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MAXIMAL NONCOMPACTNESS OF WIENER-HOPF OPERATORS

Oleksiy Karlovych[1] · **Eugene Shargorodsky**[2]

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Abstract
Let $X(\mathbb{R})$ be a separable translation-invariant Banach function space and a be a Fourier multiplier on $X(\mathbb{R})$. We prove that the Wiener-Hopf operator $W(a)$ with symbol a is maximally noncompact on the space $X(\mathbb{R}_+)$, that is, its Hausdorff measure of noncompactness, its essential norm, and its norm are all equal. This equality for the Hausdorff measure of noncompactness of $W(a)$ is new even in the case of $X(\mathbb{R}) = L^p(\mathbb{R})$ with $1 \le p < \infty$.

Keywords Wiener-Hopf operator · Essential norm · Hausdorff measure of noncompactness · Translation-invariant space · Rearrangement-invariant space

Introduction

For Banach spaces $\mathcal{X}, \mathcal{Y}$, let $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{K}(\mathcal{X}, \mathcal{Y})$ denote the sets of bounded linear and compact linear operators from $\mathcal{X}$ to $\mathcal{Y}$, respectively. We will abbreviate $\mathcal{B}(\mathcal{X}) := \mathcal{B}(\mathcal{X}, \mathcal{X})$ and $\mathcal{K}(\mathcal{X}) := \mathcal{K}(\mathcal{X}, \mathcal{X})$. The norm of an operator $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is denoted by $\|A\|_{\mathcal{B}(\mathcal{X},\mathcal{Y})}$. The essential norm of $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is defined by

$$\|A\|_{\mathcal{B}(\mathcal{X},\mathcal{Y}),\mathrm{e}} := \inf\{\|A - K\|_{\mathcal{B}(\mathcal{X},\mathcal{Y})} : \ K \in \mathcal{K}(\mathcal{X}, \mathcal{Y})\}.$$

For a bounded subset Ω of the space $\mathcal{X}$, we denote by $\chi(\Omega)$ the greatest lower bound of the set of numbers r such that Ω can be covered by a finite family of open balls of radius r. For $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, set

$$\|A\|_{\mathcal{B}(\mathcal{X},\mathcal{Y}),\chi} := \chi\left(A(B_{\mathcal{X}})\right),$$

where $B_{\mathcal{X}}$ denotes the closed unit ball in $\mathcal{X}$. The quantity $\|A\|_{\mathcal{B}(\mathcal{X},\mathcal{Y}),\chi}$ is called the Hausdorff measure of noncompactness of the operator A. It follows from the definition of the essential norm and [1, inequality (3.29)] that for every $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ one has

$$\|A\|_{\mathcal{B}(\mathcal{X},\mathcal{Y}),\chi} \le \|A\|_{\mathcal{B}(\mathcal{X},\mathcal{Y}),\mathrm{e}} \le \|A\|_{\mathcal{B}(\mathcal{X},\mathcal{Y})}. \tag{1}$$

Note that both inequalities in Eq. 1 may be strict. A simple example of an operator, for which the first inequality is strict, is given in [2, Section 2.4.11]. Moreover, there exist Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ such that

$$\sup_{A\in\mathcal{B}(\mathcal{X},\mathcal{Y})} \frac{\|A\|_{\mathcal{B}(\mathcal{X},\mathcal{Y}),\mathrm{e}}}{\|A\|_{\mathcal{B}(\mathcal{X},\mathcal{Y}),\chi}} = \infty$$

Eugene Shargorodsky contributed equally to this work

To Sergei Grudsky on the occasion of his 70th birthday.

✉ Oleksiy Karlovych
oyk@fct.unl.pt

Eugene Shargorodsky
eugene.shargorodsky@kcl.ac.uk

1 Centro de Matemática e Aplicações, Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Quita da Torre, Caparica 2829–516, Portugal

2 Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK

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(see [3, the proof of Theorem 2.5]).

An interesting example of an operator, for which the second inequality in Eq. 1 is strict, is the Cauchy singular integral operator

$$(S_\Gamma f)(t) := \frac{1}{\pi i}\int_\Gamma \frac{f(\tau)}{\tau - t}d\tau$$

on $L^2(\Gamma)$, where Γ is a smooth simple closed contour in the complex plane. In this case, $\|S_\Gamma\|_{\mathcal{B}(L^2(\Gamma)),\mathrm{e}} = \|S_\Gamma\|_{\mathcal{B}(L^2(\Gamma))}$ if and only if Γ is a circle (see [4, Corollary 5.7]).

We refer to the monographs [2, 5, 6] for the general theory of measures of noncompactness. In particular, it is well known that

$$K \in \mathcal{K}(\mathcal{X},\mathcal{Y}) \Longleftrightarrow \|K\|_{\mathcal{B}(\mathcal{X},\mathcal{Y}),\chi} = 0 \Longleftrightarrow \|K\|_{\mathcal{B}(\mathcal{X},\mathcal{Y}),\mathrm{e}} = 0$$

(for the first equivalence, see, e.g., [6, Theorem 5.29]). We will say that an operator $A \in \mathcal{B}(\mathcal{X},\mathcal{Y})$ is maximally noncompact if

$$\|A\|_{\mathcal{B}(\mathcal{X},\mathcal{Y}),\chi} = \|A\|_{\mathcal{B}(\mathcal{X},\mathcal{Y}),\mathrm{e}} = \|A\|_{\mathcal{B}(\mathcal{X},\mathcal{Y})}.$$

Let $a \in L^\infty(\mathbb{R})$ and $X(\mathbb{R})$ be a Banach function space (see [7], "Banach function spaces" section below, and also [8, Ch. 1], [9, Ch. 6]). For a set $Q \subset \mathbb{R}$ of positive measure, let e_Q be the operator of extension by 0 from Q to $\mathbb{R}$ and $X(Q)$ be the Banach space of measurable functions $f : Q \to \mathbb{C}$ such that $\|f\|_{X(Q)} := \|e_Q f\|_{X(\mathbb{R})} < \infty$.

Before discussing Wiener-Hopf operators, let us consider the multiplication operator $aI \in \mathcal{B}(X(\mathbb{R}))$. This operator is maximally noncompact. Indeed, it is easy to see that $\|aI\|_{\mathcal{B}(X(\mathbb{R}))} \le \|a\|_{L^\infty}$. On the other hand, for any $\varepsilon > 0$, there exist $c \in \mathbb{C}$ and a set of positive measure $Q \subset \mathbb{R}$ such that $|c| > \|a\|_{L^\infty} - \varepsilon/2$ and $|a - c| < \varepsilon/2$ a.e. in Q. Then,

$$\begin{aligned}\|aI\|_{\mathcal{B}(X(\mathbb{R})),\chi} &\ge \|aI\|_{\mathcal{B}(X(Q)),\chi} \ge \|cI\|_{\mathcal{B}(X(Q)),\chi} - \|(c-a)I\|_{\mathcal{B}(X(Q)),\chi}\\ &\ge |c|\|I\|_{\mathcal{B}(X(Q)),\chi} - \|(c-a)I\|_{\mathcal{B}(X(Q))} > |c| - \varepsilon/2 > \|a\|_{L^\infty} - \varepsilon\end{aligned}$$

(see [2, Theorem 1.1.6]). Since $\varepsilon > 0$ is arbitrary, one gets $\|aI\|_{\mathcal{B}(X(\mathbb{R})),\chi} \ge \|a\|_{L^\infty}$. So,

$$\|aI\|_{\mathcal{B}(X(\mathbb{R})),\chi} = \|a\|_{L^\infty} = \|aI\|_{\mathcal{B}(X(\mathbb{R}))}.$$

It is instructive to compare the above with what is known for multiplication operators on Sobolev spaces $W_p^m(\mathbb{R})$, $m \in \mathbb{N}$, $1 < p < \infty$. Suppose, for simplicity, that the support of a is a subset of $[-1, 1]$. Then, $\|aI\|_{\mathcal{B}(W_p^m(\mathbb{R}))}$ is estimated below by $\|a\|_{W_p^m}$ times a constant independent of a (see [10, Corollary 2.3.1]). On the other hand, $\|aI\|_{\mathcal{B}(W_p^m(\mathbb{R})),\chi}$ is estimated above by $\|a\|_{L^\infty}$ times a constant independent of a (see [11]). Hence, one cannot expect $aI : W_p^m(\mathbb{R}) \to W_p^m(\mathbb{R})$ to be maximally noncompact.

Let $S(\mathbb{R})$ denote the Schwartz spaces of rapidly decaying infinitely differentiable functions, let

$$(Fu)(\xi) := \widehat{u}(\xi) := \int_{\mathbb{R}} u(x)e^{-ix\xi}\,dx, \quad \xi \in \mathbb{R},$$

be the Fourier transform of $u \in S(\mathbb{R})$, and let F^{-1} denote the inverse Fourier transform on $S(\mathbb{R})$. The operators F and F^{-1} can be extended by continuity to $L^2(\mathbb{R})$, and we will use the same symbols for their extensions to $L^2(\mathbb{R})$. Let $X(\mathbb{R})$ be a Banach function space. A function $a \in L^\infty(\mathbb{R})$ is said to belong to the set $\mathcal{M}^0_{X(\mathbb{R})}$ of Fourier multipliers on $X(\mathbb{R})$ if

$$\|a\|_{\mathcal{M}^0_{X(\mathbb{R})}} := \sup\left\{\frac{\|F^{-1}aFu\|_{X(\mathbb{R})}}{\|u\|_{X(\mathbb{R})}} : u \in \left(L^2(\mathbb{R}) \cap X(\mathbb{R})\right)\setminus\{0\}\right\} < \infty.$$

The operator $F^{-1}aF$ is translation-invariant, and one can find results on maximal noncompactness of such operators in [12, Section 5]. If $a \in \mathcal{M}^0_{X(\mathbb{R})}$ and $X(\mathbb{R})$ is separable, in which case $L^2(\mathbb{R}) \cap X(\mathbb{R})$ is dense in $X(\mathbb{R})$ (see Lemma 2.2 below), then $F^{-1}aF$ can be extended by continuity to a bounded linear operator on $X(\mathbb{R})$. This operator is, in a sense, equivalent to the multiplication operator $aI : F(X(\mathbb{R})) \to F(X(\mathbb{R}))$, but maximal noncompactness of the former does not seem to follow directly from the maximal noncompactness of the operator aI on the Banach function space $X(\mathbb{R})$.

Denote by r_+ the operator of restriction from $\mathbb{R}$ to $\mathbb{R}_+ := (0, \infty)$ and by e_+ the operator of extension by 0 from $\mathbb{R}_+$ to $\mathbb{R}$. For a closed subspace $Y(\mathbb{R})$ of a Banach function space $X(\mathbb{R})$, let $Y(\mathbb{R}_+)$ be the space of all measurable functions $f : \mathbb{R}_+ \to \mathbb{C}$ such that

$$\|f\|_{Y(\mathbb{R}_+)} := \|e_+ f\|_{Y(\mathbb{R})} < \infty.$$

For $a \in \mathcal{M}^0_{X(\mathbb{R})}$, let

$$W(a)u := r_+F^{-1}aFe_+u, \quad u \in L^2(\mathbb{R}_+) \cap X(\mathbb{R}_+)$$

be the Wiener-Hopf operator with symbol a. Put

$$\|W(a)\|_{[X(\mathbb{R}_+)]} := \sup\left\{\frac{\|W(a)u\|_{X(\mathbb{R}_+)}}{\|u\|_{X(\mathbb{R}_+)}} : u \in \left(L^2(\mathbb{R}_+) \cap X(\mathbb{R}_+)\right) \setminus \{0\}\right\}.$$

Let $X_2(\mathbb{R}_+)$ be the closure of $L^2(\mathbb{R}_+) \cap X(\mathbb{R}_+)$ in $X(\mathbb{R}_+)$. Then, the bounded linear operator $W(a) : L^2(\mathbb{R}_+) \cap X(\mathbb{R}_+) \to X(\mathbb{R}_+)$ can be extended by continuity to a bounded linear operator from $X_2(\mathbb{R}_+)$ to $X(\mathbb{R}_+)$, which we will denote again by $W(a)$. Moreover,

$$\|W(a)\|_{[X(\mathbb{R}_+)]} = \|W(a)\|_{\mathcal{B}(X_2(\mathbb{R}_+), X(\mathbb{R}_+))}.$$

We are interested in the maximal noncompactness of Wiener-Hopf operators. We refer to [12–15] for maximal noncompactness of some other integral transforms.

Let us recall the results motivating our work. If $a \in \mathcal{M}^0_{L^p(\mathbb{R})}$, then

$$\|W(a)\|_{\mathcal{B}(L^p(\mathbb{R}_+))} = \|W(a)\|_{\mathcal{B}(L^p(\mathbb{R}_+)),\mathrm{e}}, \quad 1 < p < \infty. \tag{2}$$

(see [16, p. 5], where it is stated without proof for $1 \le p < \infty$, and [17, Section 9.5(b)], where the exact range for p is not specified and the given idea of the proof seems to exclude the case $p = 1$).

Equality (2) was recently extended by the first author and M. Valente to the setting of Lorentz and Orlicz spaces (see, e.g., [8, Ch. 4] for their definitions). Let $X(\mathbb{R})$ be a Lorentz space $L^{p,q}(\mathbb{R})$ with $1 < p < \infty$ and $1 \le q < \infty$ or a separable Orlicz space $L^\Phi(\mathbb{R})$ generated by a Young's function Φ satisfying

$$\lim_{t\to 0^+} \frac{\Phi(t)}{t} = 0. \tag{3}$$

It was shown in [18, Theorem 1] that if $a \in \mathcal{M}^0_{X(\mathbb{R})}$, then

$$\|W(a)\|_{\mathcal{B}(X(\mathbb{R}_+))} = \|W(a)\|_{\mathcal{B}(X(\mathbb{R}_+)),\mathrm{e}}. \tag{4}$$

It is well known that the Lebesque space $L^p(\mathbb{R})$, $1 \le p < \infty$, is the Orlicz space generated by the Young's function $\Phi(t) = t^p$ (with equal norms). So, condition (3) excludes the case of the space $L^1(\mathbb{R})$.

Note that Lebesgue, Lorentz, and Orlicz spaces are translation-invariant. So it is natural to look for extensions of the above results to the setting of arbitrary translation-invariant Banach function spaces (see "Translation-invariant subspaces of Banach function spaces" section for their definition). Let us state here a simply formulated consequence of our main result, which will be proved in the "Main result" section.

Theorem 1.1 *Suppose $X(\mathbb{R})$ is a separable translation-invariant Banach function space. If $a \in \mathcal{M}^0_{X(\mathbb{R})}$, then the Wiener-Hopf operator $W(a)$ is maximally noncompact on the space $X(\mathbb{R}_+)$, that is,*

$$\|W(a)\|_{\mathcal{B}(X(\mathbb{R}_+))} = \|W(a)\|_{\mathcal{B}(X(\mathbb{R}_+)),\mathrm{e}} = \|W(a)\|_{\mathcal{B}(X(\mathbb{R}_+)),\chi}.$$

This theorem improves equality (4) for separable Orlicz spaces because it allows to drop condition (3) (and, so, to include the space $L^1(\mathbb{R}_+)$ into consideration). What is more important is that the equality $\|W(a)\|_{\mathcal{B}(X(\mathbb{R}_+))} = \|W(a)\|_{\mathcal{B}(X(\mathbb{R}_+)),\chi}$ is new even for Lebesgue spaces $L^p(\mathbb{R}_+)$ with $1 \le p < \infty$.

The paper is organized as follows. In the "Preliminaries on Banach function spaces" section, we collect preliminaries on the class of Banach function spaces and its subclasses of translation-invariant and rearrangement-invariant spaces. For the latter subclass of spaces, we recall the definition of their Zippin (or fundamental) indices. The "Main result and its applications to Wiener-Hopf operators" section starts with our main result (Theorem 3.1) on the maximal noncompactness of the truncation $A_+ := r_+ A e_+$ of a translation-invariant operator $A : X(\mathbb{R}) \to Y(\mathbb{R})$ acting from a translation-invariant subspace $X(\mathbb{R})$ to a translation-invariant Banach function space $Y(\mathbb{R})$ with some additional properties. Further, this result is applied to prove Theorem 1.1. Finally, we show that the hypotheses of Theorem 3.1 are flexible enough to prove analogs of Theorem 1.1 for a not necessarily separable rearrangement-invariant Banach function space with the positive lower Zippin index. In particular, we state such a result for weak Lebesgue spaces $L^{p,\infty}$ with $1 < p < \infty$, which are not separable, as it is well known.

Preliminaries on Banach function spaces

Banach function spaces

The set of all Lebesgue measurable complex-valued functions on $\mathbb{R}$ is denoted by $\mathfrak{M}(\mathbb{R})$. Let $\mathfrak{M}^+(\mathbb{R})$ be the subset of functions in $\mathfrak{M}(\mathbb{R})$ whose values lie in $[0, \infty]$. The characteristic function of a measurable set $E \subset \mathbb{R}$ is denoted by $\mathbb{1}_E$ and the Lebesgue measure of E is denoted by $|E|$.

Following [7, p. 3] (see also [8, Ch. 1, Definition 1.1] and [9, Definition 6.1.5]), a mapping $\rho : \mathfrak{M}^+(\mathbb{R}) \to [0, \infty]$ is called a Banach function norm if, for all functions f, g, f_j ($j \in \mathbb{N}$) in $\mathfrak{M}^+(\mathbb{R})$, for all constants $a \geq 0$, and for all measurable subsets E of $\mathbb{R}$, the following properties hold:

$$\begin{aligned}
&(\mathrm{A1}) \quad \rho(f) = 0 \Leftrightarrow f = 0 \text{ a.e.}, \quad \rho(af) = a\rho(f), \quad \rho(f+g) \leq \rho(f) + \rho(g),\\
&(\mathrm{A2}) \quad 0 \leq g \leq f \text{ a.e.} \Rightarrow \rho(g) \leq \rho(f) \quad \text{(the lattice property)},\\
&(\mathrm{A3}) \quad 0 \leq f_j \uparrow f \text{ a.e.} \Rightarrow \rho(f_j) \uparrow \rho(f) \quad \text{(the Fatou property)},\\
&(\mathrm{A4}) \quad E \text{ is bounded} \Rightarrow \rho(\mathbb{1}_E) < \infty,\\
&(\mathrm{A5}) \quad E \text{ is bounded} \Rightarrow \int_E f(x)\,dx \leq C_E \rho(f)
\end{aligned}$$

with $C_E \in (0, \infty)$ that may depend on E and ρ but is independent of f.

When functions differing only on a set of measure zero are identified, the set $X(\mathbb{R})$ of all functions $f \in \mathfrak{M}(\mathbb{R})$ for which $\rho(|f|) < \infty$ becomes a Banach space under the norm

$$\|f\|_{X(\mathbb{R})} := \rho(|f|)$$

and under the natural linear space operations (see [7, Ch. 1, § 1, Theorem 1] or [8, Ch. 1, Theorems 1.4 and 1.6]). It is called a *Banach function space*.

If ρ is a Banach function norm, its associate norm ρ' is defined on $\mathfrak{M}^+(\mathbb{R})$ by

$$\rho'(g) := \sup\left\{\int_{\mathbb{R}} f(x)g(x)\,dx : \ f \in \mathfrak{M}^+(\mathbb{R}), \ \rho(f) \leq 1\right\}.$$

It is a Banach function norm itself (see [7, Ch. 1, § 1] or [8, Ch. 1, Theorem 2.2]). The Banach function space $X'(\mathbb{R})$ defined by the Banach function norm ρ' is called the associate space (Köthe dual) of $X(\mathbb{R})$. The Lebesgue space $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, is the archetypical example of a Banach function space. Other classical examples of Banach function spaces are Lorentz spaces $L^{p,q}(\mathbb{R})$ with $1 < p < \infty$ and $1 \leq q \leq \infty$ (see, e.g., [8, Ch. 4, Section 4]), Orlicz spaces $L^\Phi(\mathbb{R})$ (see, e.g., [8, Ch. 4, Section 8]), all other rearrangement-invariant Banach function spaces (see [8, Ch. 2]), as well as, variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R})$ (see [19]).

Remark 2.1 We note that our definition of a Banach function space is slightly different from that found in [8, Ch. 1, Definition 1.1] and [9, Definition 6.1.5]. In particular, in Axioms (A4) and (A5), we assume that the set E is a bounded set, whereas it is sometimes assumed that E merely satisfies $|E| < \infty$. It is well known that all main elements of the general theory of Banach function spaces work with (A4) and (A5) stated for bounded sets [7] (see also the discussion at the beginning of Chapter 1 on page 2 of [8] and [20]).

Density of simple functions and its consequences

Let $S_0(\mathbb{R})$ denote the set of all simple functions with compact support in $\mathbb{R}$ and $S_0(\mathbb{R}_+)$ denote the set of all simple functions with compact support in $[0, \infty)$.

Lemma 2.2 *If $X(\mathbb{R})$ is a separable Banach function space, then the sets $S_0(\mathbb{R})$, $L^2(\mathbb{R}) \cap X(\mathbb{R})$ are dense in the space $X(\mathbb{R})$, and the sets $S_0(\mathbb{R}_+)$, $L^2(\mathbb{R}_+) \cap X(\mathbb{R}_+)$ are dense in the space $X(\mathbb{R}_+)$.*

Proof Since $S_0(\mathbb{R}) \subset L^2(\mathbb{R}) \cap X(\mathbb{R})$ and $S_0(\mathbb{R}_+) \subset L^2(\mathbb{R}_+) \cap X(\mathbb{R}_+)$, it is enough to prove that $S_0(\mathbb{R})$ is dense in $X(\mathbb{R})$ and $S_0(\mathbb{R}_+)$ is dense in $X(\mathbb{R}_+)$.

Following [7, Ch. 1, § 2, Definition 3], let $X_b(\mathbb{R})$ be the closure of the set of all bounded measurable functions with compact support. It can be shown as in [8, Ch. 1, Proposition 3.10] that the closure of $S_0(\mathbb{R})$ in $X(\mathbb{R})$ coincides with $X_b(\mathbb{R})$. It follows from [7, Ch. 1, § 2, Lemma 4 and § 3, Corollary 1] (see also [8, Ch. 1, Theorem 3.11 and Corollary 5.6]) that if $X(\mathbb{R})$ is separable, then $X(\mathbb{R}) = X_b(\mathbb{R})$. So, in this case $S_0(\mathbb{R})$ is dense in $X(\mathbb{R})$.

Now, let $f \in X(\mathbb{R}_+)$. Then, $g := e_+ f \in X(\mathbb{R})$ and for every $\varepsilon > 0$ there exists $h \in S_0(\mathbb{R})$ such that $\|g - h\|_{X(\mathbb{R})} < \varepsilon$. Therefore, $r := r_+ h \in S_0(\mathbb{R}_+)$ and

$$\begin{aligned}
\|f - r\|_{X(\mathbb{R}_+)} &= \|r_+ e_+ f - r_+ h\|_{X(\mathbb{R}_+)} = \|r_+(g - h)\|_{X(\mathbb{R}_+)}\\
&= \|e_+ r_+ (g - h)\|_{X(\mathbb{R})} = \|\mathbb{1}_{(0,\infty)}(g - h)\|_{X(\mathbb{R})} \leq \|g - h\|_{X(\mathbb{R})} < \varepsilon.
\end{aligned}$$

This implies that $S_0(\mathbb{R}_+)$ is dense in $X(\mathbb{R}_+)$. □

Let us prove another result where an argument based on approximation of functions in $X'(\mathbb{R})$ by functions in $S_0(\mathbb{R})$ plays an important role (see the proof of [21, Lemma 2.10]).

Lemma 2.3 *Let $X(\mathbb{R})$ be a Banach function space and let $X'(\mathbb{R})$ be its associate space. For every $f \in X(\mathbb{R}_+)$, one has*

$$\|f\|_{X(\mathbb{R}_+)} = \sup\left\{\left|\int_{\mathbb{R}_+} f(x)s_+(x)\,dx\right| \;:\; s_+ \in S_0(\mathbb{R}_+),\ \|s_+\|_{X'(\mathbb{R}_+)} \le 1\right\}. \tag{5}$$

Proof It follows from [21, Lemma 2.10] that

$$\|f\|_{X(\mathbb{R}_+)} = \|e_+ f\|_{X(\mathbb{R})} = \sup\left\{\left|\int_{\mathbb{R}} (e_+ f)(x)s(x)\,dx\right| \;:\; s \in S_0(\mathbb{R}),\ \|s\|_{X'(\mathbb{R})} \le 1\right\}. \tag{6}$$

If $s \in S_0(\mathbb{R})$ and $\|s\|_{X'(\mathbb{R})} \le 1$, then $s_+ = r_+ s \in S_0(\mathbb{R}_+)$,

$$\|s_+\|_{X'(\mathbb{R}_+)} = \|e_+ r_+ s\|_{X'(\mathbb{R})} = \|\mathbb{1}_{(0,\infty)} s\|_{X'(\mathbb{R})} \le \|s\|_{X'(\mathbb{R})} \le 1,$$

and

$$\left|\int_{\mathbb{R}} (e_+ f)(x)s(x)\,dx\right| = \left|\int_{\mathbb{R}_+} f(x)s_+(x)\,dx\right|. \tag{7}$$

Hence,

$$\begin{aligned}&\sup\left\{\left|\int_{\mathbb{R}} (e_+ f)(x)s(x)\,dx\right| \;:\; s \in S_0(\mathbb{R}),\ \|s\|_{X'(\mathbb{R})} \le 1\right\}\\ &\le \sup\left\{\left|\int_{\mathbb{R}_+} f(x)s_+(x)\,dx\right| \;:\; s_+ \in S_0(\mathbb{R}_+),\ \|s_+\|_{X'(\mathbb{R}_+)} \le 1\right\}.\end{aligned} \tag{8}$$

On the other hand, if $s_+ \in S_0(\mathbb{R}_+)$ and $\|s_+\|_{X'(\mathbb{R}_+)} \le 1$, then $s = e_+ s_+ \in S_0(\mathbb{R})$ and

$$\|s\|_{X'(\mathbb{R})} = \|e_+ s_+\|_{X'(\mathbb{R})} = \|s_+\|_{X'(\mathbb{R}_+)} \le 1,$$

and Eq. 7 holds. Therefore,

$$\begin{aligned}&\sup\left\{\left|\int_{\mathbb{R}_+} f(x)s_+(x)\,dx\right| \;:\; s_+ \in S_0(\mathbb{R}_+),\ \|s_+\|_{X'(\mathbb{R}_+)} \le 1\right\}\\ &\le \sup\left\{\left|\int_{\mathbb{R}} (e_+ f)(x)s(x)\,dx\right| \;:\; s \in S_0(\mathbb{R}),\ \|s\|_{X'(\mathbb{R})} \le 1\right\}.\end{aligned} \tag{9}$$

Combining Eq. 6 and Eqs. 8–9, we arrive at Eq. 5. □

Translation-invariant subspaces of Banach function spaces

A closed subspace $Y(\mathbb{R})$ of a Banach function space $X(\mathbb{R})$ is said to be translation-invariant if for all $y \in \mathbb{R}$ and for all functions $u \in Y(\mathbb{R})$, one has $\tau_y u \in Y(\mathbb{R})$ and

$$\|\tau_y u\|_{Y(\mathbb{R})} = \|\tau_y u\|_{X(\mathbb{R})} = \|u\|_{X(\mathbb{R})} = \|u\|_{Y(\mathbb{R})},$$

where the translation operator τ_y is defined by $(\tau_y u)(x) := u(x - y)$ for all $x \in \mathbb{R}$. Here and in what follows, we always suppose that subspaces of Banach function spaces are equipped with the induced norms.

Note that all rearrangement-invariant Banach function spaces (see [8, Ch. 2] or "Rearrangement-invariant Banach function spaces and their Zippin indices" section below) are translation-invariant. On the other hand, in view of [19, Theorem 5.17], variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R})$ are not translation-invariant.

We will repeatedly use the fact that a Banach function space $X(\mathbb{R})$ is translation-invariant if and only if its associate space $X'(\mathbb{R})$ is translation-invariant (see [21, Lemma 2.1]).

Rearrangement-invariant Banach function spaces and their Zippin indices

Let $\mathfrak{M}_0(\mathbb{R})$ and $\mathfrak{M}_0^+(\mathbb{R})$ be the classes of a.e. finite functions in $\mathfrak{M}(\mathbb{R})$ and $\mathfrak{M}^+(\mathbb{R})$, respectively. The distribution function μ_f of $f \in \mathfrak{M}_0(\mathbb{R})$ is given by

$$\mu_f(\lambda) := |\{x \in \mathbb{R} : |f(x)| > \lambda\}|, \quad \lambda \ge 0.$$

The non-increasing rearrangement of $f \in \mathfrak{M}_0(\mathbb{R})$ is the function defined by

$$f^*(t) := \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq t\}, \quad t \geq 0.$$

We use here the standard convention that $\inf \emptyset = +\infty$. Two functions $f \in \mathfrak{M}_0(\mathbb{R})$ and $g \in \mathfrak{M}_0(\mathbb{R})$ are said to be equimeasurable if $\mu_f(\lambda) = \mu_g(\lambda)$ for all $\lambda \geq 0$.

A Banach function norm $\rho : \mathfrak{M}^+(\mathbb{R}) \to [0, \infty]$ is called rearrangement-invariant if for every pair of equimeasurable functions $f, g \in \mathfrak{M}_0^+(\mathbb{R})$, the equality $\rho(f) = \rho(g)$ holds. In that case, the Banach function space $X(\mathbb{R})$ generated by ρ is said to be a rearrangement-invariant Banach function space (or simply a rearrangement-invariant space). Lebesgue spaces $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, Orlicz spaces $L^\Phi(\mathbb{R})$, and Lorentz spaces $L^{p,q}(\mathbb{R})$ are classical examples of rearrangement-invariant Banach function spaces (see, e.g., [8] and the references therein). By [8, Ch. 2, Proposition 4.2], if a Banach function space $X(\mathbb{R})$ is rearrangement-invariant, then its associate space $X'(\mathbb{R})$ is also rearrangement-invariant.

Let $X(\mathbb{R})$ be a rearrangement-invariant Banach function space. For each $t \in [0, \infty)$, let E be a subset of $\mathbb{R}$ with $|E| = t$ and let $\varphi_X(t) = \|\mathbb{1}_E\|_{X(\mathbb{R})}$. The so-defined function φ_X is called the fundamental function of $X(\mathbb{R})$. Consider the function

$$M_X(t) := \sup_{0<s<\infty} \frac{\varphi_X(st)}{\varphi_X(s)}, \quad 0 < t < \infty.$$

The numbers

$$p_X := \sup_{0<t<1} \frac{\ln M_X(t)}{\ln t} = \lim_{t\to 0^+} \frac{\ln M_X(t)}{\ln t}, \quad q_X := \inf_{1<t<\infty} \frac{\ln M_X(t)}{\ln t} = \lim_{t\to\infty} \frac{\ln M_X(t)}{\ln t}$$

are called the lower and upper Zippin (or fundamental) indices of the space $X(\mathbb{R})$. It is well known that $0 \leq p_X \leq q_X \leq 1$ (see [8, Exercise 14 to Chapter 3] and [22, Section 4]). Simple calculations show that $\varphi_{L^p}(t) = t^{1/p}$ and hence $p_{L^p} = q_{L^p} = 1/p$ for all $1 \leq p \leq \infty$.

Main result and its applications to Wiener-Hopf operators

Main result

Let $X(\mathbb{R})$ and $Y(\mathbb{R})$ be translation-invariant subspaces of Banach function spaces. An operator $A \in \mathcal{B}(X(\mathbb{R}), Y(\mathbb{R}))$ is said to be translation invariant if $\tau_y A = A\tau_y$ for every $y \in \mathbb{R}$. For an operator $A : X(\mathbb{R}) \to Y(\mathbb{R})$, let

$$A_+ u := r_+ A e_+ u, \quad u \in X(\mathbb{R}_+).$$

For $n \in \mathbb{N}$, define

$$V_n := r_+ \tau_n e_+.$$

If $X(\mathbb{R})$ is a translation-invariant subspace of a Banach function space, then for every $f \in X(\mathbb{R}_+)$ and $n \in \mathbb{N}$, one has $\operatorname{supp} \tau_n e_+ f \subset [n, \infty)$ and

$$\begin{aligned}\|V_n f\|_{X(\mathbb{R}_+)} &= \|e_+ r_+ \tau_n e_+ f\|_{X(\mathbb{R})} = \|\mathbb{1}_{(0,\infty)} \tau_n e_+ f\|_{X(\mathbb{R})} \\ &= \|\tau_n e_+ f\|_{X(\mathbb{R})} = \|e_+ f\|_{X(\mathbb{R})} = \|f\|_{X(\mathbb{R}_+)}.\end{aligned} \tag{10}$$

If a space $Y(\mathbb{R}_+)$ is non-separable, then $S_0(\mathbb{R}_+)$ is not dense in it, but it may happen that every element of $Y(\mathbb{R}_+)$ can be approximated by elements of $S_0(\mathbb{R}_+)$ in a norm weaker than $\|\cdot\|_{Y(\mathbb{R}_+)}$. This possibility is described in the next theorem by introducing an auxiliary Banach function space $Z(\mathbb{R})$. Note that the norm $\|\cdot\|_{Y(\mathbb{R}_+)}$ is, in general, different from the restriction of the norm $\|\cdot\|_{Z(\mathbb{R}_+)}$ to $Y(\mathbb{R}_+)$.

Theorem 3.1 *Let $X(\mathbb{R})$ be a translation-invariant subspace of a Banach function space, $Y(\mathbb{R})$ and $Z(\mathbb{R})$ be translation-invariant Banach function spaces such that $Y(\mathbb{R}_+)$ is a subset of the closure of $S_0(\mathbb{R}_+)$ in $Z(\mathbb{R}_+)$. If $A \in \mathcal{B}(X(\mathbb{R}), Y(\mathbb{R}))$ is a translation-invariant operator, then $A_+ \in \mathcal{B}(X(\mathbb{R}_+), Y(\mathbb{R}_+))$ is maximally noncompact, that is,*

$$\|A_+\|_{\mathcal{B}(X(\mathbb{R}_+),Y(\mathbb{R}_+)),\chi} = \|A_+\|_{\mathcal{B}(X(\mathbb{R}_+),Y(\mathbb{R}_+)),\mathrm{e}} = \|A_+\|_{\mathcal{B}(X(\mathbb{R}_+),Y(\mathbb{R}_+))}.$$

Proof The proof is similar to the proofs of [13, Theorems 1.1–1.2] and [12, Theorem 1.1]. In view of Eq. 1, it is sufficient to prove that

$$\|A_+\|_{\mathcal{B}(X(\mathbb{R}_+),Y(\mathbb{R}_+)),\chi} \geq \|A_+\|_{\mathcal{B}(X(\mathbb{R}_+),Y(\mathbb{R}_+))}. \tag{11}$$

Take an arbitrary $\varepsilon > 0$. There exists $g \in X(\mathbb{R}_+)$ such that $\|g\|_{X(\mathbb{R}_+)} = 1$ and

$$\|A_+ g\|_{Y(\mathbb{R}_+)} > \|A_+\|_{\mathcal{B}(X(\mathbb{R}_+),Y(\mathbb{R}_+))} - \varepsilon. \tag{12}$$

It follows from Lemma 2.3 that there exists $s \in S_0(\mathbb{R}_+) \setminus \{0\}$ such that $\|s\|_{Y'(\mathbb{R}_+)} \le 1$ and

$$\left| \int_{\mathbb{R}_+} (A_+ g)(x) s(x)\, dx \right| \ge \|A_+ g\|_{Y(\mathbb{R}_+)} - \varepsilon. \tag{13}$$

Set $s_n := V_n s$. Since $Y(\mathbb{R})$ is translation-invariant, in view of [21, Lemma 2.1], so is $Y'(\mathbb{R})$. Therefore, in view of Eq. 10, one has

$$\|s_n\|_{Y'(\mathbb{R}_+)} = \|s\|_{Y'(\mathbb{R}_+)} \le 1, \quad n \in \mathbb{N}. \tag{14}$$

Making a change of variables in the left-hand side of Eq. 13, we see that for all $n \in \mathbb{N}$,

$$\left| \int_{\mathbb{R}_+} (V_n A_+ g)(x) s_n(x)\, dx \right| = \left| \int_{\mathbb{R}_+} (A_+ g)(x) s(x)\, dx \right| \ge \|A_+ g\|_{Y(\mathbb{R}_+)} - \varepsilon.$$

Since A is translation-invariant, it is easy to see that for all $n \in \mathbb{N}$,

$$\begin{aligned} V_n A_+ g &= (r_+ \tau_n e_+)(r_+ A e_+) g = r_+ \tau_n \mathbb{1}_{(0,\infty)} A e_+ g = r_+ \mathbb{1}_{(n,\infty)} \tau_n A e_+ g \\ &= \mathbb{1}_{(n,\infty)} r_+ A \tau_n e_+ g = \mathbb{1}_{(n,\infty)} r_+ A \mathbb{1}_{(0,\infty)} \tau_n e_+ g = \mathbb{1}_{(n,\infty)} (r_+ A e_+)(r_+ \tau_n e_+) g \\ &= \mathbb{1}_{(n,\infty)} A_+ V_n g. \end{aligned}$$

On the other hand, $\operatorname{supp} s_n \subset [n, \infty)$. Hence, $(A_+ V_n g) s_n = (V_n A_+ g) s_n$ a.e., and one gets

$$\left| \int_{\mathbb{R}_+} (A_+ V_n g)(x) s_n(x)\, dx \right| \ge \|A_+ g\|_{Y(\mathbb{R}_+)} - \varepsilon. \tag{15}$$

Take any finite set $\{\varphi_1, \dots, \varphi_m\} \subset Y(\mathbb{R}_+)$. Since $Y(\mathbb{R}_+)$ is a subset of the closure of $S_0(\mathbb{R}_+)$ in $Z(\mathbb{R}_+)$, and $s \in S_0(\mathbb{R}_+) \subset Z'(\mathbb{R}_+)$, there exists a set $\{\psi_1, \dots, \psi_m\} \subset S_0(\mathbb{R}_+)$ such that

$$\|\varphi_j - \psi_j\|_{Z(\mathbb{R}_+)} < \frac{\varepsilon}{\|s\|_{Z'(\mathbb{R}_+)}}, \quad j \in \{1, \dots, m\}. \tag{16}$$

Taking into account that ψ_j and s are compactly supported, we see that there exists $N \in \mathbb{N}$ such that

$$\psi_j s_N = 0, \quad j \in \{1, \dots, m\}. \tag{17}$$

Since $Z'(\mathbb{R})$ is translation-invariant (see [21, Lemma 2.1]), it follows from Eq. 10 that

$$\|s_N\|_{Z'(\mathbb{R}_+)} = \|s\|_{Z'(\mathbb{R}_+)}.$$

Then, in view of equalities (17), Hölder's inequality for the space $Z(\mathbb{R})$ (see [8, Ch. 1, Theorem 2.4]), and inequalities (16), one gets, for $j \in \{1, \dots, m\}$,

$$\begin{aligned} \left| \int_{\mathbb{R}_+} \varphi_j(x) s_N(x)\, dx \right| &= \left| \int_{\mathbb{R}_+} \big(\varphi_j(x) - \psi_j(x)\big) s_N(x)\, dx \right| \le \|\varphi_j - \psi_j\|_{Z(\mathbb{R}_+)} \|s_N\|_{Z'(\mathbb{R}_+)} \\ &< \frac{\varepsilon}{\|s\|_{Z'(\mathbb{R}_+)}} \|s\|_{Z'(\mathbb{R}_+)} = \varepsilon. \end{aligned} \tag{18}$$

Combining Eqs. 15 and 18, we see that for $j \in \{1, \dots, m\}$,

$$\begin{aligned} &\left| \int_{\mathbb{R}_+} \big((A_+ V_N g)(x) - \varphi_j(x)\big) s_N(x)\, dx \right| \\ &\quad \ge \left| \int_{\mathbb{R}_+} (A_+ V_N g)(x) s_N(x)\, dx \right| - \left| \int_{\mathbb{R}_+} \varphi_j(x) s_N(x)\, dx \right| > \|A_+ g\|_{Y(\mathbb{R}_+)} - 2\varepsilon. \end{aligned} \tag{19}$$

On the other hand, applying Hölder's inequality to the space $Y(\mathbb{R})$ (see [8, Ch. 1, Theorem 2.4]) and taking into account inequality (14), we get for $j \in \{1, \dots, m\}$,

$$\begin{aligned} \left| \int_{\mathbb{R}_+} \big((A_+ V_N g)(x) - \varphi_j(x)\big) s_N(x)\, dx \right| &\le \|A_+ V_N g - \varphi_j\|_{Y(\mathbb{R}_+)} \|s_N\|_{Y'(\mathbb{R}_+)} \\ &\le \|A_+ V_N g - \varphi_j\|_{Y(\mathbb{R}_+)}. \end{aligned} \tag{20}$$

So, we deduce from Eqs. 19, 20 and 12 that for $j \in \{1, \dots, m\}$,

$$\|A_+ V_N g - \varphi_j\|_{Y(\mathbb{R}_+)} > \|A_+ g\|_{Y(\mathbb{R}_+)} - 2\varepsilon > \|A_+\|_{\mathcal{B}(X(\mathbb{R}_+), Y(\mathbb{R}_+))} - 3\varepsilon.$$

Since $X(\mathbb{R})$ is a translation-invariant subspace of a Banach function space, it follows from Eq. 10 that

$$\|V_N g\|_{X(\mathbb{R}_+)} = \|g\|_{X(\mathbb{R}_+)} = 1.$$

So, for every finite set $\{\varphi_1, \ldots, \varphi_m\} \subset Y(\mathbb{R}_+)$, there exist an element $A_+ V_N g$ of the image of the unit ball $A_+\big(B_{X(\mathbb{R}_+)}\big)$ that lies at a distance greater than $\|A_+\|_{\mathcal{B}(X(\mathbb{R}_+),Y(\mathbb{R}_+))} - 3\varepsilon$ from every element of $\{\varphi_1, \ldots, \varphi_m\}$. This means that $A_+\big(B_{X(\mathbb{R}_+)}\big)$ cannot be covered by a finite family of open balls of radius $\|A_+\|_{\mathcal{B}(X(\mathbb{R}_+),Y(\mathbb{R}_+))} - 3\varepsilon$. Hence, for all $\varepsilon > 0$,

$$\|A_+\|_{\mathcal{B}(X(\mathbb{R}_+),Y(\mathbb{R}_+)),\chi} \geq \|A_+\|_{\mathcal{B}(X(\mathbb{R}_+),Y(\mathbb{R}_+))} - 3\varepsilon.$$

Passing to the limit as $\varepsilon \to 0+$, we arrive at Eq. 11, which completes the proof. □

Proof of theorem 1.1

If $X(\mathbb{R})$ is a translation-invariant Banach function space, then its subspace $X_2(\mathbb{R})$, being the closure of $L^2(\mathbb{R}) \cap X(\mathbb{R})$ in $X(\mathbb{R})$, is also translation-invariant. It is easy to see that if $a \in \mathcal{M}^0_{X(\mathbb{R})}$, then the Fourier multiplier operator

$$A = F^{-1} a F : X_2(\mathbb{R}) \to X(\mathbb{R}) \tag{21}$$

is translation-invariant and $A_+ = W(a) \in \mathcal{B}(X_2(\mathbb{R}_+), X(\mathbb{R}_+))$.

It follows from Lemma 2.2 that if $X(\mathbb{R})$ is separable, then $X_2(\mathbb{R}) = X(\mathbb{R})$ and $S_0(\mathbb{R}_+)$ is dense in $X(\mathbb{R}_+)$. So, applying Theorem 3.1 to A, A_+ as above and $X(\mathbb{R}) = X_2(\mathbb{R}) = Y(\mathbb{R}) = Z(\mathbb{R})$, we arrive at Theorem 1.1.

Maximal noncompactness of Wiener-Hopf operators on rearrangement-invariant Banach function spaces

Our main result is also flexible enough to treat the case of Wiener-Hopf operators on many not necessarily separable rearrangement-invariant Banach function spaces.

Theorem 3.2 *Let $X(\mathbb{R})$ be a rearrangement-invariant Banach function space with the lower Zippin index $p_X > 0$ and let $X_2(\mathbb{R})$ be the closure of $L^2(\mathbb{R}) \cap X(\mathbb{R})$ in the space $X(\mathbb{R})$. If $a \in \mathcal{M}^0_{X(\mathbb{R})}$, then the Wiener-Hopf operator $W(a) \in \mathcal{B}(X_2(\mathbb{R}_+), X(\mathbb{R}_+))$ is maximally noncompact, that is,*

$$\|W(a)\|_{\mathcal{B}(X_2(\mathbb{R}_+),X(\mathbb{R}_+))} = \|W(a)\|_{\mathcal{B}(X_2(\mathbb{R}_+),X(\mathbb{R}_+)),\mathrm{e}} = \|W(a)\|_{\mathcal{B}(X_2(\mathbb{R}_+),X(\mathbb{R}_+)),\chi}.$$

Proof Following [8, Ch. 3, Definition 1.2], for $1 < p < \infty$, let $L^1(\mathbb{R}) + L^p(\mathbb{R})$ be the collection of all measurable functions $f : \mathbb{R} \to \mathbb{C}$ that are representable as $f = g + h$ for some $g \in L^1(\mathbb{R})$ and $h \in L^p(\mathbb{R})$. For each function $f \in L^1(\mathbb{R}) + L^p(\mathbb{R})$, its norm is defined as

$$\|f\|_{L^1(\mathbb{R})+L^p(\mathbb{R})} = \inf\{\|g\|_{L^1(\mathbb{R})} + \|h\|_{L^p(\mathbb{R})} : \ f = g + h\},$$

where the infimum is taken over all representations $f = g + h$ with $g \in L^1(\mathbb{R})$ and $h \in L^p(\mathbb{R})$.

Since $p_X > 0$, there exists $p \in (1, \infty)$ such that $p_X > 1/p$. Then, it follows from [12, Lemma 5.2] that $X(\mathbb{R})$ is continuously embedded into $Z(\mathbb{R}) := L^1(\mathbb{R}) + L^p(\mathbb{R})$. Therefore, $X(\mathbb{R}_+)$ is contained in $Z(\mathbb{R}_+)$. It is shown in the proof of [12, Corollary 5.4] that $S_0(\mathbb{R})$ is dense in $Z(\mathbb{R})$. Hence, $S_0(\mathbb{R}_+)$ is dense in $Z(\mathbb{R}_+)$ (cf. the proof of Lemma 2.2). It remains to apply Theorem 3.1 to the translation-invariant operator A in Eq. 21, the Wiener-Hopf operator $A_+ = W(a) \in \mathcal{B}(X_2(\mathbb{R}_+), X(\mathbb{R}_+))$ and to the spaces $X_2(\mathbb{R})$, $X(\mathbb{R})$, and $Z(\mathbb{R})$ as above. □

Let us formulate a corollary of the above result, which does not follow from Theorem 1.1.

For $1 < p < \infty$, the Marcinkiewicz space $L^{p,\infty}(\mathbb{R})$ (the weak L^p space) consists of all measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\|f\|_{L^{p,\infty}(\mathbb{R})} := \sup_{0<t<\infty} t^{1/p-1} \int_0^t f^*(s)\, ds < \infty.$$

It is well known that $L^{p,\infty}(\mathbb{R})$ is a rearrangement-invariant Banach function space with respect to the above norm and both its Zippin indices are equal to $1/p$ (see [8, Ch. 4, Theorem 4.6] and [22, inequalities (4.14)]). Moreover, it follows from [23, Ch. II, Theorem 4.8 and Lemma 5.4] or from [9, Theorem 8.5.3] and [8, Ch. 1, Corollary 5.6] that $L^{p,\infty}(\mathbb{R})$ is nonseparable.

Corollary 3.3 *Let $1 < p < \infty$ and let $L^{p,\infty}_2(\mathbb{R})$ be the closure of the set $L^2(\mathbb{R}) \cap L^{p,\infty}(\mathbb{R})$ in the Marcinkiewicz space $L^{p,\infty}(\mathbb{R})$. If $a \in \mathcal{M}^0_{L^{p,\infty}(\mathbb{R})}$, then the Wiener-Hopf operator $W(a) \in \mathcal{B}(L^{p,\infty}_2(\mathbb{R}_+), L^{p,\infty}(\mathbb{R}_+))$ is maximally noncompact, that is,*

$$\begin{aligned}\|W(a)\|_{\mathcal{B}(L^{p,\infty}_2(\mathbb{R}_+),L^{p,\infty}(\mathbb{R}_+))} &= \|W(a)\|_{\mathcal{B}(L^{p,\infty}_2(\mathbb{R}_+),L^{p,\infty}(\mathbb{R}_+)),\mathrm{e}} \\ &= \|W(a)\|_{\mathcal{B}(L^{p,\infty}_2(\mathbb{R}_+),L^{p,\infty}_2(\mathbb{R}_+)),\chi}.\end{aligned}$$

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Declarations

Conflict of interest The authors declare no competing interests.

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