UNIVERSIDADE NOVA DE LISBOA
Faculdade de Economia

NONCOOPERATIVE EQUILIBRIUM AND CHAMBERLINIAN MONOPOLISTIC COMPETITION

Mário R. Páscoa

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.Perliminary and incomplete version (Parts I and II)

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UNIVERSIDADE NOVA DE LISBOA
Faculdade de Economia
Campo Grande, 185
1700 LISBOA

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INTRODUCTION

More than fifty years ago, Chamberlin introduced the imperishable idea that in a world of product differentiation firms of relatively insignificant size can have some monopoly power. In "The Theory of Monopolistic Competition" (1933), Chamberlin described an environment where there are many firms producing differentiated commodities, each firm being negligible, in the sense of ignoring its impact on and reactions from other firms, but still facing a downward sloping demand curve. Very few models have captured Chamberlin's perception of competition; most of the literature on product differentiation has devoted attention either to oligopolistic markets or to economies with a large number of agents and commodities while remaining within the class of perfectly competitive environments.

A general equilibrium approach to monopolistic competition was suggested long ago by Triffin (1940), who remarked the arbitrariness of grouping together firms producing imperfect substitutes and advocated with Chamberlin's approval the abandonment of the concept of group. The general equilibrium theory, however, was at the time and remains to this day dominated by the study of perfectly competitive environments.

Throughout the sixties the Arrow-Debreu literature addressed the existence and rationale of a price-taking equilibrium in an economy with a finite number of homogeneous commodities. The assumption that traders interact exclusively through a price system taken by each one of them as given was justified by the equivalence of the game-theoretic solution concept of the core with the set of Walrasian equilibria when the number of traders is large (see Debreu and Scarf (1963)) or more precisely, when the set of traders is defined as a nonatomic measure space (see Aumann (1964)).

More recently, the issue of heterogeneous commodities has been considered in the modern general equilibrium literature. In the models developed by Gabszewicz (1968), Bewley (1973) and Mas-Colell (1975), the question addressed is whether Aumann's theorem on the equivalence of the core and Walrasian equilibria in atomless markets applies to economies with infinitely many commodities. The models by Gabszewicz and Bewley consider time or uncertainty as the source of the enormous variation in commodity characteristics, but the paper by Mas-Colell applies to a situation of product differentiation and opens up implicitly the possibility of nonatomic markets conforming to Chamberlin's perception of competition. However, all attention is concentrated on environments where the perfectly competitive outcome prevails (see also Jones (1983, 1984) and Podczeck (1985)). This perspective is shared by the Bertrand noncooperative approach to perfect competition (see Jones (1985) in the context of product differentiation), which regards Chamberlin's monopolistic competition as a situation of no interest per se because it would fall into the class of environments where a sequence of Bertrand equilibria (for games with a finite but increasing number of players) fails to converge to the perfectly competitive outcome.

Chamberlin's contention that proliferation may result in a finer product differentiation without leading to perfect competition is too important to be ignored. The real world is a world of imperfect substitutes and it is by no means clear that the model of perfect competition provides the exclusive idealization of an economy with a large number of traders. Perfect competition should be regarded instead as one of the possible outcomes in a large economy under product differentiation, namely, the outcome that would prevail if consumers can find plenty of close substitutes for any brand that they like. But when there is inadequate substitutability due to a very fine grid of differentiation, the natural outcome would be Chamberlin's monopolistic competition.

The paper by Dixit and Stiglitz (1977) constitutes the first attempt to model monopolistic competition in the spirit of Chamberlin. Assuming that the representative consumer has symmetric tastes over the n differentiated commodities (a no-neighboring goods assumption), the authors show the existence of a market equilibrium where each operating firm faces a downward sloping demand curve for its distinct commodity; the symmetry of tastes implies the symmetry of demand curves across firms and in equilibrium all operating firms charge the same price and produce the same level of output. The asymmetric case presented in the last part of this paper is a very special case, where there are two groups of commodities, with perfect substitutability intergroups and symmetry of tastes intragroup.

Hart (1983) pointed out two shortcomings in the Dixit-Stiglitz model. The first shortcoming consists in the fact that the authors were not able to rule out an equilibrium where a small number of firms operate and threaten to cut prices if further entry occurs. This criticism suggests that in order to successfully model Chamberlin's monopolistic competition it is necessary to define the limiting economy, where the assumption that the number of competitors is large and each one is negligible should have a precise meaning.

The second shortcoming pointed out by Hart is related to the existence of a representative consumer. Since the price of any brand is given by the representative consumer's marginal rate of substitution between that brand and the numeraire, Hart argues that when one firm doubles its supply the price would hardly change (assuming that the MRS is a continuous function of the consumption bundle, which should hardly change when N is large). Hence, according to Hart, the notion of a representative consumer should be dropped -

in order to rule out a perfect elastic demand.

In Hart's model there is no representative consumer and the limiting economy has a countably infinite set of commodities (the set of integers). His construction of the demand facing each firm rests upon the assumption that each consumer likes only a finite number m of brands and that any m-tuple of brands is equally likely to be desired (this is Hart's version of the no-neighboring goods assumption). There is also the additional assumption that the form of the utility function is independent of which particular brands the consumer desires. These assumptions allow Hart to express the demand facing each price-setting firm in terms of the own price, the aggregate distribution on prices Q and the proportion of operating firms y. The main result is on the existence of a zero-profit noncooperative equilibrium pair (Q,y).

I consider that there are two important shortcomings in Hart's pioneering work. First, the proportion of operating firms is not well defined in the case of a countably infinite set of firms; for this reason, the limiting economy, which had not been defined in the model by Dixit and Stiglitz, is not well defined in Hart's paper. Second, Hart presents the zero-profit condition as an equilibrium condition and although this is true for his particular construction of demand curves (because under the uniformity assumptions introduced by Hart, the demand function turns out to be the same facing all firms) it will not hold in general; Chamberlin (1937), reconsidering his original position, remarked on the incompatibility of "free entry" and product differentiation and admitted that firms may enjoy monopolistic profits.

In this paper we model Chamberlin's monopolistic competition as a nonatomic Bertrand game, using an extension of the Schmeidler (1973), Ali Khan (1985) framework. Nonatomicity will give precise content to Chamberlin's assertion that "any adjustment of price...by a single producer spreads its influence over so many of his competitors that the impact felt by any one is negligible and does not lead him to any readjustment of his own situation" (Chamberlin (1933, p. 83). With a continuum of brands the model does not suffer from one of the shortcomings of Hart's model: the proportion of operating firms becomes well defined in terms of a probability measure on brands.

It is also the purpose of this paper to examine how far can Hart's assumptions on consumers' preferences be relaxed and still have the firms noncooperatively setting prices. We show that Hart's assumption that any brand is equally likely to be desired can be replaced by the weaker assumption that the fact that a consumer likes one brand does not give any information about what are the other $m-1$ desired brands (this is truly the no-neighboring goods situation). Under this weaker assumption, the demand curve varies across firms and therefore, in equilibrium, profits can be positive; this situation conforms to Chamberlin's (1937) and Triffin's (1940) contention that, in a world of competing firms selling imperfect substitutes, the fact that existing firms earn profits is not a clear indication for a set-up decision on the part of new firms. Finally, we study the possibility of relaxing Hart's assumption that each consumer likes only a finite number of brands.

Part I addresses the existence of a noncooperative equilibrium for a nonatomic Bertrand game under certain measurability and continuity conditions on the demands facing the firms. There are two results: one on the existence of a mixed strategies equilibrium and another on the existence of an approximate equilibrium in pure strategies, under an additional assumption which is an aggregation restriction on the strategic information that firms take into account. The restriction requires that the demand facing each firm depends on the other firms' strategies only through a finite number m of distributions on the strategy set, which are weighted means of the mixed strategies function. This assumption is weaker than the aggregation condition considered by Schmeidler (1973) and Ali Khan (1985).

Part II is a reformulation of Hart's model as a nonatomic Bertrand game. We examine how can the above measurability and continuity conditions be met when each consumer is assumed to like only a finite number of brands. It is shown that if consumers' tastes are suitably dispersed (more precisely, if the tastes distribution is absolutely continuous with respect to the measure on the space of commodity characteristics), then there exists a mixed strategies Bertrand equilibrium. Moreover, under a version of the no-neighboring goods assumption which is weaker than Hart's version (because it requires independence in the tastes distribution, rather than uniformity as in Hart's model) the above aggregation restriction is verified and there exists an approximate equilibrium in pure strategies.

Part III discusses Hart's (1983) contention that without the assumption that each consumer likes only a finite number of differentiated commodities, the economic environment would be perfectly competitive rather than monopolistically competitive. In this part, consumption bundles are modelled as L^{∞} functions on the Lebesgue measure space of the unit interval. It does not seem to be possible to rule out perfectly elastic demand curves facing the firms if preferences are continuous with respect to a topology at least as coarse as the Mackey topology $\tau(L^{\infty},L')$ (this argument agrees with Bewley's (1972) result). On the other hand, for a topology, finer than the Mackey topology, consumption sets will be noncompact and therefore consumers' demand correspondences cannot be shown to be nonempty valued and upper-hemi-continuous with respect to firms' strategies, in spite of the fact that these correspondences have closed graph in the product space of firms' strategies and consumption bundles. Hence, it seems to be impossible to show the existence of an equilibrium for a nonatomic Bertrand game when each consumer is allowed to be interested in infinitely many brands. This impossibility argument portrays a situation which is somehow analogous to the situation found by Gretsky and Ostroy (1984) in the context of pure exchange economies with an infinitely dimensional commodity space: under the individualistic formulation of a nonatomic exchange economy, markets will be thick (i.e., there will be many traders of each commodity) and perfect competition will prevail (see also Ostroy (1984)). Part III is still in progress.

PART I

1. The set of commodity characteristics is the unit interval $T = [0,1]$ and the space of firms is the Lebesgue measure space of the unit interval $(T, \ ,\lambda)$: each firm is negligible and produces a single and distinct commodity (identified by a point in T).

The space of consumers is a normalized nonatomic measure space $(I,\mathcal{I},\mu)$. Each consumer has tastes defined over the set T of differentiated commodities and over a homogeneous numeraire good denoted by ℓ. Consumers are endowed with ℓ, which they sell to firms to be used as the only productive input. This commodity should be thought of as either labor or money.

We assume that the firms producing these differentiated commodities do not take into account the impact of their profits on their customers' income. This assumption has been adopted in previous models of monopolistic competition (implicitly in Hart (1983) and explicitly in Jones (1985)).

2. To simplify, we assume that firms have identical cost functions in terms of the numeraire, given by $C(.) + F$, where $F > 0$ is a fixed set-up cost and the variable cost function $C(.)$ is continuous, increasing and such that average variable cost is bounded away from zero.

Each firm has two strategic variables: the price charged and the set-up decision variable. Since profit functions cannot be shown to be concave in these variables we will have to consider mixed strategies.

Let $S = P \times \{0,1\}$ be the pure strategies set, where $P \subseteq R_o^+$ is the set of chargeable prices and the binary setup variable assumes value one if the firm decides to operate and zero otherwise. The set of mixed strategies is $\mathcal{M}(S)$, the set of regular probability measures on S.

Although in this framework nonoperating firms charge prices, the construction below will make these prices irrelevant to both firms and consumers.

3. Now we will define the set of mixed strategy functions, mapping the set of firms into the set of mixed strategies.

Denote by $rba(S)$ the Banach space of regular bounded additive set functions defined on the field generated by the closed sets of S, where $\{0,1\}$ is endowed with the discrete topology and S with the product topology. Since S is normal $rba(S)$ is isometrically isomorphic to the dual of $C(S)$; symbolically, $rbs(S) \cong C(S)^*$ (see Dunford-Schwartz IV.5.2).

Let $L_w^\infty(T;rba(S))$ be the Banach space of (equivalence classes of) weak* measurable functions $f: T \to rba(S)$ such that $||f|| \in L^\infty(T,\lambda)$. Notice that $L_w^\infty(T;rba(S))$ is $\cong$ to the dual of $L'(T;C(S))$, the Banach space of Bochner integrable functions $g: T \to C(S)$ (see Dieudonne (1951), § 4.) see Appendix.

The set of mixed strategy functions is defined as

$$K = \{f \in L_w^\infty(T;rba(S)): \ f(t) \in \mathcal{M}(S) \text{ a.e.}\}.$$

This set is clearly nonempty and convex.

4. We assume that the aggregate demand facing firm t is a function of the price charged by firm t and of the mixed strategy function f:

$$D(t,.,.): P \times K \to \mathbb{R}$$

The profit function of firm t is given by

$$\pi(t,p,u,f) = u[pD(t,p,f) - c(D(t,p,f)) - F]$$

where $(p,u) \in S = P \times \{0,1\}$.

The expected profit function of firm t is defined by

$$E(t,.,.): \mathcal{M}(S) \times K \to \mathbb{R} \text{ such that}$$

$$E(t,\rho,f) = \int_S \pi(t,p,u,f)d\rho.$$

Firms maximize expected profit by choosing a mixed strategy $\rho \in \mathcal{M}(S)$.

5. The game-theoretic solution concept that we will consider is Nash's noncooperative equilibrium. This is the concept suggested by Chamberlin and Triffin for a monopolistically competitive environment, where firms are sufficiently negligible to ignore the impact of their individual actions on, and hence reactions from, the competitors.

A noncooperative equilibrium (in mixed strategies) is a function $f \in K$ such that $E(t,f(t),f) \geq E(t,\rho,f)$, $\forall \rho \in \mathcal{M}(s)$, for almost every $t \in T$.

6. On the existence of a mixed strategies equilibrium

We will show that under a compactness assumption on the set P of changeable prices and under certain measurability and continuity condition on the demand function facing the firms, there exists a mixed strategies equilibrium.

The compactness of P follows from the assumption that average costs are bounded from below and from the additional assumption that there is a uniform upper bound on prices that make revenue cover fixed costs.

Theorem 1

If (A1) $\exists \bar{p}$: $p > \bar{p} \Rightarrow \sup_{t \in T} pD(t,p,f) < F$, $\forall f \in K$

(A2) $D(.,p,f): T \to \mathbb{R}$ is $\mathcal{T}$-measurable, for any $(p,f) \in P \times K$

(A3) $D(t,.,.): P \times K \to \mathbb{R}$ is jointly continuous, with respect to the weak* topology on K

then there exists a function f in equilibrium.

7. Comments on the assumptions

Assumption (A1) is an extension of an assumption in Hart's (1983) model to the general case when the demand functions are not necessarily the same across firms.

Under (A1), the set of chargeable prices P becomes a compact interval of real numbers: $P = [\underline{p},\bar{p}]$, where $\underline{p}$ is the infimum of average costs. Then S is compact and $C(S)^* \cong rca(S)$, the Banach space of regular countably additive set functions defined on the field generated by the closed sets of S.

Now let $L_w^\infty(T;rca(S))$ be the Banach space of (equivalence classes of) weak* measurable functions $f: T \to rca(S)$ such that $||f|| \in L^\infty(T,\lambda)$. Notice that $L^\infty(T;rca(S)) \cong$ dual of $L'(T;C(S))$ and that $K = \{f \in L_w^\infty(T;rca(S)): f(t) \in \mathcal{M}(S) \text{ a.e.}\}$.

Assumption (A1) makes $C(S)$ separable and therefore $\mathcal{M}(S)$ endowed with the weak* topology is a compact metrizable space (see Dunford-Schwartz v. 7.15).

Under (A1) the set K is weak* compact (by theorem V.2 in Castaing-Valadier) and $L'(T;C(S))$ is separable; therefore the weak* topology of K is a metric topology. A sequence (f_n) in K converges weak* to $f \in K$ iff

$$\int_T \langle g,f_n\rangle \, d\lambda \to \int_T \langle g,f\rangle \, d\lambda, \quad \text{for any } g \in L'(T;C(S))$$

where $\langle g,f\rangle(t) = \int_S g(t)df(t)$.

Assumption A3 involves a certain aggregation of strategic information. In fact, the weak* continuity of D with respect to f implies that if two mixed strategy functions f_1 and f_2 are such that for any nonnegligible set E of firms the averages on E of the mixed strategies given by f_1 and f_2 are close in distribution sense, then $D(t,p,f_1)$ will be approximately equal to $D(t,p,f_2)$.

8. Proof of Theorem 1

(1) $E(.,.,f): T \times \mathcal{M}(S) \to \mathbb{R}$ is $\mathcal{T} \otimes \mathcal{B}^*(\mathcal{M}(S))$ measurable for any $f \in K$ (where $\mathcal{B}^*(\mathcal{M}(S))$ is the Borel σ-algebra generated by the weak* topology on $\mathcal{M}(S)$).

This result follows form (A2) and (A3) by Lemma III.14 in Castaing-Valadier, because $\mathcal{M}(S)$ endowed with the weak* topology is a separable metrizable space, $E(.,.,f)$ is measurable in $t \in T$ and weak* continuous in $\rho \in \mathcal{M}(S)$.

(2) $E(t,.,.): \mathcal{M}(S) \times K \to \mathbb{R}$ is jointly continuous, with respect to the weak* topologies on $\mathcal{M}(S)$ and K

In fact, let the sequence (ρ_n, f_n) converge weak* to (ρ, f) then

(a) $E(t,\rho_n,f) \to E(t,\rho,f)$, by (A3)

(b) (A3) implies also that the sequence $\{\pi(t,.,f_n)\} \subseteq C(S)$ converges continuously to $\pi(t,.,f) \in C(S)$ (see Royden, Ch. 8, problem 40). Then this sequence converges to this function uniformly on S (idem).

Hence $\forall_{\varepsilon>0} \exists_m : n \geq m \Rightarrow \sup_{s \in S} |\pi(t,s,f_n) - \pi(t,s,f)| < \varepsilon$ and therefore

$$\forall_{\varepsilon>0} \exists_m : n \geq m \Rightarrow |E(t,\rho_n,f_n) - E(t,\rho_n,f)| < \varepsilon.$$

Result (2) follows from (a) and (b).

(3) Let $B: T \times K \to 2^{\mathscr{M}(S)}$ be the individual best response correspondence defined by $B(t,f) = \{\rho \in \mathscr{M}(s): E(t,\rho,f) \geq E(t,v,f), \forall_{v \in \mathscr{M}(S)}\}$

(a) B is nonempty valued, by weak* continuity of $E(t,.,f)$ on the weak* compact set $\mathscr{M}(S)$

(b) B is convex valued, by linearity of E on $\mathscr{M}(S)$ and convexity of $\mathscr{M}(S)$

(c) B is weak* compact valued, because B is weak* closed valued

(d) $B(t,.): K \to 2^{\mathscr{M}(S)}$ has a closed graph in $K \times \mathscr{M}(S)$ with respect to the weak* topologies on K and $\mathscr{M}(S)$, by (2) above.

(e) The graph of $B(.,f): T \to 2^{\mathscr{M}(S)}$ belongs to $\mathscr{T} \otimes \mathscr{B}^*(\mathscr{M}(S))$, by (1) above and the application of Lemma III.39 in Castaing-Valadier, because $\mathscr{M}(S)$ endowed with the weak* topology is a Souslin space.

(4) Define $\alpha: K \to 2^K$, the aggregate best response correspondence, by

$$\alpha(f) = \{h \in K: h(t) \in B(t,f) \text{ a.e.}\}$$

(a) α is nonempty valued (by (3a), (3e) and Theorem III.22 in Castaing-Valadier)

(b) α is convex valued (by (3b))

(c) α is weak* compact valued (by (3a), (3b), (3c), (3e) and theorems III.37 and V.1 in Castaing-Valadier)

(d) α has closed graph in $K \times K$, with respect to the weak* topology. The proof of this claim follows the proof of claim 8 in Ali Khan (1985). Let $f_n \overset{*}{\to} f$, $h_n \overset{*}{\to} h$, where $h_n \in \alpha(f_n)$

- Suppose $\exists_{A \in \mathscr{T}}: \lambda(A) > 0$ and $h(t) \notin B(t,f)$ for $t \in A$. Let $\alpha_A(f) = \{v \in L^\infty_w(A,\lambda|_A;rca(S)): v(t) \in B(t,f) \text{ a.e. in } A\}$ Notice that $\alpha_A(f)$ is nonempty, convex and weak* closed (by (4a), (4b) and (4c), respectively) and $h|_A \notin \alpha_A(f)$. Then, by the Hahn-Banach theorem, $\exists w \in L'(T;C(S))$, $w \neq 0$ such that

$$\int_A \langle w,h\rangle \, d\lambda > \int_A \langle w,v\rangle \, d\lambda, \quad \forall \; v \in \alpha_A(f)$$

- Now $h_n \overset{*}{\to} h$ implies that $\lim_{n\to\infty} \int_A \langle w,h_n\rangle \, d\lambda = \int_A \langle w,h\rangle \, d\lambda$ and since $h_n \in L^{\infty}_w(T;rca(S))$, $\exists_{z\in L'(A,\lambda|_A)}$ such that $|\langle w,h_n\rangle| \leq z$ a.e. in A. Then, by Proposition 4.1 in Aumann (1965),

$$\int_A \langle w,h\rangle \, d\lambda \in \int_A \{\limsup_{n\to\infty} \langle w,h_n\rangle\} \, d\lambda$$

where $\limsup_{n\to\infty} \langle w(t),h_n(t)\rangle = \langle w(t), {}^*\limsup_{n\to\infty} h_n(t)\rangle$, because $\mathcal{M}(S)$ is weak* compact and $\langle w(t),.\rangle$ is weak* continuous on $\mathcal{M}(S)$.

- Now, by (3d) above, ${}^*\limsup_{n\to\infty} h_n(t) \in B(t,f)$ and therefore

$$\int_T \langle w,h\rangle \, d\lambda \in \{\int_A \langle w,v\rangle \, d\lambda, \quad v \in \alpha_A(f)\}$$

contradicting the strict inequality above.

(5) Now K is nonempty convex set and is compact for the weak* topology, which is a locally convex topology. Therefore, by the Fan-Glicksberg fixed point theorem, α has a fixed point, which is clearly an equilibrium function.

9. <u>On the existence of an approximate equilibrium in pure strategies</u>

We say that there exists an approximate equilibrium in pure strategies if $\forall_{\varepsilon>0}$ $\exists f \in K$ such that $f(t) \in \text{ext}\mathcal{M}(S)$ a.e. and

$$\left|E(t,f(t),f) - \max_{\rho\in\mathcal{M}(S)} E(t,\rho,f)\right| < \varepsilon$$

a.e. on a subset of T of measure greater than $1-\varepsilon$.

We will show that by imposing an extended Schmeidler type aggregation restriction on the demand function facing each firm we can establish the existence of an approximate equilibrium in pure strategies.

<u>Definition</u>: The <u>Gel'fand integral</u> of the weak* measurable function $h: T \to rca(S)$ over the set T is an element in $rca(S)$ denoted by $\oint h d\lambda$ such that

$$\int_S g \, d(\oint h d\lambda) = \int_T \int_S g \, dh(t) d\lambda, \quad \text{for all} \quad g \in C(S)$$

If $||h|| \in L^{\infty}(T,\lambda)$, then $\oint h d\lambda$ exists (see Diestel-Uhl (1977, p. 53 and §3 above).

The aggregation restriction that we will impose on the demand function facing each firm consists in requiring that the demand function depends on the mixed strategy function f only through a finite number m of weighted means of the mixed strategy function $\oint z_i f d\lambda$, where $z_i \in L^{\infty}(T,\lambda)$ and $\int_T z_i d\lambda = 1$, $i = 1,\ldots,m$.

Notice that since $f(t) \in \mathcal{M}(S)$ a.e., the Gel'fand integral $\oint z_i f d\lambda$ exists and is an element in $\mathcal{M}(S)$.

This aggregation restriction is inspired in Ali Khan's (1985) restriction that the payoff functions depend on the mixed strategy function f only through the mean mixed strategy $\oint_T f d\lambda$, which is the natural extension of the aggregation restriction used in Theorem 2 in Schmeidler's (1973) paper on nonatomic games with a finite number of strategies. Our restriction extends Schmeidler's weaker restriction (remark 2 in his paper) requiring that the payoff functions depend on the mixed strategy function only through a finite number of means over a collection $\{T_i\}_{i=1}^{m}$ of Lebesgue-measurable subsets of T (in fact, let $z_i = \chi_{T_i}$ and we have Schmeidler's weaker restriction).

<u>Aggregation restriction (A4)</u>: For any $p \in P$ and a.e. in T, given the function $z_i \in L^{\infty}(T,\lambda)$, $i = 1,\ldots,m$, such that $\int_T z_i d\lambda = 1$, there exists a

real valued function $\Phi(t,r,.)$ defined on the set $F = \{v \in \mathcal{M}(S)^m$: $v = (\oint z_i f d\lambda)_{i=1}^m$, for some $f \in K\}$ by $\Phi(t,p,(\oint z_i f d\lambda)_{i=1}^m) = D(t,p,f)$.

Assumption (A5): $\Phi(t,.,.): P \times F \to \mathbb{R}$ is jointly continuous on $P \times F$ a.e. in T, with respect to the weak* topology on F

Theorem 2: If Assumptions (A1) through (A5) are verified, then there exists an approximate equilibrium in pure strategies.

10. Proof of Theorem 2

(1) By Assumptions (A1), (A2) and (A3) and Theorem 1, there exists a mixed strategies equilibrium function $h \in K$.

(2) Let $B^e(t,h) = \{\rho \in \text{ext}\mathcal{M}(S): E(t,\rho,h) = \max_{v \in \mathcal{M}(S)} E(t,v,h)\}$

Notice that $B^e(t,h) = \text{ext } B(t,h)$[1] and therefore by the Krein-Mil'man theorem, $B^e(t,h) \neq \phi$ and $B(t,h) = \overline{\text{con}}^* B^e(t,h)$.

(3) Let $\hat{f}: T \to rca(S)^m$ defined by $\hat{f}_i(t) = z_i(t)f(t)$, where $z_i \in L^\infty(T,\lambda)$ and $f \in K$.

- $\hat{f}$ is strongly bounded, because $||\hat{f}(.)|| \equiv (\sum_{i=1}^m ||\hat{f}_i(.)||^2)^{1/2} \leq$ $\leq (\sum_{i=1}^m ||z_i||_\infty^2)^{1/2}$ (since $||f(.)|| = 1$ a.e.).
- $\hat{f}$ is weak* measurable, because $\langle x',\hat{f}(.)\rangle \equiv \sum_{i=1}^m \langle x_i',\hat{f}_i(.)\rangle$ where $x' \in C(S)^m$ and $\langle x_i',\hat{f}_i(.)\rangle = z_i(.) \langle x_i',f(.)\rangle$ is $\mathcal{T}$measurable.

Define the Gel'fand integral of $\hat{f}$ over the set T as an element in $rca(S)^m$ denoted by $\oint_T \hat{f} d\lambda$ such that for any $x' \in C(S)^m$,

$$\sum_{i=1}^m \int_S x_i' \, d(\oint_T \hat{f} d\lambda)_i = \int_T \sum_{i=1}^m \int_S x_i' d\hat{f}_i d\lambda$$

$\oint_T \hat{f} d\lambda$ exists because $\hat{f}$ is strongly bounded and weak* measurable.

(4) Let $A(t,h) = \{x \in rca(S)^m : x = (z_1(t)w,\ldots,z_m(t)w)$ for some $w \in B(t,h)\}$

$A^e(t,h) = \{x \in rca(s)^m : x = (z_1(t)w,\ldots,z_m(t)w)$ for some $w \in B^e(t,h)\}$

Notice that $A^e(t,h) = \text{ext } A(t,h)$ and that $A(t,h)$ is weak* compact.[2] Then by the Krein-Mil'man theorem $A(t,h) = \overline{\text{con}}^* A^e(t,h)$.

(5) Define the Gel'fand integral of a strongly bounded weak* measurable correspondence as the set of Gel'fand integrals for the strongly bounded weak* measurable selections.

<u>Claim</u>: $\oint_T A(t,h)d\lambda = \text{cl}^* \oint_T A^e(t,h)d\lambda$

<u>Proof</u>:

- Since the graph of $B(.,h)$ belongs to $\mathscr{T} \otimes \mathscr{B}^*(\mathscr{M}(S))$, it follows (by Theorem III.22 in Castaing-Valadier) that there exists a sequence (σ_n) of $(\mathscr{T},\mathscr{B}^*(\mathscr{M}(S)))$ measurable selections of $B(.,h)$ such that $B(t,h) = \overline{\{\sigma_n(t)\}}^*$, for every t.

  Let $\gamma_n: T \to rca(S)^m$ defined by $\gamma_n(t) = (z_1(t)\sigma_n(t),\ldots,z_m(t)\sigma_n(t))$

- Now $\langle x',\gamma_n(.)\rangle$ is $\mathscr{T}$-measurable, because $\forall\, x' \in C(S)^m$

  $\langle x',\gamma_n(.)\rangle = \sum_{i=1}^{m} z_i(.)\ \langle x',\sigma_n(.)\rangle$ and $\sigma_n(.)$ is scalarly measurable.

- Moreover, $\overline{\{\gamma_n(t)\}}^* = A(t,h)$ because $\subseteq$ follows immediately since $A(t,h)$ is weak* closed and to show $\supseteq$ suppose that $x \in A(t,h)$, then

$$x = (z_1(t)w,\ldots,z_m(t)w) \text{ for some } w \in B(t,h).$$

Now since $B(t,h) = \overline{\{\sigma_n(t)\}}^*$ there exists some subsequence $\sigma_{n_i}(t)$ such that $\sigma_{n_i}(t) \overset{*}{\to} w$. Let $y_{n_i} = (z_1(t)\sigma_{n_i}(t),\ldots,z_m(t)\sigma_{n_i}(t))$, then

$$\sum_{j=1}^{m} \langle x_j',(y_{n_i})_j\rangle \to \sum_{j=1}^{m} \langle x_j',x_j\rangle \text{ for any } x' \in C(S)^m$$

that is, $y_{n_i} \overset{*}{\to} x$ and therefore $x \in \overline{\{\gamma_n(t)\}}^*$.

- Then, by Theorem III.37 in Castaing-Valadier, the graph of the correspondence $A(.,h)$ belongs to $\mathcal{T} \otimes \mathcal{B}^*(rca(S)^m)$ (because $A(t,h) \subseteq \max_i ||z_i||_\infty \mathcal{M}(S)^m$).
- Hence, the graph of $A^e(.,h)$ belongs to $\mathcal{T} \otimes \mathcal{B}^*(\mathcal{M}(S))$ also because $A^e(.,h) = \text{ext } A(.,h)$ and $A(.,h)$ has weak* compact convex values in the weak* compact convex metrizable set $\max_i ||z_i||_\infty \mathcal{M}(S)^m$ (see Theorem 9.3 in Himmelberg).
- Therefore, by Corollary 1 in Ali Khan (1982) (which is based on an extension of Liapunov theorem to vector measures), the equality in the claim is established.

(6) Let $\hat{f}: T \to rca(s)^m$ be induced by $f \in K$ as in (3) and

$$\hat{E}(t,\rho,v) = E(t,\rho,f) \quad \text{where} \quad v = \oint_T \hat{f} d\lambda.$$

Denote by d the weak* metric on $\mathcal{M}(S)^m$

(a) By (5) there exists a sequence $\{\oint_T \hat{f}_n d\lambda\}$, where $\hat{f}_n$ is induced by f_n such that $f_n(t) \in B^e(t,h)$ a.e., which converges weak* to $\oint_T \hat{h} d\lambda$ (where $\hat{h}$ is induced by the mixed strategies equilibrium function h)

Moreover, $E(t,f_n(t),h) = E(t,h(t),h)$ a.e., for any n.

(b) By Assumption (A5)

$$\forall_{\varepsilon>0} \exists_{N_1}: n \geq N \Rightarrow \left|E(t,h(t),h) - \max_{\rho\in\mathcal{M}(S)} \hat{E}(t,\rho,\oint_T \hat{f}_n d\lambda)\right| < \frac{\varepsilon}{2}$$

where N_1 is independent of $t \in T_1$, for a set T_1 contained in T such that $\lambda(T_1) \geq 1 - \varepsilon/2$ (because by (A5),

$$\max_{\rho\in\mathcal{M}(S)} \hat{E}(t,\rho,\oint_T \hat{f}_n d\lambda) \to \max_{\rho\in\mathcal{M}(S)} \hat{E}(t,\rho,\oint \hat{h} d\lambda)$$

a.e. on T and therefore, given $\varepsilon > 0$, uniformly on a subset T_1 such that $\lambda(T_1) \geq 1 - \varepsilon/2$; see Dunford-Schwartz III 6.1, 6.12).

(c) Again by (A5),

$$\forall_{\varepsilon>0}\ \exists_{N_2}: n \geq N_2 \Rightarrow \left|E(t,\rho,h) - \hat{E}(t,\rho,\oint_T \hat{f}_n d\lambda)\right| < \varepsilon/2$$

where N_2 is independent of $\rho \in \mathcal{M}(S)$ and $t \in T_2$, for a set $T_2 \subseteq T$ such that $\lambda(T_2) \geq 1 - \varepsilon/2$ (because by (A5), $E(t,\cdot,\oint_T \hat{f}_n d\lambda) \to E(t,\cdot,\oint_T \hat{h} d\lambda)$ continuously on $\mathcal{M}(S)$ and for a.e. $t \in T$; therefore the convergence is uniform on $T_2 \times \mathcal{M}(S)$; see Royden Ch. 9, problem 40).

(d) Let $N = \max(N_1,N_2)$, $T_3 = T_1 \cap T_2$ (clearly nonempty for ε arbitrarily small since $\lambda(T_3) > 1-\varepsilon$). Then

$$\forall_{\varepsilon>0}\ \exists_N: n \geq N \Rightarrow \left|E(t,h(t),h) - \max_{\rho\in\mathcal{M}(s)} E(t,\rho,f_n)\right| < \varepsilon/2$$

and $\left|E(t,f_n(t),h) - E(t,f_n(t),f_n)\right| < \varepsilon/2$ where N is independent of $t \in T_3$.

(e) Then from (a) and (d) we have

$$\forall_{\varepsilon>0}\ \exists_{f\varepsilon k}: f(t) \in \text{ext}\,\mathcal{M}(S) \text{ a.e. and}$$

$$\left|E(t,f(t),f) - \max_{\rho\in\mathcal{M}(S)} E(t,\rho,f)\right| < \varepsilon$$

a.e. on a set of measure greater than $1 - \varepsilon$.

<u>Remark</u>: Two important steps in the proof above (part (6)) are to show that

(i) N_1 and N_2 are independent of $t \in T_3$ (otherwise the pure strategy function f satisfying the approximate equilibrium definition could not be found).

(ii) N_2 is independent of ρ (otherwise statement (d) would be false).

The former leads to the fact that we can only show that for a set of measure arbitrarily close to 1 (but not guaranteed to be T), almost every firm is within ε of maximizing profits, by behaving according to f. The latter depends crucially on the joint continuity of $\Phi(t,.,.)$: $P \times F \to$.

Ali Khan (1985) has two theorems on the existence of an approximate equilibrium in pure strategies (Theorems 3.4 and 3.5). In his definition of an approximate equilibrium a.e. player is within ε of maximizing a payoff function which is just assumed to be separately continuous on the player's action and on the mean action of all players (given by a Bochner integral in Theorem 3.4 and by a Gel'fand integral in Theorem 3.5). Our Theorem 2 is inspired in Theorem 3.5 in Ali Kahn; it differs from Ali Khan's result because we allowed for a weaker aggregation restriction (A4) and faced the difficulties in steps (i) and (ii) mentioned above.

Notice that if the demand function is the same for all firms then in an approximate equilibrium in pure strategies almost every firm in T will be within ε of maximizing profits.

PART 2

1. Each consumer is assumed to like only a finite number m of differentiated commodities in addition to the numeraire. A consumer type is parameterized by an m-tuple of desired brands and a continuous utility function on a compact subset E of $\mathbb{R}^{m+1}$. Notice that the order in the m-tuple is relevant.

Let $h: I \to T^m$ and $u: I \to C(E)$ be the types mappings.

<u>Assumption H1</u>: (i) h is $(\mathcal{I},\mathcal{T}^m)$-measurable and u is $(\mathcal{I},\mathcal{B}(C(E)))$-measurable; (ii) h and u are independently distributed.

Let $\nu \in \mathcal{M}(T^m)$ be the distribution of the mapping h, defined by

$$\nu(A) = \mu(\{i \in I: h(i) \in A\}) \quad \text{for} \quad A \in \mathcal{T}^m$$

and let $\rho \in \mathcal{M}(C(E))$ be the distribution of the mapping u, defined by

$$\rho(B) = \mu(\{i \in I: u(i) \in B\}) \quad \text{for} \quad B \in \mathcal{B}(C(E))$$

We will assume that consumers' tastes are dispersed over T^m; formally,

Assumption H2: ν is absolutely continuous with respect to product measure λ^m. Let $r \in L'(T^m, \lambda^m)$ be the Radon-Nikodym derivative of ν with respect to λ^m.

This assumption implies that $h(I)$ is a λ^m nonnull subset of T^m and that any λ^m null subset of T^m will be desired by a μ null set of consumers.

To simplify we assume that all consumers have the same endowment $\hat{\ell}$ of the numeraire. We also make the following assumption

Assumption H3: $u(i)$ is a strictly quasi-concave function on $\mathbb{R}^{m+1}$

2. Consider the optimization problem of a consumer of type $(t_1, \ldots, t_m, e)$, where e is an utility function satisfying the above assumptions.

$$\max\ e(x_{t}, \ldots, x_{t_m}, \hat{\ell} - \ell)$$

$$\text{s.t.} \sum_{j=1}^{m} g_{t_j} x_{t_j} \leq \ell$$

$$\ell \leq \hat{\ell}$$

where ℓ is the amount of the numeraire that consumer i sells to firms to finance the purchase of the differentiated commodities

$\hat{\ell}-\ell$ is the consumption level of the numeraire.

$$g_{t_j} = \begin{cases} p(t_j) & \text{if firm } t_j \text{ sets up} \\ & \text{arbitrarily large, otherwise.} \end{cases}$$

Let $s: T \to P \times \{0,1\}$ be the firm's play function. Then the demand for the commodity in the j^{th} position in the utility function e of a consumer who likes $(t_1,\ldots,t_m)$ is a well defined bounded and jointly continuous function on S^m (by Assumptions H1 and H3), which by H1(ii) is independent of the m-tuple $(t_1,\ldots,t_m)$. This individual demand function will be denoted by $\phi^{(j)}(e): S^m \to \mathbb{R}$.

3. <u>The derivation of the demand facing firm t^o</u>

Consider any decreasing sequence of closed intervals A_n in T containing t^o; for each A_n compute the mean demand for commodities in this interval; the limit of the sequence of mean demands as $\lambda(A_n) \to 0$ is defined as the demand facing firm t^o. We will define this concept formally.

Let us examine first the case $m = 2$ (each consumer likes only 2 differentiated commodities). Denote by $\mathscr{A}$ the class of λ^2 nonnull sets in T^2.

Define the two following set functions: $\psi^{(j)}: \mathscr{A} \to \mathbb{R}$, $\hat{\psi}^{(j)}: \mathscr{T}^2 \to \mathbb{R}$ by

$\psi^{(j)}(A\times B)$ is the <u>per consumer</u> demand for commodities in the j^{th} position in the utility function by consumers who like commodities in the set $A \times B \in \mathscr{T}^2$.

$\hat{\psi}^{(j)}(A\times B)$ is the <u>per capita</u> demand for commodities in the j^{th} position in the utility function by consumers who like commodities in the set $A \times B \in \mathscr{T}^2$.

Formally, $\psi^{(j)}(A\times B) = \hat{\psi}^{(j)}(A\times B)/\nu(A\times B)$, for $A \times B \in \mathscr{A}$

$$\hat{\psi}^{(j)}(A\times B) = \int_{C(E)} \int_{A\times B} \int_{S^2} \phi^{(j)}(e)\, df(t_1)df(t_2)r(t_1,t_2)d\lambda^2 d\rho, \quad \text{for} \quad A \times B \in \mathscr{A}^2$$

where $f \in K$ is the firms' mixed strategy function mapping T into $\mathscr{M}(S)$.

We have integrated consumers' demands with respect to the mixed strategies played by firms; $\psi^{(j)}$ and $\hat{\psi}^{(j)}$ are expected demands (with respect to the probability measures on firms' price and setup strategies).

Define the marginals of $\psi^{(j)}$ by $\psi_1^{(j)}(A) = \psi^{(j)}(A\times T)$ and $\psi_2^{(j)}(B) = \psi^{(j)}(T\times B)$. The marginals of $\hat{\psi}^{(j)}$ are defined similarly.

Notice that the per consumer (per capita) demand for a set A of commodities by consumers who like commodities in A in some position of the utility function is given by $\psi^{(1)}(A) + \psi^{(2)}(A)$ (respectively, $\hat{\psi}^{(1)}(A) + \hat{\psi}^{(2)}(A)$).

Then the mean demand for commodities in the set A (with respect to the probability measure λ on the set of commodity characteristics) is given by

$$[\psi_1^{(1)}(A)\ \nu(A\times T) + \psi_{(2)}^{(2)}(A)\ \nu(T\times A)]/\lambda(A) \equiv [\hat{\psi}_1^{(1)}(A) + \hat{\psi}_2^{(2)}(A)]/\lambda(A).$$

Hence, the demand for brand t^o is defined as the Radon-Nikodym derivative of $\hat{\psi}_1^{(1)} + \hat{\psi}_2^{(2)}$ with respect to the measure λ:

$$\lim_{A\to t^o} (\hat{\psi}_1^{(1)}(A) + \hat{\psi}_2^{(2)}(A))/\lambda(A) =$$

$$= \int_{C(E)} [\int_T \int_{S^2} \phi^{(1)}(e)\ df(t^o)\ df(t)r(t^o,t)d\lambda +$$

$$+ \int_T \int_{S^2} \phi^{(2)}(e)\ df(t)\ df(t^o)\ r(t,t^o)d\lambda]d\rho$$

We want to express the demand facing each firm as a function of the price that the firm charges and of the mixed strategies function f. This is obtained from the above by setting the mixed strategy played by t^o, $f(t^o)$, equal to the Dirac measure at the point $(p,1)$ (that is, the firm operates and charges price p with probability 1):

$$D(t^o,p,f) = \int_{C(E)} [\int_T \int_S \phi^{(1)}(e)(p,1,.)\ df(t)\ r(t^o,t)d\lambda + \tag{1}$$

$$+ \int_T \int_S \phi^{(2)}(e)(.,p,1)\ df(t)\ r(t,t^o)d\lambda]d\rho$$

Remark: $D(t^o)$ is the form considered in Theorem 1 on the existence of a mixed strategies noncooperative equilibrium (see Part I); in Hart (1983) prices of all brands were integrated with respect to the same distribution and therefore he was able to use a distribution approach. Our approach is more general and will allow us to show the existence of an approximate equilibrium in pure strategies under an aggregation restriction (of the form considered in Assumption (A4) in Part I) which is weaker than the aggregation condition required in the distribution approach used by Hart (1983). As we will see, this weaker aggregation restriction allows for a wider class of consumers' preferences and focuses on a truly no-neighboring goods assumption.

4. Properties of the demand function $D(t^o)$ and the existence of a mixed strategies equilibrium

Assumption (A1) in Part I on the existence of an upper bound for the set of chargeable prices will be met if we impose the following additional assumptions:

Assumption H4: The density of the tastes distribution ν is essentially bounded, that is, $r \in L^\infty(T^m,\lambda^m)$.

Assumption H5: The utility functions mapping $u: I \to C(E)$ is such that for any consumer $i \in I$ and for any of his/hers desired brand t, $\phi^t(i)p_n \to 0$ as $p_n \to 0$ (where $\phi^t(i)$ is consumer i's demand for brand t).

The latter implies that individual demands are price elastic but is actually stronger than that. A Cobb-Douglas utility function would be ruled out by (H5) but a function given by $u = (x_1+a_1)^{\alpha_1}(x_2+a_2)^{\alpha_2}$, where $a_1,a_2 > 0$

satisfies (H5) (because at the optimum, $x_1 p_1 = (\alpha_1/\alpha_2)(\ell - a_2 p_2) - a_1 p_1$, for $x_1 \geq 0$). Hart's (1983) statement about the existence of an upper bound on the set of prices is based upon the assumption that (H5) follows from consumer's rationality, but this does not seem to be necessarily the case for any specification of preferences. Notice that in the special case studied by Hart (1983b) Assumption (H5) is verified $(u = A(\Sigma v_i x_i)^{\alpha} + \ell$ where $0 < \alpha < 1$, $v_i \geq 0$ and by looking at expression (3.3) we see that (H5) is met).

Proposition 1: Under Assumption H, through H5, $D(t)$ given by expression (1) above satisfies assumption (A1) in Part I.

Proof:

Suppose not, i.e., $\exists_{f \epsilon k}: \sup_{t \epsilon T} pD(t,p,f) \geq F > 0$ for any $p \geq 0$. Now by Holder's inequality and (H4)

$$D(t,p,f) \leq ||r||_{\infty} \int_{C(E)} [\int_T \int_S \phi^{(1)}(e)\, df(t) d\lambda + \int_T \int_S \phi^{(2)}(e)\, df(t) d\lambda] d\rho$$

where the expression in the RHS does not depend on t.

Multiply both sides by p and let $p \to \infty$; the RHS tends to zero by (H5) and Lebesgue dominated convergence theorem; therefore, $\sup_{t \epsilon T} pD(t,p,f) \to 0$ as $p \to \infty$, a contradiction.

Proposition 2: Under Assumptions (H1), (H2) and (H3), $D(t)$ given by expression (1) above satisfies Assumption (A2) in Part I:

$$D(.,p,f) \text{ is } \mathscr{F}\text{-measurable, for any } (p;f) \in P \times K$$

(immediate, by construction as a R-N derivative with respect to λ).

<u>Proposition 3</u>: Under Assumptions (H1) through (H5), $D(t)$ given by expression (1) above satisfies Assumption (A3) in Part I:

$D(t,.,.): P \times K \to \mathbb{R}$ is jointly continuous with respect to the weak* topology on K

<u>Proof</u>

- <u>Claim</u>: $D(t^o,p,.)$ is weak* continuous on K, for any $p \in P$ in fact, let $s_{t^o} = (p,1)$ and notice that the mappings defined by

$$t \mapsto \phi^{(1)}(e)\ (s_{t^o},.)\ r(t^o,t)$$

$$t \mapsto \phi^{(2)}(e)\ (.,s_{t^o}) r(t,t^o)$$

belong to $L'(T,C(S))$. Therefore, by definition of the weak* topology on K and by Lebesgue dominated convergence theorem (applied to convergence ρ a.e. on $C(E)$) we have the result.

- <u>Claim</u>: If $p^n_{t^o} \to p_t$ and $\{f_n\}$ is a sequence of K then

$$\forall_{\delta>0}\ \exists_m: n \geq m \Rightarrow \left|D(t^o,p^n_{t^o},f_n) - D(t^o,p_{t^o},f_n)\right| < \delta$$

in fact,

- let $s^n_{t^o} = (p^n_{t^o},1)$ and $s_{t^o} = (p_{t^o},1)$; by joint continuity of $\phi^{(1)}(e)$ on S^2 it follows that $\phi^{(1)}(e)\ (s^n_{t^o},.)$ converges continuously to $\phi^{(1)}(e)\ (s_{t^o},.)$ and therefore (by Royden, Ch. 8, problem 40)

$$\forall_{\varepsilon>0}\ \exists_m: n \geq m \Rightarrow \sup_{s \in S} \left|\phi^{(1)}(e)\ (s^n_{t^o},s) - \phi^{(1)}(e)\ (s_{t^o},s)\right| < \varepsilon \qquad (2)$$

and similarly for $\phi^{(2)}(e)$.

- Now $\left|\int_S [\phi^{(1)}(e)\ (s^n_{t^o},.) - \phi^{(1)}(e)\ (s_{t^o},.)]\ df_n(t)\right| \leq$

$$\leq \sup_{s \in S} \left|\phi^{(1)}(e)\ (s^n_{t^o},s) - \phi^{(1)}(e)\ (s_{t^o},s)\right| \quad (\text{since } f_n(t)(S) = 1)$$

. and $\left|D(t^o,p^n_{t^o},f_n) - D(t^o,p_{t^o},f_n)\right| \leq$

$$\leq \int_{C(E)} \left\{\int_T r(t^o,t)\left|\int_S [\phi^{(1)}(e)\ (s^n_{t^o},.) - \phi^{(1)}(e)\ (s_{t^o},.)]\ df_n(t)\right| d\lambda + \right.$$

$$\left. + \int_T r(t,t^o)\left|\int_S [\phi^{(2)}(e)\ (.,s^n_{t^o}) - \phi^{(2)}(e)\ (.,s_{t^o})]\ df_n(t)\right| d\lambda\right\} d\rho$$

. Therefore, by (2),

$$\forall_{\varepsilon>0}\ \exists_m: n \geq m \Rightarrow \left|D(t^o,p^n_{t^o},f_n) - D(t^o,p_{t^o},f_n)\right| < \varepsilon\ (r_1(t^o) + r_2(t^o))$$

where r_1 and r_2 are the marginal densities.

Now if $r_1(t^o)$ and $r_2(t^o)$ are not both zero let $\varepsilon = \delta/(r_1(t^o) + r_2(t^o)$ otherwise the result is trivial.

Combining the results in these two claims the proof is completed.

Hence by Theorem 1 in Part I we have the following result

Corollary 1.1: Under Assumption (H1) through (H5) and for $m = 2$, there exists a mixed strategies noncooperative equilibrium for the nonatomic Bertrand game.

We will see that this result is also true for $m > 2$ (finite).

5. Before considering the general case $(m > 2)$ we shall address the existence of an approximate equilibrium in pure strategies for the above specification of preferences. Let us examine how can an aggregation restriction (of the form considered in Assumption (A4) in Part I) be imposed on the demand functions $D(t)$, $t \in T$, for all $m = 2$.

In order to have the demand facing each firm depending on the mixed strategies function f only through a common finite number of weighted means of f, we need the m weighting functions z_i to be common to all brands. Then, by inspection of the expression for $D(t^o)$, this leads to the independence assumption:

Assumption (H6): The distribution ν of the tastes mapping $h: I \to T^m$ satisfies the independence criterion $\nu = \overset{m}{\underset{i=1}{X}} \nu_i$, where ν_i is the marginal distribution of ν on the i^{th} coordinate space.

Under this assumption,

$$D(t^o;p_{t^o},f,g) = \int_{C(E)} [r_1(t^o) \int_S \phi^{(1)}(e)\ d(\oint_T r_2 f\ d\lambda) +$$

$$+ r_2(t^o) \int_S \phi^{(2)}(e)\ d(\oint_T r_1 f d\lambda)]\ d\rho$$

Then by assumption (A5) in Part I is verified: $D(t^o)$ can be rewritten as

$$\phi(t^o;p_{t^o},\oint_T r_1 f d\lambda,\ \oint_T r_2\ f d\lambda) \quad \text{(see footnote (3) also)}$$

and $\phi(t^o,.): P \times F \to \mathbb{R}$ where $F = \{\sigma \in \mathcal{M}(S)^2 : \sigma = (\oint_T z_i f d\lambda)_{i=1}^2$ for some $f \in K\}$, is jointly continuous on $P \times F$ (with respect to the weak* topology on $\mathcal{M}(S)^2$). Hence by Theorem 2 in Part I we have the following result:

Corollary 2.1: Under Assumption (H1) through (H6) and for $m = 2$, there exists an approximate equilibrium in pure strategies for the nonatomic Bertrand game.

6. Assumption (H6) is truly the no-neighboring goods assumption

Assumption (H6) states that the fact that a consumer likes brand t^o does not give any information about what are his/her other desired brands. This is plainly the no-neighboring goods (n.n.g.) assumption; Hart (1983) has the stronger assumption that the tastes distribution is uniform, but in a n.n.g. situation some goods may still be desired by more consumers than others.

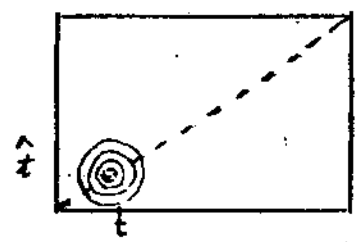
We will discuss now the two possible non-uniform n.n.g. situations. If the marginal densities r_1 and r_2 are equal then some goods will be desired by more consumers than others irrespective of their position in the utility function. On the other hand, if $r_1 \neq r_2$ then the position is relevant. The former case would allow for a larger set of continuous utility functions; in the latter, the relative position of a commodity in the utility function should have some meaning.

Notice that if $r_1 = r_2$ then $D(t)$ depends on the mixed strategy function f only through the mean mixed strategy under the tastes distribution $\oint_T \bar{r} f d\lambda$, where $\bar{r} \equiv r_1 = r_2$. If $r_1 \neq r_2$ the $D(t)$ depends on f through two weighted means, one for each marginal of the tastes distribution; this situation would occur if the order of the pair of desired brands has some meaning which is common across consumers.

Suppose that the order in the pair (t_1, t_2) implies a ranking: brand t_1 is the consumer's first choice and brand t_2 is his/her second choice. In this context, an example of the set of utility functions could be the set of functions of the form $u = (x_1+a)^{\alpha_1}(x_2+a_2)^{\alpha_2}$, where $\alpha_1 \geq \alpha_2 (a_1, a_2 > 0)$ whereas if $r_1 = r_2$ we would allow for any pair (α_1, α_2).

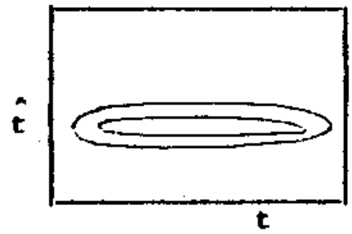
In the ranking case strategic information would be aggregated into the distribution of strategies played by firms whose products are ranked as 1st choices $(\oint_T r_1 f d\lambda)$ and the distribution of strategies played by firms where products are ranked as 2nd choices $(\oint_T r_2 f d\lambda)$. The order can have other interpretations as we associate the position in the m-tuple with a particular way in which a commodity can be used.

Let us now compare these two non-uniform n.n.g. situations with the n.g. case, by looking at the isodensity curves.



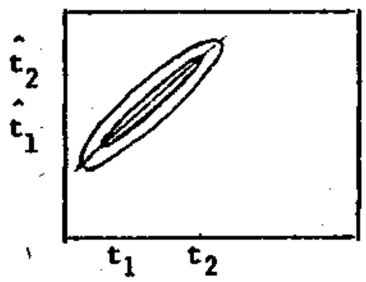
$r_1 = r_2$, $r = r_1 r_2$ (n.n.g.)

If the consumer likes t in the 1st position he will behave like anyone else with regard to his/her other desired brand (namely, favoring brands close to $\hat{t}$)



$r_1 \neq r_2$, $r = r_1 r_2$ (n.n.g.)

idem



no independence (n.g.)

If the consumer likes t_1, then he is likely to like $\hat{t}_1$ given by the regression surface.

In the first two cases all firms will be influenced by the same pair of aggregates of strategic information. In the third case it is not possible to find a finite number of "sufficient statistics" of strategic information that are common to all firms.

(The extension of corollaries 1.1 and 2.1 to the case $m > 2$ is a lengthy but straightforward inductive argument which will be presented in a forthcoming complete version of this paper.)

Footnotes

[1]To show that $B^e(t,h) = \text{ext } B(t,h)$:

- say $x \in B^e(t,h)$ and $x = \lambda x_1 + (1-\lambda)x_2$, where $0 < \lambda < 1$ $x_1, x_2 \in B(t,h)$; then since $x_1, x_2 \in \mathcal{M}(S)$ we have $x \notin \text{ext } \mathcal{M}(S)$, a contradiction.
- say $x \in \text{ext } B(t,h)$, then $x \in \text{ext } \mathcal{M}(S)$ and it follows that $x \in B^e(t,h)$.

[2]To show that $A(t,h)$ is weak* closed:

- let $x_n \in A(t,h)$, i.e., $x_n = (z_1(t)w_n, \ldots, z_m(t)w_n)$ where $w_n \in B(t,h)$ and suppose $x_n \overset{*}{\to} x$, i.e.,

$$\sum_{i=1}^{m} z_i(t) \langle x_i', w_n \rangle \to \sum_{i=1}^{m} \langle x_i', x_i \rangle,$$

  by any $x' \in C(S)^m$
- by weak* compactness of $B(t,h)$, there exists a subsequence w_{n_i} such that $\langle a, w_{n_i} \rangle \to \langle a, \bar{w} \rangle$ where $\bar{w} \in B(t,h)$, for any $a \in C(S)$.
- then $\sum_{i=1}^{m} z_i(t) \langle x_i', w_{n_i} \rangle \to \sum_{i=1}^{m} z_i(t) \langle x_i', \bar{w} \rangle$ for any $x \in C(S)^m$ and therefore $x = (z_1(t)\bar{w}, \ldots, z_m(t)\bar{w})$, that is, $x \in A(t,h)$.

[3]The aggregates of strategic information can be easily decomposed in terms of price and setup strategies.

Let $f \in K$ and define

$$f_2 : T \to \mathcal{M}(\{0,1\}) \text{ by } f_2(t)(\{a\}) = f(t)(P \times \{a\}), \ a \in \{0,1\}$$

$$f' : T \to \mathcal{M}(P) \text{ by } f'(t)(A) = f(t)(A \times \{1\})/f_2(t)(\{1\})$$

provided $f_2(t)(\{1\}) \neq 0$

To simplify assume $r_1(t) = r_2(t) \equiv \bar{r}(t)$ a.e. Then under Assumptions H1 through H6, it can be easily shown that $D(t^o,p,f)$ depends on f only through

- $\int_T \bar{r}(t)\, f_2(t)(\{0\})d\lambda$, the weighted means of the non-setup probabilities ("the proportion of non-operating firms")
- $\oint_T \bar{r}(t)\, f_2(t)(\{1\})\, f'(t)d\lambda$, the mean of the conditional (on setting-up) mixed strategies on prices, weighted by the setup marginal and by the tastes density (this is the relevant distribution of prices charged by operating firms).

<u>Mathematical Appendix</u>: The spaces $L_w^\infty(T;rba(S))$ and $L_w^\infty(T;rca(S))$

(1) Let $(T, \mathscr{T}, \lambda)$ be the Lebesgue measure space of the unit interval, E be a Banach space and E^* its topological dual. A function f mapping T into E^* is weak* measurable if $\langle x,f\rangle$ is $\mathscr{T}$-measurable for any $x \in E$. Denote by $M_w^\infty(T;E^*)$ the vector space of all weak* measurable mappings $f: T \to E^*$ such that $||f(.)|| \in L^\infty(T,\lambda)$.

(2) Consider the equivalence relation R_w on $M_w^\infty(T;E^*)$ defined by $(f,g) \in R_w$ iff $\langle x,f\rangle$ and $\langle x,g\rangle$ coincide a.e. for any $x \in E$. Denote the corresponding quotient space by $L_w^\infty(T;E^*)$ and let $f \to \dot{f}$ be the canonical mapping of $M_w^\infty(T;E^*)$ into $L^\infty(T;E^*)$. A norm on $L^\infty(T;E^*)$ is defined by

$$N_\infty(\dot{f}) = \inf \{\text{ess sup } ||g||: g \in M_w^\infty(T;E^*) \text{ and } \dot{f} = \dot{g}\}.$$

(3) $L_w^\infty(T;E^*)$ is isametrically isomorphic to the topological dual of $L'(T;E)$, the Banach-space of (equivalence classes - for the equivalence given by the equality a.e. - of) Bochner integrable functions mapping T into E (that is, $\mathscr{T}$-measurable functions $h: T \to E$ such that $\int_T ||h|| d\lambda < \infty$).

(4) If E is separable, then for any $f,g \in M_w^\infty(T;E^*)$ we have $(f,g) \in R_w$ iff f and g coincide a.e.; $L_w^\infty(T;E^*)$ is the corresponding quotient space. Clearly $N_\infty(\dot{f}) = \text{ess sup } ||f||$ for any $f \in M_w^\infty(T;E^*)$ such that $f \to \dot{f}$.

(5) Now consider the situation contemplated in Part I: $E = C(S)$, where S is normal. $L_w^\infty(T;rba(S))$ is defined as in (2) above. Under Assumption (A1), $C(S)$ is separable and therefore the remark in (3) applies to $L_w^\infty(T;rca(S))$.

The references for this Appendix are Diendonné (1951) and Tulcea and Tulcea (1969, sections IV5, VI4, 5 and 7; VII4).

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