mathematics

MDPI

Article

A New Class of Reduced-Bias Generalized Hill Estimators

Lígia Henriques-Rodrigues [1,*,†], Frederico Caeiro [2,†] and M. Ivette Gomes [3,†]

[1] School of Science and Technology and Research Center in Mathematics and Applications (CIMA), University of Évora, 7000-671 Évora, Portugal
[2] NOVA School of Science and Technology (NOVA FCT) and Center for Mathematics and Applications (CMA), NOVA University Lisbon, 2829-516 Caparica, Portugal; fac@fct.unl.pt
[3] Department of Statistics and Operational Research (DEIO), Faculty of Sciences of the University of Lisbon (FCUL) and Centre of Statistics and its Applications (CEAUL), University of Lisbon (ULisboa), 1749-016 Lisbon, Portugal; migomes@ciencias.ulisboa.pt
* Correspondence: ligiahr@uevora.pt
† These authors contributed equally to this work.

Abstract: The estimation of the extreme value index (EVI) is a crucial task in the field of statistics of extremes, as it provides valuable insights into the tail behavior of a distribution. For models with a Pareto-type tail, the Hill estimator is a popular choice. However, this estimator is susceptible to bias, which can lead to inaccurate estimations of the EVI, impacting the reliability of risk assessments and decision-making processes. This paper introduces a novel reduced-bias generalized Hill estimator, which aims to enhance the accuracy of EVI estimation by mitigating the bias.

Keywords: statistics of extremes; generalized means; reduced-bias estimators; extreme value index; asymptotic properties; Monte Carlo simulation

MSC: 62E20; 62G32; 62P12; 65C05

check for updates

Citation: Henriques-Rodrigues, L.; Caeiro, F.; Gomes, M.I. A New Class of Reduced-Bias Generalized Hill Estimators. *Mathematics* **2024**, *12*, 2866. https://doi.org/10.3390/math12182866

Academic Editor: Manuel Alberto M. Ferreira

Received: 31 July 2024
Revised: 4 September 2024
Accepted: 10 September 2024
Published: 14 September 2024

1. Introduction

Statistics of extremes is a branch of statistics dealing with inference for extreme or rare events. Due to their unpredictability, such events are a matter of enormous concern since they can produce catastrophic consequences. The areas of application are very diversified. We mention the fields of environment (extreme pollution), hydrology (river flows), meteorology (heat waves, windstorms), geology (diamond sizes), insurance (large insurance claims), telecommunications (internet traffic), public health (rates of Pneumonia), and finance (stock market crash), among others.

Extreme value theory provides the theoretical framework for building the statistical model describing extreme events. The first fundamental limit results appeared in Fréchet [1], Fisher and Tippett [2], von Mises [3], and Gnedenko [4]. More specifically, let $X_1, X_2 \ldots$ denote a sequence of independent and identically distributed random variables with a common distribution function (d.f.) $F(x) = P(X \le x)$. Assume that there exist sequences of normalizing constants $a_n > 0$, and $b_n \in \mathbb{R}$, and a non-degenerate d.f. G such that, for all x,

$$\lim_{n\to\infty} P\left(\frac{\max(X_1, X_2, \ldots, X_n) - b_n}{a_n} \le x\right) = G(x). \quad (1)$$

Then, we say that the d.f. F belongs to the max-domain of attraction of the extreme value (EV) distribution G, and we write $F \in D_M(G)$. The sequences of constants $a_n > 0$ and $b_n \in \mathbb{R}$ can be redefined in such a way that for some $\xi \in \mathbb{R}$,

$$G(x) \equiv G_\xi(x) := \begin{cases} \exp\left(-(1+\xi x)^{-1/\xi}\right),\ 1+\xi x > 0 & \text{if} \quad \xi \neq 0 \\ \exp(-\exp(-x)),\ x \in \mathbb{R} & \text{if} \quad \xi = 0. \end{cases} \quad (2)$$

The shape parameter ξ is the extreme value index (EVI), the most important parameter of extreme events, which is related to the heaviness of the right tail. It is also a cornerstone in the estimation of other parameters of extreme events (see [5–10], among others). For more details on the max domain of attraction, see Beirlant et al. [11], de Haan and Ferreira [12], and Dey and Yan [13], among others.

We consider in the present paper Pareto-type tails, i.e., $\xi > 0$ in the EV d.f. in (2). In mathematical terms, this is equivalent to assuming that the right tail or survival function $1 - F$ satisfies

$$1 - F(x) = x^{-1/\xi} L(x), \quad x \to \infty,$$

where $L(x)$ is an unspecified slowly varying function at infinity, i.e., $\lim L(tx)/L(t) = 1$, as $t \to \infty$, for all $x > 0$ (see references [12,14,15] for details on regular variation). This condition is usually called the first-order condition. If the slowly varying function L is constant, then F is the exact Pareto model. Equivalently, F is a model with a Pareto-type tail whenever the reciprocal tail quantile function U, defined by

$$U(t) = F^{\leftarrow}(1 - 1/t), \quad t \geq 1, \qquad \text{with} \qquad F^{\leftarrow}(x) = \inf\{y : F(y) \geq x\}, \tag{3}$$

is of regular variation with index ξ (denoted as $U \in RV_\xi$), i.e., for every $x > 0$,

$$\lim_{t \to \infty} \frac{U(tx)}{U(t)} = x^{\xi}. \tag{4}$$

For $\xi > 0$, the most well-known EVI-estimator is the Hill estimator [16], which is the average of the k log-excesses,

$$\hat{\xi}^H(k) := \frac{1}{k} \sum_{i=1}^{k} V_{ik}, \quad 1 \leq k < n, \tag{5}$$

with

$$V_{ik} := \ln X_{n-i+1:n} - \ln X_{n-k:n}, \tag{6}$$

where $X_{1:n} \leq X_{2:n} \leq \ldots X_{n:n}$ denote the ascending-order statistics from a random sample of size n. The consistency of the Hill estimator is achieved if the threshold $X_{n-k:n}$ is an intermediate order statistic [17], i.e., if

$$k = k(n) \to \infty \quad \text{and} \quad k/n \to 0, \text{ as} \quad n \to \infty. \tag{7}$$

The asymptotic normality of Hill's estimator has been studied, under different assumptions, by several authors (see [18–20], among others). Typically, F is a Pareto-type model and the Hill estimator has a high variance for high thresholds $X_{n-k:n}$, i.e., for small values of k and a high bias for low thresholds, i.e., for large values of k. Consequently, the mean squared error (MSE) has a very peaked pattern, making it difficult to determine the optimal k, defined as the value k_0 where the MSE reaches its minimal value. For a detailed review on estimation procedures for the EVI, see [9,21]. To overcome this trade-off problem between bias and variance, several alternative estimators are proposed in the literature. Some of these alternative estimators are based on the α-moment of the log-excesses

$$M_n^{(\alpha)}(k) := \tfrac{1}{k} \sum_{i=1}^{k} V_{ik}^{\alpha}. \tag{8}$$

We mention the class of kernel estimators introduced in Csörgő et al. [22], the moment estimator proposed by Dekkers et al. [23],

$$\hat{\xi}^{M}(k) = M_n^{(1)}(k) + 1 - \frac{1}{2}\left(1 - \frac{(M_n^{(1)}(k))^2}{M_n^{(2)}(k)}\right)^{-1}, \tag{9}$$

and the Jackknife estimators proposed in Gomes et al. [24]. Generalizations of the Hill estimator, dependent on an additional tuning parameter, can be found in Gomes and Martins [25], Groeneboom et al. [26], Beran et al. [27], Penalva et al. [28], Caeiro et al. [29], and Paulauskas and Vaičiulis [30], among others. For a comparison between several classic EVI-estimators, we refer to [31,32].

In this paper, we consider one of the aforementioned classes of generalized Hill (GH) estimators. More precisely, we consider the class of GH estimators in [25,33] based on the power mean (PM) of the normalized log-excesses,

$$\hat{\xi}^{\mathrm{PM}(\alpha)}(k) = \left(\frac{M_n^{(\alpha)}(k)}{\Gamma(\alpha+1)}\right)^{1/\alpha}, \quad \alpha > 0, \quad 1 \le k < n. \tag{10}$$

The Hill estimator is obtained by taking $\alpha = 1$ in (10) ($\hat{\xi}^{\mathrm{H}}(k) = \hat{\xi}^{\mathrm{PM}(1)}(k)$). To reduce the bias of many classic EVI-estimators, a slightly more restrictive condition than (4) is needed. More specifically, we assume that the quantile function U belongs to a subset of Hall's class [34,35],

$$U(t) = C\, t^{\xi}\left(1 + \xi\beta\, t^{\rho}/\rho + \beta'\, t^{2\rho} + o\left(t^{2\rho}\right)\right), \tag{11}$$

as $t \to \infty$, with $\beta, \beta' \neq 0$, and $\rho < 0$. This subset of Hall's class includes several well-known Pareto-type models, such as the Burr, Fréchet, Generalized Pareto, the Loglogistic, and the Student's t. Here, we introduce a new class of reduced-bias (RB) EVI-estimators, still dependent on a tuning parameter,

$$\hat{\xi}^{\mathrm{RBPM}(\alpha)}(k) = \hat{\xi}^{\mathrm{PM}(\alpha)}(k)\left(1 - \frac{1-(1-\hat{\rho})^{\alpha}}{\alpha\hat{\rho}(1-\hat{\rho})^{\alpha}}\hat{\beta}\left(\frac{n}{k}\right)^{\hat{\rho}}\right), \quad \alpha > 0, \quad 1 \le k < n, \tag{12}$$

with $(\hat{\beta}, \hat{\rho})$ adequate estimators of the second-order parameters (β, ρ) defined in (11). This class of RB estimators generalizes the well-known minimum-variance reduced-bias (MVRB) corrected Hill (CH) estimators introduced in Caeiro et al. [36]. More precisely, if $\alpha = 1$ in (12),

$$\hat{\xi}^{\mathrm{CH}}(k) \equiv \hat{\xi}^{\mathrm{RBPM}(1)}(k) := \hat{\xi}^{\mathrm{H}}(k)\left(1 - \frac{\hat{\beta}}{1-\hat{\rho}}\left(\frac{n}{k}\right)^{\hat{\rho}}\right), \quad 1 \le k < n, \tag{13}$$

with $\hat{\xi}^{\mathrm{H}}(k)$ the Hill estimator in (5). Other types of RB RVI-estimators based on the Hill estimator can be found in [37,38], among others.

The remainder of this paper is organized as follows. Section 2 is devoted to the asymptotic behavior of the EVI-estimators under study. Under a third-order framework, the asymptotic properties of the PM EVI-estimator, in (10), are presented. Next, the asymptotic non-degenerate behavior of the RBPM EVI-estimator, in (12), is derived, assuming first that the second-order parameters (β, ρ) are known and then that they are adequately estimated in the high level k_1 of a larger order than k, the number of order statistics used in the EVI-estimation. In Section 3, an asymptotic comparison at optimal levels of the new reduced-bias estimators is performed. The finite sample properties of the new estimators are assessed through a Monte Carlo simulation study presented in Section 4. Section 5 is devoted to illustrating the behavior of the estimators under study to a hydrological data set, and finally, in Section 6, the main conclusions of this study are put forward.

2. Asymptotic Behavior of the Estimators

To study the asymptotic bias of the EVI-estimators under consideration, and due to the slowly varying parts of the CDF F, we need a second-order condition. We, thus, assume the existence of an ultimately positive regular varying function A with index ρ, i.e., $|A(t)| \in RV_\rho$ [39], such that

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \xi \ln x}{A(t)} = \frac{x^\rho - 1}{\rho}, \tag{14}$$

for every $x > 0$. The second-order parameter $\rho \le 0$ rules the rate of convergence of $U(tx)/U(t)$ to x^ξ. The speed increases as $|\rho|$ increases. Algorithms for an adequate estimation of the second-order parameters (β, ρ) can be found for instance in [40,41], among others. To obtain additional information about the bias of the estimators under study, we also impose the validity of the following third-order condition: there exists a $B(t)$ function, which measures the rate of convergence in the second-order condition in (14), such that,

$$\lim_{t\to\infty} \frac{\frac{\ln U(tx) - \ln U(t) - \xi \ln x}{A(t)} - \frac{x^\rho - 1}{\rho}}{B(t)} = \frac{x^{\rho+\rho'} - 1}{\rho + \rho'}, \tag{15}$$

for all $x > 0$, with $\rho' \le 0$, a third-order parameter, and $|B(t)| \in RV_{\rho'}$. This condition holds for several Pareto-type models.

A considerable number of models that fulfill Equation (15) belong the subset of Hall's class in (11). For those models, we have

$$A(t) = \xi\beta t^\rho, \quad B(t) = \beta' t^{\rho'}, \quad \text{with} \quad \beta, \beta' \neq 0, \quad \delta = \beta'/\beta \quad \text{and} \quad \rho = \rho' < 0. \tag{16}$$

Based on Lemma 2.1 of Gomes et al. [42], we begin by introducing some notation that underlies the distributional representation of $M_n^{(\alpha)}$ in Equation (8), given the validity of the third-order condition in (15), and assuming (16). With E_i, $i = 1, 2, \ldots$, a sequence of independent and identically distributed standard exponential random variables, let us consider the notations:

$$\begin{aligned}
\mu_\alpha^{(1)} &:= \mathbb{E}[E_i{}^\alpha] = \Gamma(\alpha+1), && (17)\\
\mu_\alpha^{(2)}(\rho) &:= \mathbb{E}\left[E_i{}^{\alpha-1}\left(\frac{e^{\rho E_i} - 1}{\rho}\right)\right] = \frac{\Gamma(\alpha)}{\rho}\left(\frac{1}{(1-\rho)^\alpha} - 1\right), && (18)\\
\mu_\alpha^{(3)}(\rho) &:= \mathbb{E}\left[E_i{}^{\alpha-2}\left(\frac{e^{\rho E_i} - 1}{\rho}\right)^2\right], && (19)\\
&= \begin{cases} \frac{1}{\rho^2}\ln\frac{(1-\rho)^2}{1-2\rho}, & \alpha = 1 \\ \frac{\Gamma(\alpha)}{\rho^2(\alpha-1)}\left(\frac{1}{(1-2\rho)^{\alpha-1}} - \frac{2}{(1-\rho)^{\alpha-1}} + 1\right), & \alpha \neq 1 \end{cases}, && (20)\\
\sigma_\alpha &:= \sqrt{\mathbb{V}ar[E_i{}^\alpha]} = \sqrt{\Gamma(2\alpha+1) - \Gamma^2(\alpha+1)}, && (21)
\end{aligned}$$

and

$$\overline{\mu}_\alpha^{(j)}(\rho) := \frac{\mu_\alpha^{(j)}(\rho)}{\mu_\alpha^{(1)}}, \qquad \overline{\sigma}_\alpha := \frac{\sigma_\alpha}{\mu_\alpha^{(1)}}. \tag{22}$$

Theorem 1. *Assuming the validity of the third-order condition,* (15) *and also of* (16)*, and considering intermediate sequences $k = k_n$ satisfying* (7)*, the following distributional representation holds:*

$$M_n^{(\alpha)}(k) \stackrel{d}{=} \xi^\alpha \mu_\alpha^{(1)}\left(1 + \frac{\overline{\sigma}_\alpha}{\sqrt{k}} Z_k^{(\alpha)} + \frac{\alpha}{\xi}\overline{\mu}_\alpha^{(2)}(\rho) A(n/k) + O_p\left(\frac{A(n/k)}{\sqrt{k}}\right)\right.$$
$$\left. + \frac{1}{2\xi^2}\left(2\alpha\,\delta\overline{\mu}_\alpha^{(2)}(2\rho) + \alpha(\alpha-1)\overline{\mu}_\alpha^{(3)}(\rho)\right) A^2(n/k)(1 + o_p(1))\right),$$

where $Z_k^{(\alpha)}$ are asymptotically standard normal random variables, defined by

$$Z_k^{(\alpha)} := \frac{P_k^{(\alpha)}}{\sigma_\alpha} = \frac{\sqrt{k}}{\sigma_\alpha}\left(\frac{1}{k}\sum_{i=1}^{k} E_i^\alpha - \Gamma(\alpha+1)\right). \tag{23}$$

The proof of this theorem follows the same structure as the proof of Proposition 3.2 in [42].

Remark 1. *From the results presented in Gomes and Martins ([24]), the vector $(P_k^{(1)}, P_k^{(\alpha)})$, with $P_k^{(\alpha)}$ defined in (23), is asymptotically a bivariate random vector with zero mean value and covariance matrix,*

$$\Sigma_\alpha = \begin{bmatrix} 1 & \alpha\Gamma(\alpha+1) \\ \alpha\Gamma(\alpha+1) & \Gamma(2\alpha+1) - \Gamma^2(\alpha+1) \end{bmatrix}, \tag{24}$$

being the $\mathrm{Cov}\left(P_k^{(\alpha)}, P_k^{(\beta)}\right)$ given by $\Gamma(\alpha+\beta+1) - \Gamma(\alpha+1)\Gamma(\beta+1)$, for any pair (α, β).

In the next Theorem, we state without proof the asymptotic distributional representation of the GH estimator, $\widehat{\xi}^{PM(\alpha)}$, in (10), under a third-order framework, extending the result presented in Gomes and Martins ([24]).

Theorem 2. *Assuming the validity of the third-order condition, (15) and also of (16), and considering intermediate sequences $k = k_n$ satisfying (7), the following distributional representation holds:*

$$\begin{aligned}\widehat{\xi}^{\mathrm{PM}(\alpha)}(k) \stackrel{d}{=} \xi + \frac{\xi\,\overline{\sigma}_\alpha}{\alpha\sqrt{k}} Z_k^{(\alpha)} + \overline{\mu}_\alpha^{(2)}(\rho) A(n/k) + O_p\left(\frac{A(n/k)}{\alpha\sqrt{k}}\right) \\ + \frac{1}{2\xi}\left(2\delta\overline{\mu}_\alpha^{(2)}(2\rho) + (\alpha-1)\left(\overline{\mu}_\alpha^{(3)}(\rho) - (\overline{\mu}_\alpha^{(2)}(\rho))^2\right)\right) A^2(n/k)(1+o_p(1)),\end{aligned}$$

with

$$\begin{aligned}\frac{1}{2\xi}\left(2\delta\overline{\mu}_\alpha^{(2)}(2\rho) + (\alpha-1)\left(\overline{\mu}_\alpha^{(3)}(\rho) - (\overline{\mu}_\alpha^{(2)}(\rho))^2\right)\right) \\ = \frac{1}{2\xi\alpha^2\rho^2}\left(\frac{\alpha(1-(2-\delta)\rho)}{(1-2\rho)^\alpha} - \frac{2(1-\alpha\rho)}{(1-\rho)^\alpha} - \frac{\alpha-1}{(1-\rho)^{2\alpha}} - \delta\alpha\rho + 1\right).\end{aligned} \tag{25}$$

If $\sqrt{k}A(n/k) \to \lambda$ finite and $\sqrt{k}A^2(n/k) \to 0$, when $k \to \infty$, then

$$\sqrt{k}(\widehat{\xi}^{\mathrm{PM}(\alpha)}(k) - \xi) \xrightarrow{d} \mathcal{N}\left(\lambda\, b_{\mathrm{PM}} := \frac{\lambda}{\alpha}\frac{1-(1-\rho)^\alpha}{\rho(1-\rho)^\alpha}, v_\alpha := \frac{\xi^2}{\alpha^2}\left[\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} - 1\right]\right).$$

Remark 2. *The variance term, v_α, reaches its minimum value ξ^2 when α is equal to 1. On the other hand, the term $b_{PM} \equiv b_{PM(\alpha)}$, with ρ fixed, is positive and decreases as α increases (see [25], Figures 3 and 4).*

Asymptotic Properties of the New Reduced Bias EVI-Estimator

The properties of the RB EVI-estimator, $\widehat{\xi}^{\mathrm{RBPM}(\alpha)}$, in (12), are presented in two stages. First, we consider that the second-order parameters (β, ρ) are known. Next, we assume that we estimate externally the vector of second-order parameters in an adequate level k_1 of order statistics, with $k_1 = [n^{1-\epsilon}]$, $\epsilon \to 0$ and where $[\,.\,]$ denotes the integer part and n the size of the available sample.

Theorem 3. *Assuming the validity of the third-order condition,* (15) *and also of* (16)*, and considering intermediate sequences $k = k_n$ satisfying* (7)*,*

$$\widehat{\xi}^{\mathrm{RBPM}(\alpha)}_{(\beta,\rho)}(k) = \widehat{\xi}^{\mathrm{PM}(\alpha)}(k)\left(1 - \frac{1-(1-\rho)^{\alpha}}{\alpha\rho(1-\rho)^{\alpha}}\beta\left(\frac{n}{k}\right)^{\rho}\right), \quad \alpha > 0, \quad 1 \le k < n,$$

has a distributional representation of the type

$$\widehat{\xi}^{\mathrm{RBPM}(\alpha)}_{(\beta,\rho)}(k) \quad \overset{d}{=} \quad \xi + \frac{\xi\,\overline{\sigma}_{\alpha}}{\alpha\sqrt{k}} Z_k^{(\alpha)} + O_p\left(\frac{A(n/k)}{\alpha\sqrt{k}}\right) + b_{\mathrm{RBPM}} A^2(n/k)(1 + o_p(1)), \tag{26}$$

with

$$b_{\mathrm{RBPM}} = \frac{1}{2\xi}\Big(2\delta\,\overline{\mu}_{\alpha}^{(2)}(2\rho) + (\alpha-1)\overline{\mu}_{\alpha}^{(3)}(\rho) - (\alpha+1)(\overline{\mu}_{\alpha}^{(2)}(\rho))^2\Big). \tag{27}$$

Then, with v_α defined in Theorem 2,

- *If $\sqrt{k}A(n/k) \to \lambda$ is finite or infinite and $\sqrt{k}A^2(n/k) \to 0$, when $k \to \infty$, then*

$$\sqrt{k}(\widehat{\xi}^{\mathrm{RBPM}(\alpha)}_{(\beta,\rho)}(k) - \xi) \overset{d}{\longrightarrow} \mathcal{N}(0, v_\alpha).$$

- *If $\sqrt{k}A(n/k) \to \infty$ and $\sqrt{k}A^2(n/k) \to \lambda_A$, when $k \to \infty$, then*

$$\sqrt{k}(\widehat{\xi}^{\mathrm{RBPM}(\alpha)}_{(\beta,\rho)}(k) - \xi) \overset{d}{\longrightarrow} \mathcal{N}(\lambda_A\, b_{\mathrm{RBMP}}, v_\alpha).$$

Proof. The result follows from the distributional representation of $\widehat{\xi}^{\mathrm{PM}(\alpha)}$ presented in Theorem 2. Since $\beta\left(\frac{n}{k}\right)^{\rho} = A(n/k)/\xi$,

$$\begin{aligned}
\widehat{\xi}^{\mathrm{RBPM}(\alpha)}_{(\beta,\rho)}(k) =& \widehat{\xi}^{\mathrm{PM}(\alpha)}(k)\left(1 - \frac{\overline{\mu}_{\alpha}^{(2)}(\rho)}{\xi}A(n/k)\right) \\
\overset{d}{=}& \Bigg(\xi + \frac{\xi\,\overline{\sigma}_{\alpha}}{\alpha\sqrt{k}} Z_k^{(\alpha)} + \overline{\mu}_{\alpha}^{(2)}(\rho)A(n/k) + O_p\left(\frac{A(n/k)}{\alpha\sqrt{k}}\right) \\
&+ \left(\frac{\delta}{\xi}\overline{\mu}_{\alpha}^{(2)}(2\rho) + \frac{(\alpha-1)}{2\xi}\Big(\overline{\mu}_{\alpha}^{(3)}(\rho) - (\overline{\mu}_{\alpha}^{(2)}(\rho))^2\Big)\right)A^2(n/k)(1 + o_p(1))\Bigg) \\
&\times \left(1 - \frac{\overline{\mu}_{\alpha}^{(2)}(\rho)}{\xi}A(n/k)\right) \\
\overset{d}{=}& \xi + \frac{\xi\,\overline{\sigma}_{\alpha}}{\alpha\sqrt{k}} Z_k^{(\alpha)} + O_p\left(\frac{A(n/k)}{\alpha\sqrt{k}}\right) \\
&+ \frac{1}{2\xi}\Big(2\delta\,\overline{\mu}_{\alpha}^{(2)}(2\rho) + (\alpha-1)\overline{\mu}_{\alpha}^{(3)}(\rho) - (\alpha+1)(\overline{\mu}_{\alpha}^{(2)}(\rho))^2\Big)A^2(n/k)(1 + o_p(1)).
\end{aligned}$$

□

Despite the great variety of classes of estimators for the second-order parameters ρ and β now available in the literature, we suggest the estimators introduced in Fraga Alves et al. [43] and Gomes and Martins [44]. Alternative estimators for the second-order parameters can be found in papers such as [45–50]. The class of estimators of ρ in [43] was initially parameterized with a tuning parameter $\tau \ge 0$, but it can more generally be considered as a real number (Caeiro and Gomes [51]). It is defined as

$$\hat{\rho}(k;\tau) \equiv \hat{\rho}_\tau(k) := -\left|\frac{3(T_n(k;\tau) - 1)}{T_n(k;\tau) - 3}\right|, \tag{28}$$

where, with $M_n^{(\alpha)}(k)$ given in (8) and the notation $a^{b\tau} = b \ln a$, whenever $\tau = 0$,

$$T_n(k;\tau) := \frac{\left(M_n^{(1)}(k)\right)^{\tau} - \left(M_n^{(2)}(k)/2\right)^{\tau/2}}{\left(M_n^{(2)}(k)/2\right)^{\tau/2} - \left(M_n^{(3)}(k)/6\right)^{\tau/3}}, \quad \tau \in \mathbb{R}.$$

The theoretical and simulated results presented in Fraga Alves et al. [43] and Caeiro and Gomes [50], and their application in RB estimation, lead us to use $\tau = 0$ when $-1 \le \rho < 0$ and $\tau = 1$ when $\rho < -1$ in practical scenarios.

We consider the β-estimator first obtained in [44] and based on the scaled log-spacings

$$U_i := i(\ln X_{n-i+1:n} - \ln X_{n-i:n}), \quad 1 \le k < n.$$

On the basis of any consistent estimator $\hat{\rho}$ of the second-order parameter ρ, we consider the β-estimator, $\hat{\beta}(k;\hat{\rho})$, where, with $\rho < 0$,

$$\hat{\beta}(k;\rho) := \frac{\left(\frac{k}{n}\right)^{\rho}\left\{\left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\rho}\right)\left(\frac{1}{k}\sum_{i=1}^{k}U_i\right) - \left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\rho}U_i\right)\right\}}{\left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\rho}\right)\left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\rho}U_i\right) - \left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-2\rho}U_i\right)}. \quad (29)$$

With $(\hat{\rho}, \hat{\beta})$ generally denoting the (ρ, β)-estimators in (28) and (29), computed in the high level k_1, i.e., $(\hat{\rho}, \hat{\beta}) \equiv (\hat{\rho}(k_1), \hat{\beta}(k_1))$, we can now state the asymptotic behavior of $\widehat{\xi}^{\text{RBPM}(\alpha)}$ in (12).

Theorem 4. *Assuming the validity of the third-order condition* (15) *and also* (16), *considering intermediate sequences* $k = k_n$ *satisfying* (7), *with* $(\hat{\rho}, \hat{\beta}) \equiv (\hat{\rho}(k_1), \hat{\beta}(k_1))$, *where* k_1 *is a high level such that* $\sqrt{k_1}A(n/k_1) \to \infty$, $k = o(k_1)$ *and with* $(\hat{\rho} - \rho)\ln n = o_p(1)$, *then* $\widehat{\xi}^{RBPM(\alpha)}$ *is consistent for the estimation of* ξ. *If* $\sqrt{k}A(n/k) \to \lambda$ *is finite (possibly zero), then*

$$\sqrt{k}\left(\widehat{\xi}^{\text{RBPM}(\alpha)}(k) - \xi\right) \xrightarrow{d} \mathcal{N}(0, v_\alpha),$$

where v_α *is given in Theorem* 2. *This result still holds true for levels* k *such that* $\sqrt{k}A(n/k) \to \infty$ *and* $\sqrt{k}A^2(n/k) \to \lambda_A$, *if* $(\hat{\rho} - \rho)\ln n = o_p(1/\sqrt{k}A(n/k))$.

Proof. Defining

$$d_\alpha(\rho,\beta) := \overline{\mu}_\alpha^{(2)}(\rho)\frac{A(n/k)}{\xi} = \overline{\mu}_\alpha^{(2)}(\rho)\beta\left(\frac{n}{k}\right)^{\rho},$$

with $\overline{\mu}_\alpha^{(2)}(\rho)$ in (22), the delta method enables us to write

$$d_\alpha(\hat{\rho},\hat{\beta}) = d_\alpha(\rho,\beta) + (\hat{\beta} - \beta)\frac{\partial d_\alpha(\rho,\beta)}{\partial\beta}(1 + o_p(1)) + (\hat{\rho} - \rho)\frac{\partial d_\alpha(\rho,\beta)}{\partial\rho}(1 + o_p(1)),$$

where

$$\frac{\partial d_\alpha(\rho,\beta)}{\partial\beta} = \overline{\mu}_\alpha^{(2)}(\rho)\left(\frac{n}{k}\right)^{\rho}$$

and

$$\begin{aligned}\frac{\partial d_\alpha(\rho,\beta)}{\partial\rho} &= \frac{\partial\overline{\mu}_\alpha^{(2)}(\rho)}{\partial\rho}\beta\left(\frac{n}{k}\right)^{\rho} + \overline{\mu}_\alpha^{(2)}(\rho)\beta\left(\frac{n}{k}\right)^{\rho}\ln\left(\frac{n}{k}\right) \\ &= \overline{\mu}_\alpha^{(2)}(\rho)\beta\left(\frac{n}{k}\right)^{\rho}\left(c_\alpha(\rho) + \ln\left(\frac{n}{k}\right)\right)\end{aligned}$$

with

$$c_\alpha(\rho) = \frac{\frac{\partial}{\partial\rho}\overline{\mu}_\alpha^{(2)}(\rho)}{\overline{\mu}_\alpha^{(2)}(\rho)} = \frac{\alpha\rho - (1-\rho) + (1-\rho)^{\alpha+1}}{\rho(1-\rho)\big(1-(1-\rho)^\alpha\big)} = \frac{(1-\rho)^{\alpha+1} - (1-(\alpha+1)\rho)}{\rho(1-\rho)\big(1-(1-\rho)^\alpha\big)}.$$

Then, it follows that

$$\begin{aligned}\widehat{\xi}^{\mathrm{RBPM}(\alpha)}(k) &= \widehat{\xi}^{\mathrm{PM}(\alpha)}(k)\left(1 - \frac{1-(1-\hat{\rho})^\alpha}{\alpha\hat{\rho}(1-\hat{\rho})^\alpha}\hat{\beta}\left(\frac{n}{k}\right)^{\hat{\rho}}\right) = \widehat{\xi}^{\mathrm{PM}(\alpha)}(k)\big(1 - d_\alpha(\hat{\rho},\hat{\beta})\big) \\ &= \widehat{\xi}^{\mathrm{PM}(\alpha)}(k)\left(1 - d_\alpha(\rho,\beta) - (\hat{\beta}-\beta)\frac{\partial d_\alpha(\rho,\beta)}{\partial\beta}(1+o_p(1)) - (\hat{\rho}-\rho)\frac{\partial d_\alpha(\rho,\beta)}{\partial\rho}(1+o_p(1))\right) \\ &\overset{d}{=} \widehat{\xi}^{\mathrm{RBPM}(\alpha)}_{(\beta,\rho)}(k) - \overline{\mu}_\alpha^{(2)}(\rho)A(n/k)\left(\frac{\hat{\beta}-\beta}{\beta} + (\hat{\rho}-\rho)(c_\alpha(\rho) + \ln(n/k))\right).\end{aligned}$$

Since $\widehat{\xi}^{\mathrm{RBPM}(\alpha)}_{(\beta,\rho)}$ is consistent for the estimation of ξ, $\hat{\beta}$ and $\hat{\rho}$ are consistent for the estimation of β and ρ, respectively, and $(\hat{\rho}-\rho)\ln(n/k) = o_p(1)$,

$$\overline{\mu}_\alpha^{(2)}(\rho)A(n/k)\left(\frac{\hat{\beta}-\beta}{\beta} + (\hat{\rho}-\rho)(c_\alpha(\rho) + \ln(n/k))\right) = o_p(1),$$

and the consistency of $\widehat{\xi}^{\mathrm{RBPM}(\alpha)}(k)$ follows.

Next, given that

$$\begin{aligned}\sqrt{k}\Big(\widehat{\xi}^{\mathrm{RBPM}(\alpha)}(k) - \xi\Big) &\overset{d}{=} \sqrt{k}\Big(\widehat{\xi}^{\mathrm{RBPM}(\alpha)}_{\beta,\rho}(k) - \xi\Big) \\ &\quad - \sqrt{k}\,A(n/k)\,\overline{\mu}_\alpha^{(2)}(\rho)\left(\frac{\hat{\beta}-\beta}{\beta} + (\hat{\rho}-\rho)(c_\alpha(\rho) + \ln(n/k))\right),\end{aligned}$$

with $\hat{\beta}-\beta \overset{p}{\sim} -\beta(\hat{\rho}-\rho)\ln(n/k_1)$ ([52], Theorem 2.1); then,

$$\sqrt{k}\Big(\widehat{\xi}^{\mathrm{RBPM}(\alpha)}(k) - \xi\Big) \overset{d}{=} \sqrt{k}\Big(\widehat{\xi}^{\mathrm{RBPM}(\alpha)}_{\beta,\rho}(k) - \xi\Big) - \sqrt{k}\,A(n/k)\,\overline{\mu}_\alpha^{(2)}(\rho)(\hat{\rho}-\rho)\ln(k/k_1).$$

Thus, $\widehat{\xi}^{\mathrm{RBPM}(\alpha)}(k)$ is consistent for the estimation of ξ if $(\hat{\rho}-\rho)\ln(k/k_1) = o_p(1/A(n/k))$ and asymptotic normality holds for levels k such that $\sqrt{k}A(n/k) \to \lambda$ (finite) but also for levels k, where $\sqrt{k}A(n/k) \to \infty$ and $\sqrt{k}A^2(n/k) \to \lambda_A$, if $(\hat{\rho}-\rho)\ln(k/k_1) = o_p\Big(\frac{1}{\sqrt{k}A(n/k)}\Big)$. □

Remark 3. *Note that when $\alpha = 1$, we obtain for $\widehat{\xi}^{PM(\alpha)}(k)$ and $\widehat{\xi}^{RBPM(\alpha)}(k)$, in (10) and (12), respectively, the same asymptotic variance as $\widehat{\xi}^{CH}(k)$ in (13), which is ξ^2. The ratio between the asymptotic standard deviations of both estimators,*

$$r_\alpha := \sqrt{\frac{v_\alpha}{\xi^2}} = \frac{1}{\alpha}\sqrt{\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} - 1},$$

is always greater than or equal to 1 for $\alpha > 0$.

Let $\text{Bias}_\infty(\hat{\xi}^{\text{RBPM}(\alpha)}(k))$ represent the asymptotic bias and $\text{Var}_\infty(\hat{\xi}^{\text{RBPM}(\alpha)}(k))$ the asymptotic variance of $\hat{\xi}^{\text{RBPM}(\alpha)}(k)$, in (12). Then, based on the results presented in Theorem 3, we can derive the Asymptotic Mean Squared Error (AMSE):

$$\text{AMSE}\left(\hat{\xi}^{\text{RBPM}(\alpha)}(k)\right) = \text{Var}_\infty(\hat{\xi}^{\text{RBPM}(\alpha)}(k)) + \left(\text{Bias}_\infty(\hat{\xi}^{\text{RBPM}(\alpha)}(k))\right)^2 = \tag{30}$$

$$= \frac{\xi^2}{k\,\alpha^2}\left[\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} - 1\right] + b^2_{\text{RBPM}} A^4(n/k), \quad \text{if} \quad b_{\text{RBPM}} \neq 0, \tag{31}$$

with b_{RBPM} given in (27). The AMSE of the CH estimator is obtained by taking $\alpha = 1$:

$$\text{AMSE}\left(\hat{\xi}^{\text{CH}}(k)\right) = \frac{\xi^2}{k} + \frac{1}{\xi^2}\left(\frac{\delta}{1-2\rho} - \frac{1}{(1-\rho)^2}\right)^2 A^4(n/k). \tag{32}$$

Proposition 1. *If $b_{\text{RBPM}} \neq 0$, the threshold that minimizes the AMSE, denoted as k_0^{RBPM}, can be expressed as follows:*

$$k_0^{\text{RBPM}} = \left(\frac{\frac{1}{\alpha^2}\left[\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} - 1\right]}{(-\rho)\beta^4\left(2\delta\,\overline{\mu}_\alpha^{(2)}(2\rho) + (\alpha-1)\overline{\mu}_\alpha^{(3)}(\rho) - (\alpha+1)(\overline{\mu}_\alpha^{(2)}(\rho))^2\right)^2}\right)^{\frac{1}{1-4\rho}} n^{-\frac{4\rho}{1-4\rho}}. \tag{33}$$

Remark 4. *Note that*

$$\sqrt{k_0^{\text{RBPM}}}\,A^2(n/k_0^{\text{RBPM}}) \underset{n\to\infty}{\rightarrow} \frac{\sqrt{v_\alpha}}{\sqrt{-4\rho}\,|b_{\text{RBPM}}|}.$$

Moreover, by taking $\alpha = 1$ in (33), the value of k that minimizes the AMSE of the CH estimator is given by

$$k_0^{\text{CH}} = \left(-4\rho\left(\frac{\delta}{1-2\rho} - \frac{1}{(1-\rho)^2}\right)^2\beta^4\right)^{-\frac{1}{1-4\rho}} n^{-\frac{4\rho}{1-4\rho}}. \tag{34}$$

3. Asymptotic Comparison at Optimal Levels

Let us consider $\hat{\xi}^{\bullet}(k)$, a generic RB EVI-estimator with a distributional representation

$$\hat{\xi}^{\bullet}(k) \overset{d}{=} \xi + \frac{\sigma_\bullet}{\sqrt{k}} Z_k^{(\alpha)} + b_\bullet A^2(n/k)(1 + o_p(1)). \tag{35}$$

Let $\hat{\xi}_0^{\bullet} = \hat{\xi}_n^{\bullet}(k_0^{\bullet})$ be the RB EVI-estimator computed at its optimal level and LMSE the limiting mean square error. Using regular variation theory ([14,15]), we can ensure that whenever $b_\bullet \neq 0$, there exists a function $\varphi(n, \xi, \rho)$, which solely depends on the underlying model and not on the estimator ([31]), such that

$$\lim_{n\to\infty} \varphi(n,\xi,\rho)\,\text{AMSE}(\hat{\xi}_0^{\bullet}) = \left(\sigma_\bullet^2\right)^{-\frac{2\rho}{1-4\rho}}\left(b_\bullet^2\right)^{\frac{1}{1-4\rho}} =: \text{LMSE}(\hat{\xi}_0^{\bullet}).$$

By evaluating the ratio of LMSE for two different RB EVI-estimators, we derive a measure independent of the $\varphi(\cdot)$ function, thus leading us to consider the following efficiency measure:

$$\text{AREFF}_{(1)|(2)} := \sqrt{\frac{\text{LMSE}(\hat{\xi}_0^{(2)})}{\text{LMSE}(\hat{\xi}_0^{(1)})}} = \left(\frac{\sigma_{(2)}}{\sigma_{(1)}}\right)^{-\frac{4\rho}{1-4\rho}}\left|\frac{b_{(2)}}{b_{(1)}}\right|^{\frac{1}{1-4\rho}}. \tag{36}$$

If AREFF>1, then $\hat{\xi}^{(2)}(k)$ is more efficient than $\hat{\xi}^{(1)}(k)$; if AREFF$=1$, the two estimators are equally efficient; and if AREFF<1, then $\hat{\xi}^{(1)}(k)$ is more efficient than

$\hat{\xi}^{(2)}(k)$. With $\hat{\xi}^{(1)}(k) \equiv \hat{\xi}^{\mathrm{RBPM}(\alpha)}(k)$ and $\hat{\xi}^{(2)}(k) \equiv \hat{\xi}^{\mathrm{CH}}(k)$ ($b_{\mathrm{CH}} := \frac{1}{\xi}\left(\frac{\delta}{1-2\rho} - \frac{1}{(1-\rho)^2}\right)$, $\sigma_{\mathrm{CH}} := \xi$),

$$\mathrm{AREFF}_{\mathrm{RBPM}(\alpha)|\mathrm{CH}} = \left(\frac{1}{\alpha}\sqrt{\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} - 1}\right)^{\frac{4\rho}{1-4\rho}} \times \left|\frac{2(\alpha\rho)^2\left(\frac{\delta}{1-2\rho} - \frac{1}{(1-\rho)^2}\right)}{\frac{\alpha(1-(2-\delta)\rho)}{(1-2\rho)^\alpha} + \frac{2(1+\alpha\rho)}{(1-\rho)^\alpha} - \frac{\alpha+1}{(1-\rho)^{2\alpha}} - (1+\delta\alpha\rho)}\right|^{\frac{1}{1-4\rho}}. \quad (37)$$

Note that (37) is derived from the asymptotic formulas for the dominant components of asymptotic bias and variance.

For simplicity, we conduct the comparison only for a few Pareto-type models. For the Burr or the Generalized Pareto (GP) distributions, $\delta = 1$ and a contour plot of $\mathrm{AREFF}_{\mathrm{RBPM}(\alpha)|\mathrm{CH}}$ is presented in Figure 1. The blue lines represent the (α, ρ)-region where the two estimators are equally efficient and the grey-shaded region represents the values of the (α, ρ)-plane where $\hat{\xi}^{\mathrm{RBPM}(\alpha)}(k)$ is more efficient than $\hat{\xi}^{\mathrm{CH}}(k)$. It is apparent that the (α, ρ)-plane region where $\hat{\xi}^{\mathrm{RBPM}(\alpha)}(k)$ outperforms $\hat{\xi}^{\mathrm{CH}}(k)$ is notably large and diverse.

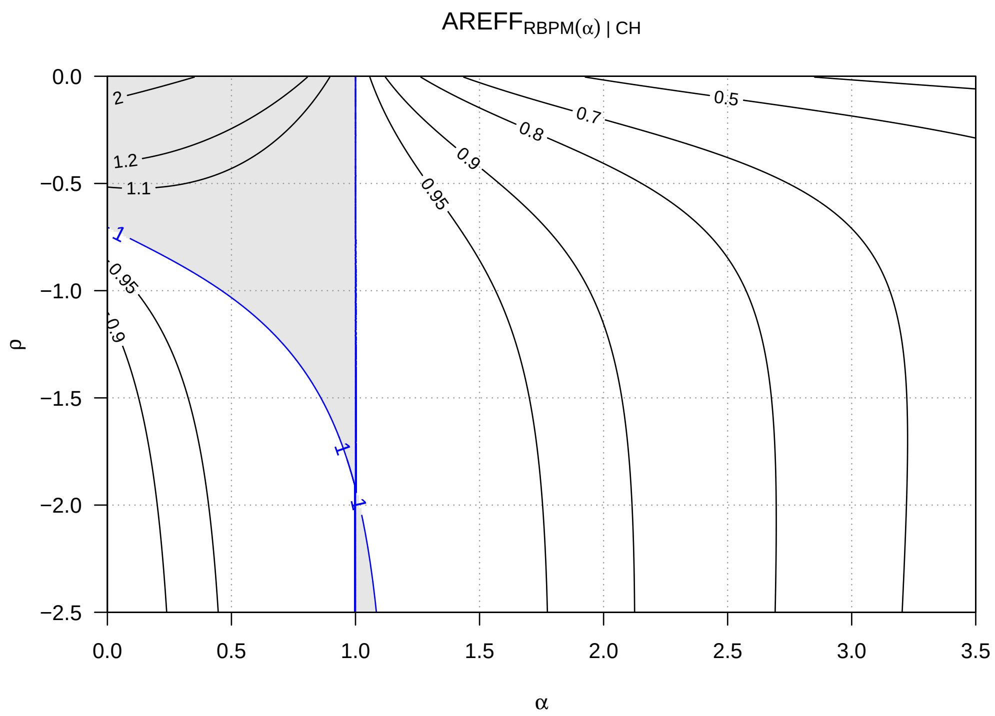

Figure 1. Contour plot of $\mathrm{AREFF}_{\mathrm{RBPM}(\alpha)|\mathrm{CH}}$, with $\delta = 1$ in the (α, ρ)-plane.

By fixing the values of (δ, ρ) it is possible to find the value of α that maximizes $\mathrm{AREFF}_{\mathrm{RBPM}(\alpha)|\mathrm{CH}}$, i.e., $\alpha_{\max} = \arg\max_\alpha \mathrm{AREFF}_{\mathrm{RBPM}(\alpha)|\mathrm{CH}}$. Considering the Pareto-type models described in Section 4 and the associated (δ, ρ) values we obtained for the Fréchet model, $\alpha_{\max} = 1.232$; for the Burr model with $\rho = -0.75$, $\alpha_{\max} = 0.464$, and $\alpha_{\max} = 0.073$, when $\rho = -0.5$; for the Half-t_1 model, $\alpha_{\max} = 0.468$; for the Half-t_2 model, $\alpha_{\max} = 0.844$; for the Half-t_3 model, $\alpha_{\max} = 1.148$, and for the Half-t_4 model, $\alpha_{\max} = 1.412$. Note that, as shown in Section 4, the use of this $\alpha_{\max}$ does not always lead to the best results because their finite sample behavior may differ.

4. Monte Carlo Simulation Study

In this section, we perform a Monte Carlo simulation study to assess the finite sample behavior of the new class of estimators $\widehat{\xi}^{\mathrm{RBPM}(\alpha)}(k)$, with selected values of α. For comparison purposes, the value $\alpha = 1$ (CH≡RBPM(1)) and the Hill estimator in (5) were included in this study. The other α values used in the simulation study were chosen based on the findings from a preliminary simulation study. The results were obtained based on 2000 simulated samples of sizes $n = 50, 100, 200, 500, 1000, 2000$ and 5000 from the following Pareto-type models:

- The Fréchet model with a d.f. given by $F(x) = \exp(-x^{-1/\xi}), \quad x > 0$, with $\xi = 0.5$ ($\rho = \rho' = -1, \delta = 5/3$).
- The Burr model, with a d.f. $F(x) = 1 - (1 + x^{-\rho/\xi})^{1/\rho}, \quad x > 0, \quad \xi > 0, \quad \rho < 0$ with $(\xi, \rho) = (0.5, -0.5)$ and $(0.5, -0.75)$ ($\rho = \rho' = -1, \delta = 1$).
- The Half-t_ν model, i.e., the absolute value of a Student's-t_ν distribution with $\nu = 2, 3, 4$, $(\xi, \rho) = (1/\nu, -2/\nu)$ ($\rho = \rho', \ \delta = \nu(\nu^2 + 4\nu + 2)/((\nu + 4)(\nu + 1))$).
- The Log-gamma model with $\xi = 0.5$ ($\rho = 0$). It is important to note that this model does not fulfill condition (11).

4.1. Results for All Values of k

For each model, the performance of the estimators is evaluated through the simulated mean value (E) and root mean squared error (RMSE) as a function of k. The results of the simulation study are provided in Figures 2–8 for samples of size $n = 500$. The dotted horizontal black line on the left plots represents the value of the EVI.

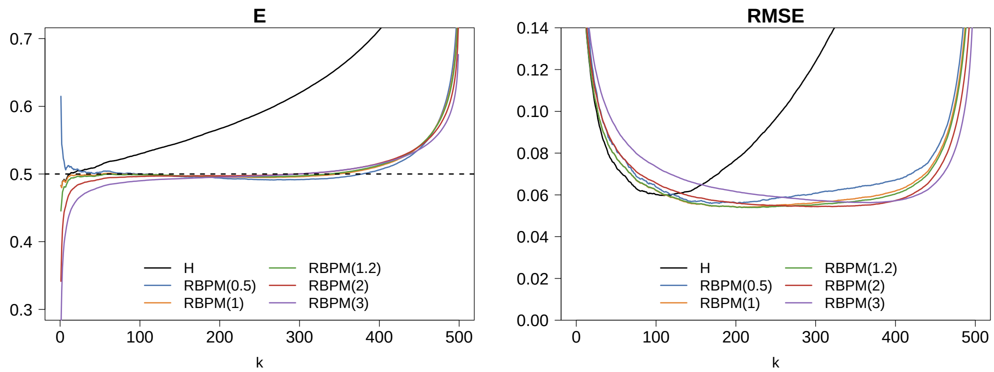

Figure 2. Simulated mean values (**left**) and RMSE (**right**) of the EVI-estimators under study for samples of size 1000 from the Fréchet model, with $\xi = 0.5$.

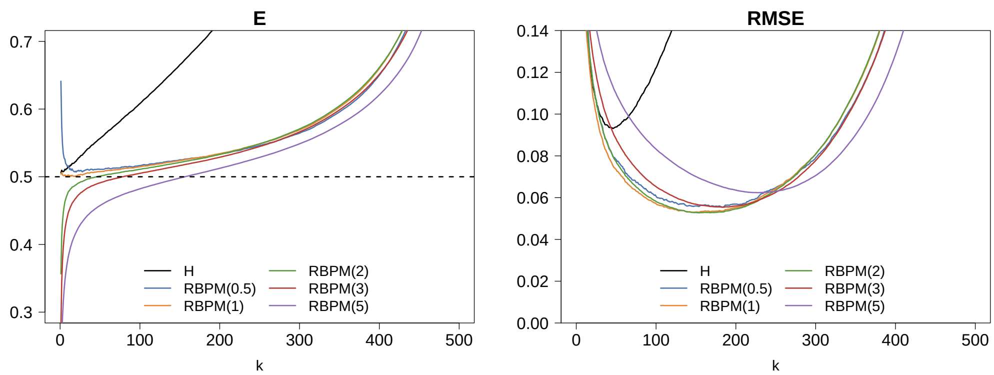

Figure 3. Simulated mean values (**left**) and RMSE (**right**) of the EVI-estimators under study for samples of size 1000 from the Burr model, with $\xi = 0.5$ and $\rho = -0.75$.

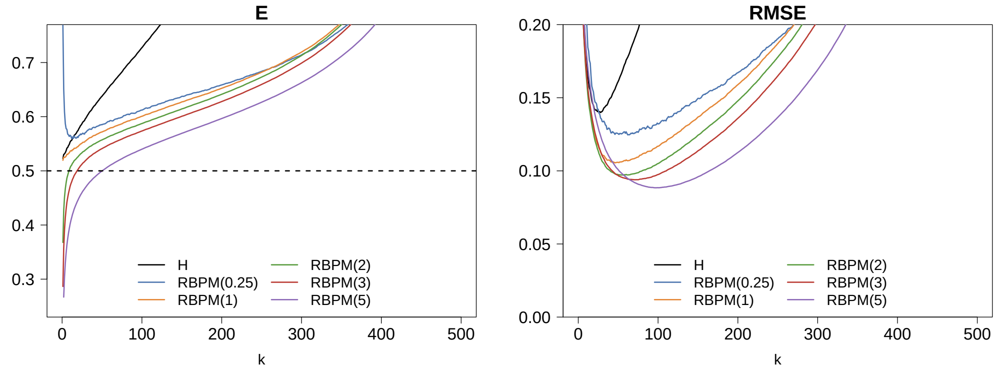

Figure 4. Simulated mean values (**left**) and RMSE (**right**) of the EVI-estimators under study for samples of size 1000 from the Burr model, with $\xi = 0.5$ and $\rho = -0.5$.

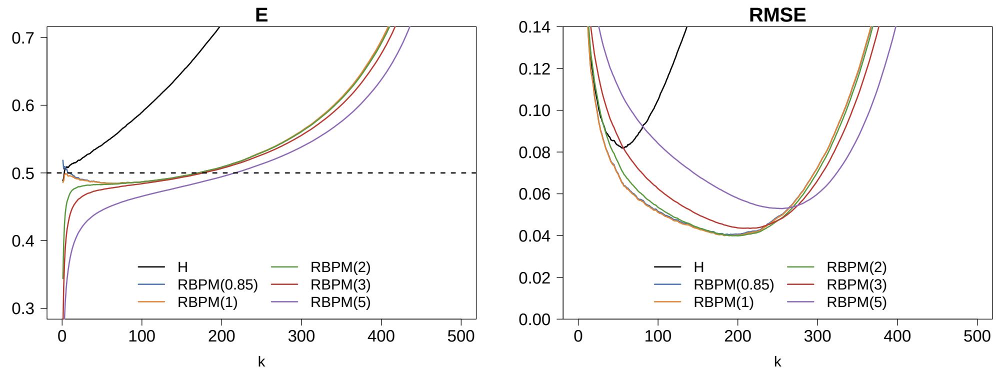

Figure 5. Simulated mean values (**left**) and RMSE (**right**) of the EVI-estimators under study for samples of size 1000 from the Half-t model, with $\nu = 2$ ($\xi = 0.5$).

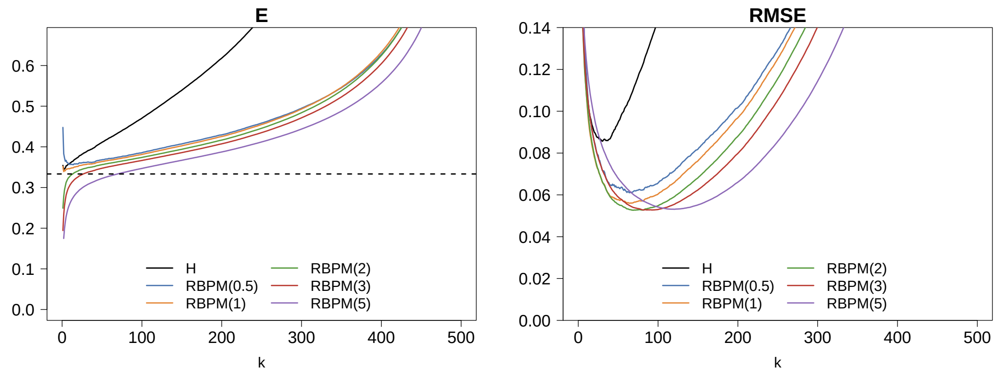

Figure 6. Simulated mean values (**left**) and RMSE (**right**) of the EVI-estimators under study for samples of size 1000 from the Half-t model, with $\nu = 3$ ($\xi = 1/3$).

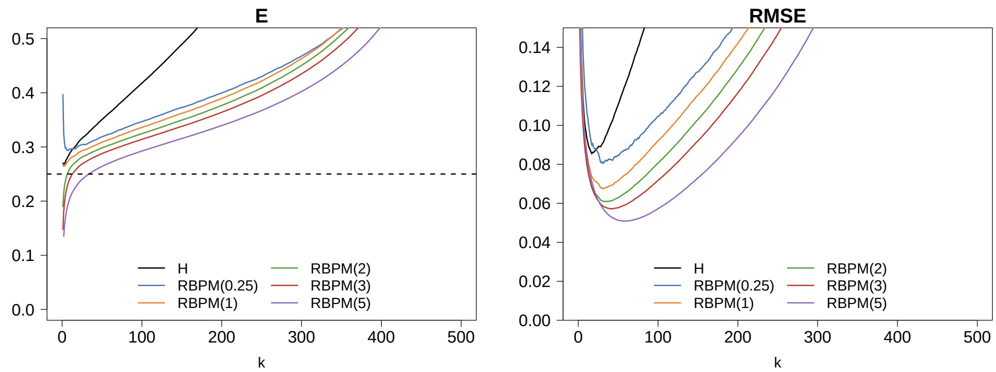

Figure 7. Simulated mean values (**left**) and RMSE (**right**) of the EVI-estimators under study for samples of size 1000 from the Half-t model, with $\nu = 4$ ($\xi = 0.25$).

From these figures, we can conclude the following:

- As expected, the finite sample behavior of the new class of RB EVI-estimators depends on the underlying model, but there is always a positive value of α that enables a clear reduction in both bias and RMSE;
- For the Fréchet model, with $\xi = 0.5$, the new RB EVI-estimators provide a stable trajectory around the true value of ξ for a wide range of k values;
- For the Burr models under study, there is also a significant reduction in bias and RMSE;
- The same holds for the Half-t models, except for the RMSE for the Half-t model with $\nu = 1$, where the Hill estimator presents the lowest RMSE;
- Even for the Log-gamma model, it is possible to find values of α that stabilize the simulated mean sample paths, also reducing the RMSE.

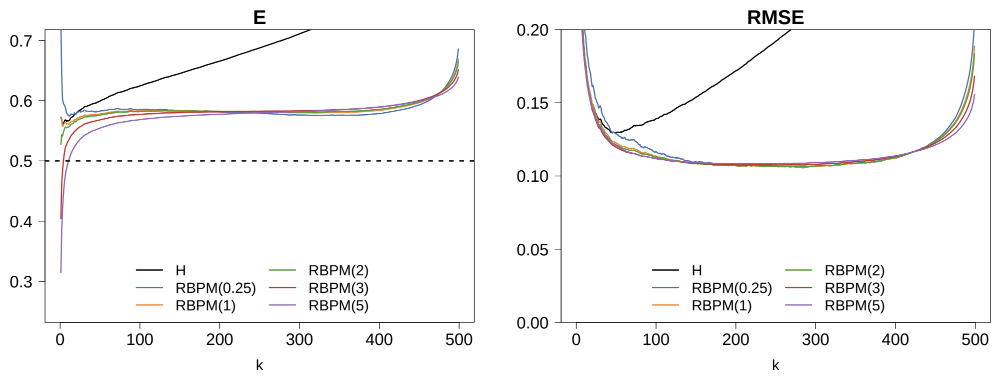

Figure 8. Simulated mean values (**left**) and RMSE (**right**) of the EVI-estimators under study for samples of size 1000 from the Log-gamma model, with $\xi = 0.5$.

4.2. Results at the Simulated Optimal k

The simulated mean values at optimal levels, aiming for minimal MSE, are summarized in Table 1 for the Pareto-type models described above. The corresponding simulated RMSE at optimal levels is presented in Table 2. The values with the smallest absolute bias and RMSE are highlighted in bold.

Table 1. Simulated mean value for the EVI-estimators under study, computed at the simulated optimal level.

	50	**100**	**200**	**500**	**1000**	**2000**	**5000**
			Fréchet model with $\xi = 0.5$ ($\rho = -1$)				
H	0.5728	0.5563	0.5414	0.5358	0.5263	0.5213	0.5132
RBPM(0.5)	0.4755	0.4745	0.4839	0.4959	0.4993	**0.5010**	**0.5010**
RBPM(1)	0.4870	0.4893	0.4879	0.4959	**0.5002**	0.5019	0.5015
RBPM(1.2)	0.4870	0.4893	0.4879	0.4959	**0.5002**	0.5019	0.5015
RBPM(2)	0.4920	**0.5038**	**0.5028**	**0.4998**	0.5010	0.5028	0.5022
RBPM(3)	**0.4975**	0.5040	0.5071	0.5082	0.5055	0.5047	0.5034
			Burr model with $(\xi, \rho) = (0.5, -0.75)$				
H	0.6189	0.5987	0.5708	0.5526	0.5446	0.5364	0.5299
RBPM(0.5)	0.5656	0.5463	0.5346	0.5296	0.5178	0.5171	0.5146
RBPM(1)	0.5556	0.5433	0.5336	0.5232	**0.5175**	0.5171	0.5134
RBPM(2)	0.5449	0.5383	0.5295	0.5220	0.5195	0.5161	**0.5129**
RBPM(3)	0.5400	0.5353	0.5304	0.5247	0.5206	**0.5154**	0.5130
RBPM(5)	**0.5260**	**0.5180**	**0.5228**	**0.5212**	0.5197	0.5178	0.5152

Table 1. *Cont.*

	50	**100**	**200**	**500**	**1000**	**2000**	**5000**
			Burr model with $(\xi, \rho) = (0.5, -0.5)$				
H	0.6878	0.6349	0.6088	0.5928	0.5776	0.5633	0.5520
RBPM(0.25)	0.6968	0.6319	0.6190	0.5924	0.5809	0.5651	0.5483
RBPM(1)	0.6344	0.6100	0.5898	0.5692	0.5634	0.5546	0.5438
RBPM(2)	0.6045	0.5861	0.5719	0.5614	0.5582	0.5466	0.5405
RBPM(3)	0.5685	0.5713	0.5578	0.5569	0.5515	0.5435	0.5378
RBPM(5)	**0.5338**	**0.5392**	**0.5365**	**0.5380**	**0.5374**	**0.5343**	**0.5326**
			Half-t model with $\nu = 2$ $(\xi = 0.5)$				
H	0.6001	0.5641	0.5568	0.5476	0.5362	0.5243	0.5232
RBPM(0.85)	0.5339	0.5217	0.5164	**0.5048**	**0.5034**	**0.5018**	**0.5007**
RBPM(1)	0.5319	0.5210	0.5164	0.5050	0.5037	0.5022	0.5011
RBPM(2)	0.5300	0.5204	**0.5148**	0.5083	0.5043	0.5032	0.5011
RBPM(3)	0.5242	0.5222	0.5177	0.5123	0.5072	0.5056	0.5031
RBPM(5)	**0.5223**	**0.5176**	0.5154	0.5156	0.5116	0.5093	0.5065
			Half-t model with $\nu = 3$ $(\xi = 1/3)$				
H	0.4304	0.4325	0.3957	0.3874	0.3769	0.3637	0.3588
RBPM(0.25)	0.4396	0.4085	0.3919	0.3744	0.3688	0.3567	0.3547
RBPM(1)	0.3991	0.3949	0.3785	0.3681	0.3636	0.3536	0.3516
RBPM(2)	0.3806	0.3811	0.3733	0.3630	0.3597	0.3544	**0.3495**
RBPM(3)	0.3747	0.3713	0.3682	0.3599	0.3581	0.3540	0.3503
RBPM(5)	**0.3504**	**0.3528**	**0.3533**	**0.3552**	**0.3541**	**0.3525**	0.3497
			Half-t model with $\nu = 4$ $(\xi = 0.25)$				
H	0.3732	0.3449	0.3266	0.3017	0.2958	0.2904	0.2800
RBPM(0.25)	0.3779	0.3434	0.3266	0.3053	0.2977	0.2845	0.2828
RBPM(1)	0.3345	0.3263	0.3109	0.2959	0.2899	0.2783	0.2754
RBPM(2)	0.3063	0.3087	0.3003	0.2891	0.2813	0.2794	0.2730
RBPM(3)	0.2975	0.2908	0.2899	0.2831	0.2796	0.2761	0.2723
RBPM(5)	**0.2615**	**0.2728**	**0.2688**	**0.2702**	**0.2684**	**0.2699**	**0.2678**
			Log-gamma with $\xi = 0.5$				
H	0.6569	0.6324	0.6172	0.5961	0.5963	0.5840	0.5759
RBPM(0.25)	**0.5110**	0.5301	0.5561	0.5743	0.5780	0.5811	0.5818
RBPM(1)	0.5189	0.5340	0.5613	0.5802	0.5862	0.5830	0.5732
RBPM(2)	0.5260	0.5405	0.5663	0.5825	0.5803	0.5762	0.5707
RBPM(3)	0.5281	0.5436	0.5659	0.5776	0.5787	0.5741	0.5702
RBPM(5)	**0.5110**	**0.5291**	**0.5490**	**0.5567**	**0.5578**	**0.5582**	**0.5582**

Table 2. Simulated RMSE for the EVI-estimators under study, computed at the simulated optimal level.

	50	**100**	**200**	**500**	**1000**	**2000**	**5000**
			Fréchet model with $\xi = 0.5$ $(\rho = -1)$				
H	0.1383	0.1051	0.0818	0.0597	0.0457	0.0365	0.0262
RBPM(0.5)	0.1205	0.0991	0.0782	0.0559	0.0418	0.0313	0.0206
RBPM(1)	0.1137	0.0957	0.0745	0.0541	**0.0406**	**0.0303**	**0.0200**
RBPM(1.2)	**0.1124**	0.0942	0.0739	**0.0540**	**0.0406**	**0.0303**	**0.0200**
RBPM(2)	0.1125	**0.0926**	**0.0732**	0.0544	0.0421	0.0315	0.0206
RBPM(3)	0.1163	0.0949	0.0758	0.0563	0.0446	0.0340	0.0224
			Burr model with $(\xi, \rho) = (0.5, -0.75)$				
H	0.2170	0.1675	0.1291	0.0931	0.0739	0.0600	0.0453
RBPM(0.5)	0.1693	0.1155	0.0837	0.0555	0.0404	0.0310	0.0234
RBPM(1)	0.1547	0.1084	0.0791	0.0530	**0.0388**	**0.0301**	**0.0227**
RBPM(2)	0.1425	0.1041	**0.0768**	**0.0527**	0.0395	0.0306	0.0229
RBPM(3)	0.1359	0.1038	0.0781	0.0554	0.0424	0.0330	0.0247
RBPM(5)	**0.1219**	**0.1003**	0.0806	0.0624	0.0508	0.0406	0.0317

Table 2. *Cont.*

	50	**100**	**200**	**500**	**1000**	**2000**	**5000**
			Burr model with $(\xi, \rho) = (0.5, -0.5)$				
H	0.3039	0.2364	0.1833	0.1399	0.1148	0.0955	0.0760
RBPM(0.25)	0.2891	0.2067	0.1615	0.1246	0.1028	0.0871	0.0693
RBPM(1)	0.2337	0.1736	0.1373	0.1055	0.0879	0.0750	0.0601
RBPM(2)	0.2020	0.1555	0.1238	0.0971	0.0814	0.0690	0.0563
RBPM(3)	0.1813	0.1444	0.1168	0.0939	0.0797	0.0678	**0.0556**
RBPM(5)	**0.1493**	**0.1239**	**0.1041**	**0.0884**	**0.0778**	**0.0672**	0.0574
			Half-t model with $\nu = 2$ ($\xi = 0.5$)				
H	0.2027	0.1511	0.1172	0.0819	0.0646	0.0517	0.0379
RBPM(0.85)	0.1426	0.0899	0.0651	0.0404	0.0268	0.0196	0.0126
RBPM(1)	0.1397	0.0885	0.0642	0.0400	**0.0265**	**0.0195**	**0.0125**
RBPM(2)	0.1303	**0.0857**	**0.0630**	**0.0399**	0.0274	0.0204	0.0133
RBPM(3)	0.1260	0.0874	0.0663	0.0435	0.0310	0.0235	0.0156
RBPM(5)	**0.1182**	0.0906	0.0733	0.0529	0.0416	0.0334	0.0242
			Half-t model with $\nu = 3$ ($\xi = 1/3$)				
H	0.1879	0.1493	0.1157	0.0856	0.0679	0.0543	0.0406
RBPM(0.25)	0.1732	0.1226	0.0915	0.0655	0.0524	0.0418	0.0315
RBPM(1)	0.1378	0.1025	0.0772	0.0560	0.0448	0.0356	0.0273
RBPM(2)	0.1197	0.0926	0.0715	**0.0527**	**0.0422**	**0.0337**	**0.0260**
RBPM(3)	0.1084	0.0873	0.0697	0.0528	0.0431	0.0348	0.0271
RBPM(5)	**0.0921**	**0.0775**	**0.0665**	0.0531	0.0464	0.0388	0.0310
			Half-t model with $\nu = 4$ ($\xi = 0.25$)				
H	0.1846	0.1468	0.1159	0.0854	0.0694	0.0560	0.0439
RBPM(0.25)	0.1826	0.1357	0.1078	0.0804	0.0669	0.0543	0.0428
RBPM(1)	0.1451	0.1116	0.0890	0.0674	0.0563	0.0460	0.0367
RBPM(2)	0.1211	0.0971	0.0788	0.0609	0.0511	0.0426	0.0342
RBPM(3)	0.1057	0.0873	0.0726	0.0571	0.0489	0.0415	**0.0339**
RBPM(5)	**0.0841**	**0.0720**	**0.0622**	**0.0508**	**0.0449**	**0.0398**	0.0341
			Log-gamma with $\xi = 0.5$				
H	0.2168	0.1820	0.1556	0.1294	0.1141	0.1035	0.0897
RBPM(0.25)	0.1805	0.1417	0.1196	**0.1059**	0.0989	0.0935	0.0869
RBPM(1)	**0.1758**	**0.1373**	**0.1169**	**0.1059**	0.1014	0.0976	0.0877
RBPM(2)	0.1810	0.1409	0.1200	0.1075	0.1015	0.0940	0.0848
RBPM(3)	0.1853	0.1452	0.1240	0.1084	0.1002	0.0928	0.0837
RBPM(5)	0.1780	0.1444	0.1243	**0.1059**	**0.0967**	**0.0903**	**0.0816**

An analysis of the provided tables leads us to the following conclusions:

- For the Fréchet model with $\xi = 0.5$, Burr model with $(\xi, \rho) = (0.5, -0.75)$, and Half-t model with $\nu = 2$, the best value of α depends on the sample size;
- For the Burr model with $(\xi, \rho) = (0.5, -0.5)$ and for the Half-t models with $\nu = 3, 4$, the value of $\alpha = 5$ seems to provide an overall good performance;
- For the Half-t model with $\nu = 1$, the best estimator in terms of minimum RMSE is the Hill estimator, but in terms of bias, the value of $\alpha = 5$, for large sample sizes, is the best choice;
- For the Log-gamma model with $\xi = 0.5$, the value of $\alpha = 0.5$ provides the smallest absolute bias and the smallest RMSE for sample sizes $n \geq 500$. For small sample sizes, the smallest RMSEs are the ones associated with the CH estimator.

As stated at the end of Section 3, the values of α that maximize the AREFF, in (37), are based on a combination of the asymptotic bias and variance. Apart from the Fréchet model where the value $\alpha_{\max} = 1.2$ provides good results in terms of simulated bias and RMSE, for the other models under study the findings of the simulations study differ from the theoretical ones. This difference may be attributed to the estimation of the second-order parameters (β, ρ) and to the fact that finite properties of the estimators may differ from the theoretical asymptotic ones.

5. Case Study: Application of New Estimators to Hydrological Data

In this section, we perform an illustration of the behavior of the new estimators in a real data set. Consider the daily flow (in m^3/s) of the Whiteadder River in Hutton, UK, within 11 years, ranging from 1 January 1995 to 31 December 2005, with a total of $n = 4018$ observations. The data were collected from https://nrfa.ceh.ac.uk/data/station/meanflow/21022 (accessed on 7 December 2015). The time series plot and the boxplot of the Whiteadder River are presented in Figure 9. From the boxplot, it is clear that the data are positively skewed (skew = 8.244), also having an excess of kurtosis of 118.598. Based on these features, one can infer that our real data set has a Pareto-type tail and these EVI-estimators can be applied to infer the heaviness of the right tail. To perform an adequate choice of ξ, we search for stability regions around a value of ξ. The larger the stability region, the better. The use of RB EVI-estimators allows us to search for stability regions for large values of k. The sample paths of the estimators under study for the Whiteadder River flow are presented in Figure 10. The Hill estimates (black line) present some stability around the value of 0.5 for a small region of k values. The CH estimates (green line) also show a small stability region. The new class of RB EVI-estimators with an appropriate choice of the tuning parameter leads to more stable sample paths when compared to the classic RB EVI-estimator in (13) (CH≡RBPM(1)), enabling the use of stability Algorithms like the ones in [41] to obtain adaptive estimates of ξ.

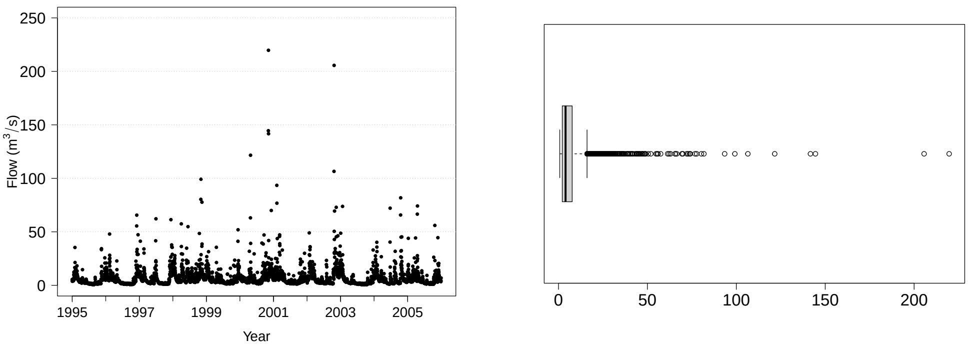

Figure 9. Daily time series (**left**) and boxplot (**right**) of the Whiteadder River flow between 1995 and 2005 ($n = 4018$).

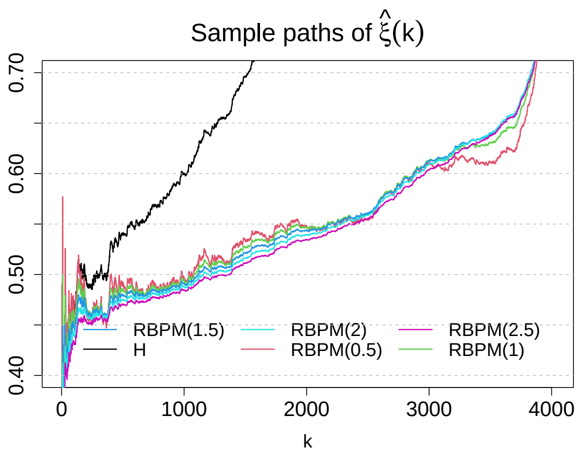

Figure 10. Sample paths of estimators under study for the Whiteadder River flow between 1995 and 2005 ($n = 4018$).

6. Conclusions

In this work, we introduce a novel class of RB EVI-estimators developed using the power mean of the normalized log-excesses and that generalizes other classes of RB EVI-estimators in the literature. We derive the asymptotic non-degenerate behavior within a third-order framework and perform an asymptotic comparison at optimal levels. Large-scale simulation experiments reveal that the parameter α generally facilitates bias reduction and enhances stability in estimates, depending on the number k of order statistics considered. Moreover, with a suitably chosen control parameter, these estimators exhibit a relatively stable RMSE pattern with k. Our findings also show significant improvements over the CH estimator. A comparison with other classes of RB EVI-estimators is currently underway and falls outside the scope of this work.

Author Contributions: Conceptualization, L.H.-R. and F.C.; methodology, L.H.-R., F.C. and M.I.G.; validation, F.C.; investigation, L.H.-R., F.C. and M.I.G.; writing—original draft preparation, L.H.-R. and F.C.; writing—review and editing, L.H.-R., F.C. and M.I.G. All authors have read and agreed to the published version of the manuscript.

Funding: Research partially supported by National Funds through FCT—Fundação para a Ciência e a Tecnologia—projects UIDB/MAT/04674/2020 (CIMA) https://doi.org/10.54499/UIDB/04674/2020, UIDB/00297/2020 https://doi.org/10.54499/UIDB/00297/2020, UIDP/00297/2020 https://doi.org/10.54499/UIDP/00297/2020 (Centro de Matemática e Aplicações), and UIDB/00006/2020 (CEAUL) https://doi.org/10.54499/UIDB/00006/2020. The first author is also supported by project 10.54499/CEECINST/00054/2018/CP1522/CT0003 https://doi.org/10.54499/CEECINST/00054/2018/CP1522/CT0003.

Data Availability Statement: The data used in this study are entirely based on simulated and open datasets. The simulation code and any relevant parameters used to generate these datasets are available from the corresponding author upon reasonable request.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Fréchet, M. Sur la loi de probabilité de l'écart maximum. *Ann. Soc. Math. Polon.* **1927**, *6*, 93–116.
2. Fisher, R.A.; Tippett, L.H.C. Limiting forms of the frequency distribution of the largest or smallest member of a sample. *Math. Proc. Camb. Philos. Soc.* **1928**, *24*, 180–190. [CrossRef]
3. Von Mises, R. La distribution de la plus grande de n valuers. *Rev. Math. Union Interbalcanique* **1936**, *1*, 141–160.
4. Gnedenko, B. Sur La Distribution Limite Du Terme Maximum D'Une Série Aléatoire. *Ann. Math.* **1943**, *44*, 423. [CrossRef]
5. Weissman, I. Estimation of parameters and large quantiles based on the k largest observations. *J. Am. Stat. Assoc.* **1978**, *73*, 812–815. [CrossRef]
6. Matthys, G.; Beirlant, J. Estimating the extreme value index and high quantiles with exponential regression models. *Stat. Sin.* **2003**, *13*, 853–880.
7. Jorion, P. *Value at Risk: The New Benchmark for Managing Market Risk*, 2nd ed.; McGraw-Hill: New York, NY, USA, 2000.
8. Girard, S. A Hill Type Estimator of the Weibull Tail-Coefficient. *Commun. Stat.—Theory Methods* **2004**, *33*, 205–234. [CrossRef]
9. Beirlant, J.; Herrmann, K.; Teugels, J. Estimation of the Extreme Value Index. In *Extreme Events in Finance*; John Wiley & Sons, Inc.: Hoboken, NJ, USA, 2016; pp. 97–115. [CrossRef]
10. Allouche, M.; El Methni, J.; Girard, S. A refined Weissman estimator for extreme quantiles. *Extremes* **2023**, *26*, 545–572. [CrossRef]
11. Beirlant, J.; Goegebeur, Y.; Segers, J.; Teugels, J.L. *Statistics of Extremes: Theory and Applications*; John Wiley & Sons: Hoboken, NJ, USA, 2004. [CrossRef]
12. de Haan, L.; Ferreira, A. *Extreme Value Theory*; Springer: New York, NJ, USA, 2006. [CrossRef]
13. Dey, D.; Yan, J. *Extreme Value Modeling and Risk Analysis: Methods and Applications*; Chapman and Hall/CRC: Boca Raton, FL, USA, 2016. [CrossRef]
14. Bingham, N.H.; Goldie, C.M.; Teugels, J.L. *Regular Variation*; *Encyclopedia of Mathematics and its Applications*; Cambridge University Press: Cambridge, MA, USA, 1987; Volume 27. [CrossRef]
15. Bingham, N. Regular variation and probability: The early years. *J. Comput. Appl. Math.* **2007**, *200*, 357–363. [CrossRef]
16. Hill, B.M. A simple general approach to inference about the tail of a distribution. *Ann. Stat.* **1975**, *3*, 1163–1174. [CrossRef]
17. Mason, D.M. Laws of large numbers for sums of extreme values. *Ann. Probab.* **1982**, *10*, 754–764. [CrossRef]
18. Haeusler, E.; Teugels, J.L. On asymptotic normality of Hill's estimator for the exponent of regular variation. *Ann. Stat.* **1985**, *13*, 743–756. [CrossRef]
19. Hsing, T. On tail index estimation using dependent data. *Ann. Stat.* **1991**, *19*, 1547–1569. [CrossRef]

20. de Haan, L.; Resnick, S. On asymptotic normality of the Hill estimator. *Stoch. Models* **1998**, *14*, 849–866. [CrossRef]
21. Gomes, M.I.; Guillou, A. Extreme value theory and statistics of univariate extremes: A review. *Int. Stat. Rev.* **2015**, *83*, 263–292. [CrossRef]
22. Csörgő, S.; Deheuvels, P.; Mason, D. Kernel estimates of the tail index of a distribution. *Ann. Stat.* **1985**, *13*, 1050–1077. [CrossRef]
23. Dekkers, A.L.M.; Einmahl, J.H.J.; de Haan, L. A Moment Estimator for the Index of an Extreme-Value Distribution. *Ann. Stat.* **1989**, *17*, 1833–1855. [CrossRef]
24. Gomes, M.I.; Martins, M.J.; Neves, M. Alternatives to a Semi-Parametric Estimator of Parameters of Rare Events–The Jackknife Methodology. *Extremes* **2000**, *3*, 207–229. [CrossRef]
25. Gomes, M.I.; Martins, M.J. Generalizations of the Hill estimator—Asymptotic versus finite sample behaviour. *J. Stat. Plan. Inference* **2001**, *93*, 161–180. [CrossRef]
26. Groeneboom, P.; Lopuhaä, H.; De Wolf, P. Kernel-type estimators for the extreme value index. *Ann. Stat.* **2003**, *31*, 1956–1995. [CrossRef]
27. Beran, J.; Schell, D.; Stehlík, M. The harmonic moment tail index estimator: Asymptotic distribution and robustness. *Ann. Inst. Stat. Math.* **2014**, *66*, 193–220. [CrossRef]
28. Penalva, H.; Gomes, M.I.; Caeiro, F.; Neves, M.M. A Couple of Non Reduced Bias Generalized Means In Extreme Value Theory: An Asymptotic Comparison. *Rev.-Stat. J.* **2020**, *18*, 281–298. [CrossRef]
29. Caeiro, F.; Mateus, A.; Soltane, L. A class of weighted Hill estimators. *Comput. Math. Methods* **2021**. [CrossRef]
30. Paulauskas, V.; Vaičiulis, M. On an improvement of Hill and some other estimators. *Lith. Math. J.* **2013**, *53*, 336–355. [CrossRef]
31. de Haan, L.; Peng, L. Comparison of tail index estimators. *Stat. Neerl.* **1998**, *52*, 60–70. [CrossRef]
32. Cabral, I.; Caeiro, F.; Gomes, M.I. On the comparison of several classical estimators of the extreme value index. *Commun. Stat.-Theory Methods* **2022**, *51*, 179–196. [CrossRef]
33. Segers, J. Residual estimators. *J. Stat. Plan. Inference* **2001**, *98*, 15–27. [CrossRef]
34. Hall, P. On Some Simple Estimates of an Exponent of Regular Variation. *J. R. Stat. Soc. Ser. B (Methodol.)* **1982**, *44*, 37–42. [CrossRef]
35. Hall, P.; Welsh, A. Adaptive estimates of parameters of regular variation. *Ann. Stat.* **1985**, *13*, 331–341. [CrossRef]
36. Caeiro, F.; Gomes, M.I.; Pestana, D. Direct reduction of bias of the classical Hill estimator. *Rev.-Stat. J.* **2005**, *3*, 113–136. [CrossRef]
37. Knight, K. A Simple Modification of the Hill Estimator with Applications to Robustness and Bias Reduction. Statistics Department, University of Toronto: Toronto, ON, Canada, 2012; *preprint*.
38. Buitendag, S.; Beirlant, J.; de Wet, T. Ridge regression estimators for the extreme value index. *Extremes* **2019**, *22*, 271–292. [CrossRef]
39. Geluk, J.; de Haan, L. *Regular Variation, Extensions and Tauberian Theorems*; CWI Tract 40, Centre for Mathematics and Computer Science: Amsterdam, The Netherlands, 1987.
40. Gomes, M.I.; Pestana, D. A sturdy reduced-bias extreme quantile (VaR) estimator. *J. Am. Stat. Assoc.* **2007**, *102*, 280–292. [CrossRef]
41. Caeiro, F.; Gomes, M.I. Threshold Selection in Extreme Value Analysis. In *Extreme Value Modeling and Risk Analysis*; Dey, D.K., Yan, J., Eds.; Chapman and Hall/CRC: Boca Raton, FL, USA, 2016; pp. 69–86. [CrossRef]
42. Gomes, M.I.; de Haan, L.; Peng, L. Semi-parametric Estimation of the Second Order Parameter in Statistics of Extremes. *Extremes* **2002**, *5*, 387–414. [CrossRef]
43. Fraga Alves, M.I.; Gomes, M.I.; de Haan, L. A new class of semi-parametric estimators of the second order parameter. *Port. Math.* **2003**, *60*, 193–213.
44. Gomes, M.I.; Martins, M.J. Asymptotically unbiased estimators of the tail index based on external estimation of the second order parameter. *Extremes* **2002**, *5*, 5–31. [CrossRef]
45. Ciuperca, G.; Mercadier, C. Semi-parametric estimation for heavy tailed distributions. *Extremes* **2010**, *13*, 55–87. [CrossRef]
46. Gomes, M.I.; Henriques-Rodrigues, L.; Pereira, H.; Pestana, D. Tail index and second-order parameters' semi-parametric estimation based on the log-excesses. *J. Stat. Comput. Simul.* **2010**, *80*, 653–666. [CrossRef]
47. Worms, J.; Worms, R. Estimation of second order parameters using probability weighted moments. *ESAIM Probab. Stat.* **2012**, *16*, 97–113. [CrossRef]
48. de Wet, T.; Goegebeur, Y.; Munch, M.R. Asymptotically unbiased estimation of a second order tail parameter. *Stat. Probab. Lett.* **2012**, *82*, 565–573. [CrossRef]
49. Deme, E.H.; Gardes, L.; Girard, S. On the Estimation of the Second Order Parameter for Heavy-Tailed Distributions. *Rev.-Stat. J.* **2013**, *11*, 277–299. [CrossRef]
50. Caeiro, F.; Gomes, M.I. Bias Reduction in the Estimation of a Shape Second-order Parameter of a Heavy Tail Model. *J. Stat. Comput. Simul.* **2015**, *85*, 3405–3419. [CrossRef]
51. Caeiro, F.; Gomes, M.I. A new class of estimators of a "scale" second order parameter. *Extremes* **2006**, *9*, 193–211. [CrossRef]
52. Caeiro, F.; Gomes, M.I.; Henriques-Rodrigues, L. Reduced-bias tail index estimators under a third order framework. *Commun. Stat.-Theory Methods* **2009**, *38*, 1019–1040. [CrossRef]