg|\, d\mu = \int_R |f|\, g\overline{\psi} d\mu.$$

Define

$$h(x) = \begin{cases} |f(x)|\,\overline{\psi(x)} & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}.$$

Then $|h| \le |f|$ on R, whence $\|h\|_X \le \|f\|_X \le 1$, and so $h \in S_X$. Hence

$$\int_R |fg|\, d\mu = \int_R hgd\mu \le \left|\int_R hgd\mu\right| \le \sup_{f\in S_X}\left|\int_R fgd\mu\right|.$$

Therefore

$$\|g\|_{X'} = \sup_{f \in S_X} \int_R |fg| \, d\mu \leq \sup_{f \in S_X} \left| \int_R fg d\mu \right|,$$

which finishes the proof. □

Recall that, for each X Banach space, the *dual space* of X is the set of all bounded linear functionals that will be denoted by X^*.

Definition 3.20. A closed linear subspace B of the dual space X^* of a Banach space X is said to be *norm-fundamental* if

$$\|f\|_X = \sup \left\{ |L(f)| : \ L \in B, \ \|L\|_{X^*} \leq 1 \right\},$$

for every $f \in X$. In other words, a norm-fundamental subspace contains enough elements of X^* to reproduce the norm of every element of X.

Theorem 3.21 (see [2, Chapter 1, Theorem 2.9])**.** *Let X be a Banach function space and let X' be its associate space. Then X' is canonically isometrically isomorphic to a norm-fundamental subspace of X^*.*

Proof. For each $g \in X'$ consider the bounded linear functional $L_g \in X^*$ defined by

$$L_g(f) = \int_R fg d\mu, \quad \forall f \in X.$$

Indeed, L_g is linear, by linearity of the integral and it is bounded, by the Hölder inequality. Furthermore, if $g \in X'$ is such that $L_g = 0$, then $L_g(f) = 0$ for every $f \in X$. But then, by the previous lemma, $\|g\|_{X'} = 0$, hence $g = 0$ a.e. Therefore the kernel of the mapping $g \mapsto L_g$ is the null space of X^*. Thus X' is isomorphic to some linear subspace of X^*, which is the image of that mapping. Once again, by the previous lemma, the map is isometric, since

$$\|L_g\|_{X^*} = \sup \left\{ |L_g(f)| : \ \|f\|_X \leq 1 \right\} = \sup \left\{ \left| \int_R fg d\mu \right| : \ \|f\|_X \leq 1 \right\} = \|g\|_{X'}.$$

Since X' is complete, the image is also complete. Since X^* is complete, the image is closed. Finally,

$$\begin{aligned} \|f\|_X &= \|f\|_{X''} \\ &= \sup \left\{ \left| \int_R fg d\mu \right| : \ \|g\|_{X'} \leq 1 \right\} \\ &= \sup \left\{ |L_g(f)| : \ \|g\|_{X'} \leq 1 \right\}, \end{aligned}$$

which leads us to the conclusion that the image of X' is norm-fundamental in X^*. □

Theorem 3.22 (see [2, Chapter 1, Proposition 2.10])**.** *If X and Y are Banach function spaces and $X \hookrightarrow Y$, then $Y' \hookrightarrow X'$.*

Proof. Since X is embedded into Y there is $c > 0$ such that, for every $f \in X$,

$$\|f\|_Y \leq c \|f\|_X .$$

It is easy to check that, for any $g \in Y'$,

$$\left\{ \int_R |fg| \, d\mu : \|f\|_X \leq 1 \right\} \subset \left\{ \int_R |fg| \, d\mu : \frac{1}{c} \|f\|_Y \leq 1 \right\}.$$

But then

$$\begin{aligned} \|g\|_{X'} &= \sup \left\{ \int_R |fg| \, d\mu : \|f\|_X \leq 1 \right\} \\ &\leq \sup \left\{ \int_R |fg| \, d\mu : \frac{1}{c} \|f\|_Y \leq 1 \right\} \\ &= \sup \left\{ c \int_R \left| \left(\frac{1}{c} f \right) g \right| d\mu : \left\| \frac{1}{c} f \right\|_Y \leq 1 \right\} \\ &= c \underbrace{\sup \left\{ \int_R |fg| \, d\mu : \|f\|_Y \leq 1 \right\}}_{\|g\|_{Y'}} . \end{aligned}$$

Hence the assertion follows. □

3.3 Absolute continuity of the norm

Throughout this section we shall use $(E_n)_{n\in\mathbb{N}}$ to denote an arbitrary sequence of measurable subsets of R. We shall write $E_n \to \emptyset$ to denote that $\chi_{E_n} \to 0$ a.e. In addition if $(E_n)_{n\in\mathbb{N}}$ is a decreasing chain, we then write $E_n \downarrow \emptyset$ a.e. instead. It is not hard to prove that $E_n \to \emptyset$ if and only if $\limsup_{n\to\infty} E_n$ is a zero measure set.

Definition 3.23. A function f in a Banach function space X is said to have *absolutely continuous norm* in X if $\|f\chi_{E_n}\|_X \to 0$ for every sequence $(E_n)_{n\in\mathbb{N}}$ satisfying $E_n \to \emptyset$ a.e. The set of all functions in X of absolutely continuous norm is denoted by X_a. If $X_a = X$, then the space X itself is said to have *absolutely continuous norm.*

The next result shows that in the preceding definition we could have restricted our attention to decreasing sequences $(E_n)_{n\in\mathbb{N}}$.

Theorem 3.24 (see [2, Chapter 1, Proposition 3.2])**.** *A function f in a Banach function space X has absolutely continuous norm if and only if $\|f\chi_{E_n}\|_X \downarrow 0$ for any $E_n \downarrow \emptyset$ a.e.*

Remark. If $1 \leq p < \infty$ then the Lebesgue space $L^p(R,\mu)$ has absolutely continuous norm, provided by the dominated convergence theorem and using Theorem 3.24. But that may not be the case when $L^\infty(R,\mu)$. It does not hold for non-atomic μ, i.e., the measure μ is non-atomic if, for each E measurable set with $\mu(E) > 0$, there is a measurable set $F \subset E$ such that $0 < \mu(F) < \mu(E)$. For instance, if we take $R = \mathbb{R}$, m to be the Lebesgue measure in $\mathbb{R}$, $f = 1$ and $E_n = [0, 1/n]$, clearly $\chi_{E_n} \to 0$ m-a.e. but $\|f\chi_{E_n}\|_\infty = \operatorname{ess\,sup}_{E_n} |f| = 1$, for all the $n \in \mathbb{N}$.

Lemma 3.25 (see [2, Chapter 1, Lemma 3.4]). *If f has absolutely continuous norm then to each $\varepsilon > 0$ there corresponds $\delta > 0$ such that $\mu(E) < \delta$ implies $\|f\chi_E\|_X < \varepsilon$.*

Proof. Suppose the contrary. Then, for some $\varepsilon > 0$, there is a sequence (E_n) of measurable sets, satisfying

$$\mu(E_n) < 2^{-n} \quad \text{and} \quad \|f\chi_{E_n}\| \geq \varepsilon,$$

for all $n \in \mathbb{N}$. Since $\sum_{n\in\mathbb{N}} \mu(E_n) \leq 1 < \infty$, by the Borel-Cantelli lemma, $E_n \to \emptyset$ a.e., and so the norm-estimate above contradicts the fact that f has absolutely continuous norm. □

Theorem 3.26 (see [2, Chapter 1, Proposition 3.5]). *A function f in a Banach function space X has absolutely continuous norm if and only if $\|f_n\|_X \downarrow 0$ for every sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions satisfying $|f| \geq f_n \downarrow 0$ a.e.*

Proof. The sufficiency is immediate by Theorem 3.24 and by taking $f_n = f\chi_{E_n}$. For the necessity, suppose f has absolutely continuous norm and $|f| \geq f_n \downarrow 0$ a.e. Take $\varepsilon > 0$ and let $(R_N)_{N\in\mathbb{N}}$ be an increasing sequence of nonzero finite measure sets whose union is R. Since the complements $Q_N = R \setminus R_N \downarrow \emptyset$ a.e., Theorem 3.24 guarantees the existence of $N \in \mathbb{N}$ such that $\|f\chi_{Q_N}\| < \varepsilon/2$. If

$$\alpha = \frac{\varepsilon}{4\,\|\chi_{R_N}\|_X},$$

for each $n \in \mathbb{N}$, let

$$E_n = \{x \in R_N : \ f_n(x) > \alpha\}.$$

Since $f_n \downarrow 0$ a.e., and R_N has finite measure, we see that $\mu(E_n) \downarrow 0$, by Corollary 2.10. Hence, Lemma 3.25 gives $\|f\chi_{E_n}\|_X < \varepsilon/4$ for sufficiently large n. For these values of n we have

$$\begin{aligned}
\|f_n\|_X &\leq \|f_n\chi_{Q_N}\|_X + \|f_n\chi_{R_N}\|_X \\
&\leq \|f_n\chi_{Q_N}\|_X + \|f_n\chi_{E_n}\|_X + \left\|f_n\chi_{R_N\setminus E_n}\right\|_X \\
&\leq \|f\chi_{Q_N}\|_X + \|f\chi_{E_n}\|_X + \alpha\,\|\chi_{R_N}\|_X \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

Hence $\|f_n\|_X \downarrow 0$ as required. □

The monotone convergence is at our disposal in every Banach function space X in form of the Fatou property. The next result exihibits X_a as the largest subspace of X for which a suitable dominated convergence theorem holds.

Theorem 3.27 (see [2, Chapter 1, Proposition 3.6]). *A function f in a Banach function space X has absolutely continuous norm if and only if the following property holds: whenever f_n (with $n = 1, 2, ...$) and g are measurable functions satisfying $|f_n| \leq |f|$ for all the n and $f_n \to g$ a.e., then $\|f_n - g\|_X \to 0$.*

Definition 3.28. A closed linear subspace Y of a Banach function space X is called an *order ideal* of X if it has the property:

$$f \in Y \text{ and } |g| \leq |f| \text{ a.e. } \implies g \in Y.$$

Clearly the trivial subspaces of X are itself order ideals of X.

Theorem 3.29 (see [2, Chapter 1, Theorem 3.8]). *The subspace X_a of functions of absolutely continuous norm is an order ideal a Banach function space X. Furthermore, if $0 \leq f_n \uparrow f$ a.e. and $f \in X_a$ then $\|f_n - f\|_X \downarrow 0$.*

Proof. We note that if f and g are measurable functions such that $|g| \leq |f|$ a.e., then $|g\chi_{E_n}| \leq |f\chi_{E_n}|$ a.e. following the inequality $\|g\chi_{E_n}\|_X \leq \|f\chi_{E_n}\|_X$, for every $E_n \to \emptyset$. Hence $g \in X_a$, whenever $f \in X_a$, which is left us to show that the subspace X_a is closed. Pick a sequence (f_n) of elements of X_a which converges to some function f in X. Then, given $\varepsilon > 0$, we shall have $\|f_N - f\|_X < \varepsilon/2$ for suitably large N. Let (E_m) be a sequence satisfying $E_m \downarrow \emptyset$. Since f_N has absolutely continuous norm, there exists M such that

$$\forall m \geq M \quad \|f_N\chi_{E_m}\|_X < \frac{\varepsilon}{2}.$$

Hence, for every $m \geq M$,

$$\begin{aligned}\|f\chi_{E_m}\|_X &\leq \|(f - f_N)\chi_{E_m}\|_X + \|f_N\chi_{E_m}\|_X \\ &\leq \|f - f_N\|_X + \|f_N\chi_{E_m}\|_X \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

and the conclusion follows. □

Definition 3.30. Let X be a Banach function space. The closure in X of the set of simple functions is denoted by X_b.

Theorem 3.31 (see [2, Chapter 1, Proposition 3.10]). *The subspace X_b is the closure in X of the set of bounded functions supported in sets of finite measure.*

Proof. We only need to show that every bounded function supported on a set with finite measure belongs to X_b. Suppose therefore that f is bounded and $E = \{x \in R : f(x) \neq 0\}$ has finite measure. Since f is bounded, there is (f_n) a sequence of simple functions that converges uniformly to f. But then,

$$\|f - f_n\chi_E\|_X = \|(f - f_n)\chi_E\|_X \leq \|f - f_n\|_\infty \underbrace{\|\chi_E\|_X}_{<\infty} \to 0,$$

as $n \to \infty$. □

Theorem 3.32 (see [2, Chapter 1, Theorem 3.11]). *The subspace X_b is an order ideal of X and $X_a \subset X_b \subset X$.*

Proof. Suppose that $f \in X_b$ and $|g| \leq |f|$ a.e. By definition, there is (f_n) a sequence of simple functions converging to f in X. Then each of the functions

$$g_n(x) = \operatorname{sgn}(g(x)) \min\{|f_n(x)|, |g(x)|\}, \quad (n = 1, 2, ...)$$

is bounded and has support in a set of finite measure. Furthermore

$$\begin{aligned} |g - g_n| &= |\operatorname{sgn}(g)|\, ||g| - \min\{|f_n|, |g|\}| \\ &= ||g| - \min\{|f_n|, |g|\}| \\ &= \max\{|g| - |f_n|, 0\} \\ &\leq \max\{|f| - |f_n|, 0\} \\ &\leq ||f| - |f_n|| \\ &\leq |f - f_n| \end{aligned}$$

so

$$\|g - g_n\|_X \leq \|f - f_n\|_X \to 0.$$

Hence $g \in X_b$ is provided by the previous theorem.

Suppose now that f has absolute continuous norm. If (R_n) is an increasing sequence of sets of finite measure whose union is R, then by the previous theorem once more, each of the functions

$$f_n(x) = \operatorname{sgn}(f(x)) \min\{|f(x)|, n\chi_{R_n}(x)\}, \quad (n = 1, 2, ...)$$

belongs to X_b. Since, for every n, $|f_n| \leq |f|$ and $f_n \to f$ a.e., it follows from Theorem 3.27 that $f_n \to f$ in X. But X_b is closed, so we have $f \in X_b$. Consequently $X_a \subset X_b$. □

The subspace X_b is always relatively large in the sense that it is a norm-fundamental subspace of $(X')^*$ (next theorem). By contrast, the subspace X_a may contain only the zero-function and so the inclusion $X_a \subset X_b$ is proper. We shall be interested in the opposite extreme when X_a and X_b coincide.

Theorem 3.33 (see [2, Chapter 1, Theorem 3.12]). *The subspace X_b of X is isometrically isomorphic to a norm-fundamental subspace of $(X')^*$.*

Proof. Since $X = X''$ and using Theorem 3.21, we may identify X, and hence X_b to a closed subspace $(X')^*$. We proceed to prove that X_b is norm-fundamental, by choosing an arbitrary $g \in X_b$ and by fixing an $\varepsilon > 0$. By the definition of the norm in X', there is f in the unit ball of X such that

$$\|g\|_{X'} - \varepsilon < \int_R |fg| d\mu. \tag{3.1}$$

Let (R_N) be any increasing sequence of sets of finite measure whose union is R, and let

$$f_N = \min\{|f|, N\}\,\chi_{R_N}, \quad (N = 1, 2, ...).$$

For all the N, it is not hard to see that f_N is a bounded function, supported on a set of finite measure and that $|f_N| \leq |f|$ a.e., hence $\|f_N\|_X \leq \|f\|_X \leq 1$. Consequently $\{f_N : N = 1, 2, ...\} \subset B = \{h \in X_b : \|h\|_X \leq 1\}$. Since $0 \leq f_N \uparrow |f|$, the monotone convergence theorem and (3.1) provide us the inequalities

$$\|g\|_{X'} \leq \sup_N \int_R |f_N g| d\mu + \varepsilon \leq \sup_{h\in B} \int_R |hg| d\mu + \varepsilon,$$

taking $\varepsilon \to 0$, and observing that the reverse inequality is an immediate consequence of Hölder's inequality, we thus obtain

$$\|g\|_{X'} = \sup_{h\in B} \int_R |hg| d\mu.$$

It is not hard to prove that the equality holds even if we pull the absolute value outside the integral, in virtue of Lemma 3.19 and using the fact that X_b is an order ideal. ☐

3.4 Duality and reflexivity

Let Y be a closed subspace of a Banach function space X which contains S, the set of simple functions. Each g in X' induces L_g a bounded linear functional on X, $f \mapsto \int fg d\mu$, and hence on Y. The map $g \mapsto L_g$ is isometric (by Theorem 3.33, using the fact that $X_b \subset Y$). Hence X' may be identified with some closed subspace of Y^*. The following theorem gives us the necessary and sufficent conditions on Y to satisfy $Y^* = X'$.

Theorem 3.34 (see [2, Chapter 1, Theorem 4.1]). *Let Y be an order ideal of a Banach function space X and suppose that Y contains S. Then $Y^* = X'$ if and only if $Y \subset X_a$. In this case,*

$$Y = X_a = X_b.$$

Proof. First, suppose that $Y \subset X_a$. As discussed above, it remains for us to show that $X' \supset Y^*$. Let $L \in Y^*$. We shall exhibit a function $g \in X'$, such that, for every $f \in Y$, we have

$$L(f) = \int_R fg d\mu.$$

Since R is σ-finite, there is pairwise disjoint subsets $(S_N)_{N\in\mathbb{N}}$, each with finite measure, and whose union is all of R. For each $N = 1, 2, ...$, let $\mathscr{A}_N$ denote the σ-algebra of all μ-measurable subsets of S_N, and define a set-function λ_N on $\mathscr{A}_N$ by,

$$\lambda_N(A) = L(\chi_A), \quad (A \in \mathscr{A}_N).$$

Note that $\lambda_N(A)$ is well-defined for all $A \in \mathscr{A}_N$, because $\chi_A \in X_b = Y$. We claim that λ_N is countably additive on $\mathscr{A}_N$. To see this, let $(A_n)_{n\in\mathbb{N}}$ be a sequence of disjoint sets from $\mathscr{A}_N$ and let

$$B_n = \bigcup_{j=1}^{n} A_j, \quad (n = 1, 2, ...), \quad A = \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n.$$

Since $A \in \mathscr{A}_N$, we have $\chi_A \in X_b = X_a$. Furthermore, it is clear that $\chi_A \geq \chi_{A\setminus B_n} \downarrow 0$, so, by the dominated convergence theorem, $\|\chi_A - \chi_{B_n}\|_X \downarrow 0$, as $n \to \infty$. The continuity and the linearity of L on Y therefore give

$$\lambda_N(A) = L(\chi_A) = \lim_{n\to\infty} L(\chi_{B_n}) = \lim_{n\to\infty} \sum_{j=1}^{n} L(\chi_{A_j}) = \sum_{n\in\mathbb{N}} \lambda_N(A_n),$$

and this establishes the σ-additivity of λ_N.

It is easy to obtain a bound for the measure by estimating

$$|\lambda_N(A)| = |L(\chi_A)| \leq \|L\|_{Y^*} \|\chi_A\|_X \leq \|L\|_{Y^*} \|\chi_{S_N}\|_X < \infty.$$

The estimate

$$|\lambda_N(A)| \leq \|L\|_{Y^*} \|\chi_{S_N}\|_X ,$$

shows also that λ_N is absolutely continuous with respect to μ, on $\mathscr{A}_N$. Hence, by the Radon-Nikodym theorem (see [1, Theorem 9.36]), there is a μ-a.e. unique function g_N defined on S_N such that

$$\lambda_N(A) = \int_A g_N d\mu.$$

Since S_N are disjoint, we may define a function g on all of R by setting $g = g_N$ on each S_N. Clearly

$$L(\chi_A) = \int_R \chi_A g d\mu,$$

for any μ-measurable set $A \subset R$. Before passing to the general case we first show that $g \in X'$. Let h be nonnegative simple function with support in some set $R_n = \bigcup_{m=1}^{n} S_m$. If we suppose for the moment that g is real-valued, then clearly $h \cdot \operatorname{sgn}(g)$ is also a simple function with support in R_n. In particular, this function is a finite linear combination of characteristic functions of μ-measurable sets. Hence,

$$\int_R |hg| d\mu = \int_R h \cdot \operatorname{sgn}(g) g d\mu = L(h \cdot \operatorname{sgn}(g)).$$

But L is bounded on Y, so

$$\int_R |hg| d\mu \leq \|L\|_{Y^*} \|h \cdot \operatorname{sgn}(g)\|_Y \leq \|L\|_{Y^*} \|h\|_X .$$

If now f is an arbitrary function in X, then we may construct a sequence $(h_n)_{n\in\mathbb{N}}$ of simple functions, each having support in R_n, such that $0 \leq h_n \uparrow |f|$ μ-a.e.

Applying the above inequality to each h_n in turn, we see from monotone convergence theorem that the left-hand sides converge to $\int |fg| d\mu$, and from the Fatou property that the right-hand sides converge to $\|L\|_{Y^*} \|f\|_X$. Hence,

$$\int_R |fg|\, d\mu \le \|L\|_{Y^*} \|f\|_X \,, \quad (f \in X).$$

This, together with the Landau's resonance theorem shows that $g \in X'$. If g is complex-valued, then the same argument is applied separately to the real and imaginary parts of g. As in the remarks preceding the statement of the theorem, we may thus regard the linear functional L_g induced by g as a member of Y^*. In this way, we can see that L and L_g coincide on all simple functions with support on some R_n. It is now fairly easy to show that L and L_g coincide on all of Y. If f is any real-valued function in Y, then, as above, there exists a sequence of simple functions h_n, $(n = 1, 2, ...)$, with the support h_n contained in R_n, such that $0 \le h_n \uparrow |f|$, μ-a.e. The functions $f_n = h_n \cdot \operatorname{sgn}(f)$ are also simple and have supports in the sets R_n, and $f_n \to f$ μ-a.e. in Y. Since $L(f_n) = L_g(f_n)$, and both functionals are continuous on Y, we conclude that $L(f) = L_g(f)$. Finally, if f is complex-valued, then both its real and imaginary parts belong to Y (because Y is an order ideal). The obvious argument therefore shows that L and L_g coincide on all of Y. As remarked previously, this establishes that $X' = Y^*$.

Conversely, assuming $X' = Y^*$, we have to show that $Y \subset X_a$. Suppose the contrary. Then there is $f \in Y$ that does not have absolutely continuous norm. Since Y is an order ideal we may suppose that $f \ge 0$. In that case, there is a sequence $E_n \downarrow \emptyset$ and $\varepsilon > 0$ such that

$$\|f\chi_{E_n}\|_X \ge \varepsilon, \quad (n = 1, 2, ...).$$

It is an immediate consequence of the dominated convergence theorem that the sets

$$G_n = \left\{ g \in X' : \int_R |f\chi_{E_n} g|\, d\mu < \frac{\varepsilon}{2} \right\}, \quad (n = 1, 2, ...)$$

cover X'. In fact, if $g \in X'$, $f\chi_{E_n} g \to 0$, μ-a.e. and

$$|f\chi_{E_n} g| \le |fg| \in L^1(R, \mu).$$

This implies that $\int_R f\chi_{E_n} g d\mu \to 0$, as $n \to \infty$. Here, we used the fact that $L^1(R, \mu)$ has absolute continuous norm. Back to our problem, since $X' = Y^*$ and since Y is an order ideal, $f\chi_{E_n} \in Y$. We therefore may regard $(G_n)_{n\in\mathbb{N}}$ as a weak* open cover of Y^*. By the Banach-Alaoglu theorem (see [5, Theorem 3.16]), there is $(G_{n(i)})_{i=1}^k$ a finite subcover of the unit ball of Y^*. Every $g \in X'$, with $\|g\|_{X'} \le 1$ therefore lies in some $G_{n(i)}$, so since $f \ge 0$, and $E_n \downarrow \emptyset$ we have

$$\int_R |f\chi_{E_n} g|\, d\mu \le \int_R \left| f\chi_{E_{n(i)}} g \right| d\mu < \frac{\varepsilon}{2},$$

for all the $n \geq N = \max\{n(1), ..., n(k)\}$. Whence, for $n \geq N$,

$$\|f\chi_{E_n}\|_X = \sup_{\|g\|_{X'} \leq 1} \int_R |f\chi_{E_n} g| \, d\mu \leq \frac{\varepsilon}{2},$$

thus a contradiction. □

Corollary 3.35 (see [2, Chapter 1, Corollary 4.2]). *If X is a Banach function space and if $X_a \supset S$, then $(X_a)^* = X'$.*

Proof. By Theorem 3.29, X_a is an order ideal. Then, by the previous theorem, putting $Y = X_a$, we arrive at the conclusion. □

Corollary 3.36 (see [2, Chapter 1, Corollary 4.3]). *Let X be a Banach function space. Then $X^* = X'$ if and only if X has absolutely continuous norm.*

Corollary 3.37 (see [2, Chapter 1, Corollary 4.4]). *A Banach function space is reflexive if and only if both X and X' have absolutely continuous norm.*

Proof. First, suppose that X and X' have absolutely continuous norm. The previous corollary and Theorem 3.18 yield

$$X^{**} = (X^*)^* = (X')^* = X'' = X,$$

which says that X has to be reflexive.

Conversely, suppose that X is reflexive. Recall that X' is a norm-fundamental subspace of X^*, by Theorem 3.21. If X' is a proper subspace of X^*, then the Hahn-Banach theorem provides a nonzero functional $F \in X^{**}$ that anihilates X'. The reflexivity of X allows us to represent F by evaluation at some $f \in X$, in which case

$$\int_R fg d\mu = F(g) = 0,$$

for all the $g \in X'$. Since X' is norm-fundamental in X^*, this implies that $f = 0$ a.e., and hence F is identically zero. From this contradiction, we conclude that $X' = X^*$, and hence X has absolutely continuous norm. From this fact, yields the equality

$$(X')^* = (X^*)^* = X^{**} = X = X'',$$

and so, by the previous corollary, X' has absolutely continous norm. □

Our goal now is to define the adjoint operator acting on certain Banach spaces. To define it consistently, we will consider the adjoint operator acting on Hilbert spaces H_1 and H_2. Let $T \colon H_1 \to H_2$ be a bounded linear operator. Then, there corresponds a unique bounded linear operator $T^* \colon H_2 \to H_1$, satisfying

$$\langle Tu, v \rangle = \langle u, T^* v \rangle,$$

for any $u \in H_1$ and for any $v \in H_2$, where $\langle \cdot, \cdot \rangle$ represents the inner product on its respective space. The operator T^* is said to be the *adjoint operator* of T.

Definition 3.38. Let X, Y be Banach spaces and $A\colon X \to Y$ a bounded linear space. Then there exists a linear operator $A'\colon Y^* \to X^*$ called *dual operator*, that yields

$$(Ax, \psi) = (x, A'\psi),$$

for all the $x \in X$ and $\psi \in Y^*$, where $(\cdot, \cdot)$ is the usual composition of functions.

It is easy to prove that $\|A\| = \|A'\|$ is verified and the uniqueness of the operator A' is also straightforward. Hence A' is bounded. Also, it is obvious that $(\lambda A)' = \lambda A'$, for any scalar λ, in constrast with $(\lambda T)^* = \overline{\lambda} T^*$, for T bounded linear operator of a Hilbert spaces.

Now, consider that X and Y are Banach function spaces with absolutely continuous norm, that is, $X' = X^*$ and $Y' = Y^*$. Further, suppose, for simplicity, that the underlying measure space of both X and Y is the same (R, μ). Consider $J_X\colon X' \to X^*$ and $J_Y\colon Y' \to Y^*$ to be the isometric isomorphism, both with the following law:

$$g \mapsto G(f) = \int_R f\overline{g}d\mu.$$

Henceforth, set $\langle f, g\rangle := \int_R f\overline{g}d\mu$. If X and Y were Hilbert spaces, say L^2, then, for each $f \in X$ and $G \in Y^*$, since $A'G \in X^*$ and $G \in Y^*$, by the Riesz representation theorem, we would have

$$(Af, G) = (f, A'G) = \left\langle f, J_X^{-1}A'G\right\rangle, \quad \text{and} \quad (Af, G) = \langle Af, g\rangle.$$

Therefore,

$$\langle Af, g\rangle = \left\langle f, J_X^{-1}A'J_Y g\right\rangle,$$

from which, we recover $A^* = J_X^{-1}A'J_Y$, by uniqueness of the adjoint operator. Returning to our case, the Riesz representation theorem is replaced by the hypothesis that the Banach function spaces are absolutely continuous, so that J_X and J_Y are invertible.

Definition 3.39. Let X, Y be Banach function spaces with absolutely continuous norm and let $A\colon X \to Y$ be a bounded linear operator. Then, the *adjoint operator* $A^*\colon Y' \to X'$ is the unique linear operator satisfying

$$\langle Af, g\rangle = \langle f, A^* g\rangle,$$

for every $f \in X'$ and $g \in Y'$.

As $A^* = J_X^{-1}A'J_Y$, we have

$$\|A^*\| = \left\|A'\right\| = \|A\| < \infty.$$

At this point, it is routine to show that:

1. $(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*$, for $\alpha, \beta \in \mathbb{C}$ and $A, B \colon X \to Y$ bounded linear operators.
2. $(AB)^* = B^* A^*$, for $A \colon X \to Y$ and $B \colon Y \to Z$ bounded linear operators.
3. $I_X^* = I_{X'}$, if $I_X \colon X \to X$ represents the identity map.

3.5 Separability

A vector space X endowed with a topology is said to be a *topological vector space* (sometimes referred as TVS) if every singleton of X is closed and if vector space operations are continuous under the endowed topology. As a consequence of that definition, the space X is Hausdorff space. Also, the translation and dilatation operators are both homeomorphism from X onto itself. A set $C \subset X$ is said to be *convex* if

$$\forall t \in [0,1] \quad tC + (1-t)C \subset C.$$

A *locally convex* topological vector space is a topological vector space such that the origin admits a *basis neighbourhood* consisting of convex sets, that is, every open neighbourhood of the origin contains a convex open neighbourhood of the origin. As a direct consequence of the "well-behavior" of translations and dilatations, every element of X admits a basis neighbourhood consisting of convex sets. (For more details check [23], pp. 6-9.)

A family $\mathscr{F}$ of seminorms of X is said to be *separating* if, for any non-zero element f of X, there is $\rho \in \mathscr{F}$ such that $\rho(f) \neq 0$.

Let X be a Banach function space and suppose Z is an order ideal of X' containing the simple functions. Then Z contains $(X')_b$ and, by Theorem 3.33, Z is a norm-fundamental subspace of X^*. The collection of seminorms

$$f \mapsto \left| \int_R fg d\mu \right|, \quad (g \in Z)$$

on X is therefore a separating family which endows X with the structure of locally convex topological space. The topology in question is called the *weak topology* on X generated by Z and is denoted by $\sigma(X, Z)$. Note that $f_n \to f$ in the $\sigma(X, Z)$-topology if and only if, for all the $g \in Z$,

$$\int_R f_n g d\mu \to \int_R fg d\mu.$$

We say that $A \subset X$ is $\sigma(X, Z)$-*bounded* if it is bounded in each of the seminorms referred above, i.e., if

$$\sup_{f \in A} \left| \int_R fg d\mu \right| < \infty,$$

for all the $g \in Z$.

Lemma 3.40 (see [2, Chapter 1, Lemma 5.1]). *Let X be a Banach function space and suppose Z is an order ideal of X' containing the simple functions. Then $A \subset X$ is $\sigma(X,Z)$-bounded if and only if it is norm-bounded in X.*

Proof. Hölder's inequality shows that norm-boundedness implies $\sigma(X,Z)$-boundedness. Conversely, suppose that A is $\sigma(X,Z)$-bounded. Then, for each $g \in Z$,

$$\sup\left\{\left|\int_R fgd\mu\right| : f \in A\right\} < \infty.$$

For each $f \in A$, associate the functional $F \in Z^*$ defined by

$$F(g) = \int_R fgd\mu, \quad (g \in Z).$$

As already referred in the remarks preceding the lemma, Z is a norm-fundamental subspace of X^* and thus $\|F\|_{Z^*} = \|f\|_X$. So, by hypothesis

$$\sup_{f\in A} |F(g)| \leq C_g < \infty,$$

for all the $g \in Z$. Hence, by the uniform boundedness principle (see [5, Theorem 2.2]),

$$\sup_{f\in A} \|f\| = \sup_{f\in A} \|F\|_{Z^*} < \infty,$$

which shows that A is norm-bounded. □

Before we proceed to the proof of $\sigma(X,Z)$-completeness of X, we shall quote a classical result from measure theory that can be further generalized to complex-valued measures.

Lemma 3.41 (Hahn-Saks theorem; see [11, Exercise 19.68, p. 339]). *Let (R,Σ,μ) be a measure space and let (ν_n) be a sequence of finite measures on Σ. Assume that for every $n \in \mathbb{N}$ the measure ν_n is absolutely continuous with respect to μ. Let ν be a functional on Σ. Assume further that for every $E \in \Sigma$ such that $\mu(E) < \infty$, one has*

$$\lim_{n\to\infty} \nu_n(E) = \nu(E).$$

Then

1. *The measures (ν_n) are uniformly absolutely continuous with respect to μ, i.e.,*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \quad \forall n \in \mathbb{N},\, \forall E \in \Sigma, \quad \mu(E) < \delta \implies \nu_n(E) < \varepsilon;$$

2. *ν is a measure, which is absolutely continuous with respect to μ.*

Theorem 3.42 (see [2, Chapter 1, Theorem 5.2]). *Let X be a Banach function space and let Z be an order ideal containing the simple functions. Then X is $\sigma(X,Z)$-complete.*

Proof. Let (f_n) be a $\sigma(X,Z)$-Cauchy sequence in X. This means that, for each $g \in Z$, $\left(\int_R f_n g d\mu\right)$ is a Cauchy sequence. Whence, for each $g \in Z$,

$$\sup_{n\in\mathbb{N}} \left| \int_R f_n g d\mu \right| < \infty,$$

from which me can conclude that (f_n) is $\sigma(X,Z)$-bounded. Hence, by the previous lemma, there is $M > 0$ such that

$$\sup_{n\in\mathbb{N}} \|f_n\|_X \leq M.$$

For each $n = 1, 2, ...$, the measure ν_n defined by

$$\nu_n(E) = \int_E f_n d\mu$$

is absolutely continuous with respect to μ (that is, $\nu_n \ll \mu$). Let (R_n) be an ascending chain of sets of finite μ-measure whose union is all of R, and fix $N \in \mathbb{N}$. By hypothesis, $(\nu_n(E))$ is Cauchy and so

$$\nu(E) := \lim_{n\to\infty} \nu_n(E)$$

exists and is finite for every μ-measurable subset E of R_N. By the Hahn-Saks theorem, the sequence (ν_n), restricted to R_N, is uniformly absolutely continuous with respect to μ, and, on R_N, the set function ν is a measure which is absolutely continuous with respect to μ. It follows, by letting $N \to \infty$, that there exists $f_0 \in L^1_{loc}(R,\mu)$ such that

$$\lim_{n\to\infty} \nu_n(E) = \nu(E) = \int_R f_0 \chi_E d\mu,$$

whenever $\mu(E) < \infty$.

If g is a simple function with support of finite measure, we therefore have

$$\lim_{n\to\infty} \int_R f_n g d\mu = \int_R f_0 g d\mu.$$

If g also satisfies $\|g\|_{X'} \leq 1$, then

$$\left| \int_R f_0 g d\mu \right| \leq \limsup_{n\to\infty} \|f_n\|_X \|g\|_{X'} \leq M.$$

But if $g \in X'$ is an arbitrary element of the unit ball, then, there is a sequence of simple functions (g_m) with support in R_m such that $\|g_m\|_{X'} \leq 1$ and $0 \leq g_m \uparrow |g|$. Using the monotone convergence,

$$\int_R |f_0|\, g_m d\mu \uparrow \int_R |f_0 g|\, d\mu,$$

as $m \to \infty$. But once again, the right-hand side limit is controlled by M. So we must have $\|f_0\|_X \leq M$. If now g is a bounded function with support of finite measure, then it can be approximated by a sequence of simple functions that converges uniformly to g and hence holds the equality

$$\lim_{n\to\infty}\int_R f_n g d\mu = \int_R f_0 g d\mu.$$

We need to show that the above equality is valid for any $g \in Z$. Fix such a function and set

$$\omega_n(E) := \int_E f_n g d\mu, \quad (n = 1, 2, ...).$$

The set functions ω_n are finite measures satisfying $\omega_n \ll \mu$. Since $g\chi_E$ belongs to Z for each measurable set E, the hypothesis implies that the sequence $(\omega_n(E))$ converges. Again, by the Hahn-Saks theorem, the sequence (ω_n) is uniformly absolutely continuous with respect to μ, that is, $\omega_n(E) \to 0$ uniformly in n as $\mu(E) \to 0$. The measure $\omega_0(E) := \int_E f_0 g d\mu$ also satisfies $\omega_0 \ll \mu$. Thus, if we define a descending chain of μ-measurable sets (E_m) by

$$\{x \in R: \ |g(x)| > m\} \cup R_m^c, \quad (m = 1, 2, ...).$$

We may notice that

$$\lim_{m\to\infty}\mu(E_m) = 0.$$

Let us prove that $\bigcap_{m\in\mathbb{N}} E_m$ is a null set. Fix $m \in \mathbb{N}$ and suppose, with view to a contradiction that there is $x \in \bigcap_{m\in\mathbb{N}} E_m$ such that it does not belong to $\{x \in R: \ |g(x)| = \infty\}$ which is known to be a null set, provided that $g \in Z \hookrightarrow \mathcal{M}_0$. Then, there is $M > 0$ such that $|g(x)| \leq M_1$. Also, there is M_2 such that $x \in R_{M_2}$. Taking $M = \max\{M_1, M_2\}$ we see that $|g(x)| \leq M$ and $x \in R_M$, equivalently $x \notin R_M^c$, that is, $x \notin E_M$, as desired. Back to our main argument, given $\varepsilon > 0$, we may choose $N \in \mathbb{N}$ such that

$$|\omega_n(E_N)| < \frac{\varepsilon}{3}, \quad (n = 0, 1, 2, ...).$$

If F_N denotes the complement of E_N, that is,

$$F_N = E_N^c = \{x \in R_N: \ |g(x)| \leq N\},$$

then $g\chi_{F_n}$ is bounded and has support of finite measure. Since

$$\lim_{n\to\infty}\int_R f_n \chi_{F_N} d\mu = \int_R f_0 g \chi_{F_N} d\mu,$$

there is an order $n_0 \in \mathbb{N}$ such that, for all the $n \geq n_0$,

$$\left|\int_{F_N} f_n g d\mu - \int_{F_N} f_0 g d\mu\right| < \frac{\varepsilon}{3}.$$

By splitting the range of integration R into F_N and E_N we therefore obtain

$$\left|\int_R f_n g d\mu - \int_R f_0 g d\mu\right| < \frac{\varepsilon}{3} + |\omega_0(E_N)| + |\omega_n(E_N)| < \varepsilon,$$

whenever $n \geq n_0$. Hence $f_n \to f_0$ in the $\sigma(X, Z)$ topology, and so X is $\sigma(X, Z)$-complete. $\square$

Corollary 3.43 (see [2, Chapter 1, Corollary 5.3]). *Every Banach function space X is $\sigma(X, X')$-complete.*

Let $\mathscr{A}$ denote the collection of all subsets of R of finite measure, where any two such sets differ by a set of μ-measure zero are identified. For instance, for Lebesgue measure on $\mathbb{R}$, the sets $\mathbb{N}$ and $\mathbb{Q}$ are two identical elements of $\mathscr{A}$. Notice that two sets, $E, F \in \mathscr{A}$, differ on a zero μ-measure set if and only if $\chi_E = \chi_F$ μ-a.e.

It can be shown that

$$d_\mu(E, F) = \int_R |\chi_E - \chi_F| d\mu, \quad (E, F \in \mathscr{A})$$

defines a complete metric space. This function d_μ is also known as Frechét-Nikodym metric. Note that

$$\int_R |\chi_E - \chi_F| d\mu = \int_R \chi_{E\Delta F} d\mu = \mu(E\Delta F),$$

where $E\Delta F = (E \cup F) \setminus (E \cap F)$. (For more details check [3], section 1.12(iii) Metric Boolean algebra, pp. 53, 54.)

A measure μ is said to be *separable* if the corresponding metric $(\mathscr{A}, d)$ is separable. For instance, μ is separable if and only if the Banach space $L^1(R, \mu)$ is separable. (For the proof, check [20] Proposition 2.3.24, p. 119.) The Lebesgue measure on $\mathbb{R}^n$ makes L^1 to be separable, so the Lebesgue measure is separable.

Theorem 3.44 (see [2, Chapter 1, Theorem 5.5]). *Let X be a Banach function space and suppose that Y is an order ideal of X containing the simple functions. Then Y is separable if and only if Y has absolutely continuous norm and μ is a separable measure.*

Proof. Suppose that Y is separable and that it contains an f_0 that does not have absolute continuous norm. Then, by Theorem 3.29 there are $\varepsilon > 0$ and $E_n \downarrow \emptyset$ such that, for any $n \in \mathbb{N}$,

$$\|f_0 \chi_{E_n}\|_X \geq \varepsilon.$$

Since X' is norm-fundamental of X^*, there are functions g_n, $(n = 1, 2, ...)$, in the unit ball of X' such that

$$\int_R |f_0 g_n \chi_{E_n}| d\mu \geq \varepsilon/2, \quad (n = 1, 2, ...).$$

The functions

$$h_n = \operatorname{sgn}\left(\overline{f_0}\right) |g_n| \chi_{E_n}, \quad (n = 1, 2...)$$

are also elements of the unit ball of X'. So Hölder's inequaliy, for Banach function spaces, gives

$$\left| \int_R f_m h_n d\mu \right| \leq \|f_m\|_X, \quad (m, n = 1, 2, ...).$$

Particularly, for each $m = 1, 2, ...$, the sequence $\left(\int_R f_m h_n d\mu\right)_{n\in\mathbb{N}}$ is bounded. Hence, by the Bolzano-Weierstrass theorem, along with diagonalization argument, we thus obtain a strictly increasing sequence $(\alpha_n)_{n\in\mathbb{N}}$, taking values on $\mathbb{N}$, such that the sequence $\left(\int_R f_m h_{\alpha_n} d\mu\right)_{n\in\mathbb{N}}$ is convergent, for any $m = 1, 2, ...$ Since $(f_m)_{m\in\mathbb{N}}$ is dense in Y it can be proved that

$$\lim_{n\to\infty} \int_R f h_{\alpha_n} d\mu$$

exists, for every $f \in Y$. When Y regarded as a subspace of $X'' = X$, this merely asserts that the sequence $(h_{\alpha_n})_{n\in\mathbb{N}}$ is $\sigma(X', Y)$-Cauchy in X'. The completeness of $\sigma(X', Y)$ is provided by Theorem 3.42. So, there is $h \in X'$ such that

$$\int_R f h_{\alpha_n} d\mu \to \int_R f h d\mu,$$

for every $f \in Y$. Recall that h_{α_n} are supported on a decreasing chain of sets E_{α_n}. Fix $E \subset R$ a finite measure set. If E does not intersect some of E_{α_n} then, by taking $f = \chi_E$ (which belongs to Y by definition), we attain

$$\int_E h d\mu = 0.$$

In fact, $h = 0$ μ-a.e. on $R \setminus E_{\alpha_n}$. Take $n \to \infty$ and we obtain that $h = 0$ μ-a.e., since $E_{\alpha_n} \downarrow \emptyset$. Overall, we have

$$\int_R f h_{\alpha_n} d\mu \to 0,$$

for every $f \in Y$. In particular,

$$0 = \lim_{n\to\infty} \int_R f_0 h_{\alpha_n} d\mu = \lim_{n\to\infty} \int_R \left| f_0 g_n \chi_{E_{\alpha_n}} \right| d\mu \geq \varepsilon/2,$$

a contradiction. So Y has absolutely continuous norm.

Next, we show that μ is separable. Let $(R_N)_{N\in\mathbb{N}}$ be an ascending chain of sets of finite μ-measure, such that their union is entirely R, and let $\mathscr{A}_N$ denote the σ-algebra of all measurable subsets of R_N, $(N = 1, 2, ...)$. By hypothesis, the characteristic function of a set belonging to any of $\mathscr{A}_N$ belongs to Y. Since Y is separable, each of its subsets is also separable (see [22, Proposition 1.6.3]). So, for every $N \in \mathbb{N}$, there is a sequence $(\chi_{E_{N,m}})_{m\in\mathbb{N}}$ which is dense in $\{\chi_E : E \in \mathscr{A}_N\} \subset Y$. We claim that

$$\mathscr{F} = \{E_{N,m} : N, m \in \mathbb{N}\}$$

is a dense set of $(\mathscr{A}, d_\mu)$, the Frechét-Nikodym metric space. This will establish the separability of μ. Let $F \in \mathscr{A}$. Then $\mu(F) < \infty$ and $F \cap R_N \uparrow F$, as $N \to \infty$. So $F \setminus R_N \downarrow \emptyset$, which is, by definition, $\chi_{F\setminus R_N} \to 0$ μ-a.e. Hence, by the dominated convergence theorem, $\mu(F \setminus R_N) \to 0$, as $N \to \infty$. For a fixed $\varepsilon > 0$, pick $N \in \mathbb{N}$ such that

$$\mu(F \setminus R_N) < \frac{\varepsilon}{2}.$$

We already know that there is a positive C_N, satisfying

$$\int_{R_N} |f| d\mu \leq C_N ||f||_X,$$

for any chosen $f \in X$. The definition of $\mathscr{F}$ gives us an $E \in \mathscr{A}_N$ such that

$$\|\chi_E - \chi_{F \cap R_N}\|_X < \frac{\varepsilon}{2C_N}.$$

Finally, altogether,

$$\begin{aligned} d_\mu(E, F) &= \int_{R_N} |\chi_E - \chi_F| \, d\mu + \int_{R \setminus R_N} |\chi_E - \chi_F| \, d\mu \\ &= \int_{R_N} |\chi_E - \chi_{F \cap R_N}| d\mu + \int_{R \setminus R_N} |0 - \chi_F| d\mu \\ &\leq C_N \|\chi_E - \chi_{F \cap R_N}\|_X + \mu(F \setminus R_N) \\ &< \varepsilon. \end{aligned}$$

Hence $\mathscr{F}$ is dense in $(\mathscr{A}, d_\mu)$, and so μ is a separable measure.

Conversely, we suppose that μ is separable and $Y \subset X_a$. Then, by Theorem 3.34, $Y = X_a = X_b$. Let $\mathscr{F}_1 = \{E_1, E_2, ...\}$ be a countable dense subset of $(\mathscr{A}, d_\mu)$. Next, we define

$$\mathscr{F} = \{E_j \cap R_N : \, j, N = 1, 2, ...\},$$

where the elements $R_1, R_2, ...$ satisfy the same conditions as in the previous implication, and also define

$$\mathscr{D} = \left\{ f \in S : \, f = \sum_{k=1}^{K} r_k \chi_{F_k}, \, r_k \in \mathbb{Q} + i\mathbb{Q}, \, F_k \in \mathscr{F}, \, K \in \mathbb{N} \right\}.$$

Obviously $\mathscr{D}$ is countable. We aim to show that Y is separable. Since $Y = X_b$, it is sufficient to prove that $\overline{\mathscr{D}} = X_b$. Now X_b is the closure of S the set of simple functions

$$g = \sum_{n=1}^{m} c_n \chi_{G_n},$$

where c_n are arbitrary scalars and G_n are arbitrary sets of finite measure. Since each c_n may be approximated arbitrarily closely by rationals, it is clear that in order to show that $\mathscr{D}$ is dense in X_b it will suffice to show that the characteristic function of an arbitrary set of finite measure can be approximated in the norm of X to any required degree of accuracy by characteristic functions of sets from $\mathscr{F}$. Suppose then that $G \in \mathscr{A}$ and fix $\varepsilon > 0$. Now χ_G belongs to $X_a = X_b$ and $\chi_{R_N \cap G} \uparrow \chi_G$, so by Theorem 3.29,

$$\|\chi_{R_N \cap G} - \chi_G\|_X < \frac{\varepsilon}{2},$$

for suitable $N \in \mathbb{N}$. Hence, it suffices to approximate $\chi_{G \cap R_N}$ by characteristic functions of sets from $\mathscr{F}$. For each $n \in \mathbb{N}$, choose $F_n \in \mathscr{F}$, such that $F_n \subset R_N$ and

$$\lim_{n \to \infty} d_\mu(G \cap R_N, F_n) = 0.$$

This merely asserts that $\chi_{F_n} \to \chi_{G \cap R_N}$ in $L^1(R, \mu)$, so we may select a subsequence $\chi_{F_{\alpha_n}} \to \chi_{G \cap R_N}$ μ-a.e. as $n \to \infty$. But all those functions are dominated by $\chi_{R_N} \in X_a$. Hence, the dominated convergence theorem shows that

$$\left\| \chi_{F_{\alpha_n}} - \chi_{G \cap R_N} \right\|_X \to 0,$$

as $n \to \infty$. As it was remarked above, this establishes the separability of Y. □

Corollary 3.45 (see [2, Chapter 1, Corollary 5.6]). *A Banach function space X is separable if and only if it has absolutely continuous norm and its underlying measure μ is separable.*

Corollary 3.46 (see [2, Chapter 1, Corollary 5.7]). *Suppose $X_a = X_b$. Then X_a is separable if and only if μ is separable.*

Corollary 3.47 (see [2, Chapter 1, Corollary 5.8]). *If the dual space X^* of a Banach function space is separable, then X is reflexive.*

Proof. Since X^* is separable, then X is so (see [16, Theorem 4.6-8]). Thus X has absolutely continuous norm. Moreover, since X^* is separable and contains X', X' is also separable (see [22, Proposition 1.6.3]), and thus has absolutely continuous norm. Then, by Corollary 3.37, X must be reflexive. □

Chapter 4

Hilbert transform on rearrangement-invariant spaces

In 1928, Marcel Riesz proved that the Hilbert transform, defined by $H\colon f(x) \mapsto \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{x-t}dt$, is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. The aim of this chapter is to present a generalization of this result to the setting of rearrangement-invariant Banach function spaces, obtained by David Boyd in 1966.

4.1 Distribution functions and decreasing rearrangements

As in the previous chapter, (R, μ) denotes a σ-finite measure space.

Definition 4.1. The *distribution function* μ_f of a function f in $\mathcal{M}_0 = \mathcal{M}_0(R, \mu)$ is given by

$$\mu_f(\lambda) := \mu\left\{x \in R\colon \, |f(x)| > \lambda\right\}, \quad (\lambda \geq 0).$$

We may notice that μ_f depends only on the absolute value of f, at each point, and μ_f may assume the value $+\infty$.

Definition 4.2. Two functions $f \in \mathcal{M}_0(R, \mu)$ and $g \in \mathcal{M}_0(S, \nu)$ are said to be *equimeasurable* if if they have the same distribution function, that is, if for all the $\lambda \geq 0$, one has $\mu_f(\lambda) = \nu_g(\lambda)$.

Theorem 4.3 (see [2, Chapter 2, Proposition 1.3])**.** *Suppose* f, g, f_n $(n = 1, 2, ...)$, *belong to* $\mathcal{M}_0 = \mathcal{M}_0(R, \mu)$ *and let* $\alpha \neq 0$. *The distribution function* μ_f *is non-negative, decreasing, and right-continuous on* $[0, \infty[$. *Furthermore,*

(I) $|g| \leq |f|$ *a.e.* $\implies \mu_g \leq \mu_f$*;*

(II) $\mu_{\alpha f}(\lambda) = \mu_f(\lambda/|\alpha|), \quad (\lambda \geq 0)$*;*

(III) $\mu_{f+g}(\lambda_1+\lambda_2) \le \mu_f(\lambda_1)+\mu_g(\lambda_2), \quad (\lambda_1,\lambda_2 \ge 0)$*;*

(IV) $\mu_{fg}(\lambda_1\lambda_2) \le \mu_f(\lambda_1)+\mu_g(\lambda_2), \quad (\lambda_1,\lambda_2 \ge 0)$*;*

(V) $|f| \le \liminf_{n\to\infty} |f_n|$ *a.e.* $\Longrightarrow$ $\mu_f \le \liminf_{n\to\infty} \mu_{f_n}$,
particularly, if $|f_n| \uparrow |f|$ *a.e., then* $\mu_{f_n} \uparrow \mu_f$*.*

Proof. It is clear that μ_f is nonnegative decreasing, in the sense that, if $0 \le \lambda_1 < \lambda_2$, then $\mu_f(\lambda_1) \ge \mu_f(\lambda_2)$. To establish the right-continuity, let

$$E(\lambda) = \{x \in R : \ |f(x)| > \lambda\}, \quad (\lambda \ge 0),$$

and fix $\lambda_0 \ge 0$. The sets $E(\lambda)$ increase, as λ decreases, and

$$E(\lambda_0) = \bigcup_{\lambda>\lambda_0} E(\lambda) = \bigcup_{n\in\mathbb{N}} E\left(\lambda_0 + \frac{1}{n}\right).$$

It follows that

$$\mu_f\left(\lambda_0+\frac{1}{n}\right) = \mu\left[E\left(\lambda_0+\frac{1}{n}\right)\right] \uparrow \mu(E(\lambda_0)) = \mu_f(\lambda_0),$$

which estabilishes the right-continuity.

The first properties (I) and (II) can be obtained by straightforth computations. For (III), take $\lambda_1, \lambda_2 \ge 0$ and $x \in R$ such that $|f(x)+g(x)| > \lambda_1+\lambda_2$. Then, at least, one of the following inequalities holds true: $|f(x)| > \lambda_1$ or $|g(x)| > \lambda_2$. Suppose the contrary. Then we have

$$\lambda_1+\lambda_2 < |f(x)+g(x)| \le |f(x)|+|g(x)| \le \lambda_1+\lambda_2,$$

an absurd. Putting in terms of sets, one gets

$$\{x\in R: \ |f(x)+g(x)| > \lambda_1+\lambda_2\} \subset \{x \in R: \ |f(x)| > \lambda_1\} \cup \{x\in R: \ |g(x)|>\lambda_2\},$$

and the desired inequality follows. The fourth property a is consequence of the following inclusion

$$\{x\in R: \ |f(x)||g(x)| > \lambda_1\lambda_2\} \subset \{x \in R: \ |f(x)| > \lambda_1\} \cup \{x\in R: \ |g(x)|>\lambda_2\}.$$

To establish the last implication, fix $\lambda \ge 0$ and let

$$E = \{x\in R: \ |f(x)|>\lambda\}, \quad E_n = \{x \in R: \ |f_n(x)| > \lambda\}, \quad (n = 1, 2, ...).$$

Clearly, $E \subset \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} E_m$. Hence,

$$\mu\left(\bigcap_{m\ge n} E_m\right) \le \inf_{m\ge n} \mu(E_m) \le \sup_{n\in\mathbb{N}} \inf_{m\ge n} \mu(E_m) = \liminf_{n\to\infty} \mu(E_n),$$

for each natural n. But $\left(\bigcap_{m\geq n} E_m\right)_{n\in\mathbb{N}}$ gives us an ascending chain, from which one concludes that

$$\mu(E) \leq \mu\left(\bigcup_{n\in\mathbb{N}}\bigcap_{m\geq n} E_m\right) = \lim_{n\to\infty}\mu\left(\bigcap_{m\geq n} E_m\right) \leq \liminf_{n\to\infty}\mu(E_n).$$

□

Remark. It is a worthwhile to formally compute the distribution function of a nonnegative simple function f. Suppose that

$$f = \sum_{j=1}^{n} a_j\chi_{E_j}, \tag{4.1}$$

for E_j pairwise disjoint subsets of R with finite measure and $a_1 > a_2 > ... > a_n > 0$. If $\lambda \geq a_1$, then clearly $\mu_f(\lambda) = 0$. If $a_1 > \lambda \geq a_2$, then the function f takes values on $]0,\infty[$ greater than λ exactly on the set E_1. Hence $\mu_f(\lambda) = \mu(E_1)$. Similarly, if $a_2 > \lambda \geq a_3$, then f exceeds λ exactly on $E_1 \cup E_2$. Hence $\mu_f(\lambda) = \mu(E_1 \cup E_2)$. In general, we have

$$\mu_f(\lambda) = \sum_{j=1}^{n} m_j\chi_{[a_{j+1},a_j[}(\lambda),$$

where $m_j = \sum_{i=1}^{j}\mu(E_i)$, and put $a_{n+1} = 0$.

Definition 4.4. Suppose f belongs to $\mathcal{M}_0(R,\mu)$. The *decreasing rearrangement* of f is the function f^* defined on $[0,\infty[$ by

$$f^*(t) := \inf\{\lambda \geq 0 : \ \mu_f(\lambda) \leq t\}, \quad (t \geq 0).$$

We use the convention that $\inf\emptyset := \infty$. Thus, if $\mu_f(\lambda) > t$ for every $\lambda \geq 0$, then $f^*(t) = \infty$. Also, if (R,μ) is a finite measure space, then the distribution function μ_f is bounded by $\mu(R)$ and so $f^*(t) = 0$ for all the $t \geq \mu(R)$. In this case, we may regard f^* as a function defined on the interval $[0,\mu(R)[$. Notice also that if μ_f happens to be strictly decreasing, then

$$f^*(\mu_f(\lambda)) = \inf\left\{\lambda' \geq 0 : \ \mu_f(\lambda') \leq \mu_f(\lambda)\right\} = \lambda,$$

so f^* is the left inverse of the distribution function μ_f.

Generally, the two following properties hold:

$$\begin{aligned} f^*(\mu_f(\lambda)) &\leq \lambda, \quad (\lambda \geq 0); \\ \mu_f(f^*(t)) &\leq t, \quad (0 \leq t < \mu(R)), \end{aligned}$$

for every $f \in \mathcal{M}_0$. The first one is obvious. For the second one, since

$$f^*(t) = \inf\{\lambda \geq 0 : \ \mu_f(\lambda) \leq t\},$$

there is a decreasing sequence $\lambda_n \downarrow f^*(t)$ such that $\mu_f(\lambda_n) \leq t$. So, by right-continuity of the distribution function, we arrive at

$$\mu_f(f^*(t)) = \lim_{n\to\infty} \mu_f(\lambda_n) \leq t.$$

Remark. Now we compute the decreasing rearrangement of the simple function f given by (4.1). We see that if $t \geq m_n$, then $f^*(t) = 0$. If $m_n > t \geq m_{n-1}$, then $x \in E_1 \cup ... \cup E_n$, thus $f^*(t) = a_n$. Generally, we have

$$f^*(t) = \sum_{j=1}^{n} a_j \chi_{[m_{j-1}, m_j[}(t), \quad (t \geq 0), \tag{4.2}$$

where we have taken $m_0 = 0$.

Furthermore, it is worthwhile to mention that f and f^* are equimeasurable. Indeed, for every $j = 1, ..., n$, if $a_{j+1} \leq \lambda < a_j$, then $s \geq 0$ can only take values such that

$$f^*(s) = a_1 \quad \vee \quad f^*(s) = a_2 \quad \vee \quad ... \quad \vee \quad f^*(s) = a_j,$$

that is,

$$s \in \bigcup_{i=1}^{j} [m_{i-1}, m_i[= [0, m_j[\,.$$

Therefore,

$$m_{f^*}(\lambda) = m\{s \geq 0 : \ f^*(s) > \lambda\} = m_j = \mu_f(\lambda).$$

In case that $\lambda \geq a_1$, since f^* cannot take values greater than a_1, we get

$$m_{f^*}(\lambda) = 0 = \mu_f(\lambda).$$

Hence f and f^* are equimeasurable. This fact remains true for an arbitrary measurable function f, which will be proved in Theorem 4.5 below.

Geometrically, in (4.2), we merely rearranged the *vertical blocks* in the graph of f in the decreasing order to obtain the decreasing rearrangement f^*. The values of f^* at the jumps are determined by the right continuity.

Although, sometimes it is more useful to section the functions into *horizontal blocks* rather than vertical ones. Let us rewrite f considering

$$F_j = \bigcup_{i=1}^{j} E_i, \quad (j = 1, ..., n),$$

so that $E_j = F_j \setminus F_{j-1}$, with $F_0 = \emptyset$. Since $a_{n+1} = 0$ and $F_0 = \emptyset$, we have

$$\begin{aligned} f &= \sum_{j=1}^{n} a_j \chi_{E_j} \\ &= \sum_{j=1}^{n} (\underbrace{a_j - a_{j+1}}_{b_j} + a_{j+1})(\chi_{F_j} - \chi_{F_{j-1}}) \\ &= \sum_{j=1}^{n} b_j \chi_{F_j} + \sum_{j=1}^{n} a_{j+1}\chi_{F_j} - a_j \chi_{F_{j-1}} \\ &= \sum_{j=1}^{n} b_j \chi_{F_j} + a_{n+1}\chi_{F_n} - a_1 \chi_{F_0} \\ &= \sum_{j=1}^{n} b_j \chi_{F_j}. \end{aligned}$$

Since $a_j = \sum_{i=j}^{n} b_i$, by (4.2),

$$f^* = \sum_{j=1}^{n}\sum_{i=j}^{n} b_i \chi_{[m_{j-1}, m_j[} = \sum_{i=1}^{n} b_i \sum_{j=1}^{i} \chi_{[m_{j-1}, m_j[} = \sum_{i=1}^{n} b_i \chi_{[0, m_i[} = \sum_{j=1}^{n} b_j \chi_{[0, \mu(F_j)[}.$$

In this case, the decreasing rearrangement is viewed as being formed by sliding blocks in each horizontal layer to form a single larger block positioned with its left-hand end against the vertical axis. We therefore conclude that if we have a simple function

$$f = \sum_{j=1}^{n} b_j \chi_{F_j}, \tag{4.3}$$

with positive coefficients and $F_1 \subset F_2 \subset \ldots \subset F_n$, then

$$f^* = \sum_{j=1}^{n} b_j \chi_{[0, \mu(F_j)[}. \tag{4.4}$$

It also establishes that, if $f \in \mathcal{M}_0^+(\mathbb{R}^+, m)$ is a decreasing simple function, then $f^* = f$. In fact, this equality holds for any decreasing function $f \in \mathcal{M}_0^+(\mathbb{R}^+, m)$.

The next theorem establishes some basic properties of the decreasing rearrangement.

Theorem 4.5 (see [10, Proposition 1.4.5]). *Suppose f, g, f_n, $(n = 1, 2, \ldots)$, belong to $\mathcal{M}_0(R, \mu)$, α is any scalar and E is a μ-measurable set. Then:*

(a) f^* is decreasing;

(b) $f^*(t) > \lambda$ if and only if $\mu_f(\lambda) > t \quad (\lambda, t \geq 0)$;

(c) $f^*(t) = m_{\mu_f}(t)$, for all the $t \geq 0$, where m denotes the Lebesgue measure on $[0, \infty[$. (This result tells us that f^* is right-continuous);

(d) $(\alpha f)^* = |\alpha| f^*$;

(e) $(f+g)^*(t_1+t_2) \leq f^*(t_1) + g^*(t_2), \quad (t_1, t_2 \geq 0)$*;*

(f) $(fg)^*(t_1+t_2) \leq f^*(t_1) g^*(t_2), \quad (t_1, t_2 \geq 0)$*;*

(g) If $|g| \leq |f|$ μ*-a.e., then* $g^*(t) \leq f^*(t)$*, for any non-negative real number* t*;*

(h) If $|f| \leq \liminf_{n\to\infty} |f_n|$ μ*-a.e., then* $f^* \leq \liminf_{n\to\infty} f_n^*$*. In particular* $f_n^* \uparrow f^*$*, whenever* $|f_n| \uparrow |f|$ μ*-a.e.;*

(i) $(\chi_E)^*(t) = \chi_{[0,\mu(E)[}(t), \quad (t \geq 0)$*;*

(j) f *and* f^* *are equimeasurable;*

(k) For $0 < p < \infty$*,* $(|f|^p)^*(t) = [f^*(t)]^p$*;*

(l) $(f\chi_E)^*(t) \leq f^*(t)\chi_{[0,\mu(E)[}(t), \quad (t \geq 0)$*;*

(m) If $0 < p < \infty$*, then* $\int_R |f|^p d\mu = p\int_0^\infty \lambda^{p-1}\mu_f(\lambda)d\lambda = \int_0^\infty [f^*(t)]^p dt$*;*

(n) $\operatorname{ess\,sup}_{x\in R} |f(x)| = \inf\{\lambda \geq 0 :\ \mu_f(\lambda) = 0\} = f^*(0)$*;*

(o) If $E \subset R$ *is a measurable set and* $g \geq 0$ *a.e., then*

$$\int_E g d\mu \leq \int_0^{\mu(E)} g^*(t)dt.$$

Proof. (a) If $0 \leq t_1 < t_2$, then

$$\{\lambda \geq 0 :\ \mu_f(\lambda) \leq t_1\} \subset \{\lambda \geq 0 :\ \mu_f(\lambda) \leq t_2\},$$

from which yields $f^*(t_1) \geq f^*(t_2)$.

(b) Notice that

$$\mu_f(\lambda) \leq t \iff \lambda \in \{s \geq 0 :\ \mu_f(s) \leq t\},$$

and so, if $\mu_f(\lambda) \leq t$, then $f^*(t) \leq \lambda$. Conversely, if $f^*(t) \leq \lambda$, then by the monotonicity of μ_f and by the discussion made above, we acquire

$$\mu_f(\lambda) \leq \mu_f(f^*(t)) \leq t.$$

(c) Let $t > 0$. Then, by (b), we have

$$m_{\mu_f}(t) = m\{\lambda \geq 0 :\ \mu_f(\lambda) > t\} = m\{\lambda \geq 0 :\ f^*(t) > \lambda\} = m([0, f^*(t)[) = f^*(t).$$

(d) Let α be any scalar and $f \in \mathcal{M}_0$. Then, for every $t \geq 0$,

$$(\alpha f)^*(t) = \inf\{\lambda \geq 0 :\ \mu_{\alpha f}(\lambda) \leq t\} = \inf\{\lambda \geq 0 :\ \mu_f(\lambda/|\alpha|) \leq t\} = |\alpha| f^*(t).$$

(e) We notice that

$$\{\lambda \geq 0 :\ \mu_f(\lambda) \leq t_1\} + \{\lambda \geq 0 :\ \mu_g(\lambda) \leq t_2\} \subset \{\lambda \geq 0 :\ \mu_{f+g}(\lambda) \leq t_1 + t_2\}.$$

This assertion is attained from applying Theorem 4.3 (III). Therefore, taking the infimum from the both sides, we get the desired inequality.

(f) Notice that, from Theorem 4.3 (IV), we obtain

$$\{\lambda \geq 0 : \mu_f(\lambda) \leq t_1\} \cdot \{\lambda \geq 0 : \mu_g(\lambda) \leq t_2\} \subset \{\lambda \geq 0 : \mu_{fg}(\lambda) \leq t_1 + t_2\},$$

and the conclusion follows by an argument analogous to that of (e).

(g) Suppose that $|g| \leq |f|$ a.e. Then, by Theorem 4.3 (I), $\mu_g \leq \mu_f$. Applying one more time the same result, we get $m_{\mu_g} \leq m_{\mu_f}$, which is $g^* \leq f^*$, by (c).

(h) As in (g), we apply two times the property (V) from Theorem 4.3, followed by (c), to achieve the desired conclusions.

(i) Let E be a measurable set. Then, for $\lambda \geq 0$, we have

$$\mu_{\chi_E}(\lambda) = \mu\{x \in R : \chi_E(x) > \lambda\} = \begin{cases} \mu(E) & \text{if } \lambda < 1 \\ 0 & \text{otherwise} \end{cases} = \mu(E)\chi_{[0,1[}(\lambda).$$

Therefore, for $t \geq 0$, we have

$$\chi_E^*(t) = \inf\{\lambda \geq 0 : \mu_{\chi_E}(\lambda) \leq t\} = \begin{cases} 1 & \text{if } t < \mu(E) \\ 0 & \text{otherwise} \end{cases} = \chi_{[0,\mu(E)[}(t).$$

(j) We are supposed to prove that

$$\mu_f(\lambda) = m_{f^*}(\lambda), \quad (\lambda \geq 0).$$

Let (f_n) be a sequence of simple functions such that $0 \leq f_n \uparrow |f|$. Based on the discussion preceding the theorem, we can state that

$$\mu_{f_n}(\lambda) = m_{f_n^*}(\lambda), \quad (\lambda \geq 0).$$

By virtue of (h), $f_n^* \uparrow f^*$. But then, by Theorem 4.3 (V), the left-hand side member tends to $\mu_f(\lambda)$ and the right-hand side member tends to $m_{f^*}(\lambda)$. Hence f and f^* are equimeasurable.

(k) It follows from

$$\mu_{|f|^p}(\lambda) = \mu\{x \in R : |f(x)|^p > \lambda\} = \mu\left\{x \in R : |f(x)| > \lambda^{1/p}\right\} = \mu_{|f|}\left(\lambda^{1/p}\right),$$

that

$$[(|f|^p)^*(t)]^{1/p} = \inf\left\{\lambda^{1/p} \geq 0 : \mu_{|f|^p}(\lambda) \leq t\right\} = \inf\left\{\lambda^{1/p} \geq 0 : \mu_{|f|}\left(\lambda^{1/p}\right) \leq t\right\} = f^*(t).$$

(l) From (g), we have $(f\chi_E)^* \leq f^*$. Hence, it is enough to show that $(f\chi_E)^*(t) = 0$, for $t \geq \mu(E)$. In fact, if $t \geq \mu(E)$, then

$$\mu_{f\chi_E}(\lambda) = \mu\{x \in E : |f(x)| > \lambda\} \leq t,$$

for any $\lambda \geq 0$. Hence, taking the infimum over all the λ, we arrive at the desired equality.

(m) Suppose that f is a non-negative, simple function written as in the remarks made above (4.1) and (4.2). Regarding that $\mu(E_j) = m_j - m_{j-1}$, $(j = 1, , ..., n)$, we have

$$\int_R |f|^p \, d\mu = \sum_{j=1}^n a_j^p \mu(E_j) = \sum_{j=1}^n a_j^p m\left([m_{j-1}, m_j[\right) = \int_0^\infty (f^*)^p(t)dt.$$

Similarly,

$$\begin{aligned} p\int_0^\infty \lambda^{p-1}\mu_f(\lambda)d\lambda = p\sum_{j=1}^n m_j \int_{a_{j+1}}^{a_j} \lambda^{p-1}d\lambda &= \sum_{j=1}^n m_j(a_j^p - a_{j+1}^p) \\ &= \sum_{j=1}^n (\underbrace{m_j - m_{j-1}}_{\mu(E_j)} + m_{j-1})(a_j^p - a_{j+1}^p) \\ &= \sum_{j=1}^n a_j^p \mu(E_j) + \sum_{j=1}^n m_{j-1}a_j^p - m_j a_{j+1}^p \\ &= \int_R |f|^p \, d\mu + \underbrace{m_0}_{=0} a_1^p - m_n \underbrace{a_{n+1}^p}_{=0}. \end{aligned}$$

The general case is done regarding (h), (V) from Theorem 4.3 and the monotone convergence.

(n) Immediate, by the definition of essential supremum.

(o) By (l) and (m), taking $p = 1$, we have

$$\int_R g\chi_E d\mu = \int_0^\infty (g\chi_E)^*(t)dt \leq \int_0^\infty g^*(t)\chi_{[0,\mu(E)[}dt.$$

□

Theorem 4.6 (Hardy-Littlewood inequality; see [2, Chapter 2, Theorem 2.2]). *If* $f, g \in \mathcal{M}_0(R, \mu)$, *then*

$$\int_R |fg| d\mu \leq \int_0^\infty f^*(s)g^*(s)ds.$$

Proof. Let f be a simple function as in (4.3), so f^* is written as in (4.4). Hence, for any $g \in \mathcal{M}^+$, if we apply Theorem 4.5 (o), we attain

$$\begin{aligned} \int_R fgd\mu = \sum_{j=1}^n b_j \int_{F_j} gd\mu \leq \sum_{j=1}^n b_j \int_0^{\mu(F_j)} g^*(t)dt \\ &= \int_0^\infty \sum_{j=1}^n b_j\chi_{[0,\mu(F_j)[}(t)g^*(t)dt \\ &= \int_0^\infty f^*(t)g^*(t)dt. \end{aligned}$$

Since f^* and g^* depend only on their respective absolute value, the general case is established by the monotone convergence. □

As a consequence of the above inequality, for every $\tilde{g} \in \mathcal{M}_0(R, \mu)$ equimeasurable with g, one has

$$\int_R |f\tilde{g}| d\mu \leq \int_0^\infty f^*(s) g^*(s) ds.$$

4.2 Rearrangement-invariant spaces

Definition 4.7. A σ-finite measure space (R, μ) is said to be *resonant* if

$$\int_0^\infty f^*(s) g^*(s) ds = \sup \left\{ \int_R |f\tilde{g}| d\mu : \ \tilde{g} \text{ equimeasurable to } g \right\},$$

for every given $f, g \in \mathcal{M}_0(R, \mu)$. The measure space is said to be *strongly resonant* if the supremmum is achieved.

Definition 4.8. A function norm ρ is said to be *rearrangement-invariant* if $\rho(f) = \rho(g)$, for every pair of equimeasurable functions $f, g \in \mathcal{M}_0^+(R, \mu)$. In that case, we say that the Banach function space $X = X(\rho)$ is a *rearrangement-invariant space*.

Observe that a Banach function space X is rearrangement-invariant if and only if, whenever f belongs to X and g is equimeasurable with f, then also g belongs to X and their norm agree.

Notice that, if $f \in L^p(R, \mu)$ and g is equimeasurable with f, then by Theorem 4.5(d),

$$f^*(t) = m_{\mu_f}(t) = m_{\mu_g}(t) = g^*(t), \quad (t \geq 0),$$

and Theorem 4.5(m) [or (n), if $p = \infty$] gives $\|f\|_{L^p(R,\mu)} = \|g\|_{L^p(R,\mu)}$, which proves that the Lebesgue space $L^p(R, \mu)$ is rearrangement-invariant.

Theorem 4.9 (Luxemburg representation theorem; see [2, Chapter 2, Theorem 4.10])**.** *Let ρ be a rearrangement-invariant function norm over a resonant measure space (R, μ). Then, there is a rearrangement-invariant function norm $\overline{\rho}$ over $(\mathbb{R}^+, m)$ such that, for every $f \in \mathcal{M}_0^+(R, \mu)$ one has*

$$\rho(f) = \overline{\rho}(f^*).$$

Furthermore, if σ is any rearrangement-invariant function norm over $(\mathbb{R}^+, m)$ which represents ρ, in the sense that, for any $f \in \mathcal{M}_0^+(R, \mu)$,

$$\rho(f) = \sigma(f^*),$$

then the associate norm ρ' of ρ is represented in the same way by the associate norm σ' of σ, that is, for each $f \in \mathcal{M}_0^+(R, \mu)$,

$$\rho'(f) = \sigma'(f^*).$$

The Luxemburg representation theorem shows in particular that the rearrangement-invariant spaces over a resonant measures space (R, μ) are completely determined by the rearrangement-invariant spaces over $(\mathbb{R}^+, m)$. The representation $\rho \to \overline{\rho}$ is in fact unique if (R, μ) is non-atomic and if it has infinite measure. In that case, the function norm $\overline{\rho}$ is defined by

$$\overline{\rho}(h) = \sup \left\{ \int_0^\infty g^*(s)h^*(s)ds : \ \rho'(g) \leq 1 \right\},$$

and satisfies the equality

$$\rho(f) = \overline{\rho}(f^*),$$

for every $f \in \mathcal{M}_0^+(R, \mu)$. The rearrangement-invariant Banach function space generated by $\overline{\rho}$ is denoted by $\overline{X}$.

4.3 Dilation operator and Boyd indices

Definition 4.10. For each $t > 0$, let E_t denote the *dilation operator* defined on $\mathcal{M}_0(\mathbb{R}^+, m)$ by

$$(E_t f)(s) = f(st), \quad (0 < s < \infty).$$

With X and $\overline{X}$ as above, let $h_X(t)$ denote the operator norm of $E_{1/t}$ as an operator from $\overline{X}$ to itself. Thus,

$$h_X(t) := \left\| E_{1/t} \right\|_{\mathcal{B}(\overline{X})}.$$

It is not immediately clear that E_t is bounded on $\overline{X}$. The next result estabilishes this fact and provides an estimate for the operator norm. But before that, we shall recall some well-known notions in functional analysis.

Definition 4.11. Let $X \subset \mathbb{R}$ and let $f \colon X \to \mathbb{R}$ be a function. Then, the function f is said to be:

- *subadditive* if X is an additive semigroup and, for every $x, y \in X$, one has

$$f(x + y) \leq f(x) + f(y);$$

- *submultiplicative* if X is a multiplicative semigroup and, for every $x, y \in X$, one has

$$f(xy) \leq f(x)f(y).$$

Theorem 4.12 (see [2, Chapter 3, Proposition 5.11])**.** *For each $t > 0$, $E_t \in \mathcal{B}(\overline{X})$. The function h_X is increasing and submultiplicative on $\mathbb{R}^+$, satisfies $h_X(1) = 1$, and*

$$h_X(t) \leq \max\{1, t\}.$$

Moreover, if X' denotes the associate space of X, then

$$h_X(t) = t h_{X'}\left(\frac{1}{t}\right).$$

Lemma 4.13 (see [2, Chapter 3, Lemma 5.8]). *Let ω be an increasing subadditive function on $\mathbb{R}$ for which $\omega(0) = 0$. Then*

$$-\omega(-s) \leq \omega(s), \quad (s \in \mathbb{R}).$$

Furthermore, there is $\alpha \geq 0$ such that

$$\alpha = \lim_{s\to\infty} \frac{\omega(s)}{s} = \inf_{s>0} \frac{\omega(s)}{s}.$$

Proof. For any $s \in \mathbb{R}$,

$$0 = \omega(0) = \omega(s + (-s)) \leq \omega(s) + \omega(-s),$$

from which the required estimate follows. Put $\alpha = \inf_{s>0} \frac{\omega(s)}{s}$. Since ω is an increasing function, $\omega(s) \geq \omega(0) = 0$, for each $s > 0$, so α must be finite. Fix $\varepsilon > 0$ and choose $t > 0$ satisfying

$$\frac{\omega(t)}{t} < \alpha + \varepsilon,$$

and select an $N \in \mathbb{N}$ big enough such that

$$(1 + 1/N)\frac{\omega(t)}{t} < \alpha + \varepsilon.$$

So, for each $s \geq Nt$, there corresponds an integer $n \geq N$ such that $nt \leq s < (n+1)t$. Using the latter inequality and the fact that ω is increasing and subadditive, we obtain $\omega(s) \leq (n+1)\omega(t)$, and so

$$\alpha \leq \frac{\omega(s)}{s} \leq \frac{(n+1)\omega(t)}{nt} \leq (1 + 1/N)\frac{\omega(t)}{t} < \alpha + \varepsilon.$$

Hence $\omega(s)/s \to \alpha$, as $s \to \infty$. □

Remark. To a submultiplicative function ψ on $\mathbb{R}^+$, we can associate ω on $\mathbb{R}$ defined by

$$\omega(s) = \log \psi(e^s), \quad (s \in \mathbb{R}).$$

Clearly ω is a subadditive function. Thus, if ψ is increasing and $\psi(1) = 1$, then it satisfies the hypotheses of the previous lemma. Then, using the substitution $t = e^s$, yields

$$\overline{\alpha}(\psi) := \inf_{t>1} \frac{\log \psi(t)}{\log t} = \lim_{t\to\infty} \frac{\log \psi(t)}{\log t}, \tag{4.5}$$

for some $0 \leq \overline{\alpha}(\psi) < \infty$.

Lemma 4.14 (see [2, Chapter 3, Lemma 5.9]). *Let ψ be an increasing submultiplicative function on $\mathbb{R}^+$ for which $\psi(1) = 1$, and let $a \in \mathbb{R}^+$. Then $\overline{\alpha}(\psi) < a$ if and only if*

$$\int_1^\infty t^{-a}\psi(t)\frac{dt}{t} < \infty.$$

Proof. If $\overline{\alpha}(\psi) < a$, then there exists $\varepsilon > 0$ such that $\overline{\alpha}(\psi) < a - \varepsilon$. By (4.5) there exists $T > 1$ such that, for all the $t \geq T$, we have

$$\frac{\log \psi(t)}{\log t} < a - \varepsilon.$$

Then, $\psi(t) < t^{a-\varepsilon}$, for $t \geq T$, so

$$\int_1^\infty t^{-a}\psi(t)\frac{dt}{t} \leq \psi(T)\int_1^T t^{-1-a}dt + \int_T^\infty t^{-1-\varepsilon}dt < \infty.$$

Conversely, if $\int_1^\infty t^{-a}\psi(t)\frac{dt}{t} < \infty$, then $s^{-a}\psi(s) < 1$ for some $s > 1$. Thus

$$\overline{\alpha}(\psi) \leq \frac{\log \psi(s)}{\log s} < \frac{\log s^a}{\log s} = a,$$

as desired. □

Definition 4.15. Let X be a rearrangement-invariant function space over an infinite, nonatomic, σ-finite measure space. The *Boyd indices* of X are the numbers $\underline{\alpha}_X$ and $\overline{\alpha}_X$ defined by

$$\underline{\alpha}_X := \sup_{0<t<1} \frac{\log h_X(t)}{\log t}, \quad \overline{\alpha}_X := \inf_{1<t<\infty} \frac{\log h_X(t)}{\log t}.$$

Theorem 4.16 (see [2, Chapter 3, Proposition 5.13])**.** *The indices $\underline{\alpha} = \underline{\alpha}_X$ and $\overline{\alpha} = \overline{\alpha}_X$ of X are given by the limits*

$$\underline{\alpha}_X = \lim_{t\downarrow 0} \frac{\log h_X(t)}{\log t}, \quad \overline{\alpha}_X = \lim_{t\to\infty} \frac{\log h_X(t)}{\log t},$$

and they satisfy

$$0 \leq \underline{\alpha} \leq \overline{\alpha} \leq 1.$$

Furthermore, the indices $\underline{\alpha}' = \underline{\alpha}_{X'}$ and $\overline{\alpha}' = \overline{\alpha}_{X'}$ of the associate space X' are given by

$$\underline{\alpha}' = 1 - \overline{\alpha}, \quad \overline{\alpha}' = 1 - \underline{\alpha}. \tag{4.6}$$

Proof. The identity

$$\frac{\log h_X(t)}{\log t} = 1 - \frac{\log h_{X'}(1/t)}{\log(1/t)} \tag{4.7}$$

is an immediate consequence of Theorem 4.12. The relations (4.6) follow from this and from the definitions of the Boyd indices. From (4.5) it follows that $\overline{\alpha}_X = \overline{\alpha}(h_X)$. In (4.7), if we take $t \downarrow 0$, the right-hand side becomes $1 - \overline{\alpha}(h_{X'})$ which is $1 - \overline{\alpha}' = \underline{\alpha}$.

Now, that $\overline{\alpha} \leq 1$, follows from Theorem 4.12. Applying (4.6) to X', it follows that $\underline{\alpha} = 1 - \overline{\alpha}' \geq 0$. Thus, it remains only to show that $\underline{\alpha} \leq \overline{\alpha}$. Since h is submultiplicative, we have $1 = h_X(1) \leq h_X(t)h_X(1/t)$. Hence, for all $t > 1$,

$$\frac{\log h_X(1/t)}{\log(1/t)} = \frac{\log\left(\frac{1}{h_X(1/t)}\right)}{\log t} \leq \frac{\log h_X(t)}{\log t}.$$

Taking $t \to \infty$, the conclusion arises. □

Remark. We note that if $X = L^p(R,\mu)$, then $h_X(t) = t^{1/p}$, from which it follows that

$$\underline{\alpha} = \overline{\alpha} = 1/p.$$

Indeed, for every $f \in \overline{X}$, we have

$$\begin{aligned} \|f\|_{\overline{X}} &= \sup\left\{\int_0^\infty f^*(s)g^*(s)ds : \|g\|_{L^{p'}(R,\mu)} \le 1\right\} \\ &= \sup\left\{\int_0^\infty f^*(s)g^*(s)ds : \|g^*\|_{L^{p'}(\mathbb{R}^+,m)} \le 1\right\} \\ &= \|f^*\|_{L^p(\mathbb{R}^+,m)}, \end{aligned}$$

and so, for $p \neq \infty$,

$$\begin{aligned} \left\|E_{1/t}f\right\|_{\overline{X}} &= \left\|(E_{1/t}f)^*\right\|_{L^p(\mathbb{R}^+,m)} \\ &= \left(\int_0^\infty |f^*(s/t)|^p ds\right)^{1/p} \\ &= \left(\int_0^\infty |f^*(\tau)|^p t d\tau\right)^{1/p} \\ &= t^{1/p} \underbrace{\left(\int_0^\infty |f^*(\tau)|^p d\tau\right)^{1/p}}_{\|f^*\|_{L^p(\mathbb{R}^+,m)}} \\ &= t^{1/p}\|f\|_{\overline{X}}, \end{aligned}$$

and for $p = \infty$,

$$\begin{aligned} \left\|E_{1/t}f\right\|_{\overline{X}} &= \left\|(E_{1/t}f)^*\right\|_{L^p(\mathbb{R}^+,m)} \\ &= \operatorname*{ess\,sup}_{s>0} f^*(s/t) \\ &= f^*(0) \\ &= \|f\|_{\overline{X}}. \end{aligned}$$

The conclusion follows.

4.4 Hilbert transform

Definition 4.17. For each f Lebesgue measurable function on $\mathbb{R}^+$, we define the linear operator

$$(Sf)(t) := \int_0^\infty \min\left\{1, \frac{s}{t}\right\} f(s)\frac{ds}{s} = \frac{1}{t}\int_0^t f(s)ds + \int_t^\infty f(s)\frac{ds}{s}.$$

Notice that, for $0 < t < t'$,

$$\min\left\{1, \frac{s}{t'}\right\} \le \min\left\{1, \frac{s}{t}\right\} \le \frac{t'}{t}\min\left\{1, \frac{s}{t'}\right\}.$$

So, if f is non-negative, it follows from the first of these inequalities that $(Sf)(t)$ is a decreasing function of t, and the two inequalities taken together show that $(Sf)(t)$ is finite for any chosen value $t > 0$ if and only if it is finite for every $t > 0$. Moreover, the monotonicity os Sf yields

$$(Sf)^* = Sf.$$

Definition 4.18. Let $f \in L^1_{loc}(\mathbb{R})$. The *Hilbert transform* Hf of f is defined by the principal-value integral

$$(Hf)(x) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} f(t)\frac{dt}{x-t} = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-t|\geq\varepsilon} f(t)\frac{dt}{x-t},$$

provided that the limit exists a.e.

Theorem 4.19 (see [2, Chapter 3, Theorem 4.8])**.** *If $f \in L^1_{loc}(\mathbb{R})$ satisfies $(Sf^*)(1) < \infty$, then the Hilbert trasnform of f, Hf, exists a.e. Furthermore,*

$$(Hf)^*(t) \leq c(Sf^*)(t), \quad (t > 0),$$

for some constant c independent of f and t.

Theorem 4.20 (see [2, Chapter 3, Proposition 4.10])**.** *If $(Sf^*)(1) < \infty$, then there is a function g equimeasurable to f such that, for any $t > 0$,*

$$(Sf^*)(t) \leq 2\pi(Hg)^*(t).$$

Proof. If $g(x) = f^*(-x)\chi_{]-\infty,0[}(x)$, $(x \in \mathbb{R})$, then clearly g is equimeasurable to f. In particular $(Sg^*)(1) = (Sf^*)(1) < \infty$, so $(Hg)(x)$ exists a.e. by Theorem 4.19. If $x > 0$, then

$$(Hg)(x) = \frac{1}{\pi}\int_0^\infty f^*(u)\frac{du}{x+u} \geq \frac{1}{2\pi}\int_0^\infty f^*(u)\min\left\{\frac{1}{x},\frac{1}{u}\right\}du = \frac{1}{2\pi}(Sf^*)(x).$$

Hence,

$$|(Hg)(x)| \geq \begin{cases} \frac{1}{2\pi}(Sf^*)(x), & x > 0,\\ 0, & x < 0,\end{cases}$$

so, taking decreasing rearrangements, we obtain the desired inequality. □

Definition 4.21. Let P_a, $(0 < a \leq 1)$, be the integral operator defined on $\mathcal{M}_0(\mathbb{R}^+, m)$ by

$$(P_a f)(t) = t^{-a}\int_0^t s^a f(s)\frac{ds}{s}, \quad (0 < t < \infty).$$

Similarly, let Q_a, $(0 \leq a < 1)$, be the integral operator defined on $\mathcal{M}_0(\mathbb{R}^+, m)$ by

$$(Q_a f)(t) = t^{-a}\int_t^\infty s^a f(s)\frac{ds}{s}, \quad (0 < t < \infty).$$

Note that Q_a is the formal adjoint of P_a when $a+b=1$. In other words, as an interchange in the order of integration shows that

$$\int_0^\infty (P_a f)(t)g(t)dt = \int_0^\infty f(t)(Q_b g)(t)dt, \tag{4.8}$$

for all f and g for which the integrals exist.

Theorem 4.22 (see [2, Chapter 3, Theorem 5.15]). *The operator P_a is bounded on $\overline{X}$ if and only if $a > \overline{\alpha}_X$, and Q_a is bounded on $\overline{X}$, if and only if $a < \underline{\alpha}_X$.*

Proof. Suppose first that P_a is bounded in $\overline{X}$. Let $f \in \overline{X}$ and $g \in \overline{X}'$ be such that

$$\|f\|_{\overline{X}} \leq 1, \quad \|g\|_{\overline{X}'} \leq 1. \tag{4.9}$$

Then $\int_0^\infty f^*(s/t)g^*(s)ds$ decreases with t and so, for each fixed $t>0$, we have

$$\begin{aligned}\int_0^\infty f^*(s/t)g^*(s)ds &= at^a \int_0^\infty f^*(s/t)g^*(s)ds \int_0^{1/t} u^{a-1}du \\ &\leq at^a \int_0^{1/t}\int_0^\infty f^*(su)g^*(s)dsu^{a-1}du \\ &= at^a \int_0^\infty g^*(s)\left(\int_0^{1/t} f^*(su)u^a \frac{du}{u}\right) ds.\end{aligned}$$

If $t>1$, we may extend the range of integration in the inner integral to $0 \leq u \leq 1$. Then, by exchange of variables, putting $v = su$, we obtain,

$$\begin{aligned}\int_0^\infty f^*(s/t)g^*(s)ds &\leq at^a \int_0^\infty g^*(s)\left(s^{-a}\int_0^s f^*(v)v^a \frac{dv}{v}\right) ds \\ &= at^a \int_0^\infty g^*(s)(P_a f^*)(s)ds \\ &\leq at^a \|P_a\|_{\mathcal{B}(\overline{X})}.\end{aligned}$$

Taking the supremum over all the f and g satisfying (4.9), we therefore have

$$h_X(t) = \left\|E_{1/t}\right\|_{\mathcal{B}(\overline{X})} \leq at^a \|P_a\|_{\mathcal{B}(\overline{X})}, \quad (t>1).$$

Hence,

$$\frac{\log h_X(t)}{\log t} \leq a + \frac{\log\left(a\|P_a\|_{\mathcal{B}(\overline{X})}\right)}{\log t} \to a,$$

as $t \to \infty$ and so it follows, from Theorem 4.16, that $\overline{\alpha}_X \leq a$. Thus, we have shown that

$$\|P_a\| := \|P_a\|_{\mathcal{B}(\overline{X})} < \infty \implies a \geq \overline{\alpha}_X. \tag{4.10}$$

We still need to obtain the strict inequality $a > \overline{\alpha}_X$. Choose $\varepsilon > 0$ sufficiently small so that $\varepsilon\|P_a\| < 1$. Then, the operator $I - \varepsilon P_a$ is invertible in $\mathcal{B}(\overline{X})$, and

$$(I-\varepsilon P_a)^{-1} = \sum_{n=0}^\infty \varepsilon^n P_a^n,$$

where the convergence is in the norm of $\mathcal{B}(\overline{X})$ (see, e.g., [24, Theorem 4.40]). The operator

$$T = P_a(I - \varepsilon P_a)^{-1} = \sum_{n=0}^{\infty} \varepsilon^n P_a^{n+1}$$

is therefore also in $\mathcal{B}(\overline{X})$. We claim that

$$(P_a^{n+1} f)(t) = \int_0^1 f(st) \frac{(\log 1/s)^n}{n!} s^{a-1} ds, \quad (n = 0, 1, ...). \tag{4.11}$$

The proof proceeds by strong induction on n. The case that $n = 0$ holds immediately from the definition of P_a, so suppose that (4.11) holds for $n = 0, 1, ..., N-1$, for some natural $N \geq 1$. Then

$$\begin{aligned}(P_a^{N+1} f)(t) = P_a(P_a^N f)(t) &= \int_0^1 (P_a^N f)(rt) r^{a-1} dr \\ &= \int_0^1 \left(\int_0^1 f(rst) \frac{(\log 1/s)^N}{N!} s^{a-1} ds \right) r^{a-1} dr,\end{aligned}$$

so, making the change of variable of $u = rs$, we have

$$(P_a^{N+1} f)(t) = \int_0^1 \left(\int_0^r f(ut) \frac{(\log r/u)^N}{N!} u^{a-1} du \right) \frac{dr}{r}.$$

Interchanging the order of integration and making the change of variable $v = r/u$, we obtain

$$\begin{aligned}(P_a^{N+1} f)(t) &= \int_0^1 \left(\int_u^1 \frac{(\log r/u)^N}{N!} \frac{dr}{r} \right) f(ut) u^{a-1} du \\ &= \int_0^1 \left(\int_1^{1/u} \frac{(\log v)^N}{N!} \frac{dv}{v} \right) f(ut) u^{a-1} du \\ &= \int_0^1 \frac{(\log 1/u)^{N+1}}{(N+1)!} f(ut) u^{a-1} du.\end{aligned}$$

This completes the induction and hence establishes (4.11). By the monotone convergence theorem, we obtain, for nonnegative functions $f \in \overline{X}$,

$$(Tf)(t) = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(\varepsilon \log 1/s)^n}{n!} \right) f(st) s^{a-1} ds = \int_0^1 f(st) s^{a-\varepsilon-1} ds.$$

By the usual device splitting a function into its positive and negative parts, we obtain this identity for all the $f \in \overline{X}$. Hence, $T = P_{a-\varepsilon}$. Since $T \in \mathcal{B}(\overline{X})$, we may therefore apply (4.10) to obtain $a - \varepsilon \geq \overline{\alpha}_X$.

Conversely, suppose that $a > \overline{\alpha}_X$. Then Lemma 4.14 shows that

$$\int_0^1 \|E_s\|_{\mathcal{B}(\overline{X})} s^{a-1} ds = \int_1^{\infty} t^{-a} h_X(t) \frac{dt}{t} < \infty. \tag{4.12}$$

Hence, if f and g satisfy (4.9),

$$\begin{aligned}\left|\int_0^\infty (P_a f)(t)g(t)dt\right| &\leq \int_0^\infty \left(\int_0^1 |f(st)|\, s^{a-1} ds\right) |g(t)|\, dt \\ &= \int_0^1 \left(\int_0^\infty |f(st)g(t)|\, dt\right) s^{a-1} ds \\ &\leq \int_0^1 \|E_s\|_{\mathcal{B}(\overline{X})}\, s^{a-1} ds.\end{aligned}$$

Using (4.12) and taking the supremum over all the f and g satisfying (4.9), we find that $P_a \in \mathcal{B}(\overline{X})$. Thus we have shown that P_a is bounded on $\overline{X}$ if and only if $a > \overline{\alpha}_X$.

If now $0 < a \leq 1$, it follows from (4.8) that Q_a is bounded on $\overline{X}$ if and only if P_{1-a} is bounded on $\overline{X}'$. By the result we have just proved, this occurs if and only if $1 - a > \overline{\alpha}_{X'}$. But $\overline{\alpha}_{X'} = 1 - \underline{\alpha}_X$. Hence, Q_a is bounded if and only if $a < \underline{\alpha}_X$. □

Theorem 4.23 (see [2, Chapter 3, Theorem 5.18]). *Let $X = X(\mathbb{R})$ be a rearrangement-invariant Banach function space on $\mathbb{R}$. Then the Hilbert transform H is bounded on X if and only if the indices of X satisfy*

$$0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1.$$

Proof. If $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1$, then, by Theorem 4.22, the operators P_1 and Q_0 are bounded on $\overline{X}$, meaning that the operator $S = P_1 + Q_0$ is bounded on $\overline{X}$. Notice that, for a fixed $f \in X$, $(Sf)(1) < \infty$. Indeed, since

$$\|f\|_{\overline{X}} = \overline{\rho}(f^*) = \|f\|_X < \infty,$$

by the boundedness of S on $\overline{X}$, $Sf \in \overline{X}$. So Sf must be finite at some point. Therefore, Sf is finite at every point. Hence, by Theorem 4.19 for every $f \in X$,

$$\|Hf\|_X = \rho(|Hf|) = \overline{\rho}\left((Hf)^*\right) \leq c\overline{\rho}(Sf^*) \leq cK\overline{\rho}(f^*) = cK\rho(|f|) = cK\, \|f\|_X\,,$$

for some $c > 0$ independent of f and $K = \|S\|_{\mathcal{B}(\overline{X})}$.

Conversely, if H is bounded on X, then Theorem 4.20 says that for each $f \in X$, there corresponds a function g equimeasurable with f such that $S(f^*) \leq 2\pi(Hg)^*$. Then

$$\overline{\rho}(Sf^*) \leq 2\pi\overline{\rho}(Hg)^* = 2\pi\, \|Hg\|_X \leq 2\pi C\, \|g\|_X = 2\pi C\, \|f\|_X = 2\pi C\overline{\rho}(f^*),$$

for all the $f \in X$. This is enough to show that S is a bounded operator on $\overline{X}$, and hence that P_1 and Q_0 are bounded on $\overline{X}$. Again, by Theorem 4.22 we conclude that $1 > \overline{\alpha}_X$ and that $0 < \underline{\alpha}_X$. □

This theorem is consistent with the conclusion that Marcel Riesz had arrived at, which is the following.

Corollary 4.24 (see [2, Chapter 3, Theorem 4.9 (a)]). *If* $1 < p < \infty$ *and if* $f \in L^p(\mathbb{R})$, *then* $Hf \in L^p(\mathbb{R})$ *and*

$$\|Hf\|_p \leq c_p \|f\|_p \,,$$

where c_p *depends only on* p.

Chapter 5

Fredholmness and invertibility on a Banach function space

5.1 C*-algebras

Definition 5.1. An algebra A over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is a vector space over $\mathbb{K}$, equipped with an additional binary operation (which we will denote by juxtaposition), satisfying the following properties:

1. $(ab)c = a(bc)$;
2. $(a+b)c = ac + bc, \quad a(b+c) = ab + ac$;
3. $(\lambda a)b = a(\lambda b) = \lambda(ab)$

for all $a, b, c \in A,\ \lambda \in \mathbb{K}$. If $\mathbb{K} = \mathbb{R}$, then A is said to be a *real algebra*. If $\mathbb{K} = \mathbb{C}$, then A is said to be a *complex algebra*. An algebra is said to be *commutative* if, for every $a, b \in A$,

$$ab = ba.$$

If $e \in A$ is an element satisfying

$$ae = ea = a,$$

for any $a \in A$, then the algebra is *unital* and e is called to be the *unit element*, alternatively denoted by I or 1.

It is a routine to show that there are at most one unit element of an algebra. Note that if $A \neq \{0\}$ is a unital algebra, then $e \neq 0$.

Definition 5.2. Let A be an algebra over $\mathbb{K}$ and $B \subset A$. Then B is said to be a *subalgebra* of A if, for every $a, b \in B$ and $\lambda \in \mathbb{K}$, we have

(i) $0 \in B$;

(ii) $\lambda a + b \in B$;

(iii) $ab \in B$.

Definition 5.3. Let A be a unital algebra. An element $a \in A$ is said to be *invertible* if there is $b \in A$ satisfying

$$ab = ba = e.$$

We call b the *inverse* of a and denote by a^{-1}, since when exists, it is uniquely determined.

Definition 5.4. A subalgebra J of an algebra A is said to be an *ideal*, if for every $a \in A$ and $j \in J$, one has

$$aj, ja \in J.$$

Definition 5.5. An algebra A is called *normed algebra* if there is a norm $\|\cdot\| : A \to \mathbb{R}$ satisfying, for every $a, b \in A$, the following property:

$$\|ab\| \leq \|a\| \, \|b\| .$$

If A is unital, we usually set $\|e\| := 1$, that is, the norm of the unit element of A is 1. If $E \subset A$, then $\operatorname{clos} E$ stands for the *closure* of E in A. We say that A is a *Banach algebra* if the normed algebra A is a complete for the fixed norm.

Definition 5.6. Let A be a Banach algebra over $\mathbb{K}$ and $E \subset A$. We denote by $\operatorname{alg}_A E$ or simply $\operatorname{alg} E$ to be the smallest closed subalgebra of A which contains E.

It is not hard to prove that, if $E \neq \emptyset$,

$$\operatorname{alg} E = \operatorname{clos} \left\{ \sum_j \gamma_j \prod_k a_{jk} : \ \gamma_j \in \mathbb{K}, \, a_{jk} \in E \right\},$$

for finite sums and ordered products.

Definition 5.7. Let A be an algebra and $*$ be a unary operation on A, $a \mapsto a^*$, called *involution*, which satisfies the next properties:

1. $(a^*)^* = a$,
2. $(\lambda a + b)^* = \overline{\lambda} a^* + b^*$,
3. $(ab)^* = b^* a^*$,

for each $a, b \in A$ and $\lambda \in \mathbb{C}$.

In this conformity, A is said to be an *algebra with involution*. The set A is said to be a **-algebra* if it is an algebra with involution and the involution is an isometry.

We say that a^* is the *adjoint element* of a.

A *Banach *-algebra* is a complete *-algebra.

Definition 5.8. A C^**-algebra* is a unital complex Banach algebra with involution, A, which satisfies

$$\|a^*a\| = \|a\|^2 ,$$

for every $a \in A$.

In particular, A is a *-algebra. Indeed,

$$\|a\|^2 = \|a^*a\| \leq \|a^*\| \, \|a\| ,$$

hence $\|a\| \leq \|a^*\|$. If we replace a by a^*, we obtain the inverse inequality.

Definition 5.9. Let A and B be C^*-algebras. A function $\Phi\colon A \to B$ is said to be a *homomorphism of C^*-algebras* if, for all the $a, b \in A$ and $\lambda \in \mathbb{C}$,

1. $\Phi(a + \lambda b) = \Phi(a) + \lambda\Phi(b)$;
2. $\Phi(ab) = \Phi(a)\Phi(b)$;
3. $\Phi(a^*) = \Phi(a)^*$;
4. $\Phi(1_A) = 1_B$.

5.2 Semi-almost periodic functions

Definition 5.10. The algebra of (uniformly) *almost periodic functions* is defined by

$$AP := \operatorname{alg}_{L^\infty(\mathbb{R})} \{e_\lambda : \lambda \in \mathbb{R}\} ,$$

where $e_\lambda(x) := e^{i\lambda x}$. In other words, it is the smallest closed subalgebra of $L^\infty(\mathbb{R})$ generated by all the functions $e_\lambda\colon \mathbb{R} \to \mathbb{C}$ with $\lambda \in \mathbb{R}$.

The set of all finite linear combination of the functions e_λ, with $\lambda \in \mathbb{R}$, is denoted by AP^0 and each element is said to be an *almost periodic polynomial*. Obviously, AP is the closure of AP^0 in $L^\infty(\mathbb{R})$. Also, it is clear that AP is a C^*-subalgebra of $L^\infty(\mathbb{R})$.

Definition 5.11. We denote by $C(\mathbb{R})$ the class of all continuous (complex-valued) functions on $\mathbb{R}$ and $BC(\mathbb{R}) := C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ stands for the bounded continuous functions on $\mathbb{R}$. We denote by $C(\dot{\mathbb{R}})$ the collection of functions $a \in BC(\mathbb{R})$ for which the limits

$$a(-\infty) := \lim_{x \to -\infty} a(x) \quad \text{and} \quad a(+\infty) := \lim_{x \to +\infty} a(x)$$

exist and coincide, denoting the common value of these two limits by $a(\infty)$. Let $C_0(\dot{\mathbb{R}})$ be the collection of all $a \in C(\dot{\mathbb{R}})$ such that $a(\infty) = 0$. Clearly, $BC(\mathbb{R}), C(\dot{\mathbb{R}})$

and $C_0(\dot{\mathbb{R}})$ are C^*-subalgebras of $L^\infty(\mathbb{R})$. A function $a \in L^\infty(\mathbb{R})$ such that the one-sided limits

$$a(x_0 - 0) := \lim_{x \to x_0 - 0} a(x), \quad a(x_0 + 0) := \lim_{x \to x_0 + 0} a(x)$$

exist, for each $x_0 \in \dot{\mathbb{R}}$, is said to be a *bounded piecewise continuous function*, and we denote the collection of all such elements by $PC := PC(\dot{\mathbb{R}})$. By convention,

$$a(\infty - 0) := a(+\infty) \quad \text{and} \quad a(\infty + 0) := a(-\infty).$$

Finally, we denote $C(\overline{\mathbb{R}}) := C(\mathbb{R}) \cap PC$ and $PC_0 := \{a \in PC : \ a(+\infty) = a(-\infty) = 0\}$. Clearly, PC is a C^*-algebra. Further, $C(\overline{\mathbb{R}})$ and PC_0 are C^*-subalgebras of PC.

Definition 5.12. We define SAP to be the smallest closed C^*-subalgebra of $L^\infty(\mathbb{R})$ generated by AP and $C(\overline{\mathbb{R}})$, i.e.,

$$SAP := \operatorname{alg}_{L^\infty(\mathbb{R})} \left(AP, C(\overline{\mathbb{R}})\right).$$

The set SAP is composed by elements called *semi-almost periodic functions*.

Definition 5.13. Let $a \in L^\infty(\mathbb{R})$. The set $\mathcal{R}(a)$ will denote the spectrum of a and we call it the *essential range* of a. We note that, if $\mu \in \mathcal{R}(a)$, then

$$\operatorname*{ess\,inf}_{x \in \mathbb{R}} |a(x) - \mu| = 0.$$

Lemma 5.14 (see [4, Theorem 1.15.]). *Let $a \in AP$ and $\mu \in \mathcal{R}(a)$. If $\varepsilon > 0$, then each set $]T, \infty[$ (and also each set $]-\infty, T[$) contains a point x such that $|a(x) - \mu| < \varepsilon$. In particular,*

$$\|a\|_\infty = \limsup_{x \to +\infty} |a(x)| = \limsup_{x \to -\infty} |a(x)|.$$

Theorem 5.15 (Sarason; see [4, Theorem 1.21]). *Let $u \in C(\overline{\mathbb{R}})$ be any function for which $u(-\infty) = 0$ and $u(+\infty) = 1$. If $a \in SAP$, then there exist $a_l, a_r \in AP$ and $a_0 \in C_0(\dot{\mathbb{R}})$ such that*

$$a = (1 - u)a_l + ua_r + a_0. \tag{5.1}$$

The functions a_l, a_r are uniquely determined by a and independent of the particular choice of u. The maps

$$a \mapsto a_l, \quad a \mapsto a_r$$

are C^*-algebra homomorphisms of SAP onto AP.

Proof. If $a \in AP$ then $a = (1 - u)a + ua + 0$ is the desired representation. For $f \in C(\overline{\mathbb{R}})$, the function $f_0 = f - (1 - u)f(-\infty) - uf(+\infty)$ belongs to $C_0(\dot{\mathbb{R}})$. Thus,

$$f = (1 - u)f(-\infty) + uf(+\infty) + f_0$$

is the seeked representation. Let $a_l, b_l, a_r, b_r \in AP$ and $a_0, b_0 \in C_0(\dot{\mathbb{R}})$. We have

$$((1-u)a_l + ua_r + a_0)\,((1-u)b_l + ub_r + b_0) = (1-u)^2 a_l b_l + u^2 a_r b_r + c_0,$$

where

$$c_0 = (1-u)u(a_l b_r + a_r b_l) + a_0\,((1-u)b_l + ub_r + b_0) + b_0\,((1-u)a_l + ua_r)\,.$$

In fact $c_0 \in C_0(\dot{\mathbb{R}})$, and as $(1-u)^2 - (1-u)$ and $u^2 - u$ belong to $C_0(\dot{\mathbb{R}})$, the product above is equal to

$$(1-u)a_l b_l + ua_r b_r + d_0,$$

where the terms $((1-u)^2 - (1-u))a_l b_l$, $(u^2 - u)a_r b_r$ and c_0 collapse into $d_0 \in C_0(\dot{\mathbb{R}})$. Since the functions of AP and $C(\overline{\mathbb{R}})$ can be written in the form (5.1), linear combinations and finite products are invariant under the representation, it follows that SAP is the closure of

$$S = \Big\{(1-u)a_l + ua_r + a_0 :\ a_l, a_r \in AP,\ a_0 \in C_0(\dot{\mathbb{R}})\Big\}\,.$$

For any $a_l, a_r \in AP$ and $a_0 \in C_0(\dot{\mathbb{R}})$ are such that $a = (1-u)a_l + ua_r + a_0$, by Lemma 5.14, we deduce that

$$\begin{aligned}\|a_l\|_\infty &= \limsup_{x\to-\infty} |a_l(x)| \\ &= \limsup_{x\to-\infty} |(1-u(x))a_l(x) + u(x)a_r(x) + a_0(x)| \\ &\le \|a\|_\infty\,,\end{aligned}$$

and, analogously, $\|a_r\|_\infty \le \|a\|_\infty$. So, if $(a^{(n)})$ is a Cauchy sequence of S, then $(a_l^{(n)})$ and $(a_r^{(n)})$ are Cauchy sequences in AP, consequently $(a_0^{(n)})$ is also a Cauchy sequence of $C_0(\dot{\mathbb{R}})$, with the norm of $L^\infty(\mathbb{R})$. Since AP and $C_0(\dot{\mathbb{R}})$ are complete, we conclude that S is a closed set.

Note that, if $a = (1-u)a_l + ua_r + a_0 = (1-u)a_l' + ua_r' + a_0'$, then by similar methods shown above, $\|a_l - a_l'\|_\infty \le \|a - a'\|_\infty = 0$ or, $a_l = a_l'$. Hence the map $a \mapsto a_l$ is well defined. It is easy to show that it is a homomorphism of C^*-algebras. Analogously, the map $a \mapsto a_r$ is a well defined homomorphism of C^*-algebras. □

Definition 5.16. Given a set E, we denote by E_N and $E_{N\times N}$ the columns of length N and the $N \times N$ matrices with entries in E, respectively. If E is a Banach or a Hilbert space, then so is E_N with the natural algebraic operations and the norm

$$\left\|(f_1, ..., f_N)^T\right\| := \left(\|f_1\|^2 + ... + \|f_N\|^2\right)^{1/2}.$$

In case E is a Banach algebra and $a \in E_{N\times N}$, we denote by aI the operator of multiplication by a on E_N. On providing $E_{N\times N}$ with natural algebraic operations and the norm

$$\|a\|_{E_{N\times N}} := \|aI\|_{\mathcal{B}(E_N)}\,,$$

where $\mathcal{B}(E_N)$ is the set of all bounded operators from E_N to itself, we make $E_{N\times N}$ a Banach algebra. If E is a C^*-algebra, then $E_{N\times N}$ is also a C^*-algebra. To avoid additional parantheses or brackets, we make use of the notation

$$L^\infty_{N\times N}(\mathbb{R}) := [L^\infty(\mathbb{R})]_{N\times N}, \quad L^2_N(\mathbb{R}) := \left[L^2(\mathbb{R})\right]_N,$$

and similar modifications. Also we denote $\mathbb{C}^N$, $\mathbb{C}^{N\times N}$ instead of $\mathbb{C}_N$, $\mathbb{C}_{N\times N}$, respectively.

5.3 Quotient and complementary spaces

We start with introducing brief definitions regarding linear algebra.

It is well-known that a subspace lying in a finite-dimensional normed space is closed. The maximal number of linearly independent vectors in a subspace M is called the *dimension* of M and is denoted by $\dim M$. If there are infinitely many independent vectors in M, then we set $\dim M = \infty$.

Now, let us recall some properties of topological complements discussed in introductory functional analysis lectures.

Definition 5.17. Let M and N be subspaces of a normed space X. The subspace M is called an *algebraic complement* of N in X, and we write $X = M \oplus N$, if $X = M + N$ and if their intersection is the null space. Furthermore, we say that M is a *topological complement* of N if it is algebraic complement of M and if they are both closed.

Definition 5.18. A *projection* on X is a linear operator $p\colon X \to X$ that is idempotent, i.e., $p^2 = p$.

Lemma 5.19 (see [24, Lemma 5.63]). *Let M and N be subspaces of a Banach space X. Suppose that $X = M \oplus N$. Then the following statements are equivalent:*

1. *The subspace M is a topological complement of N.*
2. *The subspaces M and N are closed.*
3. *The projections $p_M\colon X \to M$, with the law $m + n \mapsto m$ and $p_N\colon X \to N$, with the law $m + n \mapsto n$ are continuous.*

In a Hilbert space, every closed subspace has a topological complement (its orthogonal complement). But, in a Banach space, not every closed subspace has a topological complement. Meanwhile, in a linear space, every subspace has an algebraic complement.

Definition 5.20. Let X be a linear space and M a subspace of X. We introduce the equivalence relation on X as follows:

$$x \sim u \iff x - u \in M.$$

The equivalence classes

$$[x] = \{u \in X : x - u \in M\} = x + M$$

are elements of X/M. These elements are also known as hyperplanes. So X/M defines a linear space.

If X is normed space, then we can define a semi-norm on the space X/M as follows:

$$\|x + M\|_{X/M} := d(x, M) = \inf_{m \in M} \|x - m\| .$$

Indeed, the function $\|\cdot\|_{X/M}$ is a norm if, and only if M is closed. Furthermore, if X is a Banach space and M is a closed subspace of X, then X/M is a Banach space (see, e.g., [25, Chapter IV, Theorem 2.5]).

Theorem 5.21 (see [25, Chapter IV, Corollary 2.3])**.** *Let X and Y be linear spaces and let $A\colon X \to Y$ be a linear operator. Then $\operatorname{im} A$ is algebraically isomorphic to $X/\ker A$.*

Proof. Let us show that $\mathbf{A}\colon X/\ker A \to \operatorname{im} A$, $x + \ker A \mapsto Ax$ is an algebraic isomorphism. It is easy to show that $\mathbf{A}$ is a well-defined surjective linear operator. For injectivity, we notice that

$$\begin{aligned}
\ker \mathbf{A} &= \{x + \ker A \in X/\ker A :\ Ax = 0\} \\
&= \{x + \ker A \in X/\ker A :\ x \in \ker A\} \\
&= \{x + \ker A \in X/\ker A :\ x + \ker A = \ker A\} \\
&= \{\ker A\} .
\end{aligned}$$

Whence $\mathbf{A}$ is an isomorphism. □

Corollary 5.22 (see [25, Chapter IV, Proposition 2.2])**.** *Let X be a linear space and let M be a subspace of X. Suppose that N is an algebraic complement of M. Then X/M is algebraically isomorphic to N.*

Proof. Take $A = p_N\colon X \to N$, $m + n \mapsto n$. Then, by the previous result, $X/\ker p_N$ and $\operatorname{im} p_N$ are isomorphic. But $\operatorname{im} p_N = N$ and

$$\ker p_N = \{m + n \in M \oplus N :\ n = 0\} = M.$$

Therefore X/M and N are isomorphic. □

The next two results are known to be the isomorphism theorems, from the basic algebra course. These properties carry over the linear spaces, since the linear spaces are modules over a field (see [12, p. 168]).

Corollary 5.23 (see [12, Theorem 1.9])**.** *Let X be a linear space and let M and N be subspaces. Then $\frac{M+N}{M}$ is algebraically isomorphic to $\frac{N}{M \cap N}$.*

Proof. Consider the linear operator $A\colon N \to X/M$, such that $n \mapsto n + M$. Then

$$\operatorname{im} A = \{n + M : n \in N\} = \frac{M+N}{M},$$

and

$$\ker A = \{n \in N : n + M = M\} = \{n \in N : n \in M\} = M \cap N.$$

Hence, by Theorem 5.21 yields the desired isomorphism. ☐

Corollary 5.24 (see [12, Theorem 1.8])**.** *Let X be a linear space and let M and Z be subspaces of X satisfying $Z \subset M$. Then X/M is algebraically isomorphic to $\frac{X/Z}{M/Z}$.*

Proof. Consider the linear operator $A\colon X/Z \to X/M$, with the law $x + Z \mapsto x + M$. It is surjective, by construction, and,

$$\ker A = \{x + Z \in X/Z : x + M = M\} = \{x + Z \in X/Z : x \in M\} = M/Z.$$

Thus, by Theorem 5.21, yields the desired isomorphism. ☐

Corollary 5.25 (see [18, Corollary 7-4.9])**.** *Let X be a Banach space and let M and N be subspaces such that $M \cap N = \{0\}$ and $\dim N < \infty$. If M is closed, then $M + N$ is also closed.*

Proof. Consider the quotient map $Q_M\colon X \to X/M$, defined by the law $x \mapsto x + M$. Notice that, for every $x \in X$,

$$\|x + M\| = \inf_{m \in M} \|x - m\| \leq \|x\|,$$

hence $\|Q_M\| \leq 1$. Whence Q_M is continuous. By Corollary 5.23, $\dim \frac{M+N}{M} = \dim N < \infty$, so $\frac{M+N}{M}$ is closed. By continuity of Q_M, $(Q_M)^{-1}\left(\frac{M+N}{M}\right) = M + N$ is closed. ☐

We remark that $\|Q_M\| = 1$. In fact, for every $\varepsilon > 0$, the Riesz lemma (see, e.g., [24, Theorem 2.25]) provides us a $x \in X$ such that $\|x\| = 1$ and

$$1 - \varepsilon < d(x, M) \leq 1.$$

Therefore

$$\|Q_M\| = \sup_{\|x\|=1} \|x + M\| = 1.$$

Definition 5.26. Let M be a subspace of a Banach space X. The dimension of the quotient space X/M is called *codimension* of the subspace M, that is

$$\operatorname{codim} M := \dim X/M.$$

It is worthwile to notice that, if N is an algebraic complement of M, then $\operatorname{codim} M = \dim N$, by Corollary 5.22.

In a Hilbert space H, every closed subspace M has a topological complement $M^{\perp}$, the orthogonal complement of M. Moreover, H/M is isometrically isomorphic to $M^{\perp}$. In fact, for every $x \in H$ there are, $y \in M$ and $z \in M^{\perp}$ uniquely determined, such that $x = y + z$ (see, e.g., [24, Theorem 3.34]). Thus, Lemma 3.30 from [24] yields

$$\|x + M\| = \|z + M\| = \|z\| .$$

Now, generally for Banach spaces, not every closed subspace might be topologically complemented. The following theorems give us some conditions for which a closed subspace is topologically complemented.

Theorem 5.27 (see [9, Chapter 2, Theorem 2.1]). *Every finite-dimensional subspace M of a Banach space X is topologically complemented.*

Proof. Suppose that $(x_1, ..., x_n)$ is a basis of M and consider $\varphi_i \in X^*$ such that $\varphi_i(x_j) = \delta_{ij}$, $(i, j = 1, ..., n)$. We show that $N = \bigcap_{i=1}^{n} \ker \varphi_i$ is a complement of M. For every fixed $x \in X$, it is clear that $x - \sum_{j=1}^{n} \varphi_j(x) x_j \in N$. Whence, $M + N = X$. The subspace N is closed, since it is a finite intersection of kernels of continuous linear functionals. Now, suppose that there is $x \in M \cap N$. Then $x = \sum_{j=1}^{n} \alpha_j x_j$ and $\varphi_i(x) = 0$, for some scalars $\alpha_1, ..., \alpha_n$ and for every $i = 1, ..., n$. But then, for all the $i = 1, ..., n$, since $x \in \ker \varphi_i$,

$$\alpha_i = \sum_{j=1}^{n} \alpha_j \delta_{ij} = \sum_{j=1}^{n} \alpha_j \varphi_i(x_j) = \varphi_i(x) = 0.$$

Hence $x = 0$. Therefore N is a topological complement of M. □

Theorem 5.28 (see [9, Chapter 2, Theorem 2.2]). *Let X be a Banach space. Then every closed subspace M that has a finite codimension is topologically complemented.*

Proof. Let N be an algebraic complement of M. Then, by Corollary 5.22 $\dim N = \operatorname{codim} M < \infty$. Therefore N is closed, and the proof is complete. □

Theorem 5.29 (see [9, Chapter 2, Theorem 2.3]). *If a subspace M of a Banach space X contains a closed subspace of finite codimension, then M is closed.*

Proof. Let N be an finite-codimensional subspace of M and let L be its topological complement of N. So $X = L \oplus N$. We show that $M = (M \cap L) + N$. It is clear that $(M \cap L) + N \subset M$. Now, for every $x \in M$, there are $y \in L$ and $z \in N$ such that $x = y + z$. But $y = x - z \in L$ and, since $z \in N \subset M$, $y \in M$. Whence, $y \in M \cap L$ from which follows the other inclusion. Now, since N is closed, $M \cap L \cap N = \{0\}$ and

$$\dim(M \cap L) \leq \dim L = \operatorname{codim} N < \infty,$$

by Corollary 5.25, M must be closed. □

Definition 5.30. Let X and Y be Banach spaces. By $\mathcal{B}(X, Y)$ we denote the Banach space of all linear bounded operators mapping from X into Y, and $\mathcal{B}(X)$ denotes simply $\mathcal{B}(X, X)$. It is a Banach algebra with the usual pointwise sum and composition of functions. Let $A \in \mathcal{B}(X, Y)$. The subspace $Y/\overline{\operatorname{im} A}$ is referred to as the *cokernel* of the operator A. Every topological complement of $\overline{\operatorname{im} A}$ (provided that it exists) will also be called a cokernel of the operator A and denoted by $\operatorname{coker} A$. In fact, this definition is consistent since, if N is a topological complement of $\overline{\operatorname{im} A}$, then $Y/\overline{\operatorname{im} A}$ is topologically isomorphic to N.

By the *rank* of the operator A we understand the dimension of its image, i.e.,

$$\operatorname{rank} A := \dim \operatorname{im} A.$$

Notice that, given $A \in \mathcal{B}(X, Y)$, by Theorem 5.21, we thus have

$$\operatorname{rank} A = \dim \operatorname{im} A = \dim X/\ker A = \operatorname{codim} \ker A.$$

Also, if A has closed image in Y, then

$$\dim \operatorname{coker} A = \dim Y/\overline{\operatorname{im} A} = \operatorname{codim} \operatorname{im} A.$$

Theorem 5.31 (see [9, Chapter 4, Theorem 1.2]). *Let $A \in \mathcal{B}(X, Y)$ and let $\mathbf{A} \colon X/\ker A \to \overline{\operatorname{im} A}$, with $x + \ker A \mapsto Ax$. Then $\|\mathbf{A}\| = \|A\|$ and the linear operator $\mathbf{A}$ is a topological isomorphism if and only if $\operatorname{im} A$ is closed.*

Proof. For every $x \in X$,

$$\|x + \ker A\| = \inf_{y \in \ker A} \|x - y\| \leq \|x\|,$$

from which holds the inclusion

$$\{x \in X : \|x\| \leq 1\} \subset \{x \in X : \|x + \ker A\| \leq 1\}.$$

From here, we attain the first inequality

$$\|A\| = \sup_{\|x\| \leq 1} \|Ax\| = \sup_{\|x\| \leq 1} \|\mathbf{A}(x + \ker A)\| \leq \sup_{\|x + \ker A\| \leq 1} \|\mathbf{A}(x + \ker A)\| = \|\mathbf{A}\|.$$

For the second inequality, fix $x \in X$ and notice that

$$\|\mathbf{A}(x + \ker A)\| = \|Ax\| = \inf_{y \in \ker A} \|Ax - Ay\| \leq \|A\| \inf_{y \in \ker A} \|x - y\| = \|A\| \|x + \ker A\|.$$

Whence $\|\mathbf{A}\| \leq \|A\|$. □

Corollary 5.32. *Let M and N be topological complements of each other in a Banach space X. Then X/M is topologically isomorphic to N.*

Proof. Consider $p_N \colon X \to X$ the projection map defined by $x = m + n \mapsto n$, which is continuous, by hypothesis. It is now straightforward that $\ker p_N = M$ and $\operatorname{im} p_N = N$ and the desired conclusion arises. ☐

Definition 5.33. An operator $A \in \mathcal{B}(X,Y)$ is said to be *generalized invertible* if one can find an operator $B \in \mathcal{B}(Y,X)$ such that

$$ABA = A.$$

The operator B is referred to as a *generalized inverse operator* to A and denoted by $B = A^{(-1)}$.

Clearly, a linear bounded one-sided invertible operator is also generalized invertible. Let V be a left invertible operator from $\mathcal{B}(X)$. Obviously, the operator A from $\mathcal{B}(X \dotplus X)$ defined by

$$A := \begin{bmatrix} V & 0 \\ 0 & V^{-1} \end{bmatrix},$$

with V^{-1} being a left-inverse operator of V, is generalized invertible.

Lemma 5.34 (see [9, Chapter 4, Lemma 5.1])**.** *Let $A \in \mathcal{B}(X,Y)$ be generalized invertible. Then the operators $P_1 = AA^{(-1)}$ and $P_2 = A^{(-1)}A$ are projections, where*

$$\operatorname{im} A = \operatorname{im} P_1 \quad \textit{and} \quad \ker A = \ker P_2.$$

Proof. We first remark that, since P_1 and P_2 are compositions of bounded linear operators, $P_1 \in \mathcal{B}(Y)$, $P_2 \in \mathcal{B}(X)$, and

$$P_1^2 = \left(AA^{(-1)}A\right)A^{(-1)} = AA^{(-1)} = P_1,$$

$$P_2^2 = A^{(-1)}\left(AA^{(-1)}A\right) = A^{(-1)}A = P_2.$$

Now, notice that $y \in \operatorname{im} P_1$ if, and only if $y = P_1 y = A\left(A^{(-1)}y\right) \in \operatorname{im} A$. Whence $\operatorname{im} P_2 = \operatorname{im} A$. For the second equality, if $x \in \ker P_2$ then $A^{(-1)}Ax = 0$, so composing both sides by A, we get $Ax = 0$, which is $x \in \ker A$. The other inclusion is immediate, since $\ker A \subset \ker A^{(-1)}A = \ker P_2$. ☐

Theorem 5.35 (see [9, Chapter 4, Theorem 5.1])**.** *Necessary and sufficient conditions for the operator $A \in \mathcal{B}(X,Y)$ to be generalized invertible is that it has the following three properties:*

1. $\operatorname{im} A$ *is closed in* Y*;*
2. *the subspace* $\ker A$ *has a topological complement in* X*;*
3. *the subspace* $\operatorname{im} A$ *has a topological complement in* Y*;*

Proof. The necessity of the conditions of the theorem results from the previous lemma and from the fact that the image of a projection is topological complement of its kernel. We prove their sufficiency. Let M be a topological complement of $\ker A$ in X and N a topological complement to $\operatorname{im} A$ in Y. The operator $A|_M \colon M \to \operatorname{im} A$ is invertible. By $B \in \mathcal{B}(\operatorname{im} A, M)$ we denote the inverse operator of $A|_M$. Consider the projection $P = p_{\operatorname{im} A} \colon Y \to Y$ defined by the law $x = x_N + x_{\operatorname{im} A} \mapsto x_{\operatorname{im} A}$. Therefore, for each $x = x_M + x_{\ker A} \in X$, we have

$$ABPAx = ABAx = ABAx_M = ABA|_M x_M = Ax_M = Ax,$$

which shows that $A^{(-1)} = BP \in \mathcal{B}(Y, X)$. □

Corollary 5.36 (see [9, Chapter 4, Corollary 5.1]). *If H_1 and H_2 are Hilbert spaces, then the operator $A \in \mathcal{B}(H_1, H_2)$ is generalized invertible iff $\operatorname{im} A$ is closed in H_2.*

Corollary 5.37 (see [9, Chapter 4, Corollary 5.2]). *Every finite-rank operator from $\mathcal{B}(X, Y)$ is generalized invertible.*

Proof. Let $A \in \mathcal{B}(X, Y)$ be a finite-rank operator. Since $\dim \operatorname{im} A = \operatorname{rank} A < \infty$, $\operatorname{im} A$ is closed. Thus it has a topological complement. Furthermore, since

$$\operatorname{codim} \ker A = \operatorname{rank} A < \infty,$$

by Theorem 5.28, $\ker A$ has a topological complement. □

5.4 Compact operators

Definition 5.38. Let X be a topological space and let $S \subset X$. We say that S is a *precompact set* or a *relatively compact set* if $\overline{S}$ is compact. We say that S is *sequentially relatively compact set* if every sequence in S has a convergent subsequence in X.

On the metric spaces, compactness and sequential compactness are equivalent. We will show that those two definitions above are also equivalent on a metric space.

Lemma 5.39 (see [13, Chapter I, § 5, Corollary 2]). *Let X be a metric space and let $S \subset X$. Then S is precompact if, and only if S is sequentially relatively compact.*

Proof. For necessity, if (s_n) is a sequence of S, it is a sequence of the compact $\overline{S}$, from which we can extract a convergent subsequence.

For sufficiency, suppose that S is sequentially relatively compact and let (x_n) be a sequence in $\overline{S}$. Then, for every $n \in \mathbb{N}$, there is $s_n \in S$, such that $d(x_n, s_n) < 1/n$. By hypothesis, there is (s_{α_n}) a subsequence of (s_n) converging to some $x \in X$. By triangle inequality, we guarantee that $x_{\alpha_n} \to x$, as $n \to \infty$. □

Definition 5.40. A linear operator $A \in \mathcal{B}(X,Y)$ is said to be *compact* if the image of the unit ball is precompact.

Lemma 5.41 (see [25, Chapter II, Theorem 6.2])**.** *Let X and Y be normed spaces and let $A \in \mathcal{B}(X,Y)$. The following statements are equivalent:*

1. *A is compact.*
2. *The image of every bounded subset of X under A is precompact.*
3. *If (x_n) is a bounded sequence of X, then (Ax_n) has a convergent subsequence in Y.*

Proof. [1. ⇒ 2.] Suppose that A is a compact operator, and put

$$\mathbb{B} = \{x \in X : \|x\| \leq 1\}.$$

Let $B \subset X$ be a bounded subset of X. Then, there exists $M > 0$ such that

$$\forall x \in B \quad \|x\| \leq M.$$

Hence

$$\tilde{B} = \frac{1}{M} B \subset \mathbb{B}.$$

But then, it follows that the image of $\tilde{B}$ is contained in a compact space, so it is a precompact set. Since the compactness is invariant under finite dilatation, we conclude that the image of B is precompact.

[2. ⇒ 3.] Suppose that the image of any bounded subset of X under A is precompact and let $B = \{x_n \in X : n \in \mathbb{N}\}$ be a bounded sequence. Then, by hypothesis, the image of B is a precompact set of Y, that is, from

$$\{Ax_n \in Y : n \in \mathbb{N}\}$$

we can extract a convergent subsequence.

[3. ⇒ 1.] Suppose that, for every (x_n) bounded sequence of X, there is a convergent subsequence of (Ax_n). We will prove that

$$A\mathbb{B} = \{Ax : x \in \mathbb{B}\},$$

is a precompact set. Let (y_n) be a sequence of the later set. Then $y_n = Ax_n$ for some $x_n \in \mathbb{B}$. Since $\|x_n\| \leq 1$, for all the $n \in \mathbb{N}$, by hypothesis, there is (Ax_{α_n}) a subsequence of (Ax_n) that converges in Y. Thus (y_{α_n}) converges in Y. $\square$

It is worthwhile to notice that if $A \in \mathcal{B}(X,Y)$ with $\dim X < \infty$, since $\mathbb{B} = \{x \in X : \|x\| \leq 1\}$ is compact, its image under A is compact, which allows us to conclude that A is compact. On the other hand, every finite-rank operator is compact. This fact can easily be checked by applying the last criteria from the previous lemma alongside with the Bolzano-Weierstrass theorem.

Theorem 5.42 (see [5, Theorem 6.1])**.** *Let X and Y be Banach spaces. Then, the set $\mathcal{K}(X,Y)$ is a closed linear space of $\mathcal{B}(X,Y)$.*

Theorem 5.43 (see [5, Proposition 6.3])**.** *Let X, Y and Z be Banach spaces. If at least one of $A \in \mathcal{B}(X,Y)$ and $B \in \mathcal{B}(Y,Z)$ is compact, then $BA \in \mathcal{B}(X,Z)$ is so.*

Proof. Let (x_n) be a bounded sequence of X. If A is compact, then there is (x_{α_n}) a subsequence of (x_n) such that (Ax_{α_n}) converges in Y. By continuity of B, (BAx_{α_n}) converges in Z.

If B is compact, then, from (Ax_n) a bounded sequence of Y, we can extract a subsequence (Ax_{β_n}) such that (BAx_{β_n}) converges in Z. ☐

Theorem 5.44 (see [5, Theorem 6.4])**.** *Let X and Y be Banach spaces. Then, $A \in \mathcal{B}(X,Y)$ is compact if and only if the dual operator $A' \in \mathcal{B}(Y^*,X^*)$ is compact.*

5.5 Riesz-Schauder theory

In this section, we will study a result concerning operators of the form

$$A_\lambda = A - \lambda I \in \mathcal{B}(X)$$

where λ is a nonzero scalar and A is a compact operator on a Banach space X. It is worthwhile to remind, from linear algebra, that

$$\dim X = \dim\ker A + \dim\operatorname{im} A,$$

for any $A \in \mathcal{B}(X)$, whenever $\dim X < \infty$. This implies that

$$\dim\operatorname{coker} A = \dim X/\operatorname{im} A = \dim Z = \dim X - \dim\operatorname{im} A = \dim\ker A.$$

Indeed, from the fact that $\operatorname{im} A$ has an algebraic complement Z, in X, and from the above equality alongside with Corollary 5.22, we arrive at the desired equality.

Definition 5.45. Let X and Y be Banach spaces. By $\mathcal{K}(X,Y)$ we denote the subspace of all the compact operators. If $X = Y$, we instead write $\mathcal{K}(X)$.

Lemma 5.46 (see [25, Chapter VI, Lemma 3.1])**.** *Let X be a Banach space and $A \in \mathcal{K}(X)$ an operator with $\operatorname{rank} A < \infty$. Then, there are M a finite-dimensional subspace and N a closed subspace of X, such that*

$$X = M \oplus N, \quad A(M) \subset M, \quad N \subset \ker A.$$

Proof. Since A has finite rank, then there are $x_1, ..., x_n \in X$ such that $(Ax_1, ..., Ax_n)$ constitutes a basis of $\operatorname{im} A$. It is straightforward, by construction that

$$M = \operatorname{span}\{x_i, Ax_i : \ i = 1, ..., n\}$$

is A-invariant. On the other hand, there are linear functionals $\varphi_1, ..., \varphi_n \in (\operatorname{im} A)^*$ such that

$$\varphi_j(Ax_i) = \delta_{ij}, \quad (i, j = 1, ..., n).$$

It is clear that, for each $x \in X$,

$$x - \sum_{j=1}^{n} \varphi_j(Ax)x_j \in \ker A,$$

from which yields

$$X = M + \ker A.$$

Furthermore, since M is finite-dimensional, then there is a closed subspace N of $\ker A$, such that

$$\ker A = (M \cap \ker A) \oplus N.$$

Therefore, the two above equalities tell us that $X = M \oplus N$, which finishes the proof. ☐

Theorem 5.47 (see [25, Chapter VI, Theorem 2.1]). *Let X be a normed space and Y be a Banach space with the approximation property, i.e., there is a sequence (P_n) of elements of $\mathcal{B}(Y)$ with finite rank, such that, for every $y \in Y$,*

$$\lim_{n\to\infty} \|P_n y - y\|_Y = 0.$$

Then any compact operator $A \in \mathcal{K}(X, Y)$ is the limit in $\mathcal{B}(X, Y)$ of operators of finite rank.

Proof. Let $A \in \mathcal{K}(X, Y)$ and let (P_n) be a sequence of finite rank operators in $\mathcal{B}(Y)$ such that, for every $y \in Y$, $\|P_n y - y\|_Y \to 0$. We will show that $P_n A \to A$ in $\mathcal{B}(X, Y)$. For the sake of contradiction, suppose that does not happen. Then, there exists an increasing sequence of natural numbers (α_n) such that, for any $n \in \mathbb{N}$,

$$\|P_{\alpha_n} A - A\| \geq \varepsilon,$$

for some $\varepsilon > 0$. By definitions of norm operator and supremum, there is a sequence (x_n) of elements of the unit ball of X such that

$$\|P_{\alpha_n} Ax_n - Ax_n\|_Y > \varepsilon/2, \quad (n \in \mathbb{N}).$$

Being A a compact operator, there is an increasing sequence of natural numbers (β_n) such that (Ax_{β_n}) converges to some $y \in Y$. Also, being $(P_n y)$ a bounded in Y, by the uniform boundedness principle, (P_n) is bounded, say, by some $c > 0$. Hence, we have

$$\begin{aligned}\left\|P_{\alpha_{\beta_n}} Ax_{\beta_n} - Ax_{\beta_n}\right\|_Y &\leq \left\|P_{\alpha_{\beta_n}} Ax_{\beta_n} - P_{\alpha_{\beta_n}} y\right\|_Y + \left\|P_{\alpha_{\beta_n}} y - y\right\|_Y + \|y - Ax_{\beta_n}\|_Y \\ &\leq (c+1)\|Ax_{\beta_n} - y\|_Y + \left\|P_{\alpha_{\beta_n}} y - y\right\|_Y \to 0,\end{aligned}$$

which contradicts the fact that, for all the $n \in \mathbb{N}$, we have

$$\left\| P_{\alpha_{\beta_n}} A x_{\beta_n} - A x_{\beta_n} \right\|_Y > \varepsilon/2,$$

and this finishes the proof. $\square$

Theorem 5.48 (see [25, Chapter VI, Theorem 3.2]). *Let X be a Banach space with the approximation property, $A \in \mathcal{K}(X)$ and λ a nonzero scalar. Then, for the operator $A_\lambda = A - \lambda I$, we have:*

(i) The image of A_λ is closed in X;

(ii) $\dim \ker A_\lambda < \infty$;

(iii) $\dim \ker A_\lambda = \dim \operatorname{coker} A_\lambda$.

Proof. Firstly, suppose that A has finite rank. Then, by Lemma 5.46, there are M, N closed subspaces of X, with $\dim M < \infty$, such that

$$X = M \oplus N, \quad A(M) \subset M, \quad N \subset \ker A.$$

Then

$$\ker A_\lambda = \ker A_\lambda|_M$$

and

$$\operatorname{im} A_\lambda = \operatorname{im} A_\lambda|_M \oplus N.$$

Indeed, if $x \in \ker A_\lambda$, then $Ax = \lambda x$ and we can write $x = m + n$, with $m \in M$ and $n \in N$. Thus

$$x = \lambda^{-1} A x = \lambda^{-1} A m \in M,$$

whence yields the first equality, since the other inclusion is immediate.

For the second one, if $y \in \operatorname{im} A_\lambda$, then $y = A_\lambda x$, for some $x \in X$. But we can write x as $m + n$ with $m \in M$ and $n \in N$. Thus

$$y = A_\lambda x = A_\lambda m + A_\lambda n = A_\lambda m - \lambda n \in \operatorname{im} A_\lambda|_M \oplus N.$$

On the other hand, if $y \in \operatorname{im} A_\lambda|_M + N$, then $y = A_\lambda m + n$ for some $m \in M$ and $n \in N$. Since $n = -\lambda^{-1} A_\lambda n$, we get

$$y = A_\lambda(m - \lambda^{-1} n) \in \operatorname{im} A_\lambda,$$

from which we obtain the second inequality.

Since $\operatorname{im} A_\lambda|_M$ is finite-dimensional and N is closed, by Corollary 5.25, $\operatorname{im} A_\lambda$ is closed. It is also clear that $\ker A_\lambda$ is finite-dimensional. Moreover, since $A_\lambda|_M \in \mathcal{B}(M)$ and $\dim M < \infty$, this yields

$$\dim \ker A_\lambda|_M = \dim \operatorname{coker} A_\lambda|_M,$$

from which it follows that

$$\dim\ker A_\lambda = \dim\operatorname{coker} A_\lambda.$$

In fact, it is enough to show that

$$\dim\operatorname{coker} A_\lambda = \dim\operatorname{coker} A_\lambda|_M.$$

So, applying Corollary 5.24 alongside with the facts that M is isomorphic to X/N and $\operatorname{im} A_\lambda|_M$ is isomorphic to $\operatorname{im} A_\lambda/N$ under the same law, one has

$$\dim\operatorname{coker} A_\lambda = \dim\frac{X/N}{\operatorname{im} A_\lambda/N} = \dim\frac{M}{\operatorname{im} A_\lambda|_M} = \dim\operatorname{coker} A_\lambda|_M.$$

Now, suppose that $A \in \mathcal{K}(X)$ has non-finite rank and that X has the approximation property. Then by Theorem 5.47, there is $A_0 \in \mathcal{K}(X)$ an operator with finite rank such that

$$\|A - A_0\| < |\lambda|.$$

Then, the operator $B = I - \lambda^{-1}(A - A_0)$ is invertible in $\mathcal{B}(X)$ and

$$A_\lambda = A - \lambda I = A_0 - \lambda B = (A_0B^{-1} - \lambda I)B = (A_0B^{-1})_\lambda S.$$

From the last equality, it is not hard to show that

$$\operatorname{im} A_\lambda = \operatorname{im}(A_0B^{-1})_\lambda \quad \text{and} \quad B(\ker A_\lambda) = \ker(A_0B^{-1})_\lambda.$$

Since $\operatorname{rank} A_0 < \infty$, we can say that $\operatorname{im} A_\lambda$ is closed, $\ker A_\lambda$ is finite-dimensional and,

$$\dim\ker A_\lambda = \dim\ker(A_0B^{-1})_\lambda = \dim\operatorname{coker}(A_0B^{-1})_\lambda = \dim\operatorname{coker} A_\lambda,$$

as desired. ☐

In fact, this property remains true for a general Banach space (see, e.g., [5, Theorem 6.6]).

5.6 Fredholm operators

Definition 5.49. An operator $A \in \mathcal{B}(X,Y)$ is called a *Fredholm operator* if it has a closed image and the numbers $\dim\ker A$ and $\dim\operatorname{coker} A$ are finite. The number

$$\operatorname{Ind} A := \dim\ker A - \dim\operatorname{coker} A$$

is referred to as the *index* of the operator $A \in \mathcal{B}(X,Y)$.

By virtue of Theorem 5.35, every Fredholm operator is generalized invertible. So, the Fredholmness would be a generalization of the notion of invertibility, as expected. In fact, if $A \in \mathcal{B}(X,Y)$ is an operator acting on finite-dimensional spaces, then it is straightforward that A is a Fredholm operator. It is worthwhile to remark that generally an invertible operator $A \in \mathcal{B}(X, Y)$ acting on Banach spaces has index equal to zero. Moreover, the last theorem, with its own notations, says that A_λ is a Fredholm operator with $\operatorname{Ind} A = 0$.

Let us next provide a characterisation for the Fredholm operator.

Definition 5.50. We shall say that an operator $A \in \mathcal{B}(X, Y)$ *admits a regularization* if there exists an operator $F \in \mathcal{B}(Y, X)$ such that any of the operators $FA - I$ and $AF - I$ is compact.

The operator F is said to be a *regularizer* of A.

Theorem 5.51 (see [9, Chapter 4, Theorem 7.1])**.** *For an operator $A \in \mathcal{B}(X, Y)$ to be Fredholm it is necessary and sufficient that it admits a regularization.*

Proof. Let $F \in \mathcal{B}(Y, X)$ be a regularizer of A, i.e., $FA = I + T_1$ and $AF = I + T_2$, where T_1 and T_2 are compact operators. Since

$$\operatorname{im} A \supset \operatorname{im} AF = \operatorname{im}(I + T_2),$$

by Theorem 5.48 and by Theorem 5.29, the image of A is closed. From the same relations, it follows that

$$\ker A \subset \ker FA = \ker(I + T_1),$$

hence

$$\dim \ker A \leq \dim \ker(I + T_1) < \infty,$$

and

$$\dim \operatorname{coker} A \leq \dim \operatorname{coker}(I + T_2) < \infty.$$

To see the validity of the later, we fix $(y_1 + \operatorname{im} AF, ..., y_n + \operatorname{im} AF)$ a basis of $\operatorname{coker} AF$ and choose an arbitrary $y \in Y$, so $y + \operatorname{im} A$ is an element of $\operatorname{coker} A$. So, there are $\lambda_1, ..., \lambda_n$ scalars such that

$$\begin{aligned} y + \operatorname{im} AF &= \sum_{i=1}^{n} \lambda_i (y_i + \operatorname{im} AF) \\ &= \left(\sum_{i=1}^{n} \lambda_i y_i \right) + \operatorname{im} AF, \end{aligned}$$

from which

$$y - \sum_{i=1}^{n} \lambda_i y_i \in \operatorname{im} AF \subset \operatorname{im} A.$$

Therefore

$$y + \operatorname{im} A = \left(\sum_{i=1}^{n} \lambda_i y_i \right) + \operatorname{im} A = \sum_{i=1}^{n} \lambda_i (y_i + \operatorname{im} A) \in \operatorname{coker} A.$$

This concludes the Fredholmness of A.

Conversely, suppose that A is a Fredholm operator. Then, as it was noted before, it has a generalized inverse. Regarding to Lemma 5.34, altogether with its notations,

$$P_1 = AA^{(-1)} \quad \text{and} \quad P_2 = A^{(-1)}A,$$

we have

$$\operatorname{rank}(I - AA^{(-1)}) = \dim \underbrace{\operatorname{im}(I - P_1)}_{\ker P_1} = \dim \mathcal{B}_2 / \operatorname{im} P_1 = \dim \mathcal{B}_2 / \operatorname{im} A = \dim \operatorname{coker} A,$$

and

$$\operatorname{rank}(I - A^{(-1)}A) = \dim \operatorname{im}(I - P_2) = \dim \ker P_2 = \dim \ker A.$$

From this we conclude that $A^{(-1)}$ is regularizer of A. □

It can be easily seen that two regularizers F_1 and F_2 of an operator differ from each other only by a compact summand.

Note that, together with the last theorem, in principle the following statement has been proved.

Corollary 5.52 (see [9, Chapter 4, Theorem 7.1']). *The following assertions concerning an opeartor $A \in \mathcal{B}(X, Y)$ are equivalent:*

1. *The operator A is Fredholm.*
2. *There exist operators $F_1, F_2 \in \mathcal{B}(Y, X)$ such that the operators $F_1A - I$ and $AF_2 - I$ have finite rank.*
3. *There exist operators $F_1, F_2 \in \mathcal{B}(Y, X)$ such that the operators $F_1A - I$ and $AF_2 - I$ are compact.*
4. *There exists an operator $F \in \mathcal{B}(Y, X)$ such that the operators $AF - I$ and $FA - I$ have finite rank.*

Let X be a Banach space. It has already been shown that $\mathcal{K}(X)$ is a closed subspace of $\mathcal{B}(X)$ (Theorem 5.42), and thus $\mathcal{B}(X)/\mathcal{K}(X)$ is a Banach space. Furthermore, by Theorem 5.43, we can say that $\mathcal{K}(X)$ is an ideal of $\mathcal{K}(X)$, and thus $\mathcal{B}(X)/\mathcal{K}(X)$ is a Banach algebra.

Definition 5.53. In above configurations, the Banach algebra $\mathcal{B}(X)/\mathcal{K}(X)$ is called the *Calkin algebra*. Its elements are of the form $A + \mathcal{K}(X)$, with $A \in \mathcal{B}(X)$.

Corollary 5.54 (see [9, Chapter 4, Theorem 7.2]). *Let X be a Banach space. Then $A \in \mathcal{B}(X)$ is a Fredholm operator if and only if $A + \mathcal{K}(X)$ is invertible in the Calkin algebra.*

5.7 Towards the main problem

Definition 5.55. Let X be a Banach space and let $(A_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{B}(X)$ and $A \in \mathcal{B}(X)$. Then we say that $(A_n)_{n\in\mathbb{N}}$ converges to A, as $n \to \infty$:

- *in the norm* if $\|A_n - A\|_{\mathcal{B}(X)} \to 0$;
- *strongly* if $\|(A_n - A)f\|_X \to 0$ for every $f \in X$;
- *weakly* if $(\psi, (A_n - A)f) \to 0$ for every pair $f \in X$ and $\psi \in X^*$.

Lemma 5.56 (see [21, Lemma 1.4.1])**.** *Suppose that X is a Banach space. Further, let $(A_n)_{n\in\mathbb{N}}$ be a bounded sequence of elements in $\mathcal{B}(X)$.*

1. *If $(\psi, A_n f) \to 0$ for all f and ψ belonging to a dense subset of X and X^*, respectively, then the sequence (A_n) converges weakly to 0.*
2. *If $\|A_n f\| \to 0$ for each f belonging to a dense subset of X, then the sequence (A_n) converges strongly to 0.*

Proof. Let us prove the first statement, as the second one follows analogously. Put $M := \sup \|A_n\|$ and fix $\varepsilon > 0$. For $f \in X$ and $\psi \in X^*$ choose f^ε in the dense subset of X and ψ^ε in the dense subset of X^* such that $\|f - f^\varepsilon\| < \varepsilon$ and $\|\psi - \psi^\varepsilon\| < \varepsilon$. Then, for any $n \in \mathbb{N}$,

$$\begin{aligned}|(\psi, A_n f)| &\leq |(\psi - \psi^\varepsilon, A_n f)| + |(\psi^\varepsilon, A_n f)| \\ &\leq \|\psi - \psi^\varepsilon\| \, \|A_n f\| + |(\psi^\varepsilon, A_n(f - f^\varepsilon))| + |(\psi^\varepsilon, A_n f^\varepsilon)| \\ &\leq \varepsilon M \|f\| + (\|\psi\| + \varepsilon) M \varepsilon + |(\psi^\varepsilon, A_n f^\varepsilon)| .\end{aligned}$$

By hypothesis, there is $N \in \mathbb{N}$ such that $|(\psi^\varepsilon, A_n f^\varepsilon)| < \varepsilon$ for $n \geq N$. So, if $n \geq N$, we have

$$|(\psi, A_n f)| \leq \varepsilon M (\|f\| + \|\psi\| + \varepsilon) + \varepsilon,$$

whence, the weak convergence of $(A_n)_{n\in\mathbb{N}}$ to zero. □

Lemma 5.57 (see [21, Lemma 1.4.6])**.** *If $(A_n)_{n\in\mathbb{N}}$ a sequence of elements of $\mathcal{B}(X)$ that converges weakly to $A \in \mathcal{B}(X)$, and $K \in \mathcal{K}(X)$, then $(KA_n)_{n\in\mathbb{N}}$ converges strongly to KA.*

Proof. For simplicity, suppose that (A_n) converges weakly to the zero operator. Let $f \in X$ and put $f_n = A_n f$. Then (f_n) converges weakly to the zero operator. By the uniform boundedness principle, the sequence (f_n) is bounded. Withought loss of generality, we assume that $\|f_n\| \leq 1$, for all the $n \in \mathbb{N}$. Since K is compact, there is (f_{α_n}) a subsequence of (f_n) such that (Kf_{α_n}) converges to some $f_0 \in X$. Since K is continuous, (Kf_{α_n}) converges weakly to $K0 = 0$. So $f_0 = 0$, as the weak topology is Hausdorff. Since 0 is the only cluster point of the sequence (Kf_n) and this sequence is contained in a compact set, $\|Kf_n\| \to 0$. □

Definition 5.58. For each $h \in \mathbb{R}$ consider the linear operator U_h defined on a Banach function space X, by the law $f(x) \mapsto f(x-h)$. This operator is called a *translation operator* or, also known as a *shift operator.*

Remark. Some of the properties of this operator are straightforth in specific spaces. Let us consider $X = X(\mathbb{R})$ to be a rearrangement-invariant Banach function space with the Lebesgue measure and let $h \in \mathbb{R}$. Then, yields:

(a) The shift operator is an isometry.

(b) $(U_h)^{-1} = U_{-h}$, for $h \in \mathbb{R}$.

(c) Additionally, if $X = X_a$, then $(U_h)^* = U_{-h}$, $(h \in \mathbb{R})$.

Proof. (a) Indeed, since the Lebesgue measure is translation-invariant, we have

$$m_{U_h f}(\lambda) = m\left\{x \in \mathbb{R} : \ |f(x-h)| > \lambda\right\} = m\left(h + \{|f| > \lambda\}\right) = m\left\{|f| > \lambda\right\} = m_f(\lambda),$$

for any $h, \lambda \in \mathbb{R}$ and any $f \in X$. Therefore $\|U_h f\| = \|f\|$, by the invariance of the rearrangement.

(b) Obvious.

(c) By the uniqueness of $(U_h)^*$, it is sufficient to prove that

$$\langle U_h f, g \rangle = \langle f, U_{-h} g \rangle$$

holds true, for each $f \in X$ and $g \in X'$. Indeed, by the change of variable $t = x - h$, we get

$$\langle U_h f, g \rangle = \int_{\mathbb{R}} f(x-h)\overline{g(x)}dx = \int_{\mathbb{R}} f(t)\overline{g(t+h)}dt = \langle f, U_{-h} g \rangle .$$

□

Next, we will introduce the notion of the limit operator on Banach function spaces with absolutely continuous norm. Of course, it can be defined on arbitrary Banach spaces, but, for our purposes, it is not necessary.

Definition 5.59. Let X be a Banach function space with $X = X_a$, $A \in \mathcal{B}(X)$ and $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ a sequence of isometries. If the strong limits

$$A_{\mathcal{U}} := \operatorname*{s-lim}_{n \to \infty} \left(U_n^{-1} A U_n\right) \text{ in } \mathcal{B}(X), \quad A_{\mathcal{U}^*} := \operatorname*{s-lim}_{n \to \infty} \left(U_n^{-1} A U_n\right)^* \text{ in } \mathcal{B}(X')$$

exist, then always $(A_{\mathcal{U}})^* = A_{\mathcal{U}^*}$, and we will refer to $A_{\mathcal{U}}$ as the *limit operator* for the operator A with respect to the sequence $\mathcal{U}$. Note that usually the strong limit $A_{\mathcal{U}}$ is defined independently of the existence of the strong limit $A_{\mathcal{U}^*}$, while we need the existence of both limits for our purposes. If the limit $A_{\mathcal{U}}$ exists, then it is uniquely determined by A and $\mathcal{U}$, which justifies the notation $A_{\mathcal{U}}$.

Theorem 5.60 (see [15, Corollary 2]). *Let $\mathcal{U} = (U_{h_n})_{n\in\mathbb{N}}$ with $|h_n| \to \infty$ and let $X = X(\mathbb{R})$ be a reflexive rearrangement-invariant Banach function space with Lebesgue measure. Then $(U_{h_n})_{n\in\mathbb{N}}$ converges weakly to the zero operator.*

As a consequence of the latter theorem is that, in the same conditions, we can say that, if $K \in \mathcal{B}(X)$ is a compact operator, then $K_{\mathcal{U}} = 0$, by Lemma 5.57. Furthermore, Theorem 5.44 assures that K^* is compact, and hence

$$\begin{aligned} K_{\mathcal{U}^*} &= \operatorname*{s-lim}_{n\to\infty} \left(U_{-h_n} K U_{h_n}\right)^* \\ &= \operatorname*{s-lim}_{n\to\infty} \left(U_{h_n}\right)^* K^* \left(U_{h_n}\right)^* \\ &= \operatorname*{s-lim}_{n\to\infty} U_{-h_n} K^* U_{h_n} \\ &= 0. \end{aligned}$$

Therefore, the limit operator of K with respect to $\mathcal{U}$ is the zero operator.

Lemma 5.61 (see [14, Lemma 4.1]). *Let X be a Banach space and let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a sequence of isometries of X.*

(a) If $A \in \mathcal{B}(X)$ and $A_{\mathcal{U}}$ exists, then $\|A_{\mathcal{U}}\|_{\mathcal{B}(X)} \leq \|A\|_{\mathcal{B}(X)}$.

(b) If $A, B \in \mathcal{B}(X)$, $\alpha \in \mathbb{C}$, and if the strong limits $A_{\mathcal{U}}$ and $B_{\mathcal{U}}$ exist, then the strong limits $(\alpha A)_{\mathcal{U}}$, $(A+B)_{\mathcal{U}}$ $(AB)_{\mathcal{U}}$ also exist and

$$(\alpha A)_{\mathcal{U}} = \alpha A_{\mathcal{U}}, \quad (A+B)_{\mathcal{U}} = A_{\mathcal{U}} + B_{\mathcal{U}}, \quad (AB)_{\mathcal{U}} = A_{\mathcal{U}} B_{\mathcal{U}}.$$

(c) If $A \in \mathcal{B}(X)$ and if $(A_m)_{m\in\mathbb{N}}$ is a sequence on $\mathcal{B}(X)$ such that $(A_m)_{\mathcal{U}}$ exists for every $m \in \mathbb{N}$ and $\|A_m - A\|_{\mathcal{B}(X)} \to 0$, as $m \to \infty$, then the strong limit $A_{\mathcal{U}}$ exists and $\|A_{\mathcal{U}} - (A_m)_{\mathcal{U}}\|_{\mathcal{B}(X)} \to 0$, as $m \to \infty$.

Remark. It is worthwhile to recall that, if a Banach function space X with absolutely continuous norm and the limit operator $A_{\mathcal{U}}$ exists, then

$$A_{\mathcal{U}^*} = (A_{\mathcal{U}})^*.$$

So, Lemma 5.61 (b) yields

$$(\alpha A)_{\mathcal{U}^*} = \overline{\alpha} A_{\mathcal{U}^*}, \quad (A+B)_{\mathcal{U}^*} = A_{\mathcal{U}^*} + B_{\mathcal{U}^*}, \quad (AB)_{\mathcal{U}^*} = B_{\mathcal{U}^*} A_{\mathcal{U}^*},$$

as long as the limit operators $A_{\mathcal{U}^*}$ and $B_{\mathcal{U}^*}$ exist.

Theorem 5.62 (see [14, Theorem 4.2]). *Let X be a Banach space, let $\mathfrak{A}$ be a closed subalgebra of $\mathcal{B}(X)$, and let $\mathfrak{J}$ be a closed two-sided ideal of $\mathfrak{A}$. Suppose $A \in \mathfrak{A}$ and $\mathcal{U} = (U_n)_{n\in\mathbb{N}} \subset \mathcal{B}(X)$ is a sequence of isometries such that the limit operator $A_{\mathcal{U}}$ exists and the limit operators $J_{\mathcal{U}}$ exist and are equal to zero for all $J \in \mathfrak{J}$. If the coset $A + \mathfrak{J}$ is invertible in the quotient algebra $\mathfrak{A}/\mathfrak{J}$, then the limit operator $A_{\mathcal{U}}$ is invertible.*

Corollary 5.63. *Let X be a reflexive rearrangement-invariant Banach function space. Suppose that $A \in \mathcal{B}(X)$ and $\mathcal{U} = (U_n)_{n\in\mathbb{N}} \subset \mathcal{B}(X)$ is a sequence of isometries. If both of the strong limits $A_{\mathcal{U}}$ and $A_{\mathcal{U}^*}$ exist and if A is a Fredholm operator, then the limit operator $A_{\mathcal{U}}$ is invertible.*

Proof. By putting $\mathfrak{A} = \mathcal{B}(X)$, $\mathfrak{J} = \mathcal{K}(X)$ and using Theorem 5.60, the desired conclusion arises. ☐

Definition 5.64. For $a \in L^\infty_{N\times N}(\mathbb{R})$ and $h \in \mathbb{R}$, we define $a_h \in L^\infty_{N\times N}(\mathbb{R})$ by $a_h(x) := a(x+h)$, $x \in \mathbb{R}$.

Before proving the required result, we establish a property.

Lemma 5.65 (see [4, Lemma 10.2])**.** *If $a_1, ..., a_M \in AP^0_{N\times N}$, then there exists a real-valued sequence $h_n \to +\infty$ such that, for each $m \in \{1, ..., M\}$,*

$$\lim_{n\to\infty} \|(a_m)_{h_n} - a_m\|_\infty = 0.$$

The consequence of the last lemma will be useful in the proof of the main result, which is the following:

Theorem 5.66. *For every $a \in AP_{N\times N}$, there exists a real valued sequence $h_n \to +\infty$ such that*

$$\lim_{n\to\infty} \|(a)_{h_n} - a\|_\infty = 0.$$

Proof. It is sufficient to prove this fact for $N = 1$. For every $k \in \mathbb{N}$ and $a \in AP$, there is $a^{(k)} \in AP^0$ such that $\left\|a^{(k)} - a\right\|_\infty < 1/k$. According to the latter lemma, there is a sequence of real numbers $\left(h_n^{(k)}\right)_{n\in\mathbb{N}}$ that converges to $+\infty$ and, for some $N_k \in \mathbb{N}$, yields

$$\left\|\left(a^{(k)}\right)_{h_n^{(k)}} - a^{(k)}\right\|_\infty < 1/k,$$

for every $n \geq N_k$. On the other hand, for each $k \in \mathbb{N}$, there is $M_k \in \mathbb{N}$ satisfying

$$h_n^{(k)} > k,$$

for every $n \geq M_k$. So if we consider $(\alpha_k)_{k\in\mathbb{N}}$ to be strictly increasing sequence in $\mathbb{N}$ such that, for every $k \in \mathbb{N}$, $\alpha_k \geq \max\{N_k, M_k\}$, then the real-valued sequence $t_k := h_{\alpha_k}^{(k)}$ is such that $t_k \to +\infty$, as $k \to \infty$, and

$$\left\|\left(a^{(k)}\right)_{t_k} - a^{(k)}\right\|_\infty < \frac{1}{k}, \quad (k = 1, 2, ...).$$

Thus, for every $k \in \mathbb{N}$, we have

$$\begin{aligned}\|(a)_{t_k} - a\|_\infty &= \left\|\left(a - a^{(k)}\right)_{t_k} + \left(a^{(k)}\right)_{t_k} - a^{(k)} + a^{(k)} - a\right\|_\infty \\ &\leq \left\|\left(a - a^{(k)}\right)_{t_k}\right\|_\infty + \left\|\left(a^{(k)}\right)_{t_k} - a^{(k)}\right\|_\infty + \left\|a^{(k)} - a\right\|_\infty \\ &< 3k^{-1}.\end{aligned}$$

Therefore,

$$\lim_{k\to\infty} \|(a)_{t_k} - a\|_\infty = 0.$$

□

Let $X = X(\mathbb{R})$ be a rearrangement-invariant Banach function space on $\mathbb{R}$. Consider the *Cauchy singular integral operator*

$$(Sf)(x) := \frac{1}{\pi i}\text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{y-x} dy,$$

acting on $f \in L^1_{loc}(\mathbb{R})$. We may notice that $S = iH$ where H denotes the Hilbert transform. Hence, by Theorem 4.23, the operator S belongs to $\mathcal{B}(X)$ if and only if the Boyd indices of X satisfy

$$0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1.$$

Consider the operators $P := \frac{1}{2}(I+S)$ and $Q := \frac{1}{2}(I-S)$.

We finally arrive at the main results.

Theorem 5.67. *Let $X = X(\mathbb{R})$ be a reflexive rearrangement-invariant Banach function space with Boyd indices $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1$. If $a \in SAP_{N\times N}$ and the operator $aP + Q$ is Fredholm, then $a_l P + Q$ and $a_r P + Q$ are invertible on the space $X_N(\mathbb{R})$.*

Proof. We prove for $N = 1$. We notice that S is a bounded operator and, for any $h \in \mathbb{R}$, $U_{-h}SU_h = S$. Indeed, for every $f \in X$ and every $x \in \mathbb{R}$,

$$\begin{aligned}(U_{-h}SU_h f)(x) &= U_{-h}\frac{1}{\pi i}\text{p.v.} \int_{\mathbb{R}} \frac{(U_h f)(y)}{y-x} dy \\ &= \frac{1}{\pi i}\text{p.v.} \int_{\mathbb{R}} \frac{f(y-h)}{y-(x+h)} dy \\ &= \frac{1}{\pi i}\text{p.v.} \int_{\mathbb{R}} \frac{f(y-h)}{(y-h)-x} d(y-h) \\ &= (Sf)(x).\end{aligned}$$

Furthermore, we get

$$(U_{-h}SU_h)^* = S^*.$$

Therefore, for every sequence of isometries $\mathcal{U} = (U_{h_n})_{n\in\mathbb{N}}$, where (h_n) is an arbitrary real-valued sequence, the limit operator of $S_{\mathcal{U}}$ exists and is equal to S. The same is true for the operators P and Q.

Now, let $u \in C(\overline{\mathbb{R}})$ be such that $u(-\infty) = 0$ and $u(+\infty) = 1$. Fix $a \in SAP$. Then, by the Sarason theorem, there are $a_l, a_r \in AP$ and $a_0 \in C_0(\dot{\mathbb{R}})$ such that

$$a = (1-u)a_l + ua_r + a_0.$$

We want to prove that, for some real-valued sequence $h_n \to +\infty$, $(aI)_{\mathcal{U}} = a_r I$, where $\mathcal{U} = (U_{h_n})_{n\in\mathbb{N}}$. Since $a_r \in AP$, by Theorem 5.66, there is a sequence $h_n \to +\infty$ such that

$$\|(a_r)_{h_n} - a_r\|_\infty \to 0.$$

Therefore, for all the $f \in X$,

$$\begin{aligned} U_{-h_n} a U_{h_n} f - a_r f &= ((a)_{h_n} - a_r) f \\ &= ((1-(u)_{h_n})(a_l)_{h_n} + (u)_{h_n}(a_r)_{h_n} + (a_0)_{h_n} - a_r) f \\ &= (u)_{h_n}((a_r)_{h_n} - a_r) f + \underbrace{((1-(u)_{h_n})(a_l)_{h_n} + (a_0)_{h_n} + a_r((u)_{h_n} - 1))}_{g_n} f. \end{aligned}$$

We now notice that, by monotonicity of the norm of X,

$$\|(u)_{h_n}((a_r)_{h_n} - a_r) f\|_X \le \|u\|_\infty \|(a_r)_{h_n} - a_r\|_\infty \|f\|_X \to 0,$$

as $n \to \infty$, and since $|g_n f| \le M|f| \in X$, for some $M > 0$, and $g_n f \to 0$ a.e., Theorem 3.27 yields $\|g_n f\|_X \to 0$, by the fact that the norm of X is absolutely continuous, which is guaranteed by Corollary 3.37. Therefore,

$$\|(U_{-h_n} a U_{h_n} - a_r I) f\|_X \to 0,$$

as $n \to \infty$. From Lemma 5.61, we therefore arrive at the conclusion that

$$(aP + Q)_{\mathcal{U}} = a_r P + Q.$$

On the other hand, we have

$$U_{-h_n} \overline{a} U_{h_n} f - \overline{a_r} f = \overline{(a)_{h_n} - a_r} f,$$

for every $f \in X'$, and since the usual conjugate is norm-perserving and, by Corollary 3.37, X' has absolutely continuous norm, this yields the following:

$$\begin{aligned} (aI)_{\mathcal{U}^*} &= \operatorname*{s-lim}_{n\to\infty} (U_{-h_n}(aI)U_{h_n})^* \\ &= \operatorname*{s-lim}_{n\to\infty} U_{-h_n}(aI)^* U_{h_n} \\ &= \operatorname*{s-lim}_{n\to\infty} U_{-h_n}(\overline{a}I) U_{h_n} \\ &= \overline{a_r} I. \end{aligned}$$

Therefore, we have

$$\begin{aligned} (aP + Q)_{\mathcal{U}^*} &= P_{\mathcal{U}^*}(aI)_{\mathcal{U}^*} + Q_{\mathcal{U}^*} \\ &= P^* \overline{a}_r I + Q^*. \end{aligned}$$

This proves that the limit operator of $aP + Q$ with respect to the sequence $\mathcal{U} = (U_{h_n})_{n\in\mathbb{N}}$ exists and is equal to $a_r P + Q$. According to Corollary 5.63, $a_r P + Q$ is invertible, as long as $aP + Q$ is a Fredholm operator.

For the invertibility of the left represenative $a_l P + Q$, the proof is analogous. Indeed, it is sufficient to choose a sequence $h_n \to -\infty$ such that

$$\|(a_l)_{h_n} - a_l\|_\infty \to 0.$$

This ends our proof. ☐

Corollary 5.68. *Let $X = X(\mathbb{R})$ be a reflexive rearrangement-invariant Banach function space with Boyd indices $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1$ and let $a \in AP_{N\times N}$. Then, on $X_N(\mathbb{R})$, the operator $aP + Q$ is invertible if and only if $aP + Q$ is Fredholm.*

Proof. Since $a = (1-u)a + ua$, we have $a_l = a_r = a$. The conclusion follows from the previous theorem. ☐

These last two conclusions are new, compared to the existing literature (cf. [4, Corollary 18.11]).

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