

Bundled Discounts: Strategic Substitutes or Complements?*

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Abstract

Bundled discounts by pairs of otherwise independent firms play an increasingly important role as a strategic tool in several industries. Given that prices of firms competing for the same consumers are strategic complements, one would expect their discounts levels also to be strategic complements. However, in this paper we show that under some circumstances bundled discounts may be strategic substitutes. This occurs under vertically differentiated products where a low quality pair of producers may indeed prefer to lower its discount after an increase in the discount offered by a high quality pair of producers.

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1 Introduction

When prices are strategic complements, as is commonly accepted in the literature, one would expect that the best response of a given firm to an increase in a discount offered by a rival would be to increase its discount as well. In general, common wisdom holds that “[w]hether the competitive action is based on price, comparison advertising or coupons, a more aggressive competitive action or commitment will likely lead to a more aggressive response, and vice versa.” (Sengul et al (2012)). This implies that discounts are generally accepted as strategic complements.¹ This paper investigates whether this conventional wisdom holds good in the context of discounts offered to consumers who purchase bundles in which each component good is sold by a different and independent firm.

Bundled discounts have recently become very common: examples range from discounts in the electricity bill in the form of vouchers one can redeem at supermarket chains, offerings of cinema or amusement park tickets at gas stations or fast food chains, restaurant discounts for guests at a given hotel, car-rental discount for passengers of a given airline, and discounts for drug cocktails made up of components produced by different firms, to name just a few.

Bundled discounts by independent producers were studied by Gans and King (2006) and Brito and Vasconcelos (2014), under horizontal and vertical differentiation, respectively. These discounts have several implications: (*i*) if set in advance, discounts may work as commitment devices; (*ii*) at the time they are set, the relevant objective function (joint profit) is different from the one that is relevant for price setting in the ensuing stage (individual profit); and (*iii*) the discount is typically financed by more than one firm.

Contrary to common wisdom, we find that bundled discounts may be strategic substitutes rather than complements: under some circumstances, the best-response to an increase in the rivals’ discount may be for a pair of firms to reduce its own discount. This possibility result is illustrated within a theoretical framework based on a simplified version of Brito and Vasconcelos (2014).

Section 2 provides a brief description of the model. Section 3 presents the result. Finally, Section 4 concludes. All expressions are presented in the appendix.

¹The strategic complement/strategic substitute distinction was introduced in the literature by Bulow et al (1985).

2 The model

We consider a two-product version of Gabszewicz and Thisse (1979), where each product is sold both by a high and also by a low quality firm. Let A_X and A_Y be the high quality sellers of products X and Y , respectively, and let B_X and B_Y denote the corresponding low quality producers. In addition, let P_j and p_j denote the headline prices set by A_j and B_j , respectively, with $j = X, Y$. We assume that firm A_X is allied with A_Y and that firm B_X is allied with B_Y , meaning that a pair of allied firms offering products of similar quality level will offer a discount to those consumers who purchase both products from them. Given that there is symmetry between allied firms, we assume that they will equally finance the discount.² The discount offered by the high (resp. low) quality producers is denoted by γ_A (resp. γ_B).

The timing of the game is as follows. At a first stage, the pairs of allied firms decide simultaneously upon the discount that maximizes their joint profits. Afterwards, the four firms set their headline prices simultaneously so as to maximize individual profit.³

Consumers are assumed to always purchase one unit of each product. Moreover, consumers are uniformly distributed on a unit square where each axis measures consumers' valuation for quality of each product. A consumer with valuations for quality θ_X, θ_Y on $[0, 1]^2$ purchases from A_X and A_Y if and only if:

$$\begin{aligned}\theta_Y s &> P_Y - \gamma_A - p_Y \\ \theta_X s &> P_X - \gamma_A - p_X \\ \theta_Y s + \theta_X s &> P_X + P_Y - \gamma_A - (p_X + p_Y - \gamma_B)\end{aligned}$$

where s represents the difference between the high and the low quality versions of each product, which is assumed to be equal. The first and second inequalities ensure that, given that the consumer selects the high quality version of one of the products, it will also prefer the high quality version of the other product, because the valuation for the increment in quality more than compensates for any increase in price. The third inequality, on the other hand, ensures that the high quality pair of products generates more surplus than the low quality pair.

The demand for each of the four possible combinations of products is given by the number of consumers that verify the three relevant inequalities. We write Q_{AB} to denote the number

²We consider bundling by firms with the same quality to be the “natural” scenario. A high quality producer would probably not be interested in being allied with a low quality one as this would likely affect its reputation. Further, considering only this scenario also allows us to avoid having to determine the *optimal* way of sharing the discount in an asymmetric alliance involving a high quality and a low quality producer.

³For a justification of this timing, see Gans and King (2006) or Brito and Vasconcelos (2014).

of consumers that purchase product X from firm A and product Y from firm B , and likewise for the three remaining possible combinations.⁴

Discounts are assumed to be small. We define a small discount as a discount such that the four possible combinations of products have a positive market share for all discount levels of the competitors (if any), at the equilibrium prices. Formally, we assume that $(\gamma_A, \gamma_B) \in [0, \bar{\gamma}]^2$, where $\bar{\gamma}$ is presented in the appendix.⁵ To some extent, small discounts can be justified by the management literature that has highlighted the ineffectiveness of high discounts. Raghurir (1998) and (2004), Barat and Paswan (2005) or Wu et al. (2011) point out the possibility that a high discount may trigger negative consumer deductions about the headline price or quality, or even create negative emotions due to the implicit price discrimination involving coupons. Moreover, small discounts are empirically more relevant, as most observed discounts would most probably fall within our definition of “small discount”.⁶

The equilibrium is obtained by working backwards. For a given pair of discounts, the (individual) profit maximizing prices are obtained. Then, discounts which maximize joint profits are characterized.

3 Result

Figure 1 presents the best-response function for the pair of low quality firms. We can only characterize this function numerically and it has a downward slope: discounts are strategic substitutes.⁷

To understand why the low quality firms may prefer to lower their discount when rivals increase theirs, one needs to understand how the discount impacts the allied firms’ joint profit and how this impact changes with the rivals’ discount level.

Using $\gamma_i := \beta_i s$, joint profit for the low quality firms is

$$\Pi_B = p_Y(Q_{BB} + Q_{AB}) + p_X(Q_{BB} + Q_{BA}) - \beta_B s Q_{BB}.$$

Due to the fact that the equilibrium prices of both products of similar quality are symmetric, Π_B may be written as

$$\Pi_B = 2p_X(Q_{BB} + Q_{AB}) - \beta_B s Q_{BB}.$$

⁴The appendix presents the demand functions in detail.

⁵This is a common assumption in the literature. See Aydemir (2009) or Gans and King (2006).

⁶In our setting, the upper bound on the discount levels, $\bar{\gamma}$, represents 60% and 120% of the headline prices of the high and low quality component products of the bundle (in the no-discounting benchmark case), respectively.

⁷It can be showed that the derivative of the high quality pair of firms joint profits with respect to its discount is positive, meaning that the optimal discount for this pair of firms is the largest admissible one.

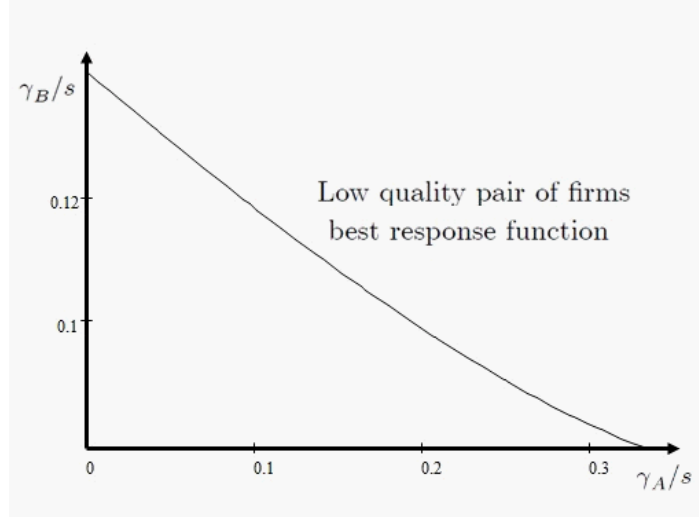


Figure 1: Low quality pair of firms best response function

The first-order condition in β_B is⁸

$$2 \left(\frac{\partial p_X}{\partial \beta_B} (Q_{BB} + Q_{AB}) + p_X \frac{\partial (Q_{BB} + Q_{AB})}{\partial \beta_B} \right) = s \left(Q_{BB} + \beta_B \frac{\partial Q_{BB}}{\partial \beta_B} \right). \quad (1)$$

Notice that all derivatives in (1) are positive: the headline price, the number of units entitled to the discount and the low quality firms' aggregate sales increase in β_B . Price p_X increases with β_B for two reasons: (i) a higher discount offered by the low quality producers increases the demand for each low quality product; and (ii) the introduction of the discount can be interpreted as a unit cost, partially incurred by each firm, for the units entitled to the discount. Further, despite the fact that p_X increases with the discount, the “net” bundle price decreases with it, leading to a higher demand for the bundle and also for all low quality products.

The left-hand side in (1) is the discount's marginal benefit. As the headline price will increase ($\frac{\partial p_X}{\partial \beta_B} > 0$), the margin per unit is higher, excluding the discount “cost”. Total sales will also increase ($\frac{\partial (Q_{BB} + Q_{AB})}{\partial \beta_B} > 0$). Hence, the discount's marginal benefit comes from both an increase in price and in sales.

The right-hand side in (1) is the discount's marginal cost. It also includes two effects: (i) all units that were previously sold as a bundle (Q_{BB}) now entitle the corresponding consumers to a higher discount; and (ii) there will be an increase in the sales entitled to the discount ($\frac{\partial Q_{BB}}{\partial \beta_B} > 0$).

The optimal discount is obtained when the marginal revenue equals marginal cost. Now, to understand how allied firms should react to different rival's discounts we need to establish how

⁸Note that changing the discount will affect prices and quantities in the ensuing stage.

marginal benefit and cost change with β_A . Differentiating (1) with respect to β_A we obtain:

$$2 \left(\frac{\partial^2 p_X}{\partial \beta_B \partial \beta_A} (Q_{BB} + Q_{AB}) + \frac{\partial p_X}{\partial \beta_B} \frac{\partial (Q_{BB} + Q_{AB})}{\partial \beta_A} + p_X \frac{\partial^2 (Q_{BB} + Q_{AB})}{\partial \beta_B \partial \beta_A} + \frac{\partial p_X}{\partial \beta_A} \frac{\partial (Q_{BB} + Q_{AB})}{\partial \beta_B} \right) - s \left(\frac{\partial Q_{BB}}{\partial \beta_A} + \beta_B \frac{\partial^2 Q_{BB}}{\partial \beta_B \partial \beta_A} \right)$$

Consider first the impact of the rivals' discount on marginal benefit. The headline price increases more with own discount in the presence of the rivals' discount, but this increase impacts a smaller number of original units demanded. Additionally, although the number of additional units demanded is now higher, these may be sold at a lower or higher price than in the absence of a rival discount. Despite the fact that the net result is potentially ambiguous, the positive effects dominate, implying that there is an incentive to increase own discount when rivals increase theirs.

As for the marginal cost, when rivals increase the discount Q_{BB} becomes larger. However, fewer additional units will be sold by the low quality pair of firms as a result of a higher discount. Although the net effect is again potentially ambiguous, the aggregate effect is positive.

The following table summarizes these effects and also highlights how they change with β_A .

Term:	Marginal Benefit				Marginal Cost	
	1st		2nd		1st	2nd
Sign of derivative wrt β_A	$\frac{\partial p_X}{\partial \beta_B}$	$(Q_{BB} + Q_{AB})$	p_X	$\frac{\partial (Q_{BB} + Q_{AB})}{\partial \beta_B}$	Q_{BB}	$\beta_B \frac{\partial Q_{BB}}{\partial \beta_B}$
	> 0	< 0	$?$	> 0	> 0	< 0
	$\underbrace{\hspace{10em}}_{> 0}$		$\underbrace{\hspace{10em}}_{> 0}$		$\underbrace{\hspace{10em}}_{> 0}$	

Hence, if discounts are strategic substitutes, it must be that the increase in marginal cost dominates the increase in marginal benefit. Perhaps counter-intuitively, when the high quality rivals' increase their discount, the sales of the low quality bundle increase: increasing the high quality bundle discount, increases the high quality headline prices, leading some consumers who were purchasing products of different qualities to switch to the low quality bundle. This then makes it more expensive to increase the low quality bundle discount, as the number of consumers entitled to it becomes larger.⁹

⁹Notice that the impact of the discount on headline prices would not exist if discounts and prices were set simultaneously, as one would expect to be the case if discounts were instead offered by a multi-product firm.

4 Conclusions

Bundled discounts play an increasingly important role as a strategic tool. In many instances, the discounts are given by pairs of independent firms who agree to cut prices charged to consumers who purchase from both of them. Given that prices of firms competing for the same consumers are strategic complements, one would expect these discounts to be strategic complements as well. However, in some circumstances, they may be strategic substitutes.

In this paper, we analyze the case of competition between independent vertically differentiated producers offering bundled discounts. We find that the low quality pair of producers may prefer to lower their discount after an increase in the discount offered by the high quality producers. The reason is that the demand for their bundle may actually increase with the rivals' discount, making their own discount very expensive. As the high quality discount increases, so will the corresponding headline prices. This means that consumers who were previously purchasing products of different quality may prefer to switch to the low quality bundle and benefit from the associated discount. In response, this may lead the latter pair of producers to prefer to decrease their discount.

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Appendix

Demand Functions

With no discounts, consumers purchase product $i = X, Y$ from firm A_i if and only if

$$V + \theta_i s_A - P_i > V + \theta_i s_B - p_i \Leftrightarrow \theta_i > \theta_i^* := \frac{P_i - p_i}{s}$$

where V is the reservation price.

Assume the high quality firms introduce discount γ_A . Then, a consumer characterized by:

- $\theta_X > \theta_X^* = \frac{P_X - p_X}{s}$ and $\theta_Y > \theta_Y^* = \frac{P_Y - p_Y}{s}$ will still purchase from A_X, A_Y .

- $\theta_X > \theta_X^* = \frac{P_X - p_X}{s}$ and $\theta_Y < \theta_Y^* = \frac{P_Y - p_Y}{s}$ will purchase A_X, A_Y if

$$\theta_X s_A - P_X + \theta_Y s_A - P_Y + \gamma_A > \theta_X s_A - P_X + \theta_Y s_B - p_Y \Leftrightarrow \theta_Y > \theta_Y^{*a} := \frac{P_Y - p_Y - \gamma_A}{s}.$$

- $\theta_X < \theta_X^* = \frac{P_X - p_X}{s}$ and $\theta_Y < \theta_Y^* = \frac{P_Y - p_Y}{s}$ will purchase A_X, A_Y if

$$\theta_X s_A - P_X + \theta_Y s_A - P_Y + \gamma_A > \theta_X s_B - p_X + \theta_Y s_B - p_Y \Leftrightarrow \theta_X > \frac{P_X - p_X + P_Y - p_Y - \gamma_A}{s} - \theta_Y.$$

- $\theta_X < \theta_X^* = \frac{P_X - p_X}{s}$ and $\theta_Y > \theta_Y^* = \frac{P_Y - p_Y}{s}$ will purchase A_X, A_Y if

$$\theta_X s_A - P_X + \theta_Y s_A - P_Y + \gamma_A > \theta_X s_B - p_X + \theta_Y s_A - P_Y \Leftrightarrow \theta_X > \theta_X^{*a} = \frac{P_X - p_X - \gamma_A}{s}.$$

Assume additionally that the low quality firms introduce discount γ_B . Then, a consumer characterized by:

- $\theta_X < \theta_X^{*a}$ and $\theta_Y > \theta_Y^*$ will purchase B_X, B_Y if

$$\theta_X s_B - p_X + \theta_Y s_B - p_Y + \gamma_B > \theta_X s_B - p_X + \theta_Y s_A - p_Y \Leftrightarrow \theta_Y < \theta_Y^{*b} := \frac{P_Y - p_Y + \gamma_B}{s}.$$

- $\theta_X > \theta_X^*$ and $\theta_Y < \theta_Y^{*a}$ will purchase B_X, B_Y if

$$\theta_X s_B - p_X + \theta_Y s_B - p_Y + \gamma_B > \theta_X s_A - p_X + \theta_Y s_B - p_Y \Leftrightarrow \theta_X < \theta_X^{*b} := \frac{P_X - p_X + \gamma_B}{s}.$$

- $\theta_X > \theta_X^{*a}$ and $\theta_Y > \theta_Y^{*a}$ and $\theta_X > \frac{P_X - p_X + P_Y - p_Y - \gamma_A}{s} - \theta_Y$ will purchase B_X, B_Y if

$$\begin{aligned} \theta_X s_B - p_X + \theta_Y s_B - p_Y + \gamma_B &> \theta_X s_A - p_X + \theta_Y s_A - p_Y + \gamma_A \Leftrightarrow \\ \theta_X &< \frac{P_X - p_X + P_Y - p_Y + \gamma_B - \gamma_A}{s} - \theta_Y = \theta_X^{*a} + \theta_Y^{*b} - \theta_Y. \end{aligned}$$

Demand functions, resulting from the relevant areas in the (θ_X, θ_Y) –space, are:

$$\begin{aligned} Q_{A_X, B_Y} &= \left(1 - \frac{P_X - p_X + \gamma_B}{s}\right) \left(\frac{P_Y - p_Y - \gamma_A}{s}\right), \\ Q_{B_X, A_Y} &= \left(1 - \frac{P_Y - p_Y + \gamma_B}{s}\right) \left(\frac{P_X - p_X - \gamma_A}{s}\right), \\ Q_{A_X, A_Y} &= \left(1 - \frac{P_Y - p_Y - \gamma_A}{s}\right) \left(1 - \frac{P_X - p_X - \gamma_A}{s}\right) - \frac{(\gamma_A + \gamma_B)^2}{2s^2}, \\ Q_{B_X, B_Y} &= \left(\frac{P_Y - p_Y + \gamma_B}{s}\right) \left(\frac{P_X - p_X + \gamma_B}{s}\right) - \frac{(\gamma_A + \gamma_B)^2}{2s^2}. \end{aligned}$$

The profit functions at the pricing stage are:

$$\begin{aligned} \Pi_X &= P_X (Q_{A_X B_Y} + Q_{A_X A_Y}) - \frac{\gamma_A}{2} (Q_{A_X A_Y}) \\ \Pi_Y &= P_Y (Q_{A_X A_Y} + Q_{B_X A_Y}) - \frac{\gamma_A}{2} (Q_{A_X A_Y}) \\ \pi_X &= p_X (Q_{B_X B_Y} + Q_{B_X A_Y}) - \frac{\gamma_B}{2} (Q_{B_X B_Y}) \\ \pi_Y &= p_Y (Q_{B_X B_Y} + Q_{A_X B_Y}) - \frac{\gamma_B}{2} (Q_{B_X B_Y}) \end{aligned}$$

Using $\gamma_i := \beta_i s$ and solving the system of first-order-conditions one obtains:

$$\begin{aligned} P_X &= \frac{12\beta_A + 6\beta_B + 6\beta_A^2 + \beta_A^3 + \beta_B^3 + 5\beta_A\beta_B + 4\beta_A\beta_B^2 + 4\beta_A^2\beta_B + 8}{2(5\beta_A + 5\beta_B + 6)}s \\ P_Y &= \frac{12\beta_A + 6\beta_B + 6\beta_A^2 + \beta_A^3 + \beta_B^3 + 5\beta_A\beta_B + 4\beta_A\beta_B^2 + 4\beta_A^2\beta_B + 8}{2(5\beta_A + 5\beta_B + 6)}s \\ p_X &= \frac{2\beta_A + 6\beta_B + \beta_A^3 + 6\beta_B^2 + \beta_B^3 + 5\beta_A\beta_B + 4\beta_A\beta_B^2 + 4\beta_A^2\beta_B + 4}{2(5\beta_A + 5\beta_B + 6)}s \\ p_Y &= \frac{2\beta_A + 6\beta_B + \beta_A^3 + 6\beta_B^2 + \beta_B^3 + 5\beta_A\beta_B + 4\beta_A\beta_B^2 + 4\beta_A^2\beta_B + 4}{2(5\beta_A + 5\beta_B + 6)}s \end{aligned}$$

and

$$\begin{aligned}
Q_{A_X, B_Y} &= \frac{(\beta_B + 5\beta_A\beta_B + 3\beta_A^2 + 2\beta_B^2 - 4)(\beta_A + 5\beta_A\beta_B + 2\beta_A^2 + 3\beta_B^2 - 2)}{(5\beta_A + 5\beta_B + 6)^2}, \\
Q_{B_X, A_Y} &= \frac{(\beta_B + 5\beta_A\beta_B + 3\beta_A^2 + 2\beta_B^2 - 4)(\beta_A + 5\beta_A\beta_B + 2\beta_A^2 + 3\beta_B^2 - 2)}{(5\beta_A + 5\beta_B + 6)^2}, \\
Q_{A_X, A_Y} &= \frac{\left(\begin{aligned} &96\beta_A + 80\beta_B + 128\beta_A\beta_B + 68\beta_A^2 - 12\beta_A^3 + 62\beta_B^2 - 17\beta_A^4 - 7\beta_B^4 \\ &-8\beta_A\beta_B^2 - 20\beta_A^2\beta_B - 40\beta_A\beta_B^3 - 60\beta_A^3\beta_B - 76\beta_A^2\beta_B^2 + 32 \end{aligned} \right)}{2(5\beta_A + 5\beta_B + 6)^2}, \\
Q_{B_X, B_Y} &= \frac{\left(\begin{aligned} &40\beta_A + 48\beta_B + 88\beta_A\beta_B + 38\beta_A^2 + 52\beta_B^2 - 7\beta_A^4 - 12\beta_B^3 - 17\beta_B^4 \\ &-20\beta_A\beta_B^2 - 8\beta_A^2\beta_B - 60\beta_A\beta_B^3 - 40\beta_A^3\beta_B - 76\beta_A^2\beta_B^2 + 8 \end{aligned} \right)}{2(5\beta_A + 5\beta_B + 6)^2}.
\end{aligned}$$

For all segments of demand to be positive, one needs to assume that:

$$(4 - 5\beta_A\beta_B - 3\beta_A^2 - 2\beta_B^2 - \beta_B) \geq 0 \quad (2)$$

$$(2 - 5\beta_A\beta_B - 3\beta_B^2 - 2\beta_A^2 - \beta_A) \geq 0 \quad (3)$$

where (3) implies (2). Assuming that both discounts cannot exceed $\bar{\gamma} = s\bar{\beta}$, it follows that (3) holds for any discount if and only if $(2 - 5\bar{\beta}\bar{\beta} - 3\bar{\beta}^2 - 2\bar{\beta}^2 - \bar{\beta}) = (2\bar{\beta} + 1)(2 - 5\bar{\beta}) \geq 0$. Hence, we assume throughout the paper that $(\gamma_A, \gamma_B) \in [0, \bar{\gamma}]$, with $\bar{\gamma} = \frac{2}{5}s$.

Comparative Statics

The derivatives referred in Section 3 are:¹⁰

$$\begin{aligned}
\frac{\partial(Q_{B_X, B_Y} + Q_{A_X, B_Y})}{\partial\beta_A} &= \frac{(6\beta_A\beta_B - 5\beta_B + 9\beta_A^2 + 5\beta_A^3 - 3\beta_B^2 + 5\beta_A\beta_B^2 + 10\beta_A^2\beta_B - 4)}{(5\beta_A + 5\beta_B + 6)^2} < 0 \\
\frac{\partial(Q_{B_X, B_Y} + Q_{A_X, B_Y})}{\partial\beta_B} &= \frac{(5\beta_A - 6\beta_A\beta_B + 3\beta_A^2 - 9\beta_B^2 - 5\beta_B^3 - 10\beta_A\beta_B^2 - 5\beta_A^2\beta_B + 2)}{(5\beta_A + 5\beta_B + 6)^2} > 0 \\
\frac{\partial^2(Q_{B_X, B_Y} + Q_{A_X, B_Y})}{\partial\beta_B\partial\beta_A} &= \frac{\left(\begin{aligned} &108\beta_A + 86\beta_B + 180\beta_A\beta_B + 90\beta_A^2 + 25\beta_A^3 + 90\beta_B^2 + 25\beta_B^3 \\ &+ 75\beta_A\beta_B^2 + 75\beta_A^2\beta_B + 40 \end{aligned} \right)}{(5\beta_A + 5\beta_B + 6)^3} > 0 \\
\frac{\partial Q_{B_X, B_Y}}{\partial\beta_B} &= \frac{\left(\begin{aligned} &184\beta_A + 192\beta_B - 80\beta_A\beta_B + 6\beta_A^2 - 140\beta_A^3 - 108\beta_B^2 \\ &- 65\beta_A^4 - 234\beta_B^3 - 85\beta_B^4 - 630\beta_A\beta_B^2 - 536\beta_A^2\beta_B - 320\beta_A\beta_B^3 \\ &- 280\beta_A^3\beta_B - 450\beta_A^2\beta_B^2 + 104 \end{aligned} \right)}{(5\beta_A + 5\beta_B + 6)^3} > 0 \\
\frac{\partial Q_{B_X, B_Y}}{\partial\beta_A} &= \frac{\left(\begin{aligned} &128\beta_A + 124\beta_B - 78\beta_A\beta_B - 84\beta_A^3 - 100\beta_B^2 - 35\beta_A^4 - 170\beta_B^3 \\ &- 65\beta_B^4 - 446\beta_A\beta_B^2 - 360\beta_A^2\beta_B - 230\beta_A\beta_B^3 - 170\beta_A^3\beta_B \\ &- 300\beta_A^2\beta_B^2 + 80 \end{aligned} \right)}{(5\beta_A + 5\beta_B + 6)^3} > 0 \\
\frac{\partial^2 Q_{B_X, B_Y}}{\partial\beta_A\partial\beta_B} &= \frac{\begin{aligned} &-1768\beta_A - 2440\beta_B - 5572\beta_A\beta_B - 2550\beta_A^2 - 1560\beta_A^3 \\ &- 2560\beta_B^2 - 325\beta_A^4 - 1560\beta_B^3 - 325\beta_B^4 - 4460\beta_A\beta_B^2 - 4460\beta_A^2\beta_B \\ &- 1300\beta_A\beta_B^3 - 1300\beta_A^3\beta_B - 1950\beta_A^2\beta_B^2 - 456 \end{aligned}}{(5\beta_A + 5\beta_B + 6)^4} < 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial P_X}{\partial\beta_A} &= \frac{72\beta_A + 60\beta_B + 108\beta_A\beta_B + 48\beta_A^2 + 10\beta_A^3 + 49\beta_B^2 + 15\beta_B^3 + 40\beta_A\beta_B^2 + 35\beta_A^2\beta_B + 32}{2(5\beta_A + 5\beta_B + 6)^2} > 0 \\
\frac{\partial p_X}{\partial\beta_B} &= \frac{50\beta_A + 72\beta_B + 108\beta_A\beta_B + 49\beta_A^2 + 15\beta_A^3 + 48\beta_B^2 + 10\beta_B^3 + 35\beta_A\beta_B^2 + 40\beta_A^2\beta_B + 16}{2(5\beta_A + 5\beta_B + 6)^2} > 0 \\
\frac{\partial p_X}{\partial\beta_A} &= \frac{(10\beta_B + 48\beta_A\beta_B + 18\beta_A^2 + 10\beta_A^3 + 19\beta_B^2 + 15\beta_B^3 + 40\beta_A\beta_B^2 + 35\beta_A^2\beta_B - 8)}{2(5\beta_A + 5\beta_B + 6)^2} < 0 \\
\frac{\partial^2 p_X}{\partial\beta_B\partial\beta_A} &= \frac{\left(\begin{aligned} &338\beta_A + 178\beta_B + 430\beta_A\beta_B + 270\beta_A^2 + 75\beta_A^3 + \\ &270\beta_B^2 + 75\beta_B^3 + 225\beta_A\beta_B^2 + 225\beta_A^2\beta_B + 140 \end{aligned} \right)}{2(5\beta_A + 5\beta_B + 6)^3} > 0
\end{aligned}$$

¹⁰The signs of some derivatives were established numerically, under the assumption of small discounts.

Moreover,

$$\begin{aligned}
\frac{\partial^2 p_X}{\partial \beta_B \partial \beta_A} (Q_{BB} + Q_{AB}) + \frac{\partial p_X}{\partial \beta_B} \frac{\partial (Q_{BB} + Q_{AB})}{\partial \beta_A} &> 0 \\
\frac{\partial p_X}{\partial \beta_A} \frac{\partial (Q_{BB} + Q_{AB})}{\partial \beta_B} + p_X \frac{\partial^2 (Q_{BB} + Q_{AB})}{\partial \beta_B \partial \beta_A} &> 0 \\
\frac{\partial Q_{BB}}{\partial \beta_A} + \beta_B \frac{\partial^2 Q_{BB}}{\partial \beta_B \partial \beta_A} &> 0
\end{aligned}$$

At the discount stage, the objective function is

$$\Pi_B = \pi_X + \pi_Y = p_X (Q_{B_X B_Y} + Q_{B_X A_Y}) + p_Y (Q_{B_X B_Y} + Q_{A_X B_Y}) - \gamma_B (Q_{B_X B_Y})$$

The first-order-conditions are:

$$\frac{\partial \Pi_B}{\partial \beta_B} = \frac{\left(\begin{array}{l} -20\beta_B^6 - \beta_B^5 (105\beta_A - 129) - \beta_B^4 (195\beta_A^2 - 615\beta_A - 430) \\ -\beta_B^3 (140\beta_A^3 - 1197\beta_A^2 - 1526\beta_A - 100) \\ -\beta_B^2 (504 - 1836\beta_A^2 - 1155\beta_A^3 - 120\beta_A) \\ -\beta_B (528\beta_A - 32\beta_A^2 - 872\beta_A^3 - 534\beta_A^4 - 45\beta_A^5 + 160) \\ -88\beta_A^2 + 132\beta_A^4 + 90\beta_A^5 + 15\beta_A^6 + 32 \end{array} \right)^s}{2(5\beta_A + 5\beta_B + 6)} = 0$$

Figure 1 presents this function.