Nonlinear Differ. Equ. Appl.
(2025) 32:126

https://doi.org/10.1007/s00030-025-01129-4

Nonlinear Differential Equations
and Applications NoDEA

Optimal control of a class of third-grade-Voigt equations

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Abstract. This work aims to understand the velocity tracking control problem for a class of non-Newtonian fluids. We introduce the third-grade-Voigt equations in the two-dimensional torus $\mathbb{T}^2$ and prove the existence and uniqueness of the solution. Then, we show the existence and uniqueness of solution to the corresponding linearized state equation and adjoint equation. Additionally, we provide a suitable stability result for the state equation and demonstrate that the Gateaux derivative of the control-to-state mapping agrees with the solution of the linearized state equation. Next, we establish the first order optimality conditions and show the existence of an optimal solution. Ultimately, we are able to provide a uniqueness result for the coupled system consisting of the adjoint equation, the state equation, and the first order optimality condition. Therefore, under appropriate conditions on the data, the uniqueness of the optimal solution holds.

Mathematics Subject Classification. 76A05, 76D03, 93D20.

Keywords. Third-grade-Voigt equations, Optimal control, PDEs.

1. Introduction

1.1. Motivation

Newtonian fluids characterized by Newton's law of viscosity have been the focus of most research in fluid dynamics. Nevertheless, a large number of real, industrial, or physiological fluids exhibiting nonlinear viscoelastic behavior does not obey the Newton's equation of viscosity. These fluids, called

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This work is funded by national funds through the FCT – Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 https://doi.org/10.54499/UIDB/00297/2020 and UIDP/00297/2020 https://doi.org/10.54499/UIDP/00297/2020 (Center for Mathematics and Applications).

Birkhäuser

non-Newtonian fluids, include geological flows, blood, and other natural biological fluids. They also arise during the processing of polymers, coating, colloidal suspensions and emulsions, inkjet printing, and other processes (see e.g. [15,16,33]).

In recent years, special attention has been devoted to non-Newtonian fluids of differential type because it may be related to shallow water models, the viscous Camassa and Holm equation, geodesic motion on the volume-preserving diffeomorphism group (see [10,14,22]). Additionally, it has been found to be helpful in turbulence theory, as evidenced by [13]. Unlike Newtonian fluids, the rheological relation of fluids of grade n, $n > 1$, is a nonlinear relations, and its constitutive law reads

$$\mathbb{S} = -p\mathrm{I} + \mathrm{F}(\mathrm{A}_1(\boldsymbol{v}), \cdots, \mathrm{A}_n(\boldsymbol{v})),$$

where $\mathbb{S}$ is the Cauchy stress tensor and F is an isotropic polynomial function of degree n subject to the usual requirement of material frame indifference, $\boldsymbol{v}$ is fluid's velocity field and $\mathrm{A}_n, n \geq 1$, are the Rivlin-Ericksen kinematic tensors (see [35]), which are defined by

$$\begin{cases} \mathrm{A}_1(\boldsymbol{v}) = \nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T, \\ \mathrm{A}_n(\boldsymbol{v}) = \dfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{A}_{n-1}(\boldsymbol{v}) + \mathrm{A}_{n-1}(\boldsymbol{v})(\nabla\boldsymbol{v}) + (\nabla\boldsymbol{v})^T\mathrm{A}_{n-1}(\boldsymbol{v}), \quad n = 2, 3, \cdots, \end{cases}$$

where $\dfrac{\mathrm{d}}{\mathrm{d}t} = \dfrac{\partial}{\partial t} + (\boldsymbol{v}\cdot\nabla)$ which is also known as material derivative.

In particular the constitutive law of third-grade fluids is given by the following equation

$$\begin{aligned}\mathbb{S} = &-p\mathrm{I} + \mu\mathrm{A}_1(\boldsymbol{v}) + \alpha_1\mathrm{A}_2(\boldsymbol{v}) + \alpha_2\mathrm{A}_1^2(\boldsymbol{v}) + \beta_1\mathrm{A}_3(\boldsymbol{v}) + \beta_2(\mathrm{A}_1(\boldsymbol{v})\mathrm{A}_2(\boldsymbol{v}) \\ &+ \mathrm{A}_2(\boldsymbol{v})\mathrm{A}_1(\boldsymbol{v})) + \beta_3\mathrm{Tr}(\mathrm{A}_1^2(\boldsymbol{v}))\mathrm{A}_1(\boldsymbol{v})\end{aligned}$$

where μ is the viscosity and $(\alpha_i)_{1,2}$, $(\beta_i)_{1,2,3}$ are material moduli. The momentum equations are given by

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \mathrm{div}(\mathbb{S}).$$

In accordance with [16], the fluid motion is compatible with thermodynamic under the following restriction of the parameters

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta}, \quad \beta_1 = \beta_2 = 0, \beta_3 = \beta \geq 0. \tag{1.1}$$

The case $\beta = 0$ corresponds to the second-grade fluids; in addition, if $\alpha_1 = \alpha_2 = 0$ and $\beta = 0$, we recover the Navier Stokes equations (NSE). From mathematical point of view, fluids of grade n are modeled by an hierarchy of partial differential equations with increasing complexity and more nonlinear terms, when comparing with Newtonian (grade 1) or second-grade fluids models, third-grade fluid models are much more complex and require more involved analysis. Although the second-grade fluid models are mathematically more tractable, when testing relevant non-Newtonian fluids, rheologists have concluded that the second-grade model fails to account for crucial rheological features such as shear thickening and shear thinning effects (see, for example, [16] and references therein).

Let us briefly outline the physical concept of third-grade fluids. While Newtonian fluids possess a shear stress tensor proportional to the rate-of-deformation tensor ($\frac{1}{2}\mathrm{A}_1(\boldsymbol{v})$) and second-grade fluids include terms up to quadratic order in the rate-of-deformation tensor, the shear stress tensor of third-grade fluids contains specific cubic (third-order) terms of this tensor. These higher-order terms are physically significant because they capture strong nonlinear effects observed at high shear rates. Moreover, because these third-order terms are highly sensitive to directional interactions within the flow, they allow the stress to respond differently depending on the orientation of the deformation, thereby accurately modeling anisotropic effects. In addition these higher-order contributions introduce complex couplings between different components of the flow, enabling the model to reproduce phenomena such as shear thickening, shear thinning, and other non-Newtonian behaviors that cannot be described by Newtonian or second-grade fluid models.

In order to comprehend and elucidate the properties of various nanofluids, a number of simulation studies have been conducted by using third-grade fluid models (see [19,32,34] and references therein). Let us recall that nanofluids are engineered colloidal suspensions of nanoparticles (usually composed of metals, oxides, carbides, or carbon nanotubes) in a base fluid, such as water, ethylene glycol, or oil. These nanofluids have higher thermal conductivity than the base fluid and have great potential for use in a variety of technological applications, such as heat transfer, microelectronics, fuel cells, pharmaceutical processes, hybrid-powered engines, engine cooling, vehicle thermal management, etc. Therefore, studying the kinematic properties of the third-grade fluids is of paramount importance in applications. From a mathematical perspective, the analysis of third-grade fluid equations or close models with similar strong nonlinearities is also very challenging. The third-grade fluid equations are written by

$$\begin{cases} \partial_t(z(\boldsymbol{v})) - \mu\Delta\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)z(\boldsymbol{v}) + \displaystyle\sum_{j=1}^{2} z(\boldsymbol{v})^j\nabla v^j - (\alpha_1+\alpha_2)\mathrm{div}((\mathrm{A}(\boldsymbol{v}))^2) \\ \quad -\beta\mathrm{div}[\mathrm{Tr}(\mathrm{A}(\boldsymbol{v})\mathrm{A}(\boldsymbol{v})^T)\mathrm{A}(\boldsymbol{v})] = -\nabla p + \boldsymbol{f}, \\ \qquad\qquad\qquad \mathrm{div}\ \boldsymbol{v} = 0, \end{cases} \tag{1.2}$$

where $z(\boldsymbol{v}) := \boldsymbol{v} - \alpha_1\Delta\boldsymbol{v}$ and $\mathrm{A}(\boldsymbol{v}) := \mathrm{A}_1(\boldsymbol{v})$, μ denotes the viscosity of the fluid and α_1, α_2, β are material moduli verifying (1.1), p stands the pressure and $\boldsymbol{f}$ denotes an external force.

From practical perspective, the optimal control of the evolution of third-grade fluids velocity field is crucial to develop optimal flows that can be successfully employed and implemented in the industry. Referring to mathematical studies, the optimal control of flows governed by the incompressible third-grade fluid equations with $\alpha_1 > 0$, on bounded domain D with Navier-slip boundary conditions has been studied in the article [38]. Where the authors have considered the divergence-free initial data with $\mathbb{H}^3(D)$-regularity which ensures that the solution of system (1.2) (with $\alpha_1 > 0$) belongs to $\mathrm{L}^\infty(0,T;\mathbb{H}^3(D))$.

The $\mathbb{H}^3(D)$-regularity of solution of system (1.2) is a requirement to obtain the existence and uniqueness of solution to the associated linearized system.

1.2. Scope of the article

The present work intends to contribute to the understanding of the control problem for the class of incompressible third-grade fluid equations (1.2) with $\alpha_1 = 0$, whose evolution equations are

$$\begin{cases} \partial_t \boldsymbol{v} - \mu\Delta\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \alpha \mathrm{div}((\mathrm{A}(\boldsymbol{v}))^2) - \beta\mathrm{div}(|\mathrm{A}(\boldsymbol{v})|^2\mathrm{A}(\boldsymbol{v})) = \boldsymbol{f} - \nabla\mathbf{P}; \\ \mathrm{div}\ \boldsymbol{v} = 0. \end{cases} \tag{1.3}$$

The well-posedness of system (1.3) has been addressed in the articles in [20,24] for the square integrable and divergence-free initial data. The authors in [20] also establish the existence of regular solution for divergence-free and $\mathbb{H}^1$-initial data. It is worth to observe that although the term $\mathrm{div}(|\mathrm{A}(\boldsymbol{v})|^2\mathrm{A}(\boldsymbol{v}))$ is nonlinear, it has a good symmetry, which gives an advantage in the study of state equation; however, when dealing with the control problem, this term is a source of difficulties. In fact, the corresponding term in the linearized equation loses the symmetry, and more regularity of the state solution is required to estimate the nonlinear terms in order to establish the well-posedness of linearized system corresponding to (1.3), which consequently impacts the deduction of the first order optimality conditions to this class of fluids.

Aligned with the regularization models available in the literature for the NSE ([31,44]), here we propose a similar regularization to the Navier-Stokes-Voigt equations for third-grade fluids with property $\alpha_1 = 0$, occupying the two-dimensional torus $\mathbb{T}^2$. Accordingly, the following regularization model (1.3) is named third-grade-Voigt (TGV) equations

$$\begin{cases} \partial_t(\boldsymbol{v} - \gamma\Delta\boldsymbol{v}) = \boldsymbol{f} - \nabla\mathbf{P} + \mu\Delta\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \alpha\mathrm{div}((\mathrm{A}(\boldsymbol{v}))^2) & \\ \qquad + \beta\mathrm{div}(|\mathrm{A}(\boldsymbol{v})|^2\mathrm{A}(\boldsymbol{v})) & \text{in } \mathbb{T}^2\times(0,\infty), \\ \mathrm{div}\ \boldsymbol{v} = 0 & \text{in } \mathbb{T}^2\times[0,\infty), \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x) & \text{in } \mathbb{T}^2. \end{cases} \tag{1.4}$$

For $\mathrm{L} > 0$, we identify the two-dimensional torus $\mathbb{T}^2 = \left(\frac{\mathbb{R}}{\mathrm{L}\mathbb{Z}}\right)^2$, with the rectangle $[0,\mathrm{L}]^2$ with periodic boundary conditions (see [17, Chapter 3]), namely $\boldsymbol{v}(\cdot,\cdot)$, $\mathbf{P}(\cdot,\cdot)$ and $\boldsymbol{f}(\cdot,\cdot)$ satisfy the following periodic conditions:

$$\begin{aligned} \boldsymbol{v}(x+\mathrm{L}e_i,\cdot) = \boldsymbol{v}(x,\cdot),\ \mathbf{P}(x+\mathrm{L}e_i,\cdot) = \mathbf{P}(x,\cdot) \text{ and } \boldsymbol{f}(x+\mathrm{L}e_i,\cdot) \\ = \boldsymbol{f}(x,\cdot), \quad x\in\mathbb{R}^2, \quad i\in\{1,2\}, \end{aligned} \tag{1.5}$$

where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$. The constants appearing in system (1.3) γ, μ, α and β are positive constants.

The main goal of this article is to solve a tracking control problem for TGV equations, in the two-dimensional torus. More precisely, we wish to minimize the following cost functional

$$\mathrm{J}(\boldsymbol{f},\boldsymbol{v}) = \frac{1}{2}\int_0^T \|\boldsymbol{v}(t) - \boldsymbol{v}_d(t)\|_2^2\mathrm{d}t + \frac{\lambda}{2}\int_0^T \|\boldsymbol{f}(t)\|_2^2\mathrm{d}t,$$

where $\boldsymbol{v}_d \in \mathrm{L}^2(0,T;\dot{\mathbb{L}}^2(\mathbb{T}^2))$ (see Subsection 2.1 below for the functional spaces) corresponds to a desired target vector field, $\lambda \geq 0$ sets the intensity of the cost, the control acts through the external force $\boldsymbol{f}$, and the vector field $\boldsymbol{v}$ is constrained to satisfy the dynamic equation (1.4). We consider pertinent to mention that in a forthcoming paper, we will propose an approximation scheme to the velocity tracking problem for third-grade flows (1.3), relying on the optimal solution to the TGV equations.

1.3. A brief literature

Comparing the system (1.4) with Navier-Stokes-Voigt equations (see [44]), the constant $\gamma > 0$ known as the relaxation time, sets the time required for a viscoelastic fluid relax from a deformed state back to its equilibrium configuration. So, the reaction of materials that display entire intermediate range of properties between an elastic solid and a viscous fluid, such as polymers, have been modeled in applications using NSV equations. These materials can return to a former state when the shear stress is removed, indicating that they have some memory (see [4]). The incompressible NSE perturbed by the relaxation term $\gamma\partial_t\Delta\boldsymbol{v}$ were also referred to as Kelvin Voigt by the author of [31], who also examined the equations in great detail (see [28]). However, neither Kelvin nor Voigt have proposed a stress-strain relation, or system of governing equations for viscoelastic fluids, as noted in the study [44] (see [23] also). Nevertheless, the most crucial mathematical characteristic of NSV equations, as noted by Ladyzhenskaya (see [26]), is that the term $\gamma\partial_t\Delta\boldsymbol{v}$ works as a regularization of the 3D incompressible NSE, guaranteeing the global solvability of the associated problem (see [30,43,44]). Sharing the same idea, we consider that an important feature of the TGV is the guarantee of solving the velocity tracking control problem.

Let us mention, for instance, that controlling the turbulence within a flow that was governed by or tracked the flows' velocity was the main problem, see [1,21] and its references. Applications for directing a velocity field to a desired velocity field over time are numerous in science and engineering, including combustion, chemical reaction flows, and design difficulties, among others (see for instance [18]). The tracking control challenges for Newtonian flows are well documented. The authors of [18] developed an optimality system for the optimal solutions to the optimal control problem of tracking the velocity for Navier-Stokes flows in bounded two-dimensional domains with bounded distributed controls. In [42], a second-order sufficient optimality requirement was then established. A boundary optimal control for two-dimensional NSE was examined by the authors in [12]. The control acts on the boundary via an injection-suction device. An optimal control issue for two-dimensional NSE with measure-valued controls was examined by the authors in [11]. We would also mention some works [2,25,29,39] which establish the optimal control problem for some different Newtonian and non-Newtonian fluid equations.

Remark 1.1. (1). It is remarkable to mention that one requires $|\alpha| \leq \sqrt{2\mu\beta}$ and $|\alpha| \leq \sqrt{\frac{\mu\beta}{2}}$ for the existence of solution of system (1.3) when initial data belongs to square integrable Lebesgue space and first order Sobolev space, respectively. However, in our analysis, due to regularizing term $\gamma\partial_t\Delta\boldsymbol{v}$, we do not require such kind of extra conditions on parameters.

(2). As already mentioned, for the optimal control problem of tracking the velocity for system (1.2) with $\alpha_1 > 0$, the authors in [38] used $\mathrm{L}^\infty(0,T;\mathbb{H}^3)$ regularity of solution of system (1.2). However, in our analysis, $\mathrm{L}^\infty(0,T;\mathbb{H}^{2+\varepsilon})$ (for some $\varepsilon > 0$) regularity of solution of system (1.4) is enough to track the velocity for system (1.4), see also Remark 3.1 below.

1.4. Organization of of the article

The article is organized as follows: In Section 2, we state appropriate functional spaces and some operators to obtain the abstract formulation of system (1.4). Next, we formulate the control problem and state the main results of this work. The well-posedness of the state equation (1.4) is established in Section 3. The existence and uniqueness of the solution to the linearized state equation are demonstrated in Section 4. We establish a stability result for the state equation in Section 5, which will be a crucial component in Section 6, particularly, it will help to examine the Gateaux differentiability of the control-to-state mapping. We write the adjoint equations and discuss the existence and uniqueness of the solution in Section 7. We establish a duality relation between the adjoint state and the solution of the linearized equation in Section 8. We then determine the first order optimality condition and demonstrate the existence of the control problem solution in the same section. Lastly, the proof of the uniqueness of the solution to the coupled system for a significant cost intensity has been accomplished in Section 9, which deals with the quadratic Lagrangian.

2. Mathematical formulation and preliminary results

The goal of this work is to study the well-posedness and control the state dynamics of deterministic third-grade-Voigt equations.

2.1. Notations and function spaces

The unknowns in system (1.4) are the vector field $\boldsymbol{v}$ and the scalar field $\mathbf{P}$, and $\boldsymbol{f}$ is a given external forcing (which is considered time-dependent):

$$
\begin{aligned}
&\boldsymbol{v}: \mathbb{T}^2\times[0,\infty)\to\mathbb{R}^2,\\
&\mathbf{P}: \mathbb{T}^2\times[0,\infty)\to\mathbb{R},\\
&\boldsymbol{f}: \mathbb{T}^2\times[0,\infty)\to\mathbb{R}^2\\
&(x,t)\mapsto\{v_i(x,t)\}_{i=1}^2,\\
&(x,t)\mapsto\mathbf{P}(x,t),\\
&(x,t)\mapsto\{f_i(x,t)\}_{i=1}^2.
\end{aligned}
$$

Let us considere $m\in\mathbb{N}$ and $p>0$. The usual norms for scalar (resp. vector) valued functions in the classical Lebesgue and Sobolev spaces $\mathrm{L}^p(\mathbb{T}^2)$ (resp. $\mathbb{L}^p(\mathbb{T}^2)$), $\mathrm{W}^{m,p}(\mathbb{T}^2)$ (resp. $\mathbb{W}^{m,p}(\mathbb{T}^2)$) and $\mathrm{H}^m(\mathbb{T}^2)$ (resp. $\mathbb{H}^m(\mathbb{T}^2)$) will be denoted by $\|\cdot\|_p$, $\|\cdot\|_{\mathrm{W}^{m,p}}$ (resp. $\|\cdot\|_{\mathbb{W}^{m,p}}$) and $\|\cdot\|_{\mathrm{H}^m}$ (resp. $\|\cdot\|_{\mathbb{H}^m}$), respectively. For $m=0$, we set $\|\cdot\|_{\mathrm{W}^{m,p}} = \mathrm{L}^p(\mathbb{T}^2)$ (resp. $\|\cdot\|_{\mathbb{W}^{m,p}} = \mathbb{L}^p(\mathbb{T}^2)$).

Let $\dot{\mathrm{C}}^\infty(\mathbb{T}^2;\mathbb{R}^2)$ denote the space of all infinitely differentiable functions defined on $\mathbb{T}^2$ with values in $\mathbb{R}^2$ such that $\int_{\mathbb{T}^2} \boldsymbol{v}(x)\mathrm{d}x = \mathbf{0}$. We notice that the functions in $\dot{\mathrm{C}}^\infty(\mathbb{T}^2;\mathbb{R}^2)$ satisfy the periodic boundary condition $\boldsymbol{v}(x+\mathrm{L}e_i) = \boldsymbol{v}(x)$. The Sobolev space $\dot{\mathbb{H}}^s(\mathbb{T}^2)$, $s\in\mathbb{N}_0 =: \mathbb{N}\cup\{0\}$, is defined as the completion of $\dot{\mathrm{C}}^\infty(\mathbb{T}^2;\mathbb{R}^2)$ with respect to the following Sobolev norm

$$\|\boldsymbol{v}\|_{\mathbb{H}^s} := \left(\sum_{0\leq|\alpha|\leq s} \|\mathrm{D}^\alpha \boldsymbol{v}\|^2_{\mathbb{L}^2(\mathbb{T}^2)}\right)^{1/2}.$$

According to Proposition 5.39, [36], we have

$$\dot{\mathbb{H}}^s(\mathbb{T}^2) = \left\{\boldsymbol{v} : \boldsymbol{v} = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} \boldsymbol{v}_{\boldsymbol{k}}\, e^{2\pi i \boldsymbol{k}\cdot\boldsymbol{x}/\mathrm{L}}, \boldsymbol{v}_0 = \mathbf{0},\ \bar{\boldsymbol{v}}_{\boldsymbol{k}} = \boldsymbol{v}_{-\boldsymbol{k}},\ \|\boldsymbol{v}\|^2_{\dot{\mathbb{H}}^s} := \sum_{\boldsymbol{k}\in\mathbb{Z}^2} |\boldsymbol{k}|^{2s}|\boldsymbol{v}_{\boldsymbol{k}}|^2 < \infty\right\},$$

and from Proposition 5.38, [36], we infer that $\|\cdot\|_{\dot{\mathbb{H}}^s}$ defines a norm on the space $\|\cdot\|_{\dot{\mathbb{H}}^s}$, which is equivalent to the induced standard Sobolev norm $\|\cdot\|_{\mathbb{H}^s}$ on $\dot{\mathbb{H}}^s(\mathbb{T}^2)$. The zero mean condition provides the well-known *Poincaré inequality*,

$$\lambda_1 \int_{\mathbb{T}^2} |\boldsymbol{v}(x)|^2 \mathrm{d}x \leq \int_{\mathbb{T}^2} |\nabla \boldsymbol{v}(x)|^2 \mathrm{d}x, \text{ for any } \boldsymbol{v}\in\dot{\mathbb{H}}^1(\mathbb{T}^2), \tag{2.1}$$

where $\lambda_1 = \frac{4\pi^2}{\mathrm{L}^2}$ (Lemma 5.40, [36]). Let us define $\mathscr{V} := \{\boldsymbol{v}\in\dot{\mathrm{C}}^\infty(\mathbb{T}^2;\mathbb{R}^2) : \nabla\cdot\boldsymbol{v} = 0\}$. We denote by $\mathbb{H}$ the closure of $\mathscr{V}$ in the Lebesgue spaces $\dot{\mathbb{L}}^2(\mathbb{T}^2)$; and by $\mathbb{V}$ the closure of $\mathscr{V}$ in the Sobolev space $\dot{\mathbb{H}}^1(\mathbb{T}^2)$.

The spaces $\mathbb{H}$ and $\mathbb{V}$ are Hilbert spaces with inner products defined by

$$(\boldsymbol{u},\boldsymbol{v}) := \int_{\mathbb{T}^2} \boldsymbol{u}(x)\cdot\boldsymbol{v}(x)\mathrm{d}x = \sum_{i=1}^2 \int_{\mathbb{T}^2} u_i(x)v_i(x)\,\mathrm{d}x, \ \text{ for all } \boldsymbol{u},\boldsymbol{v}\in\mathbb{L}^2(\mathbb{T}^2), \tag{2.2}$$

$$(\boldsymbol{u},\boldsymbol{v})_{\mathbb{V}} := (\nabla\boldsymbol{u},\nabla\boldsymbol{v}), \ \text{ for all } \boldsymbol{u},\boldsymbol{v}\in\mathbb{V}. \tag{2.3}$$

We denote the corresponding norms by $\|\cdot\|_2$ and $\|\cdot\|_{\mathbb{V}}$, respectively.

Let us introduce the Banach space $(\mathbb{X} := \mathbb{W}^{1,4}(\mathbb{T}^2)\cap\mathbb{V}, \|\cdot\|_{\mathbb{X}})$ with $\|\cdot\|_{\mathbb{X}} := \|\cdot\|_{\mathbb{W}^{1,4}} + \|\cdot\|_{\mathbb{V}}$. We recall that $\mathbb{W}^{1,4}(\mathbb{T}^2)$ endowed with the usual norm $\|\cdot\|_{\mathbb{W}^{1,4}}$

$$\|\boldsymbol{w}\|^4_{\mathbb{W}^{1,4}} = \int_{\mathbb{T}^2} |\boldsymbol{w}(x)|^4\mathrm{d}x + \int_{\mathbb{T}^2} |\nabla\boldsymbol{w}(x)|^4\mathrm{d}x$$

is a Banach space.

We represent by $\langle\cdot,\cdot\rangle$ the duality relation between the spaces $\mathbb{V}$ and its dual $\mathbb{V}'$ as well as $\mathbb{X}$ and its dual $\mathbb{X}'$.

Next, let us introduce the scalar product between two matrices $A : B = \mathrm{Tr}(AB^T)$ and denote $|A|^2 := A : A$. The divergence of a matrix $A\in\mathscr{M}_{2\times2}(E)$

is given by $\left\{\operatorname{div}(A)_i\right\}_{i=1}^2 = \left\{\sum_{j=1}^{2} \partial_j a_{ij}\right\}_{i=1}^{2}$. We recall that

$$(A, B) = \int_{\mathbb{T}^2} A(x) : B(x)\, \mathrm{d}x; \qquad \text{for all } A, B \in \mathscr{M}_{2\times 2}(\mathbb{L}^2(\mathbb{T}^2)).$$

Throughout the article, we denote by C generic constant, which may vary from line to line.

2.2. Linear and nonlinear operators

Let $\mathbb{P} : \dot{\mathbb{L}}^2(\mathbb{T}^2) \to \mathbb{H}$ be the Helmholtz-Hodge (or Leray) orthogonal projection (cf. [3]). We define the Stokes operator

$$\mathscr{A}\boldsymbol{u} := -\mathbb{P}\Delta\boldsymbol{u},\ \boldsymbol{u} \in \mathrm{D}(\mathscr{A}),$$

where $\mathrm{D}(\mathscr{A}) = \left\{\boldsymbol{u} \in \dot{\mathbb{H}}^2(\mathbb{T}^2) : \nabla\cdot\boldsymbol{u} = 0\right\}$. It should be noted that $\mathbb{P}$ and Δ commutes in a torus, see [37, Lemma 2.9]. Given $\boldsymbol{u} \in \mathrm{D}(\mathscr{A})$ with Fourier expansion $\boldsymbol{u} = \sum\limits_{\boldsymbol{k}\in\mathbb{Z}^2} e^{2\pi i\boldsymbol{k}\cdot\boldsymbol{x}/\mathrm{L}}\boldsymbol{u}_{\boldsymbol{k}}$, one obtains

$$-\Delta\boldsymbol{u} = \frac{4\pi^2}{\mathrm{L}^2}\sum_{\boldsymbol{k}\in\mathbb{Z}^2} e^{2\pi i\boldsymbol{k}\cdot\boldsymbol{x}/\mathrm{L}}|\boldsymbol{k}|^2\boldsymbol{u}_{\boldsymbol{k}}.$$

Since $\mathscr{A}^{-1}$ is a compact self-adjoint operator in $\mathbb{H}$, there exists an orthonormal basis in $\mathbb{H}$ of eigenfunctions $\{\boldsymbol{w}_i\}_{i\in\mathbb{N}} \subset \dot{\mathrm{C}}^\infty(\mathbb{T}^2;\mathbb{R}^2)$ of the Stokes operator; namely we have $\mathscr{A}\boldsymbol{w}_i = \lambda_i\boldsymbol{w}_i$, for $i = 1, 2, \ldots$, where the eigenvalues of $\mathscr{A}$ verify $0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$. Note that $\lambda_1 = \frac{4\pi^2}{\mathrm{L}^2}$ is the smallest eigenvalue of $\mathscr{A}$ appearing in the Poincaré inequality (2.1).

The following relation holds

$$\lambda_i(\boldsymbol{w}_i, \boldsymbol{v}) = (\mathscr{A}\boldsymbol{w}_i, \boldsymbol{v}) = (-\Delta\boldsymbol{w}_i, \boldsymbol{v}) = (\nabla\boldsymbol{w}_i, \nabla\boldsymbol{v}) = (\boldsymbol{w}_i, \boldsymbol{v})_{\mathbb{V}}, \qquad \text{for all } \boldsymbol{v} \in \mathbb{V}. \tag{2.4}$$

Hence $\{\boldsymbol{w}_i\}_{i\in\mathbb{N}}$ is an orthogonal basis in $\mathbb{V}$.

We recall that the operators $\mathscr{A}^\lambda$, $\lambda \in \mathbb{R}$, are well defined and

$$\mathrm{D}(\mathscr{A}^{s/2}) = \left\{\boldsymbol{u} \in \dot{\mathbb{H}}^s(\mathbb{T}^2) : \nabla\cdot\boldsymbol{u} = 0\right\}.$$

In addition it can be shown that there exists a positive constant C such that $\|\mathscr{A}^{s/2}\boldsymbol{u}\|_2 = C\|\boldsymbol{u}\|_{\dot{\mathbb{H}}^s}$, for all $\boldsymbol{u} \in \mathrm{D}(\mathscr{A}^{s/2})$, $s \geq 0$ (see [36, Chapter 6]). Note that the operator $\mathscr{A}$ is a non-negative self-adjoint operator in $\mathbb{H}$ with a compact resolvent and

$$|(\mathscr{A}\boldsymbol{u}, \boldsymbol{v})| \leq C\|\boldsymbol{u}\|_{\mathbb{V}}\|\boldsymbol{v}\|_{\mathbb{V}}, \quad \text{for all } \ \boldsymbol{u} \in \mathrm{D}(\mathscr{A}),\ \boldsymbol{v} \in \mathbb{V}, \ \text{then } \ \|\mathscr{A}\boldsymbol{u}\|_{\mathbb{V}'} \leq \|\boldsymbol{u}\|_{\mathbb{V}}. \tag{2.5}$$

Therefore, there exists a unique extension of $\mathscr{A}$ denoted by the same symbol verifying

$$\mathscr{A} : \mathbb{V} \to \mathbb{V}', \qquad \langle\mathscr{A}\boldsymbol{u}, \boldsymbol{v}\rangle := (\nabla\boldsymbol{u}, \nabla\boldsymbol{v}), \qquad \forall\boldsymbol{u},\ \boldsymbol{v} \in \mathbb{V}.$$

Next, we define the trilinear form $b(\cdot,\cdot,\cdot):\mathbb{V}\times\mathbb{V}\times\mathbb{V}\to\mathbb{R}$ by

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=\int_{\mathbb{T}^2}(\boldsymbol{u}(x)\cdot\nabla)\boldsymbol{v}(x)\cdot\boldsymbol{w}(x)\mathrm{d}x=\sum_{i,j=1}^{2}\int_{\mathbb{T}^2}u_i(x)\frac{\partial v_j(x)}{\partial x_i}w_j(x)\mathrm{d}x.$$

If $\boldsymbol{u},\boldsymbol{v}$ are such that the linear map $b(\boldsymbol{u},\boldsymbol{v},\cdot)$ is continuous on $\mathbb{V}$, the corresponding element of $\mathbb{V}'$ is denoted by $\mathscr{B}(\boldsymbol{u},\boldsymbol{v})$. We represent $\mathscr{B}(\boldsymbol{v})=\mathscr{B}(\boldsymbol{v},\boldsymbol{v})=\mathbb{P}(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$. Using an integration by parts, it is immediate that

$$\begin{cases} b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v})=0, \quad \text{for all } \ \boldsymbol{u},\boldsymbol{v}\in\mathbb{V},\\ b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=-b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}), \quad \text{for all } \ \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathbb{V}.\end{cases} \tag{2.6}$$

For $\boldsymbol{u},\boldsymbol{v}\in\mathbb{V}$, we have

$$\begin{aligned}|\langle\mathscr{B}(\boldsymbol{u})-\mathscr{B}(\boldsymbol{v}),\boldsymbol{w}\rangle|&\leq|b(\boldsymbol{u},\boldsymbol{u}-\boldsymbol{v},\boldsymbol{w})|+|b(\boldsymbol{u}-\boldsymbol{v},\boldsymbol{v},\boldsymbol{w})|\leq\big[\|\boldsymbol{u}\|_4\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|_2\\ &\quad+\|\boldsymbol{u}-\boldsymbol{v}\|_4\|\nabla\boldsymbol{v}\|_2\big]\|\boldsymbol{w}\|_4\end{aligned} \tag{2.7}$$

for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathbb{V}$. Thus the operator $\mathscr{B}(\cdot):\mathbb{V}\to\mathbb{V}'$ is locally Lipschitz. In addition, the operator $\mathscr{B}$ enjoys the following important orthogonality property (see [41, Lemma 3.1, p. 404]):

$$(\mathscr{B}(\boldsymbol{v},\boldsymbol{v}),\mathscr{A}\boldsymbol{v})=b(\boldsymbol{v},\boldsymbol{v},\mathscr{A}\boldsymbol{v})=0, \text{ for any } \boldsymbol{v}\in\mathrm{D}(\mathscr{A}). \tag{2.8}$$

Let us now define the operator $\mathscr{J}(\boldsymbol{v}):=-\mathbb{P}\mathrm{div}(\mathrm{A}(\boldsymbol{v})\mathrm{A}(\boldsymbol{v}))$. Note that for $\boldsymbol{v}\in\mathbb{X}$, we have

$$\|\mathscr{J}(\boldsymbol{v})\|_{\mathbb{X}'}\leq C\|\boldsymbol{v}\|_{\mathbb{X}}^2,$$

and hence the map $\mathscr{J}(\cdot):\mathbb{X}\to\mathbb{X}'$. For $\boldsymbol{u},\boldsymbol{v}\in\mathbb{X}$, we have

$$\begin{aligned}|\langle\mathscr{J}(\boldsymbol{u})-\mathscr{J}(\boldsymbol{v}),\boldsymbol{w}\rangle|&=\bigg|\frac{1}{2}\int_{\mathbb{T}^2}[\mathrm{A}(\boldsymbol{u}-\boldsymbol{v})\mathrm{A}(\boldsymbol{u})+\mathrm{A}(\boldsymbol{v})\mathrm{A}(\boldsymbol{u}-\boldsymbol{v})]:\\ &\qquad\mathrm{A}(\boldsymbol{w})\mathrm{d}x\bigg|\leq C\big[\|\mathrm{A}(\boldsymbol{u})\|_4+\|\mathrm{A}(\boldsymbol{v})\|_4\big]\|\mathrm{A}(\boldsymbol{u}-\boldsymbol{v})\|_4\|\mathrm{A}(\boldsymbol{w})\|_2,\end{aligned} \tag{2.9}$$

for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathbb{X}$. Thus the operator $\mathscr{J}(\cdot):\mathbb{X}\to\mathbb{X}'$ is locally Lipschitz. In addition, due to divergence-free condition, we obtain $\mathrm{Tr}([\mathrm{A}(\boldsymbol{v})]^3)=0$, for any $\boldsymbol{v}\in\mathbb{X}$. Therefore, we have

$$\langle\mathscr{J}(\boldsymbol{v}),\boldsymbol{v}\rangle=\frac{1}{2}\int_{\mathbb{T}^2}\mathrm{Tr}([\mathrm{A}(\boldsymbol{v}(x))]^3)\mathrm{d}x=0, \text{ for any } \boldsymbol{v}\in\mathbb{X}.$$

Finally, we define the operator $\mathscr{K}(\boldsymbol{v}):=-\mathbb{P}\mathrm{div}(|\mathrm{A}(\boldsymbol{v})|^2\mathrm{A}(\boldsymbol{v}))$. It is immediate that

$$\langle\mathscr{K}(\boldsymbol{v}),\boldsymbol{v}\rangle=\frac{1}{2}\|\mathrm{A}(\boldsymbol{v})\|_4^4.$$

Note that for $\boldsymbol{v}\in\mathbb{X}$, we have

$$\|\mathscr{K}(\boldsymbol{v})\|_{\mathbb{X}'}\leq C\|\boldsymbol{v}\|_{\mathbb{X}}^3,$$

and hence the map $\mathscr{K}(\cdot):\mathbb{X}\to\mathbb{X}'$. For $\boldsymbol{u},\boldsymbol{v}\in\mathbb{X}$, we infer

$$\begin{aligned}|\langle\mathscr{K}(\boldsymbol{u})-\mathscr{K}(\boldsymbol{v}),\boldsymbol{w}\rangle|&=\bigg|\frac{1}{2}\int_{\mathbb{T}^2}[\mathrm{A}(\boldsymbol{u}-\boldsymbol{v}):\mathrm{A}(\boldsymbol{u})+\mathrm{A}(\boldsymbol{v}):\mathrm{A}(\boldsymbol{u}-\boldsymbol{v})]\mathrm{A}(\boldsymbol{u}):\mathrm{A}(\boldsymbol{w})\mathrm{d}x\\ &\qquad+\frac{1}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v})|^2\mathrm{A}(\boldsymbol{u}-\boldsymbol{v}):\mathrm{A}(\boldsymbol{w})\mathrm{d}x\bigg|\end{aligned}$$

$$\leq C\big[\|\mathrm{A}(\boldsymbol{u})\|_4^2+\|\mathrm{A}(\boldsymbol{v})\|_4^2\big]\|\mathrm{A}(\boldsymbol{u}-\boldsymbol{v})\|_4\|\mathrm{A}(\boldsymbol{w})\|_4, \quad (2.10)$$

for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathbb{X}$. Thus the operator $\mathscr{K}(\cdot):\mathbb{X}\to\mathbb{X}'$ is locally Lipschitz.

Let us complete this subsection by recalling the following continuous Sobolev embeddings from [7, Theorem B] (see also [6, Theorem 2.20, p.44]) which will be used frequently in the sequel:

(1) For any given $0<\varepsilon<1$, we have

$$\mathbb{W}^{2+\varepsilon,2}(\mathbb{T}^2)\hookrightarrow\mathbb{W}^{2,\frac{2}{1-\varepsilon}}(\mathbb{T}^2). \quad (2.11)$$

(2) For any given $\delta>0$, we have

$$\mathbb{W}^{2+\delta,2}(\mathbb{T}^2)\hookrightarrow\mathbb{W}^{1,\infty}(\mathbb{T}^2),\quad \text{that is,}\quad \|\boldsymbol{u}\|_{\mathbb{W}^{1,\infty}}\leq S^\delta\|\boldsymbol{u}\|_{\mathbb{W}^{2+\delta,2}}, \quad (2.12)$$

for $\boldsymbol{u}\in\mathbb{W}^{2+\delta,2}(\mathbb{T}^2)$, where S^δ is a positive constant.

(3) For any given $1\leq p<\infty$, we have

$$\mathbb{W}^{2,2}(\mathbb{T}^2)\hookrightarrow\mathbb{W}^{1,p}(\mathbb{T}^2),\quad \text{that is,}\quad \|\boldsymbol{u}\|_{\mathbb{W}^{1,p}}\leq S_p\|\boldsymbol{u}\|_{\mathbb{W}^{2,2}}, \quad (2.13)$$

for $\boldsymbol{u}\in\mathbb{W}^{2,2}(\mathbb{T}^2)$, where S_p is a positive constant.

2.3. Control problem

Let us denote time $\mathscr{T}$ such that the unique solution $\boldsymbol{v}$ to the state equation (1.4) in the sense of Definition 3.1 below belongs to $\mathrm{L}^\infty(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))$ (for some $\varepsilon>0$).

Our main goal is to control the solution of the system (1.4) by a distributed force $\boldsymbol{f}$. The control variables $\boldsymbol{f}$ belong to the set $\mathscr{F}_{ad}$ of admissible controls, which is defined as a nonempty bounded closed convex subset of $\mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{H}}^\varepsilon(\mathbb{T}^2))$. In other words

$$\mathscr{F}_{ad}:=\{\boldsymbol{f}\in\mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{H}}^\varepsilon(\mathbb{T}^2)):\ \|\boldsymbol{f}\|_{\mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{H}}^\varepsilon(\mathbb{T}^2))}\leq K\}, \quad (2.14)$$

where, one can consider $0<K<+\infty$ for $\varepsilon=1$ as we have global solution (see Theorem 3.3) whereas for $0<\varepsilon<1$, one has to consider $K>0$ such that any $\boldsymbol{f}\in\mathscr{F}_{ad}$ and $\boldsymbol{v}_0\in\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}})$ satisfy (3.50) with $t=\mathscr{T}$.

Remark 2.1. Note that, for $0<\varepsilon<1$, $\mathscr{T}\leq T^*$ when $\boldsymbol{v}_0\in\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}})$ and $\boldsymbol{f}\in\mathscr{F}_{ad}$, where $T^*=T^*(\varepsilon,\boldsymbol{v}_0,K)$ is the time obtained in Theorem 3.2 and is independent of $\boldsymbol{f}\in\mathscr{F}_{ad}$. Moreover, $\mathscr{T}$ can be chosen any arbitrary time when $\boldsymbol{v}_0\in\mathrm{D}(\mathscr{A}^{\frac{3}{2}})$ and $\boldsymbol{f}\in\mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{H}}^1(\mathbb{T}^2))$, see Theorem 3.3.

We consider the cost functional $\mathrm{J}:\mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{H}}^\varepsilon(\mathbb{T}^2))\times\mathrm{L}^\infty(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))\to\mathbb{R}^+$ given by

$$\mathrm{J}(\boldsymbol{f},\boldsymbol{v})=\frac{1}{2}\int_0^{\mathscr{T}}\|\boldsymbol{v}(t)-\boldsymbol{v}_d(t)\|_2^2\mathrm{d}t+\frac{\lambda}{2}\int_0^{\mathscr{T}}\|\boldsymbol{f}(t)\|_2^2\mathrm{d}t, \quad (2.15)$$

where $\boldsymbol{v}_d\in\mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{L}}^2(\mathbb{T}^2))$ corresponds to a desired target field and any $\lambda\geq0$. The control problem reads

$$\min_{\boldsymbol{f}\in\mathscr{F}_{ad}}\Big\{\mathrm{J}(\boldsymbol{f},\boldsymbol{v})\ :\ \boldsymbol{v}\text{ is the solution of (1.4) with force }\boldsymbol{f}.\Big\}. \quad (2.16)$$

Remark 2.2. In this article, the Lagrangian $\mathfrak{L}$ is given by

$$\mathfrak{L}(\cdot,\boldsymbol{f},\boldsymbol{v})=\frac{1}{2}\|\boldsymbol{v}-\boldsymbol{v}_d\|_2^2+\frac{\lambda}{2}\|\boldsymbol{f}\|_2^2.$$

Therefore, we have

$$\int_0^{\mathscr{T}}(\nabla_{\boldsymbol{v}}\mathfrak{L}(t,\boldsymbol{f}(t),\boldsymbol{v}(t)),\boldsymbol{y}(t))\mathrm{d}t=\int_0^{\mathscr{T}}(\boldsymbol{y}(t),\boldsymbol{v}(t)-\boldsymbol{v}_d(t))\mathrm{d}t,$$
$$\int_0^{\mathscr{T}}(\nabla_{\boldsymbol{f}}\mathfrak{L}(t,\boldsymbol{f}(t),\boldsymbol{v}(t)),\boldsymbol{g}(t))\mathrm{d}t=\lambda\int_0^{\mathscr{T}}(\boldsymbol{g}(t),\boldsymbol{f}(t))\mathrm{d}t,$$

for any $\boldsymbol{y}\in\mathrm{L}^\infty(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))$ and $\boldsymbol{g}\in\mathscr{F}_{ad}$.

2.4. Main results

Our first main result shows the existence of a solution to the control problem, and establishes the first order optimality conditions.

Theorem 2.1. *Assume that* $\boldsymbol{v}_0\in\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}})$ *for some* $\varepsilon>0$. *Then the control problem (2.16) admits, at least, one optimal solution*

$$(\widetilde{\boldsymbol{f}},\widetilde{\boldsymbol{v}})\in\mathscr{F}_{ad}\times\big(\mathrm{L}^\infty(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))\cap\mathrm{H}^1(0,\mathscr{T};\mathbb{V})\big),\tag{2.17}$$

where $\widetilde{\boldsymbol{v}}$ *is the unique solution of (3.1) with* $\boldsymbol{f}=\widetilde{\boldsymbol{f}}$. *Moreover, there exists a unique solution* $\boldsymbol{p}$ *of (7.1) with* $\boldsymbol{g}=\nabla_{\boldsymbol{v}}\mathfrak{L}(\cdot,\widetilde{\boldsymbol{f}},\widetilde{\boldsymbol{v}})$, *such that if* $\widetilde{\boldsymbol{z}}$ *is the solution of (4.1) for* $\boldsymbol{v}=\widetilde{\boldsymbol{v}}$ *and* $\boldsymbol{\psi}=\boldsymbol{\psi}-\widetilde{\boldsymbol{f}}$, *the following duality property*

$$\int_0^{\mathscr{T}}(\boldsymbol{\psi}(t)-\widetilde{\boldsymbol{f}}(t),\widetilde{\boldsymbol{p}}(t))\mathrm{d}t=\int_0^{\mathscr{T}}(\nabla_{\boldsymbol{v}}\mathfrak{L}(t,\widetilde{\boldsymbol{f}}(t),\widetilde{\boldsymbol{v}}(t)),\widetilde{\boldsymbol{z}}(t))\mathrm{d}t,\tag{2.18}$$

and the following optimality condition hold

$$\int_0^{\mathscr{T}}(\boldsymbol{\psi}(t)-\widetilde{\boldsymbol{f}}(t),\widetilde{\boldsymbol{p}}(t)+\nabla_{\boldsymbol{f}}\mathfrak{L}(t,\widetilde{\boldsymbol{f}}(t),\widetilde{\boldsymbol{v}}(t)))\mathrm{d}t\geq 0.\tag{2.19}$$

An additional step in the study of the control problem relies on the analysis of the solutions of the coupled system constituted by the state equation (1.4), the adjoint equation (7.1) and the optimality relation (2.19). Our next result goes in this direction and establishes an uniqueness result for the solutions of the coupled system for (2.16).

Theorem 2.2. *Let* $\lambda>2\Gamma_2\widetilde{\lambda}\left[\Gamma+\frac{4}{\mu}|\alpha|[S_4]^2\Gamma_1+\frac{112\beta}{\mu}[S_4]^2S^\varepsilon\kappa\Gamma_1\right]$, *where* Γ_1, $\Gamma_2,\widetilde{\lambda},\Gamma,\mu,\alpha,\beta,S_4,S^\varepsilon$ *and* κ *are given in (5.3), (5.4), (9.3), (9.4), (1.4), (1.4), (1.4), (2.13) (for* $p=4$*), (2.12) (for* $\delta=\varepsilon$*) and (9.2), respectively. Then, the optimal control problem (2.16) has a unique solution.*

3. Existence and uniqueness of global solutions of state equation

In this section, we establish the existence of unique global solution to system (1.4). Let us write the abstract formulation to system (1.4) by taking the projection $\mathbb{P}$ as:

$$\begin{cases}\partial_t\left(\boldsymbol{v}(t)+\gamma\mathscr{A}\boldsymbol{v}(t)\right)+\mu\mathscr{A}\boldsymbol{v}(t)+\mathscr{B}(\boldsymbol{v}(t))+\alpha\mathscr{J}(\boldsymbol{v}(t))+\beta\mathscr{K}(\boldsymbol{v}(t))=\mathbb{P}\boldsymbol{f}(t),\ a.e.t\in(0,\infty),\\ \boldsymbol{v}(0)=\boldsymbol{v}_0,\end{cases}\tag{3.1}$$

where $\boldsymbol{f} \in \mathrm{L}^2(0,T;\dot{\mathbb{L}}^2(\mathbb{T}^2))$ and $\boldsymbol{v}_0 \in \mathrm{D}(\mathscr{A})$. We establish the existence of unique global solution to system (3.1) in the following sense:

Definition 3.1. A function $\boldsymbol{v} \in \mathrm{C}([0,T];\mathbb{V}) \cap \mathrm{L}^\infty(0,T;\mathrm{D}(\mathscr{A}))$ with $\partial_t \boldsymbol{v} \in \mathrm{L}^2(0,T;\mathbb{V})$, is called a *solution* to system (3.1), if for $\boldsymbol{f} \in \mathrm{L}^2(0,T;\dot{\mathbb{L}}^2(\mathbb{T}^2))$ and $\boldsymbol{v}_0 \in \mathrm{D}(\mathscr{A})$, it satisfies

(i) for any $\boldsymbol{\phi} \in \mathbb{V}$,

$$\langle \partial_t[\mathrm{I}+\gamma\mathscr{A}]\boldsymbol{v}(t), \boldsymbol{\phi}\rangle = -\left\langle \mu\mathscr{A}\boldsymbol{v}(t) + \mathscr{B}(\boldsymbol{v}(t)) + \alpha\mathscr{J}(\boldsymbol{v}(t)) + \beta\mathscr{K}(\boldsymbol{v}(t)) - \boldsymbol{f}(t), \boldsymbol{\phi}\right\rangle, \tag{3.2}$$

for a.e. $t \in [0,T]$;

(ii) the initial data:

$$\boldsymbol{v}(0) = \boldsymbol{v}_0 \ \text{ in } \ \mathrm{D}(\mathscr{A}).$$

Next theorem is the main result of this section which provides the existence and uniqueness of global weak solutions to system (3.1) satisfying energy equality.

Theorem 3.1. *For $T>0$, $\boldsymbol{v}_0 \in \mathrm{D}(\mathscr{A})$ and $\boldsymbol{f} \in \mathrm{L}^2(0,T;\dot{\mathbb{L}}^2(\mathbb{T}^2))$, there exists a unique solution $\boldsymbol{v}$ to system (3.1) in the sense of Definition 3.1 which satisfies the following energy equality:*

$$\|\boldsymbol{v}(t)\|_2^2 + \gamma\|\nabla\boldsymbol{v}(t)\|_2^2 + 2\mu\int_0^t \|\nabla\boldsymbol{v}(s)\|_2^2\mathrm{d}s + \beta\int_0^t \|\mathrm{A}(\boldsymbol{v}(s))\|_4^4\mathrm{d}s = \|\boldsymbol{v}_0\|_2^2 + \gamma\|\nabla\boldsymbol{v}_0\|_2^2 + 2\int_0^t (\boldsymbol{f}(s),\boldsymbol{v}(s))\mathrm{d}s, \tag{3.3}$$

for all $t \in [0,T]$.

Proof of Theorem 3.1. Let us choose and fix $T>0$. The proof is divided into the following four steps.

Step I. *Finite-dimensional approximation.* Here, we consider the basis $\{\boldsymbol{w}_i\}_{i\in\mathbb{N}}$ of eigenfunctions of the Stokes operator, which is orthonormal in $\mathbb{H}$ and orthogonal in $\mathbb{V}$, and set $\mathbb{H}_n := \operatorname{span}\{\boldsymbol{w}_j : j=1,\ldots,n\}$, $n \in \mathbb{N}$. Let us denote by P_n the orthogonal projection of $\mathbb{V}'$ to $\mathbb{H}_n$, that is, for every $\boldsymbol{x} \in \mathbb{V}'$, we write $\mathrm{P}_n\boldsymbol{x} = \sum\limits_{j=1}^n \langle \boldsymbol{x}, \boldsymbol{w}_j\rangle \boldsymbol{w}_j$. Since every element $\boldsymbol{x} \in \mathbb{H}$ induces a functional $\boldsymbol{x}^* \in \mathbb{V}'$ by the formula $\langle \boldsymbol{x}^*, \boldsymbol{y}\rangle = (\boldsymbol{x},\boldsymbol{y})$, $\boldsymbol{y} \in \mathbb{V}$, then $\mathrm{P}_n|_{\mathbb{H}}$ corresponds to the orthogonal projection from $\mathbb{H}$ onto $\mathbb{H}_n$. Hence, for all $\boldsymbol{x} \in \mathbb{H}$, $\mathrm{P}_n\boldsymbol{x} = \sum\limits_{i=1}^n (\boldsymbol{x},\boldsymbol{w}_i)\,\boldsymbol{w}_i$ and $\|\mathrm{P}_n\boldsymbol{x} - \boldsymbol{x}\|_2 \to 0$ as $n \to \infty$. Due to relation (2.4), for all $x \in \mathbb{V}$, $\mathrm{P}_n\boldsymbol{x}$ coincides with the orthogonal projection with respect to the inner product $(\cdot,\cdot)_{\mathbb{V}}$ in $\mathbb{V}$, then we also have $\|\mathrm{P}_n\boldsymbol{x} - \boldsymbol{x}\|_{\mathbb{V}} \to 0$ as $n \to \infty$. Let us define $\mathscr{A}_n\cdot := \mathrm{P}_n\mathscr{A}\cdot$, $\mathscr{B}_n(\cdot) := \mathrm{P}_n\mathscr{B}(\cdot)$, $\mathscr{J}_n(\cdot) := \mathrm{P}_n\mathscr{J}(\cdot)$, $\mathscr{K}_n(\cdot) := \mathrm{P}_n\mathscr{K}(\cdot)$ and $\boldsymbol{f}_n := \mathrm{P}_n[\mathbb{P}\boldsymbol{f}]$. For each $n \in \mathbb{N}$, we search for a approximate solution of the form

$$\boldsymbol{v}^n(x,t) := \sum_{k=1}^n g_k^n(t)\boldsymbol{w}_k(x),$$

where $g_1^n(t),\dots,g_k^n(t)$ are unknown scalar functions of t such that it satisfies the following finite-dimensional system of ordinary differential equations in $\mathbb{H}_n$:

$$\left\{\begin{aligned}\partial_t\boldsymbol{v}^n(t)+\gamma\partial_t\mathscr{A}\boldsymbol{v}^n(t)&=-\mu\mathscr{A}_n\boldsymbol{v}^n(t)-\mathscr{B}_n(\boldsymbol{v}^n(t))-\alpha\mathscr{J}_n(\boldsymbol{v}^n(t))-\beta\mathscr{K}_n(\boldsymbol{v}^n(t))+\boldsymbol{f}_n(t),\\ \boldsymbol{v}^n(0)&=\boldsymbol{v}_{0n},\end{aligned}\right. \tag{3.4}$$

for a.e. $t\in[0,T]$, where $\boldsymbol{v}_{0n}=\mathrm{P}_n[\boldsymbol{v}_0]$. Since $\mathscr{G}_n(\cdot):=\mu\mathscr{A}_n\cdot+\mathscr{B}_n(\cdot)+\alpha\mathscr{J}_n(\cdot)+\beta\mathscr{K}_n(\cdot)$ is locally Lipschitz (see (2.5), (2.7), (2.9) and (2.10) above), therefore, using the Carathéodory's existence theorem, there exists a local maximal solution $\boldsymbol{v}^n\in\mathrm{C}([0,T^*];\mathbb{H}_n)$, for some $0<T^*\le T$ of the system (3.4) and uniqueness is immediate from the local Lipschitz property. The time T^* can be extended to T by establishing the uniform energy estimates of the solutions satisfied by the system (3.4).

Step II. *A priori estimates.*

Taking the inner product with $\boldsymbol{v}^n$ in $(3.4)_1$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\boldsymbol{v}^n(t)\|_2^2+\gamma\|\nabla\boldsymbol{v}^n(t)\|_2^2\right]=-\mu\|\nabla\boldsymbol{v}^n(t)\|_2^2-\frac{\beta}{2}\|\mathrm{A}(\boldsymbol{v}^n(t))\|_4^4+(\boldsymbol{f}(t),\boldsymbol{v}^n(t)), \tag{3.5}$$

for a.e. $t\in[0,T]$. Next, using Hölder's and Young's inequalities, we estimate

$$\left|(\boldsymbol{f},\boldsymbol{v}^n)\right|\le\|\boldsymbol{f}\|_2\|\boldsymbol{v}^n\|_2\le C\|\boldsymbol{f}\|_2\|\nabla\boldsymbol{v}^n\|_2\le C\|\boldsymbol{f}\|_2^2+\frac{\mu}{2}\|\nabla\boldsymbol{v}^n\|_2^2. \tag{3.6}$$

Combining (3.5)-(3.6), we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\boldsymbol{v}^n(t)\|_2^2+\gamma\|\nabla\boldsymbol{v}^n(t)\|_2^2\right]+\mu\|\nabla\boldsymbol{v}^n(t)\|_2^2+\beta\|\mathrm{A}(\boldsymbol{v}^n(t))\|_4^4\le C\|\boldsymbol{f}(t)\|_2^2, \tag{3.7}$$

for a.e. $t\in[0,T]$, Hence we can write for all $t\in[0,T]$

$$\begin{aligned}&\|\boldsymbol{v}^n(t)\|_2^2+\gamma\|\nabla\boldsymbol{v}^n(t)\|_2^2+\mu\int_0^t\|\nabla\boldsymbol{v}^n(s)\|_2^2\mathrm{d}s+\frac{\beta}{2}\int_0^t\|\mathrm{A}(\boldsymbol{v}^n(s))\|_4^4\mathrm{d}s\\ &\le\|\boldsymbol{v}^n(0)\|_2^2+\gamma\|\nabla\boldsymbol{v}^n(0)\|_2^2+C\int_0^t\|\boldsymbol{f}(s)\|_2^2\mathrm{d}s\\ &\le\|\boldsymbol{v}(0)\|_2^2+\gamma\|\nabla\boldsymbol{v}(0)\|_2^2+C\int_0^T\|\boldsymbol{f}(s)\|_2^2\mathrm{d}s:=K_1(T).\end{aligned} \tag{3.8}$$

Using the fact that $\boldsymbol{f}\in\mathrm{L}^2(0,T;\dot{\mathbb{L}}^2(\mathbb{T}^2))$, we have from (3.8) that

$$\{\boldsymbol{v}^n\}_{n\in\mathbb{N}}\text{ is a bounded sequence in }\mathrm{L}^\infty(0,T;\mathbb{V})\cap\mathrm{L}^4(0,T;\mathbb{W}^{1,4}(\mathbb{T}^2)). \tag{3.9}$$

Next, taking the inner product with $\mathscr{A}\boldsymbol{v}^n$ in $(3.4)_1$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\nabla\boldsymbol{v}^n(t)\|_2^2+\gamma\|\mathscr{A}\boldsymbol{v}^n(t)\|_2^2\right]\\ &=-\mu\|\mathscr{A}\boldsymbol{v}^n(t)\|_2^2-\underbrace{b(\boldsymbol{v}^n(t),\boldsymbol{v}^n(t),\mathscr{A}\boldsymbol{v}^n(t))}_{=0\text{ by }(2.8).}-\alpha\left\langle\mathrm{div}[(\mathrm{A}(\boldsymbol{v}^n(t)))^2],\mathscr{A}\boldsymbol{v}^n(t)\right\rangle\end{aligned}$$

$$-\beta\left\langle \mathrm{div}|\mathrm{A}(\boldsymbol{v}^n(t))|^2\mathrm{A}(\boldsymbol{v}^n(t)), \mathscr{A}\boldsymbol{v}^n(t)\right\rangle + (\boldsymbol{f}(t), \mathscr{A}\boldsymbol{v}^n(t)), \tag{3.10}$$

for a.e. $t \in [0,T]$. Let us denote $[\mathrm{A}(\boldsymbol{v})]_{ij} := a_{ij}$ which will be used later. Next, using integration by parts, and Hölder's and Young's inequalities, we estimate the terms of the right hand side of (3.10) as (see [9] for more details)

$$\begin{aligned}
&\left|\alpha\left\langle \mathrm{div}[(\mathrm{A}(\boldsymbol{v}^n))^2], \mathscr{A}\boldsymbol{v}^n\right\rangle\right| \leq |\alpha|\|\mathrm{div}[(\mathrm{A}(\boldsymbol{v}^n)^2]\|_2\|\mathscr{A}\boldsymbol{v}^n\|_2\\
&\leq \frac{\beta}{16}\|\mathrm{div}[(\mathrm{A}(\boldsymbol{v}^n)^2]\|_2^2 + \frac{4\alpha^2}{\beta}\|\mathscr{A}\boldsymbol{v}^n\|_2^2\\
&= \frac{\beta}{16}\int_{\mathbb{T}^2}|\mathrm{div}[(\mathrm{A}(\boldsymbol{v}^n(x))^2|^2 + \frac{4\alpha^2}{\beta}\|\mathscr{A}\boldsymbol{v}^n\|_2^2\\
&\leq \frac{\beta}{16}\sum_{m=1}^2\int_{\mathbb{T}^2}|\partial_m\mathrm{A}(\boldsymbol{v}^n(x))\mathrm{A}(\boldsymbol{v}^n(x)) + \mathrm{A}(\boldsymbol{v}^n(x))\partial_m\mathrm{A}(\boldsymbol{v}^n(x))|^2\mathrm{d}x + \frac{4\alpha^2}{\beta}\|\mathscr{A}\boldsymbol{v}^n\|_2^2\\
&\leq \frac{\beta}{8}\sum_{m=1}^2\left[\int_{\mathbb{T}^2}|\partial_m\mathrm{A}(\boldsymbol{v}^n(x))\mathrm{A}(\boldsymbol{v}^n(x))|^2 + \int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}^n(x))\partial_m\mathrm{A}(\boldsymbol{v}^n(x))|^2\right]\mathrm{d}x\\
&\quad + \frac{4\alpha^2}{\beta}\|\mathscr{A}\boldsymbol{v}^n\|_2^2\\
&\leq \frac{\beta}{4}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}^n(x))|^2|\nabla\mathrm{A}(\boldsymbol{v}^n(x))|^2\mathrm{d}x + \frac{4\alpha^2}{\beta}\|\mathscr{A}\boldsymbol{v}^n\|_2^2,
\end{aligned}\tag{3.11}$$

$$\begin{aligned}
&-\beta\left\langle \mathrm{div}\left[|\mathrm{A}(\boldsymbol{v}^n)|^2\mathrm{A}(\boldsymbol{v}^n)\right], \mathscr{A}\boldsymbol{v}^n\right\rangle\\
&= -\frac{\beta}{2}\sum_{m=1}^2\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}^n(x))|^2[\mathrm{A}(\boldsymbol{v}^n(x)) : \partial_m^2\mathrm{A}(\boldsymbol{v}^n(x))]\mathrm{d}x\\
&= -\frac{\beta}{2}\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}[a_{ij}(x)]^2a_{kl}(x)\partial_m^2a_{kl}(x)\mathrm{d}x\\
&= -\frac{\beta}{2}\sum_{m=1}^2\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}^n(x))|^2|\partial_m\mathrm{A}(\boldsymbol{v}^n(x))|^2\mathrm{d}x - \beta\sum_m\int_{\mathbb{T}^2}\left[\sum_{i,j}a_{ij}(x)\partial_m a_{ij}(x)\right]^2\mathrm{d}x\\
&= -\frac{\beta}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}^n(x))|^2|\nabla\mathrm{A}(\boldsymbol{v}^n(x))|^2\mathrm{d}x - \frac{\beta}{4}\int_{\mathbb{T}^2}\left|\nabla\left(|\mathrm{A}(\boldsymbol{v}^n(x))|^2\right)\right|^2\mathrm{d}x,
\end{aligned}\tag{3.12}$$

$$|(\boldsymbol{f}, \mathscr{A}\boldsymbol{v}^n)| \leq \|\boldsymbol{f}\|_2\|\mathscr{A}\boldsymbol{v}^n\|_2 \leq \frac{\mu}{2}\|\mathscr{A}\boldsymbol{v}^n\|_2^2 + \frac{1}{2\mu}\|\boldsymbol{f}\|_2^2. \tag{3.13}$$

Combining (3.10)-(3.13), we deduce

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\nabla\boldsymbol{v}^n(t)\|_2^2 + \gamma\|\mathscr{A}\boldsymbol{v}^n(t)\|_2^2\right] + \mu\|\mathscr{A}\boldsymbol{v}^n(t)\|_2^2 + \frac{\beta}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}^n(x))|^2|\nabla\mathrm{A}(\boldsymbol{v}^n(x))|^2\mathrm{d}x\\
&+ \frac{\beta}{2}\int_{\mathbb{T}^2}\left|\nabla\left(|\mathrm{A}(\boldsymbol{v}^n(x))|^2\right)\right|^2\mathrm{d}x\\
&\leq \frac{1}{\mu}\|\boldsymbol{f}(t)\|_2^2 + C\left[\|\nabla\boldsymbol{v}^n(t)\|_2^2 + 1\right]\left[\|\nabla\boldsymbol{v}^n(t)\|_2^2 + \gamma\|\mathscr{A}\boldsymbol{v}^n(t)\|_2^2\right],
\end{aligned}\tag{3.14}$$

for a.e. $t \in [0,T]$, which by an application of Gronwall's inequality, gives for all $t \in [0,T]$

$$\begin{aligned}
&\|\nabla\boldsymbol{v}^n(t)\|_2^2 + \gamma\|\mathscr{A}\boldsymbol{v}^n(t)\|_2^2 + \mu\int_0^t\|\mathscr{A}\boldsymbol{v}^n(s)\|_2^2\mathrm{d}s\\
&\leq \left[\|\nabla\boldsymbol{v}^n(0)\|_2^2 + \gamma\|\mathscr{A}\boldsymbol{v}^n(0)\|_2^2 + \frac{1}{\mu}\int_0^t\|\boldsymbol{f}(s)\|_2^2\mathrm{d}s\right]\exp\left\{C\int_0^t\left[\|\nabla\boldsymbol{v}^n(s)\|_2^2 + 1\right]\mathrm{d}s\right\}\\
&\leq \left[C\|\mathscr{A}\boldsymbol{v}(0)\|_2^2 + \frac{1}{\mu}\int_0^T\|\boldsymbol{f}(s)\|_2^2\mathrm{d}s\right]\exp\left\{C\left[\frac{2K_1(T)}{\beta} + T\right]\right\} := K_2(T),
\end{aligned}\tag{3.15}$$

where we have used (3.8) in the final inequality. Using the fact that $\boldsymbol{v}_0 \in \mathrm{D}(\mathscr{A})$ and $\boldsymbol{f} \in \mathrm{L}^2(0,T;\dot{\mathbb{L}}^2(\mathbb{T}^2))$, we have from (3.15) that

$$\{\boldsymbol{v}^n\}_{n\in\mathbb{N}} \text{ is a bounded sequence in } \mathrm{L}^\infty(0,T;\mathrm{D}(\mathscr{A})). \tag{3.16}$$

Now, taking the inner product with $\partial_t \boldsymbol{v}^n$ in $(3.4)_1$, we obtain

$$\begin{aligned}&\|\partial_t\boldsymbol{v}^n(t)\|_2^2+\gamma\|\partial_t\nabla\boldsymbol{v}^n(t)\|_2^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\mu\|\nabla\boldsymbol{v}^n(t)\|_2^2+\frac{\beta}{2}\|\mathrm{A}(\boldsymbol{v}^n(t))\|_4^4\right]\\&\quad=-b(\boldsymbol{v}^n(t),\boldsymbol{v}^n(t),\partial_t\boldsymbol{v}^n(t))-\alpha\left\langle\mathrm{div}[(\mathrm{A}(\boldsymbol{v}^n(t)))^2],\partial_t\boldsymbol{v}^n(t)\right\rangle+(\boldsymbol{f}(t),\partial_t\boldsymbol{v}^n(t)),\end{aligned} \tag{3.17}$$

for a.e. $t\in[0,T]$. Next, using integration by parts, and Hölder's, Sobolev and Young's inequalities, we estimate the terms of the right hand side of (3.17) as

$$\begin{aligned}\left|b(\boldsymbol{v}^n,\boldsymbol{v}^n,\partial_t\boldsymbol{v}^n)\right|&\leq\|\boldsymbol{v}^n\|_4\|\nabla\boldsymbol{v}^n\|_4\|\partial_t\boldsymbol{v}^n\|_2\\&\leq C\|\nabla\boldsymbol{v}^n\|_2\|\mathrm{A}(\boldsymbol{v}^n)\|_4\|\partial_t\boldsymbol{v}^n\|_2\\&\leq\frac{1}{4}\|\partial_t\boldsymbol{v}^n\|_2^2+C\|\mathrm{A}(\boldsymbol{v}^n)\|_4^2\|\nabla\boldsymbol{v}^n\|_2^2\\&\leq\frac{1}{4}\|\partial_t\boldsymbol{v}^n\|_2^2+C\big[\|\mathrm{A}(\boldsymbol{v}^n)\|_4^4+1\big]\|\nabla\boldsymbol{v}^n\|_2^2\\&\leq\frac{1}{4}\|\partial_t\boldsymbol{v}^n\|_2^2+C\big[\|\mathrm{A}(\boldsymbol{v}^n)\|_4^4+1\big]\left[\mu\|\nabla\boldsymbol{v}^n\|_2^2+\frac{\beta}{2}\|\mathrm{A}(\boldsymbol{v}^n)\|_4^4\right],\end{aligned} \tag{3.18}$$

$$\begin{aligned}\left|\alpha\left\langle\mathrm{div}[(\mathrm{A}(\boldsymbol{v}^n))^2],\partial_t\boldsymbol{v}^n\right\rangle\right|&=\left|\alpha\int_{\mathbb{T}^2}(\mathrm{A}(\boldsymbol{v}^n(x)))^2:\partial_t\nabla\boldsymbol{v}^n(x)\mathrm{d}x\right|\\&\leq|\alpha|\|\mathrm{A}(\boldsymbol{v}^n)\|_4^2\|\partial_t\nabla\boldsymbol{v}^n\|_2\leq\frac{\alpha^2}{2\gamma}\|\mathrm{A}(\boldsymbol{v}^n)\|_4^4+\frac{\gamma}{2}\|\partial_t\nabla\boldsymbol{v}^n\|_2^2,\end{aligned} \tag{3.19}$$

$$|(\boldsymbol{f},\partial_t\boldsymbol{v}^n)|\leq\|\boldsymbol{f}\|_2\|\partial_t\boldsymbol{v}^n\|_2\leq\frac{1}{4}\|\partial_t\boldsymbol{v}^n\|_2^2+\|\boldsymbol{f}\|_2^2. \tag{3.20}$$

Combining (3.17)-(3.20), we deduce

$$\begin{aligned}&\|\partial_t\boldsymbol{v}^n(t)\|_2^2+\gamma\|\partial_t\nabla\boldsymbol{v}^n(t)\|_2^2+\frac{\mathrm{d}}{\mathrm{d}t}\left[\mu\|\nabla\boldsymbol{v}^n(t)\|_2^2+\frac{\beta}{2}\|\mathrm{A}(\boldsymbol{v}^n(t))\|_4^4\right]\\&\leq2\|\boldsymbol{f}(t)\|_2^2+C\big[\|\mathrm{A}(\boldsymbol{v}^n(t))\|_4^4+1\big]\left[\mu\|\nabla\boldsymbol{v}^n(t)\|_2^2+\frac{\beta}{2}\|\mathrm{A}(\boldsymbol{v}^n(t))\|_4^4\right],\end{aligned} \tag{3.21}$$

for a.e. $t\in[0,T]$, which by an application of Gronwall's inequality, gives for all $t\in[0,T]$

$$\begin{aligned}&\int_0^t\|\partial_t\boldsymbol{v}^n(s)\|_2^2\mathrm{d}s+\gamma\int_0^t\|\partial_t\nabla\boldsymbol{v}^n(s)\|_2^2\mathrm{d}s+\mu\|\nabla\boldsymbol{v}^n(t)\|_2^2+\frac{\beta}{2}\|\mathrm{A}(\boldsymbol{v}^n(t))\|_4^4\\&\leq\left[\mu\|\boldsymbol{v}^n(0)\|_{\mathbb{V}}^2+\frac{\beta}{2}\|\mathrm{A}(\boldsymbol{v}^n(0))\|_4^4+2\int_0^t\|\boldsymbol{f}(s)\|_2^2\,\mathrm{d}s\right]\exp\left\{C\int_0^t\left[\|\mathrm{A}(\boldsymbol{v}^n(s))\|_4^4+1\right]\mathrm{d}s\right\}\\&\leq\left[\mu\|\boldsymbol{v}(0)\|_{\mathbb{V}}^2+C\|\mathscr{A}(\boldsymbol{v}(0))\|_2^4+2\int_0^T\|\boldsymbol{f}(s)\|_2^2\mathrm{d}s\right]\times\exp\left\{C\left[\frac{2K_1(T)}{\beta}+T\right]\right\}:=K_3(T),\end{aligned} \tag{3.22}$$

where we have used (3.8) in the final inequality. Using the fact that $\boldsymbol{f} \in \mathrm{L}^2(0,T;\dot{\mathbb{L}}^2(\mathbb{T}^2))$, we have from (3.22) that

$$\{\partial_t \boldsymbol{v}^n\}_{n\in\mathbb{N}} \text{ is a bounded sequence in } \mathrm{L}^2(0,T;\mathbb{V}). \tag{3.23}$$

Note that, from (3.4), we have

$$\begin{aligned}
&\int_0^T \|\mathscr{G}_n(\boldsymbol{v}^n(s))\|_{\mathbb{V}'}^2 \mathrm{d}s \\
&\leq \int_0^T \|\mu\mathscr{A}_n\boldsymbol{v}^n(s) + \mathscr{B}_n(\boldsymbol{v}^n(s)) + \alpha\mathscr{J}_n(\boldsymbol{v}^n(s)) + \beta\mathscr{K}_n(\boldsymbol{v}^n(s))\|_{\mathbb{V}'}^2 \mathrm{d}s \\
&\leq \int_0^T \|\partial_t\boldsymbol{v}^n(s) + \gamma\partial_t\mathscr{A}\boldsymbol{v}^n(s) - \boldsymbol{f}_n(s)\|_{\mathbb{V}'}^2 \mathrm{d}s \\
&\leq 3\int_0^T \|\partial_t\boldsymbol{v}^n(s)\|_{\mathbb{V}'}^2 \mathrm{d}s + 3\gamma^2\int_0^T \|\partial_t\mathscr{A}\boldsymbol{v}^n(s)\|_{\mathbb{V}'}^2 \mathrm{d}s + 3\int_0^T \|\boldsymbol{f}_n(s)\|_{\mathbb{V}'}^2 \mathrm{d}s \\
&\leq C\int_0^T \|\partial_t\nabla\boldsymbol{v}^n(s)\|_2^2 \mathrm{d}s + C\int_0^T \|\boldsymbol{f}(s)\|_2^2 \mathrm{d}s \leq CK_3(T) + C\int_0^T \|\boldsymbol{f}(s)\|_2^2 \mathrm{d}s,
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
\int_0^T \|\partial_t(\mathrm{I}+\gamma\mathscr{A})\boldsymbol{v}^n(s)\|_{\mathbb{V}'}^2 \mathrm{d}s &\leq 2\int_0^T \|\partial_t\boldsymbol{v}^n(s)\|_{\mathbb{V}'}^2 \mathrm{d}s + 2\gamma^2\int_0^T \|\partial_t\mathscr{A}\boldsymbol{v}^n(s)\|_{\mathbb{V}'}^2 \mathrm{d}s \\
&\leq C\int_0^T \|\partial_t\nabla\boldsymbol{v}^n(s)\|_2^2 \mathrm{d}s \leq CK_3(T),
\end{aligned} \tag{3.25}$$

where we have used (3.22), which implies

$$\{\mathscr{G}_n(\boldsymbol{v}^n)\}_{n\in\mathbb{N}} \text{ and } \{\partial_t(\mathrm{I}+\gamma\mathscr{A})\boldsymbol{v}^n\}_{n\in\mathbb{N}} \text{ is a bounded sequence in } \mathrm{L}^2(0,T;\mathbb{V}'). \tag{3.26}$$

Step III. *Weak*, weak and strong limits.* Using (3.9), (3.16), (3.23), (3.26) and the *Banach-Alaoglu theorem*, we infer the existence of an element $\boldsymbol{v} \in \mathrm{L}^\infty(0,T;\mathrm{D}(\mathscr{A})) \cap \mathrm{L}^4(0,T;\mathbb{W}^{1,12}(\mathbb{T}^2))$ with $\partial_t(\mathrm{I}+\gamma\mathscr{A})\boldsymbol{v} \in \mathrm{L}^2(0,T;\mathbb{V}')$ and $\mathscr{G}_0 \in \mathrm{L}^2(0,T;\mathbb{V}')$ such that

$$\left.\begin{aligned}
\boldsymbol{v}^n &\xrightarrow{w^*} \boldsymbol{v} && \text{in} && \mathrm{L}^\infty(0,T;\mathrm{D}(\mathscr{A})),\\
\boldsymbol{v}^n &\xrightarrow{w} \boldsymbol{v} && \text{in} && \mathrm{L}^2(0,T;\mathrm{D}(\mathscr{A})) \cap \mathrm{L}^4(0,T;\mathbb{W}^{1,12}(\mathbb{T}^2)),\\
\partial_t(\mathrm{I}+\gamma\mathscr{A})\boldsymbol{v}^n &\xrightarrow{w} \partial_t(\mathrm{I}+\gamma\mathscr{A})\boldsymbol{v} && \text{in} && \mathrm{L}^2(0,T;\mathbb{V}'),\\
\mathscr{G}_n(\boldsymbol{v}^n) &\xrightarrow{w} \mathscr{G}_0 && \text{in} && \mathrm{L}^2(0,T;\mathbb{V}'),
\end{aligned}\right\} \tag{3.27}$$

along a subsequence (still denoted by the same symbol). In addition, since $\boldsymbol{v}^n \in \mathrm{L}^2(0,T;\mathrm{D}(\mathscr{A}))$, $\partial_t\boldsymbol{v}^n \in \mathrm{L}^2(0,T;\mathbb{H})$, in view of continuous embeddings $\mathrm{D}(\mathscr{A}) \subset \mathbb{V} \subset \mathbb{H}$ with compact embedding $\mathrm{D}(\mathscr{A}) \subset \mathbb{V}$ and Aubin-Lions compactness lemma, we also have (along a subsequence, still denoted by the same symbol)

$$\boldsymbol{v}^n \to \boldsymbol{v} \quad \text{in} \quad \mathrm{L}^2(0,T;\mathbb{V}). \tag{3.28}$$

Step IV. *Passing* $n \to \infty$ *in* (3.4). Let us now show that $\mathscr{G}_0 = \mathscr{G}(\boldsymbol{v}) := \mu\mathscr{A}\boldsymbol{v} + \mathscr{B}(\boldsymbol{v}) + \alpha\mathscr{J}(\boldsymbol{v}) + \beta\mathscr{K}(\boldsymbol{v})$ in $\mathrm{L}^2(0,T;\mathbb{V}')$. For $\boldsymbol{\phi} \in \mathrm{L}^2(0,T;\mathbb{V})$, we consider

$$\begin{aligned}&\int_0^T \langle \mathscr{G}_n(\boldsymbol{v}^n(t)) - \mathscr{G}(\boldsymbol{v}(t)), \boldsymbol{\phi}(t)\rangle \,\mathrm{d}t\\ &\quad= \underbrace{\int_0^T \langle \mathscr{G}(\boldsymbol{v}^n(t)), \mathrm{P}_n\boldsymbol{\phi}(t) - \boldsymbol{\phi}(t)\rangle \,\mathrm{d}t}_{:=G_1(n)} + \underbrace{\int_0^T \langle \mathscr{G}(\boldsymbol{v}^n(t)) - \mathscr{G}(\boldsymbol{v}(t)), \boldsymbol{\phi}(t)\rangle \,\mathrm{d}t}_{:=G_2(n)}.\end{aligned} \tag{3.29}$$

We first show that $\lim\limits_{n\to\infty} G_1(n) = 0$. Since $\mathrm{P}_n\boldsymbol{\phi} \to \boldsymbol{\phi}$ in $\mathrm{L}^2(0,T;\mathbb{V})$, it is enough to show that $\int_0^T \|\mathscr{G}(\boldsymbol{v}^n(t))\|_{\mathbb{V}'}^2 \mathrm{d}t$ is bounded above by a constant which is independent of n. Consider,

$$\begin{aligned}&\int_0^T \|\mathscr{G}(\boldsymbol{v}^n(t))\|_{\mathbb{V}'}^2 \mathrm{d}t\\ &\quad= \int_0^T \|\mu\mathscr{A}\boldsymbol{v}^n(t) + \mathscr{B}(\boldsymbol{v}^n(t)) + \alpha\mathscr{J}(\boldsymbol{v}^n(t)) + \beta\mathscr{K}(\boldsymbol{v}^n(t))\|_{\mathbb{V}'}^2 \mathrm{d}t\\ &\quad\le 4\mu^2\int_0^T \|\mathscr{A}\boldsymbol{v}^n(t)\|_{\mathbb{V}'}^2 \mathrm{d}t + 4\int_0^T \|\mathscr{B}(\boldsymbol{v}^n(t))\|_{\mathbb{V}'}^2 \mathrm{d}t + 4\alpha^2\int_0^T \|\mathscr{J}(\boldsymbol{v}^n(t))\|_{\mathbb{V}'}^2 \mathrm{d}t\\ &\qquad + 4\beta^2\int_0^T \|\mathscr{K}(\boldsymbol{v}^n(t))\|_{\mathbb{V}'}^2 \mathrm{d}t\\ &\quad\le C\bigg[\int_0^T \|\nabla\boldsymbol{v}^n(t)\|_2^2 \mathrm{d}t + \int_0^T \|\boldsymbol{v}^n(t)\|_4^4 \mathrm{d}t + \int_0^T \|\mathrm{A}(\boldsymbol{v}^n(t))\|_4^4 \mathrm{d}t + \int_0^T \|\mathrm{A}(\boldsymbol{v}^n(t))\|_6^6 \mathrm{d}t\bigg]\\ &\quad\le C\bigg[\int_0^T \|\nabla\boldsymbol{v}^n(t)\|_2^2 \mathrm{d}t + \int_0^T \|\nabla\boldsymbol{v}^n(t)\|_2^4 \mathrm{d}t + \int_0^T \|\mathrm{A}(\boldsymbol{v}^n(t))\|_4^4 \mathrm{d}t + \int_0^T \|\mathscr{A}\boldsymbol{v}^n(t)\|_2^6 \mathrm{d}t\bigg]\\ &\quad\le C\bigg[\int_0^T \|\nabla\boldsymbol{v}^n(t)\|_2^2 \mathrm{d}t + \sup_{t\in[0,T]}\|\nabla\boldsymbol{v}^n(t)\|_2^2\int_0^T \|\nabla\boldsymbol{v}^n(t)\|_2^2 \mathrm{d}t + \int_0^T \|\mathrm{A}(\boldsymbol{v}^n(t))\|_4^4 \mathrm{d}t\\ &\qquad + \sup_{t\in[0,T]}\|\mathscr{A}\boldsymbol{v}^n(t)\|_2^4\int_0^T \|\mathscr{A}\boldsymbol{v}^n(t)\|_2^2 \mathrm{d}t\bigg]\\ &\quad\le C\Big[K_1(T) + \{K_1(T)\}^2 + \{K_2(T)\}^3\Big],\end{aligned} \tag{3.30}$$

where we have used continuous Sobolev embedding $\mathrm{D}(\mathscr{A}) \hookrightarrow \mathbb{W}^{1,p}(\mathbb{T}^2)$ (for $1 \le p < \infty$), (3.8) and (3.15). Hence, we deduce $\lim\limits_{n\to\infty} G_1(n) = 0$.

Let us now show that $\lim\limits_{n\to\infty} G_2(n) = 0$. Consider

$$\begin{aligned}&\left|\int_0^T \langle \mathscr{G}(\boldsymbol{v}^n(t)) - \mathscr{G}(\boldsymbol{v}(t)), \boldsymbol{\phi}(t)\rangle \,\mathrm{d}t\right|\\ &\le C\int_0^T \Big[|(\nabla(\boldsymbol{v}^n(t) - \boldsymbol{v}(t)), \nabla\boldsymbol{\phi}(t))| + |b(\boldsymbol{v}^n(t) - \boldsymbol{v}(t), \boldsymbol{\phi}(t), \boldsymbol{v}^n(t))|\\ &\quad + |b(\boldsymbol{v}(t), \boldsymbol{\phi}(t), \boldsymbol{v}^n(t) - \boldsymbol{v}(t))|\\ &\quad + |\langle \mathrm{A}(\boldsymbol{v}^n(t) - \boldsymbol{v}(t))\mathrm{A}(\boldsymbol{v}(t)), \nabla\boldsymbol{\phi}(t)\rangle| + |\langle \mathrm{A}(\boldsymbol{v}^n(t))\mathrm{A}(\boldsymbol{v}^n(t) - \boldsymbol{v}(t)), \nabla\boldsymbol{\phi}(t)\rangle|\\ &\quad + |\langle [\mathrm{A}(\boldsymbol{v}^n(t) - \boldsymbol{v}(t)) : \mathrm{A}(\boldsymbol{v}(t))]\mathrm{A}(\boldsymbol{v}^n(t)), \nabla\boldsymbol{\phi}(t)\rangle|\\ &\quad + |\langle [\mathrm{A}(\boldsymbol{v}^n(t) - \boldsymbol{v}(t)) : \mathrm{A}(\boldsymbol{v}^n(t))]\mathrm{A}(\boldsymbol{v}^n(t)), \nabla\boldsymbol{\phi}(t)\rangle|\\ &\quad + |\langle |\mathrm{A}(\boldsymbol{v}(t))|^2\mathrm{A}(\boldsymbol{v}^n(t) - \boldsymbol{v}(t)), \nabla\boldsymbol{\phi}(t)\rangle|\Big]\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_0^T\bigg[\|\nabla(\boldsymbol{v}^n(t)-\boldsymbol{v}(t))\|_2+\|\boldsymbol{v}^n(t)-\boldsymbol{v}(t)\|_2\|\boldsymbol{v}^n(t)\|_\infty+\|\boldsymbol{v}(t)\|_\infty\|\boldsymbol{v}^n(t)-\boldsymbol{v}(t)\|_2\\
&\quad+\|\mathrm{A}(\boldsymbol{v}(t))\|_3\|\mathrm{A}(\boldsymbol{v}^n(t)-\boldsymbol{v}(t))\|_6+\|\mathrm{A}(\boldsymbol{v}^n(t))\|_3\|\mathrm{A}(\boldsymbol{v}^n(t)-\boldsymbol{v}(t))\|_6\\
&\quad+\|\mathrm{A}(\boldsymbol{v}(t))\|_6\|\mathrm{A}(\boldsymbol{v}^n(t))\|_6\|\mathrm{A}(\boldsymbol{v}^n(t)-\boldsymbol{v}(t))\|_6+\|\mathrm{A}(\boldsymbol{v}^n(t))\|_6^2\|\mathrm{A}(\boldsymbol{v}^n(t)-\boldsymbol{v}(t))\|_6\\
&\quad+\|\mathrm{A}(\boldsymbol{v}(t))\|_6^2\|\mathrm{A}(\boldsymbol{v}^n(t)-\boldsymbol{v}(t))\|_6\bigg]\|\nabla\boldsymbol{\phi}(t)\|_2\mathrm{d}t\\
&\leq C\int_0^T\bigg[\|\nabla(\boldsymbol{v}^n(t)-\boldsymbol{v}(t))\|_2+\|\boldsymbol{v}^n(t)-\boldsymbol{v}(t)\|_2\|\mathscr{A}\boldsymbol{v}^n(t)\|_2+\|\mathscr{A}\boldsymbol{v}(t)\|_2\|\boldsymbol{v}^n(t)-\boldsymbol{v}(t)\|_2\\
&\quad+\|\mathscr{A}\boldsymbol{v}(t)\|_2\|\mathscr{A}(\boldsymbol{v}^n(t)-\boldsymbol{v}(t))\|_2^{\frac{5}{6}}\|\boldsymbol{v}^n(t)-\boldsymbol{v}(t)\|_2^{\frac{1}{6}}+\|\mathscr{A}\boldsymbol{v}^n(t)\|_2\|\mathscr{A}(\boldsymbol{v}^n(t)\\
&\quad-\boldsymbol{v}(t))\|_2^{\frac{5}{6}}\|\boldsymbol{v}^n(t)-\boldsymbol{v}(t)\|_2^{\frac{1}{6}}\\
&\quad+\|\mathscr{A}\boldsymbol{v}(t)\|_2\|\mathscr{A}\boldsymbol{v}^n(t)\|_2\|\mathscr{A}(\boldsymbol{v}^n(t)-\boldsymbol{v}(t))\|_2^{\frac{5}{6}}\|\boldsymbol{v}^n(t)-\boldsymbol{v}(t)\|_2^{\frac{1}{6}}\\
&\quad+\|\mathscr{A}\boldsymbol{v}^n(t)\|_2^2\|\mathscr{A}(\boldsymbol{v}^n(t)-\boldsymbol{v}(t))\|_2^{\frac{5}{6}}\|\boldsymbol{v}^n(t)-\boldsymbol{v}(t)\|_2^{\frac{1}{6}}\\
&\quad+\|\mathscr{A}\boldsymbol{v}(t)\|_2^2\|\mathscr{A}(\boldsymbol{v}^n(t)-\boldsymbol{v}(t))\|_2^{\frac{5}{6}}\|\boldsymbol{v}^n(t)-\boldsymbol{v}(t)\|_2^{\frac{1}{6}}\bigg]\|\nabla\boldsymbol{\phi}(t)\|_2\mathrm{d}t, \qquad (3.31)
\end{aligned}
$$

where we have used Hölder's inequality, the following Gagliardo-Nirenberg inequality

$$
\|\mathrm{A}(\boldsymbol{y}(t))\|_6\leq C\|\mathscr{A}(\boldsymbol{y}(t))\|_2^{\frac{5}{6}}\|\boldsymbol{y}(t)\|_2^{\frac{1}{6}},\ \text{ for all }\boldsymbol{y}\in\mathrm{D}(\mathscr{A}), \qquad (3.32)
$$

and continuous Sobolev embedding $\mathrm{D}(\mathscr{A})\hookrightarrow\mathbb{W}^{1,p}(\mathbb{T}^2)$ (for any $1\leq p<\infty$). In view of (3.16), (3.28) and the fact that $\boldsymbol{v}\in\mathrm{L}^\infty(0,T;\mathrm{D}(\mathscr{A}))$, we infer from (3.31) that $\lim\limits_{n\to\infty}G_2(n)=0$. Therefore, from (3.29), we obtain $\mathscr{G}_0=\mathscr{G}(\boldsymbol{v})$.

Also, we have $\boldsymbol{f}_n\to\mathbb{P}\boldsymbol{f}$ in $\mathrm{L}^2(0,T;\mathbb{H})$. Therefore, on passing to limit as $n\to\infty$ in (3.4), the limit $\boldsymbol{v}(\cdot)$ satisfies:

$$
\partial_t(\mathrm{I}+\gamma\mathscr{A})\boldsymbol{v}=-\mathscr{G}(\boldsymbol{v})+\mathbb{P}\boldsymbol{f}, \qquad \text{in} \qquad \mathrm{L}^2(0,T;\mathbb{V}'). \qquad (3.33)
$$

Next, we notice that [40, Lemma 1.2, p.176], $(\mathrm{I}+\gamma\mathscr{A})^{\frac{1}{2}}\boldsymbol{v}\in\mathrm{L}^2(0,T;\mathbb{V})$ and $\partial_t(\mathrm{I}+\gamma\mathscr{A})^{\frac{1}{2}}\boldsymbol{v}\in\mathrm{L}^2(0,T;\mathbb{V}')$ imply $(\mathrm{I}+\gamma\mathscr{A})^{\frac{1}{2}}\boldsymbol{v}\in\mathrm{C}([0,T];\mathbb{H})$, the real-valued function $t\mapsto\|(\mathrm{I}+\gamma\mathscr{A})^{\frac{1}{2}}\boldsymbol{v}(t)\|_2^2$ is absolutely continuous and the following equality is satisfied:

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\mathrm{I}+\gamma\mathscr{A})^{\frac{1}{2}}\boldsymbol{v}(t)\|_2^2=\left\langle\partial_t(\mathrm{I}+\gamma\mathscr{A})^{\frac{1}{2}}\boldsymbol{v}(t),(\mathrm{I}+\gamma\mathscr{A})^{\frac{1}{2}}\boldsymbol{v}(t)\right\rangle, \qquad \text{for a.e. } t\in[0,T]. \qquad (3.34)
$$

This also implies

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\boldsymbol{v}(t)\|_2^2+\gamma\|\nabla\boldsymbol{v}(t)\|_2^2\right]=\langle\partial_t(\mathrm{I}+\gamma\mathscr{A})\boldsymbol{v}(t),\boldsymbol{v}(t)\rangle=\langle-\mathscr{G}(\boldsymbol{v}(t))+\mathbb{P}\boldsymbol{f}(t),\boldsymbol{v}(t)\rangle, \qquad (3.35)
$$

for a.e. $t\in[0,T]$, which also implies (3.3) immediately. In addition, the condition (ii) in the Definition 3.1 also makes sense.

Step V. *Uniqueness:* Define $\mathfrak{V}=\boldsymbol{v}_1-\boldsymbol{v}_2$, where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are two solutions of system (3.1) in the sense of Definition 3.1. Then $\mathfrak{V}\in\mathrm{C}([0,T];\mathbb{V})\cap$

$\mathrm{L}^{\infty}(0,T;\mathrm{D}(\mathscr{A}))$ and satisfies

$$\begin{cases} \partial_t(\mathrm{I}+\gamma\mathscr{A})\mathfrak{V}(t) = -\left[\mathscr{G}(\boldsymbol{v}_1(t)) - \mathscr{G}(\boldsymbol{v}_2(t))\right], \\ \mathfrak{V}(0) = \mathbf{0}, \end{cases} \tag{3.36}$$

in the weak sense. We know that

$$\langle \nu\mathscr{A}\boldsymbol{v}_1 - \nu\mathscr{A}\boldsymbol{v}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle = \nu\|\nabla(\boldsymbol{v}_1-\boldsymbol{v}_2)\|_2^2. \tag{3.37}$$

Now, using (2.6), Hölder's, Sobolev and Young's inequalities, we estimate

$$\begin{aligned} &|\langle \mathscr{B}(\boldsymbol{v}_1) - \mathscr{B}(\boldsymbol{v}_2), \boldsymbol{v}_1-\boldsymbol{v}_2\rangle| \\ &\leq |b(\boldsymbol{v}_1-\boldsymbol{v}_2, \boldsymbol{v}_1-\boldsymbol{v}_2, \boldsymbol{v}_2| + |b(\boldsymbol{v}_2, \boldsymbol{v}_1-\boldsymbol{v}_2, \boldsymbol{v}_1-\boldsymbol{v}_2)| \\ &\leq 2\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_4\|\nabla(\boldsymbol{v}_1-\boldsymbol{v}_2)\|_2\|\boldsymbol{v}_2\|_4 \leq C\|\nabla\boldsymbol{v}_2\|_2\|\nabla(\boldsymbol{v}_1-\boldsymbol{v}_2)\|_2^2 \\ &\leq \frac{\nu}{2}\|\nabla(\boldsymbol{v}_1-\boldsymbol{v}_2)\|_2^2 + C\|\nabla\boldsymbol{v}_2\|_2^2\|\nabla(\boldsymbol{v}_1-\boldsymbol{v}_2)\|_2^2. \end{aligned} \tag{3.38}$$

From [20, Equation (2.21)], we have

$$\begin{aligned} &|\alpha\langle \mathscr{J}(\boldsymbol{v}_1) - \mathscr{J}(\boldsymbol{v}_2), \boldsymbol{v}_1-\boldsymbol{v}_2\rangle| \\ &\leq C\|\nabla(\boldsymbol{v}_1-\boldsymbol{v}_2)\|_2^2 + \frac{\beta}{4}\int_{\Omega}|\mathrm{A}(\boldsymbol{v}_1-\boldsymbol{v}_2)|^2(|\mathrm{A}(\boldsymbol{v}_1)|^2 + |\mathrm{A}(\boldsymbol{v}_2)|^2). \end{aligned} \tag{3.39}$$

Now, from [20, Equation (2.13)], we have

$$\begin{aligned} &\beta\langle \mathscr{K}(\boldsymbol{v}_1) - \mathscr{K}(\boldsymbol{v}_2), \boldsymbol{v}_1-\boldsymbol{v}_2\rangle \\ &= \frac{\beta}{2}\int_{\Omega}(|\mathrm{A}(\boldsymbol{v}_1)|^2 - |\mathrm{A}(\boldsymbol{v}_2)|^2)^2 + \frac{\beta}{2}\int_{\Omega}|\mathrm{A}(\boldsymbol{v}_1-\boldsymbol{v}_2)|^2(|\mathrm{A}(\boldsymbol{v}_1)|^2 + |\mathrm{A}(\boldsymbol{v}_2)|^2). \end{aligned} \tag{3.40}$$

Now, taking the inner product with $\mathfrak{V}(\cdot)$ in $(3.36)_1$ and using (3.37)-(3.40), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\mathfrak{V}(t)\|_2^2 + \gamma\|\nabla\mathfrak{V}(t)\|_2^2\right] \leq C\left[1 + \|\nabla\boldsymbol{v}_2(t)\|_2^2\right]\|\nabla\mathfrak{V}(t)\|_2^2, \tag{3.41}$$

for a.e. $t \in [0,T]$. An application of Gronwall's inequality and the fact that $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathrm{L}^2(0,T;\mathbb{V})$ and $\mathfrak{V}(0) = \mathbf{0}$ give $\boldsymbol{v}_1(t) = \boldsymbol{v}_2(t)$, for all $t \in [0,T]$ in $\mathbb{V}$, which proves the uniqueness. This completes the proof. □

Theorem 3.2. *Assume that, for some* $0 < \varepsilon < 1$, $\boldsymbol{v}_0 \in \mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}})$ *and* $\boldsymbol{f} \in \mathrm{L}^2(0,T;\dot{\mathbb{H}}^{\varepsilon}(\mathbb{T}^2))$. *Then, there exists a time* $T^* = T^*(\varepsilon, \boldsymbol{v}_0, \boldsymbol{f})$ *such that the unique solution* $\boldsymbol{v}$ *to system (3.1) belongs to* $\mathrm{L}^{\infty}(0,T^*;\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))$.

Proof. The following result has been taken from [5] which will be used in the sequel. Let us define

$$\mathfrak{D}^s \equiv \mathfrak{D}^s_{\mathbb{T}^2} \equiv (-\Delta)^{\frac{s}{2}}$$

denote the Fourier multiplier operators on $\mathbb{T}^2$ with multipliers $(\frac{2\pi|\boldsymbol{k}|}{L})^s$ and recall the fractional Leibniz rule from [5] (see also [27]) for the operator $\mathfrak{D}$. For any $s > 0$ and $f, g \in \mathrm{D}(\mathfrak{D}^s)$, we have

$$\|\mathfrak{D}^s(fg)\|_2 \leq C_s\left[\|\mathfrak{D}^s f\|_2\|g\|_{\infty} + \|\mathfrak{D}^s g\|_2\|f\|_{\infty}\right], \tag{3.42}$$

where $C_s > 0$ is a constant depending only on s. Also, note that, for $\boldsymbol{v} \in \mathrm{D}(\mathscr{A}^s)$ and $s > 0$, we have $\mathfrak{D}^s\boldsymbol{v} = \mathscr{A}^s\boldsymbol{v}$, consequently

$$\|\mathfrak{D}^s\boldsymbol{v}\|_2 = \|\mathscr{A}^s\boldsymbol{v}\|_2.$$

Taking the inner product with $\mathscr{A}^{1+\varepsilon}\boldsymbol{v}^n$ in $(3.4)_1$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\mathscr{A}^{\frac{1+\varepsilon}{2}}\boldsymbol{v}^n(t)\|_2^2 + \gamma\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n(t)\|_2^2\right] + \mu\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n(t)\|_2^2\\
&= -b(\boldsymbol{v}^n(t), \boldsymbol{v}^n(t), \mathscr{A}^{1+\varepsilon}\boldsymbol{v}^n(t)) - \alpha\left(\mathrm{div}[(\mathrm{A}(\boldsymbol{v}^n(t)))^2], \mathscr{A}^{1+\varepsilon}\boldsymbol{v}^n(t)\right)\\
&- \beta\left(\mathrm{div}\left[|\mathrm{A}(\boldsymbol{v}^n(t))|^2\mathrm{A}(\boldsymbol{v}^n(t))\right], \mathscr{A}^{1+\varepsilon}\boldsymbol{v}^n(t)\right)\\
&\quad + (\boldsymbol{f}_n(t), \mathscr{A}^{1+\varepsilon}\boldsymbol{v}^n(t)),
\end{aligned}\tag{3.43}$$

for a.e. $t \in [0, T]$. Next we estimate all the terms on the right hand side of the equation (3.43) one by one. Using (3.42), (2.12), and Hölder's and Young's inequalities, we estimate

$$\begin{align*}
&|b(\boldsymbol{v}^n, \boldsymbol{v}^n, \mathscr{A}^{1+\varepsilon}\boldsymbol{v}^n)|\\
&= |\big((\boldsymbol{v}^n\cdot\nabla)\boldsymbol{v}^n, \mathscr{A}^{1+\varepsilon}\boldsymbol{v}^n\big)| = |\big(\mathrm{div}(\boldsymbol{v}^n\otimes\boldsymbol{v}^n), \mathfrak{D}^{1+\varepsilon}\boldsymbol{v}^n\big)|\\
&= |\big(\mathfrak{D}^{\frac{\varepsilon}{2}}\mathrm{div}(\boldsymbol{v}^n\otimes\boldsymbol{v}^n), \mathfrak{D}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\big)| \leq \|\mathfrak{D}^{\frac{\varepsilon}{2}}\mathrm{div}(\boldsymbol{v}^n\otimes\boldsymbol{v}^n)\|_2\|\mathfrak{D}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2\\
&\leq C\|\mathfrak{D}^{\frac{1+\varepsilon}{2}}(\boldsymbol{v}^n\otimes\boldsymbol{v}^n)\|_2\|\mathfrak{D}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2 \leq C\|\mathfrak{D}^{\frac{1+\varepsilon}{2}}\boldsymbol{v}^n\|_2\|\boldsymbol{v}^n\|_\infty\|\mathfrak{D}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2\\
&\leq C\|\mathfrak{D}^{\frac{1+\varepsilon}{2}}\boldsymbol{v}^n\|_2\|\mathfrak{D}\boldsymbol{v}^n\|_2\|\mathfrak{D}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2\\
&\leq \frac{\mu}{8}\|\mathfrak{D}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2^2 + C\|\mathfrak{D}\boldsymbol{v}^n\|_2^2\|\mathfrak{D}^{\frac{1+\varepsilon}{2}}\boldsymbol{v}^n\|_2^2\\
&\leq \frac{\mu}{8}\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2^2 + C\|\mathscr{A}\boldsymbol{v}^n\|_2^2\left[1 + \|\mathscr{A}^{\frac{1+\varepsilon}{2}}\boldsymbol{v}^n\|_2^2 + \gamma\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2^2\right]^2,\\
(\boldsymbol{f}_n, \mathscr{A}^{1+\varepsilon}\boldsymbol{v}^n) &= (\mathscr{A}^{\frac{\varepsilon}{2}}\boldsymbol{f}_n, \mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n) \leq \|\mathscr{A}^{\frac{\varepsilon}{2}}\boldsymbol{f}_n\|_2\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2\\
&\leq \|\boldsymbol{f}\|_{\dot{\mathbb{H}}^\varepsilon}\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2 \leq \frac{\mu}{8}\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2^2 + C\|\boldsymbol{f}\|_{\dot{\mathbb{H}}^\varepsilon}^2, \tag{3.44}\\
&|\alpha\left(\mathrm{div}[(\mathrm{A}(\boldsymbol{v}^n))^2], \mathscr{A}^{1+\varepsilon}\boldsymbol{v}^n\right)|\\
&= |\alpha\left(\mathrm{div}[(\mathrm{A}(\boldsymbol{v}^n))^2], \mathfrak{D}^{1+\varepsilon}\boldsymbol{v}^n\right)|\\
&= \left|\frac{\alpha}{2}\left(\mathfrak{D}^{\frac{1+\varepsilon}{2}}[(\mathrm{A}(\boldsymbol{v}^n))^2], \mathfrak{D}^{\frac{1+\varepsilon}{2}}\mathrm{A}(\boldsymbol{v}^n)\right)\right|\\
&\leq C\|\mathfrak{D}^{\frac{1+\varepsilon}{2}}[(\mathrm{A}(\boldsymbol{v}^n))^2]\|_2\|\mathfrak{D}^{\frac{1+\varepsilon}{2}}\mathrm{A}(\boldsymbol{v}^n)\|_2\\
&\leq C\|\mathfrak{D}^{\frac{1+\varepsilon}{2}}\mathrm{A}(\boldsymbol{v}^n)\|_2\|\mathrm{A}(\boldsymbol{v}^n)\|_\infty\|\mathfrak{D}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2\\
&\leq C\|\mathfrak{D}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2^3 = C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2^3\\
&\leq C\left[1 + \|\mathscr{A}^{\frac{1+\varepsilon}{2}}\boldsymbol{v}^n\|_2^2 + \gamma\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2^2\right]^2, \tag{3.45}\\
&|\beta\left(\mathrm{div}[|\mathrm{A}(\boldsymbol{v}^n)|^2\mathrm{A}(\boldsymbol{v}^n)], \mathscr{A}^{1+\varepsilon}\boldsymbol{v}^n\right)|\\
&= |\beta\left(\mathrm{div}[|\mathrm{A}(\boldsymbol{v}^n)|^2\mathrm{A}(\boldsymbol{v}^n)], \mathfrak{D}^{1+\varepsilon}\boldsymbol{v}^n\right)|\\
&= |\frac{\beta}{2}\left(\mathfrak{D}^{\frac{1+\varepsilon}{2}}\left[|\mathrm{A}(\boldsymbol{v}^n)|^2\mathrm{A}(\boldsymbol{v}^n)\right], \mathfrak{D}^{\frac{1+\varepsilon}{2}}\mathrm{A}(\boldsymbol{v}^n)\right)|\\
&\leq C\|\mathfrak{D}^{\frac{1+\varepsilon}{2}}\left[|\mathrm{A}(\boldsymbol{v}^n)|^2\mathrm{A}(\boldsymbol{v}^n)\right]\|_2\|\mathfrak{D}^{\frac{1+\varepsilon}{2}}\mathrm{A}(\boldsymbol{v}^n)\|_2\\
&\leq C\left[\|\mathfrak{D}^{\frac{1+\varepsilon}{2}}\left[|\mathrm{A}(\boldsymbol{v}^n)|^2\right]\|_2\|\mathrm{A}(\boldsymbol{v}^n)\|_\infty + \|\mathfrak{D}^{\frac{1+\varepsilon}{2}}\mathrm{A}(\boldsymbol{v}^n)\|_2\||\mathrm{A}(\boldsymbol{v}^n)|^2\|_\infty\right]\|\mathfrak{D}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2
\end{align*}$$

$$
\begin{aligned}
&\leq C\Big[\|\mathfrak{D}^{\frac{1+\varepsilon}{2}}\left[|\mathrm{A}(\boldsymbol{v}^n)|^2\right]\|_2\|\mathfrak{D}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2 \\
&\leq C\|\mathfrak{D}^{\frac{1+\varepsilon}{2}}\left[|\mathrm{A}(\boldsymbol{v}^n)|^2\right]\|_2\|\mathfrak{D}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2^2 \\
&\leq C\|\mathfrak{D}^{\frac{1+\varepsilon}{2}}\mathrm{A}(\boldsymbol{v}^n)\|_2\|\mathrm{A}(\boldsymbol{v}^n)\|_\infty\|\mathfrak{D}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2^2 \\
&\leq C\|\mathfrak{D}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2^4 \leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2^4 \\
&\leq C\Big[1+\|\mathscr{A}^{\frac{1+\varepsilon}{2}}\boldsymbol{v}^n\|_2^2+\gamma\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n\|_2^2\Big]^2. \qquad (3.46)
\end{aligned}
$$

Now, define $\mathbf{Y}^\varepsilon(t) = 1+\|\mathscr{A}^{\frac{1+\varepsilon}{2}}\boldsymbol{v}^n(t)\|_2^2+\gamma\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n(t)\|_2^2$ so that $\mathbf{Y}^\varepsilon(t)$ satisfies (in view of above estimates)

$$
\frac{\mathrm{d}\mathbf{Y}^\varepsilon(t)}{\mathrm{d}t} \leq C\|\boldsymbol{f}(t)\|_{\dot{\mathbb{H}}^\varepsilon}^2 + C\left[1+\|\mathscr{A}\boldsymbol{v}^n(t)\|_2^2\right]\mathbf{Y}^\varepsilon(t)^2. \qquad (3.47)
$$

Therefore, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\{\mathbf{Y}^\varepsilon(t)\}^{-1}\right] &= -\frac{1}{\{\mathbf{Y}^\varepsilon(t)\}^2}\frac{\mathrm{d}\mathbf{Y}^\varepsilon(t)}{\mathrm{d}t} \geq \\
&\quad -\frac{C}{\{\mathbf{Y}^\varepsilon(t)\}^2}\left[\|\boldsymbol{f}(t)\|_{\dot{\mathbb{H}}^\varepsilon}^2+\left[1+\|\mathscr{A}\boldsymbol{v}^n(t)\|_2^4\right]\{\mathbf{Y}^\varepsilon(t)\}^2\right] \\
&\geq -C\|\boldsymbol{f}(t)\|_{\dot{\mathbb{H}}^\varepsilon}^2 - C\left[1+\|\mathscr{A}\boldsymbol{v}^n(t)\|_2^2\right], \qquad (3.48)
\end{aligned}
$$

which implies

$$
\{\mathbf{Y}^\varepsilon(t)\}^{-1} \geq \{\mathbf{Y}^\varepsilon(0)\}^{-1} - C\int_0^t\|\boldsymbol{f}(s)\|_{\dot{\mathbb{H}}^\varepsilon}^2\mathrm{d}s - C\int_0^t\left[1+\|\mathscr{A}\boldsymbol{v}^n(s)\|_2^2\right]\mathrm{d}s. \qquad (3.49)
$$

In view of (3.15) and the fact that $\boldsymbol{f}\in\mathrm{L}^2(0,T;\dot{\mathbb{H}}^\varepsilon(\mathbb{T}^2))$, we ensure that there exists a time $T^* = T^*(\varepsilon,\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}_0\|_2,\|\boldsymbol{f}\|_{\mathrm{L}^2(0,T;\dot{\mathbb{H}}^\varepsilon)}) > 0$ such that for any $t\in[0,T^*]$

$$
C\int_0^t\|\boldsymbol{f}(s)\|_{\dot{\mathbb{H}}^\varepsilon}^2\mathrm{d}s + C\int_0^t\left[1+\|\mathscr{A}\boldsymbol{v}^n(s)\|_2^2\right]\mathrm{d}s < \frac{1}{2}\{\mathbf{Y}^\varepsilon(0)\}^{-1}. \qquad (3.50)
$$

Hence, we conclude

$$
\mathbf{Y}^\varepsilon(t) \leq 2\mathbf{Y}^\varepsilon(0), \qquad (3.51)
$$

for all $t\in[0,T^*]$, equivalently

$$
\begin{aligned}
\|\mathscr{A}^{\frac{1+\varepsilon}{2}}\boldsymbol{v}^n(t)\|_2^2+\gamma\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n(t)\|_2^2 &\leq 2(1+\|\mathscr{A}^{\frac{1+\varepsilon}{2}}\boldsymbol{v}^n(0)\|_2^2+\gamma\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}^n(0)\|_2^2) \\
&\leq 2(1+\|\mathscr{A}^{\frac{1+\varepsilon}{2}}\boldsymbol{v}(0)\|_2^2+\gamma\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}(0)\|_2^2), \qquad (3.52)
\end{aligned}
$$

for all $t\in[0,T^*]$. This gives that $\{\boldsymbol{v}^n\}_{n\in\mathbb{N}}$ is a bounded sequence in $\mathrm{L}^\infty(0,T^*;\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))$, consequently the unique solution $\boldsymbol{v}$ to system (3.1) belongs to $\mathrm{L}^\infty(0,T^*;\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))$. This completes the proof. □

Remark 3.1. Using the local well-posedness result provided by Theorem 3.2, we are able to formulate and solve the optimal control problem locally in time for $\boldsymbol{v}_0\in D(\mathscr{A}^{1+\frac{\varepsilon}{2}})$ and $\boldsymbol{f}\in\mathrm{L}^2(0,T^*;\dot{\mathbb{H}}^\varepsilon(\mathbb{T}^2))$, $0<\varepsilon<1$. The formulation of the global (in time) control problem for $\boldsymbol{v}_0\in D(\mathscr{A}^{1+\frac{\varepsilon}{2}})$ and

$\boldsymbol{f} \in \mathrm{L}^2(0,T^*;\dot{\mathbb{H}}^{\varepsilon}(\mathbb{T}^2))$, $0<\varepsilon<1$, relies on the existence of the global solution in $\mathrm{L}^{\infty}(0,T;\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))$, which is not an easy issue, due to strong nonlinearities in $(3.1)_1$. Nevertheless, we should emphasize that by assuming the existence of global (in time) solution in $\mathrm{L}^{\infty}(0,T;\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))$, $0<\varepsilon<1$, all the analysis performed for local solutions can be extended with minor adjustments to global ones in order to solve the control problem and state the first order optimality conditions.

In the more regular case $\boldsymbol{v}_0 \in \mathrm{D}(\mathscr{A}^{\frac{3}{2}})$ and $\boldsymbol{f} \in \mathrm{L}^2(0,T;\dot{\mathbb{H}}^1(\mathbb{T}^2))$, we will show that the state equation has a global (in time) solution $\boldsymbol{v} \in \mathrm{L}^{\infty}(0,T;\mathrm{D}(\mathscr{A}^{\frac{3}{2}}))$, and solve the optimal control problem globally in time.

Theorem 3.3. *Assume that $\boldsymbol{v}_0 \in \mathrm{D}(\mathscr{A}^{\frac{3}{2}})$ and $\boldsymbol{f} \in \mathrm{L}^2(0,T;\dot{\mathbb{H}}^1(\mathbb{T}^2))$. Then, for any given $T>0$, the unique solution $\boldsymbol{v}$ to system (3.1) belongs to $\mathrm{L}^{\infty}(0,T;\mathrm{D}(\mathscr{A}^{\frac{3}{2}}))$.*

Proof. Let us recall the following inequality from [8, Proposition 1] which will be used in the sequel: let $\varepsilon_0>0$, then for any $\boldsymbol{v} \in \dot{\mathbb{H}}^{1+\varepsilon_0}(\mathbb{T}^2)$

$$\|\boldsymbol{v}\|_{\infty} \leq \frac{C_*}{\sqrt{\varepsilon_0}}\|\boldsymbol{v}\|_{\dot{\mathbb{H}}^{1+\varepsilon_0}}, \tag{3.53}$$

where C_* is a universal constant independent of ε_0. Taking the inner product with $\mathscr{A}^2\boldsymbol{v}^n$ in $(3.4)_1$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\mathscr{A}\boldsymbol{v}^n(t)\|_2^2+\gamma\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n(t)\|_2^2\right]+\mu\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n(t)\|_2^2\\
&= -b(\boldsymbol{v}^n(t),\boldsymbol{v}^n(t),\mathscr{A}^2\boldsymbol{v}^n(t)) - \alpha\left(\mathrm{div}[(\mathrm{A}(\boldsymbol{v}^n(t)))^2],\mathscr{A}^2\boldsymbol{v}^n(t)\right)\\
&\quad - \beta\left(\mathrm{div}\left[|\mathrm{A}(\boldsymbol{v}^n(t))|^2\mathrm{A}(\boldsymbol{v}^n(t))\right],\mathscr{A}^2\boldsymbol{v}^n(t)\right)\\
&\quad + (\boldsymbol{f}_n(t),\mathscr{A}^2\boldsymbol{v}^n(t)),
\end{aligned}\tag{3.54}$$

for a.e. $t\in[0,T]$. Next we estimate all the terms on the right hand side of the equation (3.54) one by one. Using integration by parts, Hölder's, interpolation and Young's inequalities and (3.53), we estimate for $0<\varepsilon_0\leq 1$

$$\begin{aligned}
&|b(\boldsymbol{v}^n,\boldsymbol{v}^n,\mathscr{A}^2\boldsymbol{v}^n)|\\
&= |((\boldsymbol{v}^n\cdot\nabla)\boldsymbol{v}^n,\mathscr{A}^2\boldsymbol{v}^n)| = |(\mathrm{div}(\boldsymbol{v}^n\otimes\boldsymbol{v}^n),\mathfrak{D}^2\boldsymbol{v}^n)|\\
&= |(\mathfrak{D}^{\frac{1}{2}}\mathrm{div}(\boldsymbol{v}^n\otimes\boldsymbol{v}^n),\mathfrak{D}^{\frac{3}{2}}\boldsymbol{v}^n)| \leq \|\mathfrak{D}^{\frac{1}{2}}\mathrm{div}(\boldsymbol{v}^n\otimes\boldsymbol{v}^n)\|_2\|\mathfrak{D}^{\frac{3}{2}}\boldsymbol{v}^n\|_2\\
&\leq C\|\mathfrak{D}(\boldsymbol{v}^n\otimes\boldsymbol{v}^n)\|_2\|\mathfrak{D}^{\frac{3}{2}}\boldsymbol{v}^n\|_2 \leq C\|\mathfrak{D}(\boldsymbol{v}^n\otimes\boldsymbol{v}^n)\|_2\|\mathfrak{D}^{\frac{3}{2}}\boldsymbol{v}^n\|_2\\
&\leq C\|\mathfrak{D}\boldsymbol{v}^n\|_2\|\boldsymbol{v}^n\|_{\infty}\|\mathfrak{D}^{\frac{3}{2}}\boldsymbol{v}^n\|_2 \leq C\|\mathscr{A}\boldsymbol{v}^n\|_2^2\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n\|_2\\
&\leq \frac{\mu}{8}\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n\|_2^2 + C\|\mathscr{A}\boldsymbol{v}^n\|_2^4,
\end{aligned}\tag{3.55}$$

$$\begin{aligned}
&(\boldsymbol{f}_n,\mathscr{A}^2\boldsymbol{v}^n)\\
&= (\mathscr{A}^{\frac{1}{2}}\boldsymbol{f}_n,\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n)\\
&\leq \|\mathscr{A}^{\frac{1}{2}}\boldsymbol{f}_n\|_2\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n\|_2\\
&\leq \|\boldsymbol{f}\|_{\mathbb{H}^1}\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n\|_2 \leq \frac{\mu}{8}\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n\|_2^2 + C\|\boldsymbol{f}\|_{\mathbb{H}^1}^2,
\end{aligned}\tag{3.56}$$

$$|\alpha\left(\mathrm{div}[(\mathrm{A}(\boldsymbol{v}^n))^2],\mathscr{A}^2\boldsymbol{v}^n\right)|$$

$$
\begin{aligned}
&= \left|\frac{\alpha}{2}\left(\mathfrak{D}[(\mathrm{A}(\boldsymbol{v}^n))^2], \mathfrak{D}\mathrm{A}(\boldsymbol{v}^n)\right)\right| \leq C\|\mathfrak{D}[(\mathrm{A}(\boldsymbol{v}^n))^2]\|_2\|\mathfrak{D}^{\frac{3}{2}}\boldsymbol{v}^n\|_2 \\
&\leq C\|\mathfrak{D}\mathrm{A}(\boldsymbol{v}^n)\|_2\|\mathrm{A}(\boldsymbol{v}^n)\|_\infty\|\mathfrak{D}^{\frac{3}{2}}\boldsymbol{v}^n\|_2 + C\sum_m\|\partial_m\mathrm{A}(\boldsymbol{v}^n)\|_4^2\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n\|_2 \\
&\leq \frac{C}{\sqrt{\varepsilon_0}}\|\mathscr{A}^{1+\varepsilon_0}\boldsymbol{v}^n\|_2\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n\|_2^2 + C\|\mathscr{A}\boldsymbol{v}^n\|_2\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n\|_2^2 \leq \frac{C}{\sqrt{\varepsilon_0}}\|\mathscr{A}\boldsymbol{v}^n\|_2^{1-\varepsilon_0}\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n\|_2^{2+\varepsilon_0} \\
&\leq \frac{C}{\sqrt{\varepsilon_0}}\|\mathscr{A}\boldsymbol{v}^n\|_2^{1-\varepsilon_0}\left[\|\mathscr{A}\boldsymbol{v}^n\|_2^2 + \gamma\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n\|_2^2\right]^{1+\frac{\varepsilon_0}{2}}, \qquad (3.57)
\end{aligned}
$$

$$
\begin{aligned}
&\beta\left(\mathrm{div}|\mathrm{A}(\boldsymbol{v}^n)|^2\mathrm{A}(\boldsymbol{v}^n), \mathscr{A}^2\boldsymbol{v}^n\right) \\
&= -\frac{\beta}{2}\left(|\mathrm{A}(\boldsymbol{v}^n)|^2\mathrm{A}(\boldsymbol{v}^n), \mathfrak{D}^2\mathrm{A}(\boldsymbol{v}^n)\right) \\
&= -\beta\sum_{i,j,k,l,m}\int_{\mathbb{T}^2} a_{ij}^n(x)\partial_m a_{ij}^n(x)a_{kl}^n(x)\partial_m\mathscr{A}a_{kl}(x)dx \\
&\quad -\frac{\beta}{2}\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}[a_{ij}^n(x)]^2\partial_m a_{kl}^n(x)\partial_m\mathscr{A}a_{kl}(x)dx \\
&= \beta\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}\Big[\partial_m a_{ij}^n(x)\partial_m a_{ij}^n(x)a_{kl}^n(x)\mathscr{A}a_{kl}(x) + a_{ij}^n(x)\partial_m^2 a_{ij}^n(x)a_{kl}^n(x)\mathscr{A}a_{kl}(x) \\
&\qquad + a_{ij}^n(x)\partial_m a_{ij}^n(x)\partial_m a_{kl}^n(x)\mathscr{A}a_{kl}(x)\Big]dx \\
&\quad + \frac{\beta}{2}\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}\left[2a_{ij}^n(x)\partial_m a_{ij}^n(x)\partial_m a_{kl}^n(x)\mathscr{A}a_{kl}(x) + [a_{ij}^n(x)]^2\partial_m^2 a_{kl}^n(x)\mathscr{A}a_{kl}(x)\right]dx \\
&\leq \beta\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}\Big[\partial_m a_{ij}^n(x)\partial_m a_{ij}^n(x)a_{kl}^n(x)\mathscr{A}a_{kl}(x) \\
&\qquad + 2a_{ij}^n(x)\partial_m a_{ij}^n(x)\partial_m a_{kl}^n(x)\mathscr{A}a_{kl}(x)\Big]dx \\
&\leq C\sum_m\int_{\mathbb{T}^2}|\partial_m\mathrm{A}(\boldsymbol{v}^n(x))|^2|\mathrm{A}(\boldsymbol{v}^n(x))||\mathfrak{D}\mathrm{A}(\boldsymbol{v}^n(x))|dx \\
&\leq C\sum_m\|\partial_m\mathrm{A}(\boldsymbol{v}^n)\|_4^2\|\mathrm{A}(\boldsymbol{v}^n)\|_\infty\|\mathfrak{D}\mathrm{A}(\boldsymbol{v}^n)\|_2 \\
&\leq C\|\mathrm{A}(\boldsymbol{v}^n)\|_\infty\|\mathscr{A}\boldsymbol{v}^n\|_2\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n\|_2^2 \leq \frac{C}{\sqrt{\varepsilon_0}}\|\mathscr{A}\boldsymbol{v}^n\|_2^{2-\varepsilon_0}\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n\|_2^{2+\varepsilon_0} \\
&\leq \frac{C}{\sqrt{\varepsilon_0}}\|\mathscr{A}\boldsymbol{v}^n\|_2^{2-\varepsilon_0}\left[\|\mathscr{A}\boldsymbol{v}^n\|_2^2 + \gamma\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n\|_2^2\right]^{1+\frac{\varepsilon_0}{2}}. \qquad (3.58)
\end{aligned}
$$

Define $\mathbf{Y}(t) = 1 + \|\mathscr{A}\boldsymbol{v}^n(t)\|_2^2 + \gamma\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n(t)\|_2^2$ so that, in view of the above estimates, $\mathbf{Y}(t)$ satisfies

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{Y}(t)}{\mathrm{d}t} &\leq C\|\mathscr{A}\boldsymbol{v}^n\|_2^4 + C\|\boldsymbol{f}\|_{\mathbb{H}^1}^2 + \frac{C}{\sqrt{\varepsilon_0}}\left[\|\mathscr{A}\boldsymbol{v}^n\|_2^{1-\varepsilon_0} + \|\mathscr{A}\boldsymbol{v}^n\|_2^{2-\varepsilon_0}\right](\mathbf{Y}(t)-1)^{1+\frac{\varepsilon_0}{2}} \\
&\leq C\|\boldsymbol{f}\|_{\mathbb{H}^1}^2 + \frac{C}{\sqrt{\varepsilon_0}}\left[1 + \|\mathscr{A}\boldsymbol{v}^n\|_2^2\right]\left\{1 + (\mathbf{Y}(t)-1)^{1+\frac{\varepsilon_0}{2}}\right\} \\
&\leq C\|\boldsymbol{f}\|_{\mathbb{H}^1}^2 + \frac{C}{\sqrt{\varepsilon_0}}\left[1 + \|\mathscr{A}\boldsymbol{v}^n\|_2^2\right]\mathbf{Y}(t)^{1+\frac{\varepsilon_0}{2}}, \qquad (3.59)
\end{aligned}
$$

where we have used that $1 + (y-1)^{\frac{3}{2}} \leq y^{\frac{3}{2}}$ for any $y \geq 1$. Therefore, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left[\{\mathbf{Y}(t)\}^{-\frac{\varepsilon_0}{2}}\right] = -\frac{\varepsilon_0}{2\mathbf{Y}(t)^{1+\frac{\varepsilon_0}{2}}}\frac{\mathrm{d}\mathbf{Y}(t)}{\mathrm{d}t}
$$

$$
\begin{aligned}
&\geq -\frac{\varepsilon_0}{2\mathbf{Y}(t)^{1+\frac{\varepsilon_0}{2}}}\left[C\|\boldsymbol{f}(t)\|^2_{\mathbb{H}^1}+\frac{C}{\sqrt{\varepsilon_0}}\left[1+\|\mathscr{A}\boldsymbol{v}^n(t)\|^2_2\right]\mathbf{Y}(t)^{1+\frac{\varepsilon_0}{2}}\right]\\
&\geq -C\sqrt{\varepsilon_0}\left[1+\|\boldsymbol{f}(t)\|^2_{\mathbb{H}^1}+\|\mathscr{A}\boldsymbol{v}^n(t)\|^2_2\right], \qquad (3.60)
\end{aligned}
$$

which implies

$$
\{\mathbf{Y}(t)\}^{-\frac{\varepsilon_0}{2}} \geq \{\mathbf{Y}(0)\}^{-\frac{\varepsilon_0}{2}} - C\sqrt{\varepsilon_0}\int_0^t \left[1+\|\boldsymbol{f}(s)\|^2_{\mathbb{H}^1}+\|\mathscr{A}\boldsymbol{v}^n(s)\|^2_2\right]\mathrm{d}s. \tag{3.61}
$$

In view of (3.15) and the fact that $\boldsymbol{f}\in \mathrm{L}^2(0,T;\dot{\mathbb{H}}^1(\mathbb{T}^2))$, we infer that, for any given time T, there exists $0<\varepsilon_*=\varepsilon_*(T)\leq 1$ such that for any $t\in[0,T]$

$$
C\int_0^t \left[1+\|\boldsymbol{f}(s)\|^2_{\mathbb{H}^1}+\|\mathscr{A}\boldsymbol{v}^n(s)\|^2_2\right]\mathrm{d}s < 2{\varepsilon_0}^{-\frac{1}{2}}\{\mathbf{Y}(0)\}^{-\frac{\varepsilon_0}{2}}. \tag{3.62}
$$

Note that such an ε_* exists as the limit when $\varepsilon_0\to 0$ of the left-hand side is $+\infty$. Moreover, one can choose ε_* small enough which will be suitable for any given $T>0$. Hence, we obtain

$$
\{\mathbf{Y}(t)\}^{-\frac{\varepsilon_*}{2}} \geq \frac{\{\mathbf{Y}(0)\}^{-\frac{\varepsilon_*}{2}}}{2}, \tag{3.63}
$$

for $t\in[0,T]$, equivalently

$$
\begin{aligned}
\|\mathscr{A}\boldsymbol{v}^n(t)\|^2_2+\gamma\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n(t)\|^2_2 &\leq 4^{\frac{1}{\varepsilon_*}}(1+\|\mathscr{A}\boldsymbol{v}^n(0)\|^2_2+\gamma\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}^n(0)\|^2_2)\\
&\leq 4^{\frac{1}{\varepsilon_*}}(1+\|\mathscr{A}\boldsymbol{v}(0)\|^2_2+\gamma\|\mathscr{A}^{\frac{3}{2}}\boldsymbol{v}(0)\|^2_2),
\end{aligned} \tag{3.64}
$$

for $t\in[0,T]$. This gives that $\{\boldsymbol{v}^n\}_{n\in\mathbb{N}}$ is a bounded sequence in $\mathrm{L}^\infty(0,T;\mathrm{D}(\mathscr{A}^{\frac{3}{2}}))$, consequently the unique solution $\boldsymbol{v}$ to system (3.1) belongs to $\mathrm{L}^\infty(0,T;\mathrm{D}(\mathscr{A}^{\frac{3}{2}}))$. This completes the proof. □

4. Linearized state equation

In the present section, we establish well-posedness results for the linearized state equation. Let us denote by $\mathscr{T}$ the time horizon such that the unique solution $\boldsymbol{v}$ to the state equation (3.1) in the sense of Definition 3.1 exists and belongs to $\mathrm{L}^\infty(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))$ (for some $\varepsilon>0$). Also, recall that $\mathrm{A}(\boldsymbol{v})=\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^T$ and $[\mathrm{A}(\boldsymbol{v})]_{ij}:=a_{ij}$.

Remark 4.1. Taking into account Theorem 3.2, we have $\mathscr{T}\leq T^*$ for $\boldsymbol{v}_0\in\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}})$ and $\boldsymbol{f}\in\mathrm{L}^2(0,T;\dot{\mathbb{H}}^\varepsilon(\mathbb{T}^2))$, $0<\varepsilon<1$. On the other hand, due to Theorem 3.3, $\mathscr{T}$ is a positive arbitrary time for $\boldsymbol{v}_0\in\mathrm{D}(\mathscr{A}^{\frac{3}{2}})$ and $\boldsymbol{f}\in\mathrm{L}^2(0,T;\dot{\mathbb{H}}^1(\mathbb{T}^2))$.

Let $\boldsymbol{\psi}:\mathbb{T}^2\times(0,\mathscr{T})\to\mathbb{R}^2$ be a force such that $\boldsymbol{\psi}\in\mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{L}}^2(\mathbb{T}^2))$ and consider the following system:

$$\begin{cases}\partial_t(\boldsymbol{z}-\gamma\Delta\boldsymbol{z})=\mathbb{P}\boldsymbol{\psi}-\nabla\pi+\mu\Delta\boldsymbol{z}-(\boldsymbol{v}\cdot\nabla)\boldsymbol{z}-(\boldsymbol{z}\cdot\nabla)\boldsymbol{v}+\alpha\mathrm{div}(\mathrm{A}(\boldsymbol{v})\mathrm{A}(\boldsymbol{z}) \\ \qquad+\mathrm{A}(\boldsymbol{z})\mathrm{A}(\boldsymbol{v}))+\beta\big[\mathrm{div}\big(|\mathrm{A}(\boldsymbol{v})|^2\mathrm{A}(\boldsymbol{z})\big)\big] \\ \qquad+2\beta\mathrm{div}\big((\mathrm{A}(\boldsymbol{z}):\mathrm{A}(\boldsymbol{v}))\mathrm{A}(\boldsymbol{v})\big), & \text{in } \mathbb{T}^2\times(0,\mathscr{T}], \\ \mathrm{div}\,\boldsymbol{z}=0, & \text{in } \mathbb{T}^2\times[0,\mathscr{T}], \\ \boldsymbol{z}(x,0)=\mathbf{0}, & \text{in } \mathbb{T}^2,\end{cases} \tag{4.1}$$

with periodic boundary conditions, where $\boldsymbol{z}=(z^1,z^2)$ and π are unknown vector and scalar fields, respectively, and $\mathrm{A}(\boldsymbol{z})=\nabla\boldsymbol{z}+(\nabla\boldsymbol{z})^T$. Applying the projection $\mathbb{P}$ to the system (4.1), we obtain

$$\begin{cases}\partial_t(\boldsymbol{z}+\gamma\mathscr{A}\boldsymbol{z})=\mathbb{P}\boldsymbol{\psi}-\mu\mathscr{A}\boldsymbol{z}-\mathscr{B}(\boldsymbol{v},\boldsymbol{z})-\mathscr{B}(\boldsymbol{z},\boldsymbol{v})+\alpha\mathbb{P}\big[\mathrm{div}\big(\mathrm{A}(\boldsymbol{v})\mathrm{A}(\boldsymbol{z})+\mathrm{A}(\boldsymbol{z})\mathrm{A}(\boldsymbol{v})\big)\big] \\ \qquad+\beta\mathbb{P}\big[\mathrm{div}\big(|\mathrm{A}(\boldsymbol{v})|^2\mathrm{A}(\boldsymbol{z})\big)\big]+2\beta\mathbb{P}\big[\mathrm{div}\big((\mathrm{A}(\boldsymbol{z}):\mathrm{A}(\boldsymbol{v}))\mathrm{A}(\boldsymbol{v})\big)\big], \\ \boldsymbol{z}(0)=\mathbf{0},\end{cases} \tag{4.2}$$

We will establish the well-posedness of system (4.2) in the following sense:

Definition 4.1. A function $\boldsymbol{z}\in\mathrm{C}([0,\mathscr{T}];\mathbb{V})\cap\mathrm{L}^\infty(0,\mathscr{T};\mathrm{D}(\mathscr{A}))$ with $\partial_t\boldsymbol{z}\in\mathrm{L}^2(0,\mathscr{T};\mathbb{V})$, is called a *solution* to system (4.2), if for $\boldsymbol{\psi}\in\mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{L}}^2(\mathbb{T}^2))$, it satisfies

(i) for any $\boldsymbol{\phi}\in\mathbb{V}$,

$$\begin{aligned}&\langle\partial_t[\mathrm{I}+\gamma\mathscr{A}]\boldsymbol{z}(t),\boldsymbol{\phi}\rangle\\&=\big\langle\boldsymbol{\psi}(t)-\mu\mathscr{A}\boldsymbol{z}(t)-\mathscr{B}(\boldsymbol{v}(t),\boldsymbol{z}(t))-\mathscr{B}(\boldsymbol{z}(t),\boldsymbol{v}(t))\\&\quad+\alpha\mathrm{div}\big(\mathrm{A}(\boldsymbol{v}(t))\mathrm{A}(\boldsymbol{z}(t))+\mathrm{A}(\boldsymbol{z}(t))\mathrm{A}(\boldsymbol{v}(t))\big)\\&\quad+\beta\mathrm{div}\big(|\mathrm{A}(\boldsymbol{v}(t))|^2\mathrm{A}(\boldsymbol{z}(t))\big)+2\beta\mathrm{div}\big((\mathrm{A}(\boldsymbol{z}(t)):\mathrm{A}(\boldsymbol{v}(t)))\mathrm{A}(\boldsymbol{v}(t))\big),\boldsymbol{\phi}\big\rangle,\end{aligned} \tag{4.3}$$

for a.e. $t\in[0,\mathscr{T}]$;

(ii) the initial data:

$$\boldsymbol{z}(0)=\mathbf{0}.$$

The next theorem is the main result of this section, which provides the existence and uniqueness of global solutions to system (4.3) in the sense of Definition 4.1 satisfying energy equality.

Theorem 4.1. *There exists a unique solution $\boldsymbol{z}$ to system (4.3) in the sense of Definition 4.1 which satisfies the following energy equality:*

$$\begin{aligned}&\|\boldsymbol{z}(t)\|_2^2+\gamma\|\nabla\boldsymbol{z}(t)\|_2^2+2\mu\int_0^t\|\nabla\boldsymbol{z}(s)\|_2^2\mathrm{d}s+\beta\int_0^t\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}(x,s))|^2|\mathrm{A}(\boldsymbol{z}(x,s))|^2\mathrm{d}x\mathrm{d}s\\&=2\int_0^t(\boldsymbol{\psi}(s),\boldsymbol{z}(s))\mathrm{d}s-2\int_0^t b(\boldsymbol{z}(s),\boldsymbol{v}(s),\boldsymbol{z}(s))\mathrm{d}s\\&\quad+\alpha\int_0^t\int_{\mathbb{T}^2}[\mathrm{A}(\boldsymbol{v}(x,s))\mathrm{A}(\boldsymbol{z}(x,s))]:\mathrm{A}(\boldsymbol{z}(x,s))\mathrm{d}x\mathrm{d}s\\&\quad+\alpha\int_0^t\int_{\mathbb{T}^2}[\mathrm{A}(\boldsymbol{z}(x,s))\mathrm{A}(\boldsymbol{v}(x,s))]:\mathrm{A}(\boldsymbol{z}(x,s))\mathrm{d}x\mathrm{d}s\end{aligned}$$

$$- 2\beta \int_0^t \int_{\mathbb{T}^2} [\mathrm{A}(\boldsymbol{v}(x,s)) : \mathrm{A}(\boldsymbol{z}(x,s))]^2 \,\mathrm{d}x\mathrm{d}s, \tag{4.4}$$

for all $t \in [0, \mathscr{T}]$.

Proof of Theorem 4.1. The proof is divided into the following four steps.
Step I. *Finite-dimensional approximation.* Let us consider the following approximate equation for system (4.2) on the finite-dimensional space $\mathbb{H}_n$:

$$\begin{cases} \partial_t(\boldsymbol{z}^n(t) + \gamma \mathscr{A} \boldsymbol{z}^n(t)) = \mathrm{P}_n \mathbb{P} \boldsymbol{\psi}(t) - \mu \mathrm{P}_n \mathscr{A} \boldsymbol{z}^n(t) - \mathrm{P}_n \mathscr{B}(\boldsymbol{v}(t), \boldsymbol{z}^n(t)) - \mathrm{P}_n \mathscr{B}(\boldsymbol{z}^n(t), \boldsymbol{v}(t)) \\ \qquad\qquad + \alpha \mathrm{P}_n \mathbb{P}\big[\mathrm{div}\big(\mathrm{A}(\boldsymbol{v}(t))\mathrm{A}(\boldsymbol{z}^n(t)) + \mathrm{A}(\boldsymbol{z}^n(t))\mathrm{A}(\boldsymbol{v}(t))\big)\big] \\ \qquad\qquad + \beta \mathrm{P}_n \mathbb{P}\big[\mathrm{div}\big(|\mathrm{A}(\boldsymbol{v}(t))|^2 \mathrm{A}(\boldsymbol{z}^n(t))\big)\big] + 2\beta \mathrm{P}_n \mathbb{P}\big[\mathrm{div}\big((\mathrm{A}(\boldsymbol{z}^n(t)) : \\ \qquad\qquad \mathrm{A}(\boldsymbol{v}(t)))\mathrm{A}(\boldsymbol{v}(t))\big)\big], \\ \boldsymbol{z}^n(0) = \boldsymbol{0}, \end{cases} \tag{4.5}$$

for a.e. $t \in [0, \mathscr{T}]$. Note that system (4.5) defines a system of linear ordinary differential equations, which has a unique local solution $\boldsymbol{z}^n \in \mathrm{C}([0, \mathscr{T}_n]; \mathbb{H}_n)$, for some $0 < \mathscr{T}_n \leq \mathscr{T}$. The following a priori estimates show that the time $\mathscr{T}_n$ can be extended to time $\mathscr{T}$.
Step II. *A priori estimates.* Taking the inner product with $\boldsymbol{z}^n$ to $(4.5)_1$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\|\boldsymbol{z}^n(t)\|_2^2 + \gamma \|\nabla \boldsymbol{z}^n(t)\|_2^2] + \mu \|\nabla \boldsymbol{z}^n(t)\|_2^2 + \frac{\beta}{2}\||\mathrm{A}(\boldsymbol{v}(t))|\mathrm{A}(\boldsymbol{z}^n(t))\|_2^2 \\ &\quad + \beta \int_{\mathbb{T}^2} (\mathrm{A}(\boldsymbol{z}^n(x,t)) : \mathrm{A}(\boldsymbol{v}(x,t)))^2 \mathrm{d}x \\ &= (\boldsymbol{\psi}(t), \boldsymbol{z}^n(t)) + b(\boldsymbol{z}^n(t), \boldsymbol{z}^n(t), \boldsymbol{v}(t)) - \alpha \left\langle \mathrm{A}(\boldsymbol{v}(t))\mathrm{A}(\boldsymbol{z}^n(t)) \right. \\ &\quad \left. + \mathrm{A}(\boldsymbol{z}^n(t))\mathrm{A}(\boldsymbol{v}(t))), \nabla \boldsymbol{z}^n(t) \right\rangle, \end{aligned} \tag{4.6}$$

for a.e. $t \in [0, \mathscr{T}]$. Let us estimate the terms on the right hand side of (4.6) using integration by parts, Hölder's, Agmon's and Young's inequalities as follows:

$$\begin{aligned} |b(\boldsymbol{z}^n, \boldsymbol{z}^n, \boldsymbol{v})| &\leq \|\boldsymbol{z}^n\|_2 \|\nabla \boldsymbol{z}^n\|_2 \|\boldsymbol{v}\|_\infty \leq C \|\boldsymbol{z}^n\|_2 \|\nabla \boldsymbol{z}^n\|_2 \|\mathscr{A}\boldsymbol{v}\|_2 \leq C \|\boldsymbol{z}^n\|_2^2 \|\mathscr{A}\boldsymbol{v}\|_2^2 \\ &\quad + \frac{\mu}{2}\|\nabla \boldsymbol{z}^n\|_2^2, \end{aligned} \tag{4.7}$$

$$|\alpha \left\langle \mathrm{A}(\boldsymbol{v})\mathrm{A}(\boldsymbol{z}^n) + \mathrm{A}(\boldsymbol{z}^n)\mathrm{A}(\boldsymbol{v})), \nabla \boldsymbol{z}^n \right\rangle| \leq \frac{\beta}{2} \int_{\mathbb{T}^2} |\mathrm{A}(\boldsymbol{v}(x))|^2 |\mathrm{A}(\boldsymbol{z}^n(x))|^2 \mathrm{d}x + C \|\nabla \boldsymbol{z}^n\|_2^2, \tag{4.8}$$

$$|(\boldsymbol{\psi}, \boldsymbol{z}^n)| \leq \|\boldsymbol{\psi}\|_2 \|\boldsymbol{z}^n\|_2 \leq C \|\boldsymbol{\psi}\|_2^2 + C \|\boldsymbol{z}^n\|_2^2. \tag{4.9}$$

Combining (4.6)-(4.9), we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\|\boldsymbol{z}^n(t)\|_2^2 + \gamma \|\nabla \boldsymbol{z}^n(t)\|_2^2] + \frac{\mu}{2}\|\nabla \boldsymbol{z}^n(t)\|_2^2 \\ &\leq C \|\boldsymbol{\psi}(t)\|_2^2 + C \left\{1 + \|\mathscr{A}\boldsymbol{v}(t)\|_2^2\right\} [\|\boldsymbol{z}^n(t)\|_2^2 + \gamma \|\nabla \boldsymbol{z}^n(t)\|_2^2], \end{aligned} \tag{4.10}$$

for a.e. $t \in [0, \mathscr{T}]$, and the Gronwall inequality implies

$$\|\boldsymbol{z}^n(t)\|_2^2 + \gamma \|\nabla \boldsymbol{z}^n(t)\|_2^2 + \mu \int_0^t \|\nabla \boldsymbol{z}^n(s)\|_2^2 \mathrm{d}s \leq C e^{Ct + C\int_0^t \|\mathscr{A}\boldsymbol{v}(s)\|_2^2 \mathrm{d}s} \int_0^t \|\boldsymbol{\psi}(s)\|_2^2 \mathrm{d}s, \tag{4.11}$$

for all $t \in [0, \mathscr{T}]$. Using the fact that $\boldsymbol{\psi} \in \mathrm{L}^2(0, \mathscr{T}; \dot{\mathbb{L}}^2(\mathbb{T}^2))$ and $\boldsymbol{v} \in \mathrm{L}^\infty(0, \mathscr{T}; \mathrm{D}(\mathscr{A}))$, we have from (4.11) that

$$\{\boldsymbol{z}^n\}_{n\in\mathbb{N}} \text{ is a bounded sequence in } \mathrm{L}^\infty(0, \mathscr{T}; \mathbb{V}). \tag{4.12}$$

Next, taking the inner product with $\mathscr{A}\boldsymbol{z}^n$ to $(4.5)_1$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\|\nabla\boldsymbol{z}^n(t)\|_2^2 + \gamma\|\mathscr{A}\boldsymbol{z}^n(t)\|_2^2] + \mu\|\mathscr{A}\boldsymbol{z}^n(t)\|_2^2 \\
&= (\boldsymbol{\psi}(t), \mathscr{A}\boldsymbol{z}^n(t)) - b(\boldsymbol{v}(t), \boldsymbol{z}^n(t), \mathscr{A}\boldsymbol{z}^n(t)) - b(\boldsymbol{z}^n(t), \boldsymbol{v}(t), \mathscr{A}\boldsymbol{z}^n(t)) \\
&\quad + \alpha\int_{\mathbb{T}^2} \mathrm{div}[\mathrm{A}(\boldsymbol{v}(x,t))\mathrm{A}(\boldsymbol{z}^n(x,t))] \cdot \mathscr{A}\boldsymbol{z}^n(x,t)\mathrm{d}x \\
&\quad + \alpha\int_{\mathbb{T}^2} \mathrm{div}[\mathrm{A}(\boldsymbol{z}^n(x,t))\mathrm{A}(\boldsymbol{v}(x,t))] \cdot \mathscr{A}\boldsymbol{z}^n(x,t)\mathrm{d}x \\
&\quad + \beta\int_{\mathbb{T}^2} \mathrm{div}[|\mathrm{A}(\boldsymbol{v}(x,t))|^2\mathrm{A}(\boldsymbol{z}^n(x,t)))] \cdot \mathscr{A}\boldsymbol{z}^n(x,t)\mathrm{d}x \\
&\quad + 2\beta\int_{\mathbb{T}^2} \mathrm{div}[(\mathrm{A}(\boldsymbol{z}^n(x,t)) : \mathrm{A}(\boldsymbol{v}(x,t)))\mathrm{A}(\boldsymbol{v}(x,t))] \cdot \mathscr{A}\boldsymbol{z}^n(x,t))\mathrm{d}x,
\end{aligned} \tag{4.13}$$

for a.e. $t \in [0, \mathscr{T}]$. We denote $[\mathrm{A}(\boldsymbol{z}^n)]_{ij} := b_{ij}^n$ which will be used later. Let us estimate the terms on the right hand side of (4.13) using integration by parts, (2.11), (2.12), (2.13), Hölder's, Agmon's and Young's inequalities as follows:

$$\begin{aligned}
|b(\boldsymbol{v}, \boldsymbol{z}^n, \mathscr{A}\boldsymbol{z}^n)| &\leq \|\boldsymbol{v}\|_\infty\|\nabla\boldsymbol{z}^n\|_2\|\mathscr{A}\boldsymbol{z}^n\|_2 \leq C\|\mathscr{A}\boldsymbol{v}\|_2\|\mathscr{A}\boldsymbol{z}^n\|_2^2 \\
&\leq C\|\mathscr{A}\boldsymbol{v}\|_2^2\|\mathscr{A}\boldsymbol{z}^n\|_2^2 + \frac{\mu}{10}\|\mathscr{A}\boldsymbol{z}^n\|_2^2, \qquad (4.14) \\
|b(\boldsymbol{z}^n, \boldsymbol{v}, \mathscr{A}\boldsymbol{z}^n)| &\leq \|\boldsymbol{z}^n\|_\infty\|\nabla\boldsymbol{v}\|_2\|\mathscr{A}\boldsymbol{z}^n\|_2 \leq C\|\mathscr{A}\boldsymbol{v}\|_2\|\mathscr{A}\boldsymbol{z}^n\|_2^2 \\
&\leq C\|\mathscr{A}\boldsymbol{v}\|_2^2\|\mathscr{A}\boldsymbol{z}^n\|_2^2 + \frac{\mu}{10}\|\mathscr{A}\boldsymbol{z}^n\|_2^2, \qquad (4.15) \\
|(\boldsymbol{\psi}, \mathscr{A}\boldsymbol{z}^n)| &\leq \|\boldsymbol{\psi}\|_2\|\mathscr{A}\boldsymbol{z}^n\|_2 \leq C\|\boldsymbol{\psi}\|_2^2 + \frac{\mu}{10}\|\mathscr{A}\boldsymbol{z}^n\|_2^2, \qquad (4.16)
\end{aligned}$$

$$\begin{aligned}
&\alpha\int_{\mathbb{T}^2} \mathrm{div}[\mathrm{A}(\boldsymbol{v}(x))\mathrm{A}(\boldsymbol{z}^n(x))] \cdot \mathscr{A}\boldsymbol{z}^n(x)\mathrm{d}x \\
&= -\frac{\alpha}{2}\int_{\mathbb{T}^2} [\mathrm{A}(\boldsymbol{v}(x))\mathrm{A}(\boldsymbol{z}^n(x))] : \mathscr{A}\mathrm{A}(\boldsymbol{z}^n(x))\mathrm{d}x \\
&= \frac{\alpha}{2}\sum_{i,j,k,m}\int_{\mathbb{T}^2} a_{ik}(x)b_{kj}^n(x)\partial_m^2 b_{ij}^n(x)\mathrm{d}x \\
&= -\frac{\alpha}{2}\sum_{i,j,k,m}\int_{\mathbb{T}^2} \partial_m a_{ik}(x)b_{kj}^n(x)\partial_m b_{ij}^n(x)\mathrm{d}x \\
&\quad - \frac{\alpha}{2}\sum_{i,j,k,m}\int_{\mathbb{T}^2} a_{ik}(x)\partial_m b_{kj}^n(x)\partial_m b_{ij}^n(x)\mathrm{d}x \\
&\leq \frac{|\alpha|}{2}\|\mathrm{A}(\boldsymbol{z}^n)\|_{\frac{2}{\varepsilon}}\|\boldsymbol{v}\|_{\mathbb{W}^{2,\frac{2}{1-\varepsilon}}}\|\boldsymbol{z}^n\|_{\dot{\mathbb{H}}^2} + \frac{|\alpha|}{2}\|\mathrm{A}\boldsymbol{v}\|_\infty\|\boldsymbol{z}^n\|_{\dot{\mathbb{H}}^2}^2 \\
&\leq C\|\boldsymbol{v}\|_{\dot{\mathbb{H}}^{2+\varepsilon}}\|\boldsymbol{z}^n\|_{\dot{\mathbb{H}}^2}^2 \leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2\|\mathscr{A}\boldsymbol{z}^n\|_2^2
\end{aligned}$$

$$\leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2\|\mathscr{A}\boldsymbol{z}^n\|_2^2+\frac{\mu}{10}\|\mathscr{A}\boldsymbol{z}^n\|_2^2, \tag{4.17}$$

$$\begin{aligned}
&\alpha\int_{\mathbb{T}^2}\mathrm{div}[\mathrm{A}(\boldsymbol{z}^n(x))\mathrm{A}(\boldsymbol{v}(x))]\cdot\mathscr{A}\boldsymbol{z}^n(x)\mathrm{d}x\\
&=-\frac{\alpha}{2}\int_{\mathbb{T}^2}[\mathrm{A}(\boldsymbol{z}^n(x))\mathrm{A}(\boldsymbol{v}(x))]:\mathscr{A}\mathrm{A}(\boldsymbol{z}^n(x))\mathrm{d}x\\
&=\frac{\alpha}{2}\sum_{i,j,k,m}\int_{\mathbb{T}^2}b_{ik}^n(x)a_{kj}(x)\partial_m^2 b_{ij}^n(x)\mathrm{d}x\\
&=-\frac{\alpha}{2}\sum_{i,j,k,m}\int_{\mathbb{T}^2}\partial_m b_{ik}^n(x)a_{kj}(x)\partial_m b_{ij}^n(x)\mathrm{d}x\\
&\quad-\frac{\alpha}{2}\sum_{i,j,k,m}\int_{\mathbb{T}^2}b_{ik}^n(x)\partial_m a_{kj}(x)\partial_m b_{ij}^n(x)\mathrm{d}x\\
&\leq\frac{|\alpha|}{2}\|\mathrm{A}\boldsymbol{v}\|_\infty\|\boldsymbol{z}^n\|_{\dot{\mathbb{H}}^2}^2+\frac{|\alpha|}{2}\|\mathrm{A}(\boldsymbol{z}^n)\|_{\frac{2}{\varepsilon}}\|\boldsymbol{v}\|_{\mathbb{W}^{2,\frac{2}{1-\varepsilon}}}\|\boldsymbol{z}^n\|_{\dot{\mathbb{H}}^2}\\
&\leq C\|\boldsymbol{v}\|_{\dot{\mathbb{H}}^{2+\varepsilon}}\|\boldsymbol{z}^n\|_{\dot{\mathbb{H}}^2}^2\leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2\|\mathscr{A}\boldsymbol{z}^n\|_2^2
\end{aligned}$$

$$\leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2\|\mathscr{A}\boldsymbol{z}^n\|_2^2+\frac{\mu}{10}\|\mathscr{A}\boldsymbol{z}^n\|_2^2, \tag{4.18}$$

$$\begin{aligned}
&\beta\int_{\mathbb{T}^2}\mathrm{div}[|\mathrm{A}(\boldsymbol{v}(x))|^2\mathrm{A}(\boldsymbol{z}^n(x))]\cdot\mathscr{A}\boldsymbol{z}^n(x)\mathrm{d}x\\
&=\frac{\beta}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}(x))|^2\mathrm{A}(\boldsymbol{z}^n(x))):\Delta\mathrm{A}(\boldsymbol{z}^n(x))\mathrm{d}x\\
&=\frac{\beta}{2}\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}(a_{ij}(x))^2 b_{kl}^n(x)\partial_m^2 b_{kl}^n(x)\mathrm{d}x\\
&=\underbrace{-\frac{\beta}{2}\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}(a_{ij}(x))^2(\partial_m b_{kl}^n(x))^2\mathrm{d}x}_{\leq 0.}-\beta\\
&\quad\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}a_{ij}(x)\partial_m a_{ij}(x)b_{kl}^n(x)\partial_m b_{kl}^n(x)\mathrm{d}x\\
&\leq\beta\|\mathrm{A}\boldsymbol{v}\|_\infty\|\mathrm{A}\boldsymbol{z}^n\|_{\frac{2}{\varepsilon}}\|\boldsymbol{v}\|_{\mathbb{W}^{2,\frac{2}{1-\varepsilon}}}\|\boldsymbol{z}^n\|_{\dot{\mathbb{H}}^2}
\end{aligned}$$

$$\leq C\|\boldsymbol{v}\|_{\dot{\mathbb{H}}^{2+\varepsilon}}^2\|\boldsymbol{z}^n\|_{\dot{\mathbb{H}}^2}^2\leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2\|\mathscr{A}\boldsymbol{z}^n\|_2^2, \tag{4.19}$$

$$\begin{aligned}
&2\beta\int_{\mathbb{T}^2}\mathrm{div}[(\mathrm{A}(\boldsymbol{z}^n(x)):\mathrm{A}(\boldsymbol{v}(x)))\mathrm{A}(\boldsymbol{v}(x))]\cdot\mathscr{A}\boldsymbol{z}^n(x))\mathrm{d}x\\
&=\beta\int_{\mathbb{T}^2}(\mathrm{A}(\boldsymbol{z}^n(x)):\mathrm{A}(\boldsymbol{v}(x)))(\mathrm{A}(\boldsymbol{v}(x)):\Delta\mathrm{A}(\boldsymbol{z}^n(x)))\mathrm{d}x\\
&=\beta\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}a_{ij}(x)b_{ij}^n(x)a_{kl}(x)\partial_m^2 b_{kl}^n(x)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \underbrace{-\beta\sum_{i,j,k,l,m}\int_{\mathbb{T}^2} a_{ij}(x)\partial_m b^n_{ij}(x)a_{kl}(x)\partial_m b^n_{kl}(x)\mathrm{d}x}_{\leq 0} - \beta \\
&\quad \sum_{i,j,k,l,m}\int_{\mathbb{T}^2}\partial_m a_{ij}(x) b^n_{ij}(x) a_{kl}(x)\partial_m b^n_{kl}(x)\mathrm{d}x\\
&\qquad -\beta \sum_{i,j,k,l,m}\int_{\mathbb{T}^2} a_{ij}(x) b^n_{ij}(x)\partial_m a_{kl}(x)\partial_m b^n_{kl}(x)\mathrm{d}x\\
&\leq \beta\|\mathrm{A}\boldsymbol{v}\|_\infty \|\mathrm{A}\boldsymbol{z}^n\|_{\frac{2}{\varepsilon}} \|\boldsymbol{v}\|_{\mathbb{W}^{2,\frac{2}{1-\varepsilon}}}\|\boldsymbol{z}^n\|_{\dot{\mathbb{H}}^2}\\
&\leq C\|\boldsymbol{v}\|^2_{\dot{\mathbb{H}}^{2+\varepsilon}}\|\boldsymbol{z}^n\|^2_{\dot{\mathbb{H}}^2} \leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2\|\mathscr{A}\boldsymbol{z}^n\|_2^2. \qquad (4.20)
\end{aligned}
$$

Combining (4.14)-(4.20), it yields

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\|\nabla\boldsymbol{z}^n(t)\|_2^2+\gamma\|\mathscr{A}\boldsymbol{z}^n(t)\|_2^2]+\frac{\mu}{2}\|\mathscr{A}\boldsymbol{z}^n(t)\|_2^2\\
&\leq C\|\boldsymbol{\psi}(t)\|_2^2 + C\left\{\|\mathscr{A}\boldsymbol{v}(t)\|_2^2+\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}(t)\|_2^2\right\}\|\mathscr{A}\boldsymbol{z}^n(t)\|_2^2\\
&\leq C\|\boldsymbol{\psi}(t)\|_2^2 + C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}(t)\|_2^2\left[\|\nabla\boldsymbol{z}^n(t)\|_2^2+\gamma\|\mathscr{A}\boldsymbol{z}^n(t)\|_2^2\right], \qquad (4.21)
\end{aligned}
$$

for a.e. $t\in[0,\mathscr{T}]$, and the Gronwall inequality implies

$$
\begin{aligned}
&\|\nabla\boldsymbol{z}^n(t)\|_2^2+\gamma\|\mathscr{A}\boldsymbol{z}^n(t)\|_2^2+\mu\int_0^t\|\mathscr{A}\boldsymbol{z}^n(s)\|_2^2\mathrm{d}s\\
&\leq Ce^{C\int_0^t\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}(s)\|_2^2\mathrm{d}s}\int_0^t\|\boldsymbol{\psi}(s)\|_2^2\mathrm{d}s, \qquad (4.22)
\end{aligned}
$$

for all $t\in[0,\mathscr{T}]$. Using the fact that $\boldsymbol{\psi}\in\mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{L}}^2(\mathbb{T}^2))$ and $\boldsymbol{v}\in\mathrm{L}^2(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))$, we have from (4.22) that

$$
\{\boldsymbol{z}^n\}_{n\in\mathbb{N}} \text{ is a bounded sequence in } \mathrm{L}^\infty(0,\mathscr{T};\mathrm{D}(\mathscr{A})). \qquad (4.23)
$$

Taking the inner product with $\partial_t\boldsymbol{z}^n$ to $(4.5)_1$ and using integration by parts, we obtain

$$
\begin{aligned}
&\|\partial_t\boldsymbol{z}^n(t)\|_2^2+\gamma\|\partial_t\nabla\boldsymbol{z}^n(t)\|_2^2+\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\boldsymbol{z}^n(t)\|_2^2\\
&=(\boldsymbol{\psi}(t),\partial_t\boldsymbol{z}^n(t))-b(\boldsymbol{v}(t),\boldsymbol{z}^n(t),\partial_t\boldsymbol{z}^n(t))-b(\boldsymbol{z}^n(t),\boldsymbol{v}(t),\partial_t\boldsymbol{z}^n(t))\\
&\quad -\alpha\big(\mathrm{A}(\boldsymbol{v}(t))\mathrm{A}(\boldsymbol{z}^n(t)),\partial_t\nabla\boldsymbol{z}^n(t)\big)-\alpha\big(\mathrm{A}(\boldsymbol{z}^n(t))\mathrm{A}(\boldsymbol{v}(t)),\partial_t\nabla\boldsymbol{z}^n(t)\big)\\
&-\beta\big(|\mathrm{A}(\boldsymbol{v}(t))|^2\mathrm{A}(\boldsymbol{z}^n(t)),\partial_t\nabla\boldsymbol{z}^n(t)\big)\\
&\quad -2\beta\big((\mathrm{A}(\boldsymbol{z}^n(t)):\mathrm{A}(\boldsymbol{v}(t)))(\mathrm{A}(\boldsymbol{v}(t)),\partial_t\nabla\boldsymbol{z}^n(t))\big), \qquad (4.24)
\end{aligned}
$$

for a.e. $t\in[0,\mathscr{T}]$. Let us estimate the terms on the right hand side of (4.24) using (2.12), Hölder's, Agmon's and Young's inequalities as follows:

$$
\begin{aligned}
&|b(\boldsymbol{v},\boldsymbol{z}^n,\partial_t\boldsymbol{z}^n)|\\
&\leq\|\boldsymbol{v}\|_\infty\|\nabla\boldsymbol{z}^n\|_2\|\partial_t\boldsymbol{z}^n\|_2\leq C\|\mathscr{A}\boldsymbol{v}\|_2\|\nabla\boldsymbol{z}^n\|_2\|\partial_t\boldsymbol{z}^n\|_2
\end{aligned}
$$

$$\leq C\|\mathscr{A}\boldsymbol{v}\|_2^2\|\nabla\boldsymbol{z}^n\|_2^2+\frac{1}{6}\|\partial_t\boldsymbol{z}^n\|_2^2, \tag{4.25}$$

$$\begin{aligned}
&|b(\boldsymbol{z}^n,\boldsymbol{v},\partial_t\boldsymbol{z}^n)|\\
&\leq \|\boldsymbol{z}^n\|_2\|\nabla\boldsymbol{v}\|_\infty\|\partial_t\boldsymbol{z}^n\|_2\leq C\|\mathscr{A}\boldsymbol{v}\|_2\|\nabla\boldsymbol{z}^n\|_2\|\partial_t\boldsymbol{z}^n\|_2\\
&\leq C\|\mathscr{A}\boldsymbol{v}\|_2^2\|\nabla\boldsymbol{z}^n\|_2^2+\frac{1}{6}\|\partial_t\boldsymbol{z}^n\|_2^2,
\end{aligned} \tag{4.26}$$

$$|(\boldsymbol{\psi},\partial_t\boldsymbol{z}^n)|\leq\|\boldsymbol{\psi}\|_2\|\partial_t\boldsymbol{z}^n\|_2\leq\frac{3}{2}\|\boldsymbol{\psi}\|_2^2+\frac{1}{6}\|\partial_t\boldsymbol{z}^n\|_2^2, \tag{4.27}$$

$$\begin{aligned}
&\big|\alpha\big(\mathrm{A}(\boldsymbol{v})\mathrm{A}(\boldsymbol{z}^n),\partial_t\nabla\boldsymbol{z}^n\big)\big|\\
&\leq|\alpha|\|\mathrm{A}(\boldsymbol{v})\|_\infty\|\mathrm{A}(\boldsymbol{z}^n)\|_2\|\partial_t\nabla\boldsymbol{z}^n\|_2\leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2\|\nabla\boldsymbol{z}^n\|_2\|\partial_t\nabla\boldsymbol{z}^n\|_2\\
&\leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2\|\nabla\boldsymbol{z}^n\|_2^2+\frac{\gamma}{8}\|\partial_t\nabla\boldsymbol{z}^n\|_2^2,
\end{aligned} \tag{4.28}$$

$$\begin{aligned}
&\big|\alpha\big(\mathrm{A}(\boldsymbol{z}^n)\mathrm{A}(\boldsymbol{v}),\partial_t\nabla\boldsymbol{z}^n\big)\big|\\
&\leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2\|\nabla\boldsymbol{z}^n\|_2^2+\frac{\gamma}{8}\|\partial_t\nabla\boldsymbol{z}^n\|_2^2,
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
&\big|\beta\big(|\mathrm{A}(\boldsymbol{v})|^2\mathrm{A}(\boldsymbol{z}^n),\partial_t\nabla\boldsymbol{z}^n\big)\big|\\
&\leq\beta\|\mathrm{A}(\boldsymbol{v})\|_\infty^2\|\mathrm{A}(\boldsymbol{z}^n)\|_2\|\partial_t\nabla\boldsymbol{z}^n\|_2\leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2\|\nabla\boldsymbol{z}^n\|_2\|\partial_t\nabla\boldsymbol{z}^n\|_2\\
&\leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^4\|\nabla\boldsymbol{z}^n\|_2^2+\frac{\gamma}{8}\|\partial_t\nabla\boldsymbol{z}^n\|_2^2,
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
&\big|2\beta\big((\mathrm{A}(\boldsymbol{z}^n):\mathrm{A}(\boldsymbol{v}))(\mathrm{A}(\boldsymbol{v}),\partial_t\nabla\boldsymbol{z}^n)\big)\big|\\
&\leq 2\beta\|\mathrm{A}(\boldsymbol{v})\|_\infty^2\|\mathrm{A}(\boldsymbol{z}^n)\|_2\|\partial_t\nabla\boldsymbol{z}^n\|_2\leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^4\|\nabla\boldsymbol{z}^n\|_2^2+\frac{\gamma}{8}\|\partial_t\nabla\boldsymbol{z}^n\|_2^2.
\end{aligned} \tag{4.31}$$

Combining (4.24)-(4.31), we obtain

$$\begin{aligned}
&\|\partial_t\boldsymbol{z}^n(t)\|_2^2+\gamma\|\partial_t\nabla\boldsymbol{z}^n(t)\|_2^2+\mu\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\boldsymbol{z}^n(t)\|_2^2\leq 3\|\boldsymbol{\psi}(t)\|_2^2\\
&\quad+C\left[\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}(t)\|_2^2+\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}(t)\|_2^4\right]\|\nabla\boldsymbol{z}^n(t)\|_2^2,
\end{aligned} \tag{4.32}$$

for a.e. $t\in[0,\mathscr{T}]$, and the Gronwall inequality implies

$$\begin{aligned}
&\mu\|\nabla\boldsymbol{z}^n(t)\|_2^2+\int_0^t\left[\|\partial_t\boldsymbol{z}^n(s)\|_2^2+\gamma\|\partial_t\nabla\boldsymbol{z}^n(s)\|_2^2\right]\mathrm{d}s\\
&\leq 3e^{C\int_0^t[\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}(s)\|_2^2+\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}(s)\|_2^4]\mathrm{d}s}\int_0^t\|\boldsymbol{\psi}(s)\|_2^2\mathrm{d}s,
\end{aligned} \tag{4.33}$$

for all $t\in[0,\mathscr{T}]$. Using the fact that $\boldsymbol{\psi}\in\mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{L}}^2(\mathbb{T}^2))$ and $\boldsymbol{v}\in\mathrm{L}^\infty(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))$, we have from (4.33) that

$$\{\partial_t\boldsymbol{z}^n\}_{n\in\mathbb{N}}\text{ is a bounded sequence in }\mathrm{L}^2(0,\mathscr{T};\mathbb{V}). \tag{4.34}$$

Note that, similar to (3.25), in view of (4.33), we have

$$\{\partial_t(\mathrm{I}+\gamma\mathscr{A})\boldsymbol{z}^n\}_{n\in\mathbb{N}}\text{ is a bounded sequence in }\mathrm{L}^2(0,\mathscr{T};\mathbb{V}') \tag{4.35}$$

Step III. *Weak and strong limits, and passing $n\to\infty$ in system* (4.5). Using (4.12), (4.23), (4.35), the *Banach-Alaoglu theorem* and *Aubin-Lions compactness lemma* (similar to (3.28)), we infer the existence of an element $\boldsymbol{z}\in$

$\mathrm{L}^{\infty}(0,\mathscr{T};\mathrm{D}(\mathscr{A}))$ with $\partial_t(\mathrm{I}+\gamma\mathscr{A})\boldsymbol{z}\in\mathrm{L}^2(0,\mathscr{T};\mathbb{V}')$ such that

$$\left.\begin{aligned}\boldsymbol{z}^n &\xrightarrow{w^*} \boldsymbol{z} && \text{in} \quad \mathrm{L}^{\infty}(0,\mathscr{T};\mathrm{D}(\mathscr{A})),\\ \partial_t(\mathrm{I}+\gamma\mathscr{A})\boldsymbol{z}^n &\xrightarrow{w} \partial_t(\mathrm{I}+\gamma\mathscr{A})\boldsymbol{z} && \text{in} \quad \mathrm{L}^2(0,\mathscr{T};\mathbb{V}'),\\ \boldsymbol{z}^n &\to \boldsymbol{z} && \text{in} \quad \mathrm{L}^2(0,\mathscr{T};\mathbb{V}),\end{aligned}\right\} \tag{4.36}$$

along a subsequence (still denoted by the same symbol). Making use of weak and strong convergence (4.36), $\boldsymbol{v}\in\mathrm{L}^{\infty}(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))$ (for some $\varepsilon>0$) and strong convergence $\mathrm{P}_n\mathbb{P}\boldsymbol{\psi}\to\mathbb{P}\boldsymbol{\psi}$ in $\mathrm{L}^2(0,\mathscr{T};\mathbb{H})$), one can easily pass the limit $n\to\infty$ in (4.5) and obtain

$$\begin{aligned}&\partial_t(\boldsymbol{z}+\gamma\mathscr{A}\boldsymbol{z})\\ &\quad=\mathbb{P}\boldsymbol{\psi}-\mu\mathscr{A}\boldsymbol{z}-\mathscr{B}(\boldsymbol{v},\boldsymbol{z})-\mathscr{B}(\boldsymbol{z},\boldsymbol{v})+\alpha\mathbb{P}\big[\mathrm{div}\big(\mathrm{A}(\boldsymbol{v})\mathrm{A}(\boldsymbol{z})+\mathrm{A}(\boldsymbol{z})\mathrm{A}(\boldsymbol{v})\big)\big] \qquad (4.37)\\ &\qquad+\beta\mathbb{P}\big[\mathrm{div}\big(|\mathrm{A}(\boldsymbol{v})|^2\mathrm{A}(\boldsymbol{z})\big)\big]+2\beta\mathbb{P}\big[\mathrm{div}\big((\mathrm{A}(\boldsymbol{z}):\mathrm{A}(\boldsymbol{v}))\mathrm{A}(\boldsymbol{v})\big)\big], \quad \text{in} \quad \mathrm{L}^2(0,\mathscr{T};\mathbb{V}').\end{aligned} \tag{4.38}$$

Next, since $(\mathrm{I}+\gamma\mathscr{A})^{\frac{1}{2}}\boldsymbol{z}\in\mathrm{L}^2(0,\mathscr{T};\mathbb{V})$ and $\partial_t(\mathrm{I}+\gamma\mathscr{A})^{\frac{1}{2}}\boldsymbol{z}\in\mathrm{L}^2(0,\mathscr{T};\mathbb{V}')$, it implies from [40, Lemma 1.2, p.176] that $(\mathrm{I}+\gamma\mathscr{A})^{\frac{1}{2}}\boldsymbol{z}\in\mathrm{C}([0,\mathscr{T}];\mathbb{H})$, the real-valued function $t\mapsto\|(\mathrm{I}+\gamma\mathscr{A})^{\frac{1}{2}}\boldsymbol{z}(t)\|_2^2$ is absolutely continuous and the following equality is satisfied:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\mathrm{I}+\gamma\mathscr{A})^{\frac{1}{2}}\boldsymbol{z}(t)\|_2^2=\left\langle\partial_t(\mathrm{I}+\gamma\mathscr{A})^{\frac{1}{2}}\boldsymbol{z}(t),(\mathrm{I}+\gamma\mathscr{A})^{\frac{1}{2}}\boldsymbol{z}(t)\right\rangle, \quad \text{for a.e. } t\in[0,\mathscr{T}]. \tag{4.39}$$

This also implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg[\|\boldsymbol{z}(t)\|_2^2+\gamma\|\nabla\boldsymbol{z}(t)\|_2^2\bigg]=\langle\partial_t(\mathrm{I}+\gamma\mathscr{A})\boldsymbol{z}(t),\boldsymbol{z}(t)\rangle, \tag{4.40}$$

for a.e. $t\in[0,\mathscr{T}]$, which provides (4.4) immediately. In addition, the condition (ii) in the Definition 4.1 also makes sense.

Step IV. *Uniqueness:* Define $\mathfrak{z}:=\boldsymbol{z}_1-\boldsymbol{z}_2$, where $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are two solutions of system (4.3) in the sense of Definition 4.1. Since, system (4.3) is linear in $\boldsymbol{z}$, therefore $\mathfrak{z}$ is also a solution of system (4.3) with $\boldsymbol{\psi}=\mathbf{0}$ which gives

$$\begin{aligned}&\|\mathfrak{z}(t)\|_2^2+\gamma\|\nabla\mathfrak{z}(t)\|_2^2+2\mu\int_0^t\|\nabla\mathfrak{z}(s)\|_2^2\mathrm{d}s+\beta\int_0^t\||\mathrm{A}(\boldsymbol{v}(s))|\mathrm{A}(\mathfrak{z}(s))\|_2^2\mathrm{d}s\\ &\qquad+2\beta\int_0^t\int_{\mathbb{T}^2}[\mathrm{A}(\boldsymbol{v}(x,s)):\mathrm{A}(\mathfrak{z}(x,s))]^2\,\mathrm{d}x\mathrm{d}s\\ &\quad=-2\int_0^t b(\mathfrak{z}(s),\boldsymbol{v}(s),\mathfrak{z}(s))\mathrm{d}s+\alpha\int_0^t\int_{\mathbb{T}^2}[\mathrm{A}(\boldsymbol{v}(x,s))\mathrm{A}(\mathfrak{z}(x,s))]:\mathrm{A}(\mathfrak{z}(x,s))\mathrm{d}x\mathrm{d}s\\ &\qquad+\alpha\int_0^t\int_{\mathbb{T}^2}[\mathrm{A}(\mathfrak{z}(x,s))\mathrm{A}(\boldsymbol{v}(x,s))]:\mathrm{A}(\mathfrak{z}(x,s))\mathrm{d}x\mathrm{d}s,\end{aligned} \tag{4.41}$$

for all $t\in[0,\mathscr{T}]$. A reasoning similar to that used to obtain (4.10) and $\mathfrak{z}(0)=\mathbf{0}$ yield

$$\|\mathfrak{z}(t)\|_2^2+\gamma\|\nabla\mathfrak{z}(t)\|_2^2\leq C\int_0^t\left\{1+\|\mathscr{A}\boldsymbol{v}(s)\|_2^2\right\}[\|\mathfrak{z}(s)\|_2^2+\gamma\|\nabla\mathfrak{z}(s)\|_2^2]\mathrm{d}s, \tag{4.42}$$

for all $t \in [0, \mathscr{T}]$. Hence, an application of Gronwall's inequality and the fact that $\boldsymbol{v} \in \mathrm{L}^\infty(0, \mathscr{T}; \mathrm{D}(\mathscr{A}))$ give $\boldsymbol{z}_1(t) = \boldsymbol{z}_2(t)$, for all $t \in [0, \mathscr{T}]$, then the uniqueness holds, which completes the proof. $\square$

5. A stability result

This section's primary goal is to prove a stability result for the solution of the state equation. This step is essential for studying the control-to-state mapping's Gateaux derivative. For any $\rho > 0$, let us define

$$\boldsymbol{z}_\rho := \frac{\boldsymbol{v}_\rho - \boldsymbol{v}}{\rho}, \tag{5.1}$$

where $\boldsymbol{v}$ is the solution of system (3.1) and $\boldsymbol{v}_\rho$ is the solution of system (3.1) with $\boldsymbol{f}$ replaced by $\boldsymbol{f} + \rho\boldsymbol{\psi}$ in the sense of Definition 3.1. Then, $\boldsymbol{z}_\rho$ satisfies the following system:

$$\begin{cases} \partial_t\left(\boldsymbol{z}_\rho(t) + \gamma\mathscr{A}\boldsymbol{z}_\rho(t)\right) + \mu\mathscr{A}\boldsymbol{z}_\rho(t) + \rho\mathscr{B}(\boldsymbol{z}_\rho(t)) + \mathscr{B}(\boldsymbol{v}(t), \boldsymbol{z}_\rho(t)) + \mathscr{B}(\boldsymbol{z}_\rho(t), \boldsymbol{v}(t)) \\ -\alpha\rho\mathbb{P}\mathrm{div}[\mathrm{A}(\boldsymbol{z}_\rho(t))\mathrm{A}(\boldsymbol{z}_\rho(t))] - \alpha\mathbb{P}\mathrm{div}[\mathrm{A}(\boldsymbol{z}_\rho(t))\mathrm{A}(\boldsymbol{v}(t))] - \alpha\mathbb{P}\mathrm{div}[\mathrm{A}(\boldsymbol{v}(t))\mathrm{A}(\boldsymbol{z}_\rho(t))] \\ \qquad -\beta\rho^2\mathbb{P}\mathrm{div}[|\mathrm{A}(\boldsymbol{z}_\rho(t))|^2\mathrm{A}(\boldsymbol{z}_\rho(t))] - \beta\rho\mathbb{P}\mathrm{div}[|\mathrm{A}(\boldsymbol{z}_\rho(t))|^2\mathrm{A}(\boldsymbol{v}(t))] \\ \quad -2\beta\rho\mathbb{P}\mathrm{div}[(\mathrm{A}(\boldsymbol{z}_\rho(t)) : \mathrm{A}(\boldsymbol{v}(t)))\mathrm{A}(\boldsymbol{z}_\rho(t))] - 2\beta\mathbb{P}\mathrm{div}[(\mathrm{A}(\boldsymbol{z}_\rho(t)) : \mathrm{A}(\boldsymbol{v}(t)))\mathrm{A}(\boldsymbol{v}(t))] \\ \qquad\qquad -\beta\mathbb{P}\mathrm{div}[|\mathrm{A}(\boldsymbol{v}(t))|^2\mathrm{A}(\boldsymbol{z}_\rho(t))] = \mathbb{P}\boldsymbol{\psi}(t), \\ \qquad\qquad \boldsymbol{z}_\rho(0) = \mathbf{0}, \end{cases} \tag{5.2}$$

for a.e. $t \in [0, T]$, in $\mathbb{V}'$. Next lemma provides the convergence of $\boldsymbol{v}_\rho$ towards $\boldsymbol{v}$ as $\rho \to 0$.

Remark 5.1. Let us emphasize that $\boldsymbol{v}_\rho$, $\boldsymbol{v} \in \mathrm{L}^\infty(0, T; \mathrm{D}(\mathscr{A}))$, $\rho > 0$, for given $T > 0$ and therefore system (5.2) make sense globally in time.

Lemma 5.1. *For some $\varepsilon > 0$, let $\boldsymbol{v} \in \mathrm{L}^\infty(0, \mathscr{T}; \mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))$, where $\mathscr{T}$ is the time discussed in Remark 2.1. Then, there exist two constants $\Gamma_1 = \Gamma_1(\alpha, \beta, \mu, \gamma, T)$ and $\Gamma_2 = \Gamma_2(\alpha, \beta, \mu, \gamma, T)$ (independent of ρ) such that $\boldsymbol{z}_\rho$ satisfies*

$$\|\boldsymbol{z}_\rho(t)\|_2^2 + \gamma\|\nabla\boldsymbol{z}_\rho(t)\|_2^2 + \mu\int_0^t \|\nabla\boldsymbol{z}_\rho(s)\|_2^2\mathrm{d}s \leq [\Gamma_1]^2\int_0^{\mathscr{T}} \|\boldsymbol{\psi}(t)\|_2^2\mathrm{d}t := \widetilde{\Gamma}_1, \tag{5.3}$$

$$\|\nabla\boldsymbol{z}_\rho(t)\|_2^2 + \gamma\|\mathscr{A}\boldsymbol{z}_\rho(t)\|_2^2 + \mu\int_0^t \|\mathscr{A}\boldsymbol{z}_\rho(s)\|_2^2\mathrm{d}s \leq [\Gamma_2]^2\int_0^{\mathscr{T}} \|\boldsymbol{\psi}(t)\|_2^2\mathrm{d}t := \widetilde{\Gamma}_2, \tag{5.4}$$

for all $t \in [0, \mathscr{T}]$.

Proof of Lemma 5.1. We formally present an a-priori energy estimate that corresponds to system (5.2). Standard Faedo-Galerkin's approximation can be used to provide a rigorous justification as it has been used in the proof of Theorems 3.1 and 4.1. Taking the inner product in (5.2) by $\boldsymbol{z}_\rho$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\boldsymbol{z}_\rho(t)\|_2^2 + \gamma\|\nabla\boldsymbol{z}_\rho(t)\|_2^2\right] + \mu\|\nabla\boldsymbol{z}_\rho(t)\|_2^2 + \frac{\beta\rho^2}{2}\|\mathrm{A}(\boldsymbol{z}_\rho(t))\|_4^4$$

$$
\begin{aligned}
&+\beta\int_{\mathbb{T}^2}[\mathrm{A}(\boldsymbol{v}(x,t)):\mathrm{A}(\boldsymbol{z}_\rho(x,t))]^2\mathrm{d}x\\
&+\frac{\beta}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}(x,t))|^2|\mathrm{A}(\boldsymbol{z}_\rho(x,t))|^2\mathrm{d}x\\
&=-b(\boldsymbol{z}_\rho(t),\boldsymbol{v}(t),\boldsymbol{z}_\rho(t))-\frac{\alpha\rho}{2}\int_{\mathbb{T}^2}\mathrm{A}(\boldsymbol{z}_\rho(x,t))\mathrm{A}(\boldsymbol{z}_\rho(x,t)):\mathrm{A}(\boldsymbol{z}_\rho(x,t))\mathrm{d}x\\
&\quad-\frac{\alpha}{2}\int_{\mathbb{T}^2}\mathrm{A}(\boldsymbol{z}_\rho(x,t))\mathrm{A}(\boldsymbol{v}(x,t)):\mathrm{A}(\boldsymbol{z}_\rho(x,t))\mathrm{d}x-\frac{\alpha}{2}\int_{\mathbb{T}^2}\mathrm{A}(\boldsymbol{v}(x,t))\mathrm{A}(\boldsymbol{z}_\rho(x,t))\\
&\quad:\mathrm{A}(\boldsymbol{z}_\rho(x,t))\mathrm{d}x\\
&\quad-\frac{3\beta\rho}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho(x,t))|^2[\mathrm{A}(\boldsymbol{v}(x,t)):\mathrm{A}(\boldsymbol{z}_\rho(x,t))]\mathrm{d}x+(\boldsymbol{\psi}(t),\boldsymbol{z}_\rho(t)), \qquad (5.5)
\end{aligned}
$$

for a.e. $t\in[0,\mathscr{T}]$. Now, we estimate the terms of right hand side of (5.5) using (2.12), Hölder's, Agmon's and Young's inequalities as follows:

$$
\begin{aligned}
&|b(\boldsymbol{z}_\rho,\boldsymbol{v},\boldsymbol{z}_\rho)|=|b(\boldsymbol{z}_\rho,\boldsymbol{z}_\rho,\boldsymbol{v})|\leq\|\boldsymbol{v}\|_\infty\|\boldsymbol{z}_\rho\|_2\|\nabla\boldsymbol{z}_\rho\|_2\\
&\leq C\|\mathscr{A}\boldsymbol{v}\|_2\|\boldsymbol{z}_\rho\|_2\|\nabla\boldsymbol{z}_\rho\|_2\leq C_\mu\|\mathscr{A}\boldsymbol{v}\|_2^2\|\boldsymbol{z}_\rho\|_2^2+\frac{\mu}{4}\|\nabla\boldsymbol{z}_\rho\|_2^2, \qquad (5.6)\\
&\left|\frac{\alpha\rho}{2}\int_{\mathbb{T}^2}\mathrm{A}(\boldsymbol{z}_\rho(x))\mathrm{A}(\boldsymbol{z}_\rho(x)):\mathrm{A}(\boldsymbol{z}_\rho(x))\mathrm{d}x\right|\\
&\leq\frac{|\alpha|\rho}{2}\|\mathrm{A}(\boldsymbol{z}_\rho)\|_4^2\|\mathrm{A}(\boldsymbol{z}_\rho)\|_2\leq\frac{\beta\rho^2}{8}\|\mathrm{A}(\boldsymbol{z}_\rho)\|_4^4+C_{\alpha,\beta}\|\nabla\boldsymbol{z}_\rho\|_2^2, \qquad (5.7)\\
&\left|\frac{\alpha}{2}\int_{\mathbb{T}^2}\mathrm{A}(\boldsymbol{z}_\rho(x))\mathrm{A}(\boldsymbol{v}(x)):\mathrm{A}(\boldsymbol{z}_\rho(x))\mathrm{d}x\right|\\
&\leq\frac{\beta}{8}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}(x))|^2|\mathrm{A}(\boldsymbol{z}_\rho(x))|^2\mathrm{d}x+C_{\alpha,\beta}\|\nabla\boldsymbol{z}_\rho\|_2^2, \qquad (5.8)\\
&\left|\frac{\alpha}{2}\int_{\mathbb{T}^2}\mathrm{A}(\boldsymbol{v}(x))\mathrm{A}(\boldsymbol{z}_\rho(x)):\mathrm{A}(\boldsymbol{z}_\rho(x))\mathrm{d}x\right|\\
&\leq\frac{\beta}{8}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}(x))|^2|\mathrm{A}(\boldsymbol{z}_\rho(x))|^2\mathrm{d}x+C_{\alpha,\beta}\|\nabla\boldsymbol{z}_\rho\|_2^2, \qquad (5.9)\\
&\left|\frac{3\beta\rho}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho(x))|^2[\mathrm{A}(\boldsymbol{v}(x)):\mathrm{A}(\boldsymbol{z}_\rho(x))]\mathrm{d}x\right|\\
&\leq\frac{3\beta\rho}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho(x))|^3|\mathrm{A}(\boldsymbol{v}(x))|\mathrm{d}x\\
&\leq\frac{3\beta\rho}{2}\|\mathrm{A}(\boldsymbol{z}_\rho)\|_4^2\|\mathrm{A}(\boldsymbol{v})\|_\infty\|\mathrm{A}(\boldsymbol{z}_\rho)\|_2\\
&\leq\frac{\beta\rho^2}{8}\|\mathrm{A}(\boldsymbol{z}_\rho)\|_4^4+\frac{9\beta}{2}\|\mathrm{A}(\boldsymbol{v})\|_\infty^2\|\mathrm{A}(\boldsymbol{z}_\rho)\|_2^2\\
&\leq\frac{\beta\rho^2}{8}\|\mathrm{A}(\boldsymbol{z}_\rho)\|_4^4+C_\beta\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2\|\nabla\boldsymbol{z}_\rho\|_2^2, \qquad (5.10)\\
|(\boldsymbol{\psi},\boldsymbol{z}_\rho)|&\leq\|\boldsymbol{\psi}\|_2\|\boldsymbol{z}_\rho\|_2\leq C\|\boldsymbol{\psi}\|_2\|\nabla\boldsymbol{z}_\rho\|_2\\
&\leq C_\mu\|\boldsymbol{\psi}\|_2^2+\frac{\mu}{4}\|\nabla\boldsymbol{z}_\rho\|_2^2. \qquad (5.11)
\end{aligned}
$$

We infer by combining (5.5)-(5.11)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\boldsymbol{z}_\rho(t)\|_2^2+\gamma\|\nabla\boldsymbol{z}_\rho(t)\|_2^2\right]+\mu\|\nabla\boldsymbol{z}_\rho(t)\|_2^2+\frac{\beta\rho^2}{2}\|\mathrm{A}(\boldsymbol{z}_\rho(t))\|_4^4\\&\leq C_\mu\|\boldsymbol{\psi}(t)\|_2^2+C_{\alpha,\beta,\mu,\gamma}\left\{\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}(t)\|_2^2+1\right\}\left[\|\boldsymbol{z}_\rho(t)\|_2^2+\gamma\|\nabla\boldsymbol{z}_\rho(t)\|_2^2\right],\end{aligned}\tag{5.12}$$

for a.e. $t\in[0,\mathscr{T}]$, and in view of Gronwall's inequality, we obtain (5.3).

Next, taking the inner product in (5.2) by $\mathscr{A}\boldsymbol{z}_\rho$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\nabla\boldsymbol{z}_\rho(t)\|_2^2+\gamma\|\mathscr{A}\boldsymbol{z}_\rho(t)\|_2^2\right]+\mu\|\mathscr{A}\boldsymbol{z}_\rho(t)\|_2^2\\&=(\boldsymbol{\psi}(t),\mathscr{A}\boldsymbol{z}_\rho(t))\\&\quad-\rho\underbrace{b(\boldsymbol{z}_\rho(t),\boldsymbol{z}_\rho(t),\mathscr{A}\boldsymbol{z}_\rho(t))}_{=0\ by\ (2.8).}-b(\boldsymbol{v}(t),\boldsymbol{z}_\rho(t),\mathscr{A}\boldsymbol{z}_\rho(t))-b(\boldsymbol{z}_\rho(t),\boldsymbol{v}(t),\mathscr{A}\boldsymbol{z}_\rho(t))\\&\quad+\frac{\alpha\rho}{2}\int_{\mathbb{T}^2}\mathrm{A}(\boldsymbol{z}_\rho(x,t))\mathrm{A}(\boldsymbol{z}_\rho(x,t)):\Delta\mathrm{A}(\boldsymbol{z}_\rho(x,t))\mathrm{d}x\\&\quad+\frac{\alpha}{2}\int_{\mathbb{T}^2}\mathrm{A}(\boldsymbol{z}_\rho(x,t))\mathrm{A}(\boldsymbol{v}(x,t)):\Delta\mathrm{A}(\boldsymbol{z}_\rho(x,t))\mathrm{d}x\\&\quad+\frac{\alpha}{2}\int_{\mathbb{T}^2}\mathrm{A}(\boldsymbol{v}(x,t))\mathrm{A}(\boldsymbol{z}_\rho(x,t)):\Delta\mathrm{A}(\boldsymbol{z}_\rho(x,t))\mathrm{d}x\\&\quad+\frac{\beta\rho^2}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho(x,t))|^2[\mathrm{A}(\boldsymbol{z}_\rho(x,t)):\Delta\mathrm{A}(\boldsymbol{z}_\rho(x,t))]\mathrm{d}x\\&\quad+\frac{\beta\rho}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho(x,t))|^2[\mathrm{A}(\boldsymbol{v}(x,t)):\Delta\mathrm{A}(\boldsymbol{z}_\rho(x,t))]\mathrm{d}x\\&\quad+\frac{\beta}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}(x,t))|^2[\mathrm{A}(\boldsymbol{z}_\rho(x,t)):\Delta\mathrm{A}(\boldsymbol{z}_\rho(x,t))]\mathrm{d}x\\&\quad+\beta\rho\int_{\mathbb{T}^2}[\mathrm{A}(\boldsymbol{z}_\rho(x,t)):\mathrm{A}(\boldsymbol{v}(x,t))][\mathrm{A}(\boldsymbol{z}_\rho(x,t)):\Delta\mathrm{A}(\boldsymbol{z}_\rho(x,t))]\mathrm{d}x\\&\quad+\beta\int_{\mathbb{T}^2}[\mathrm{A}(\boldsymbol{z}_\rho(x,t)):\mathrm{A}(\boldsymbol{v}(x,t))][\mathrm{A}(\boldsymbol{v}(x,t)):\Delta\mathrm{A}(\boldsymbol{z}_\rho(x,t))]\mathrm{d}x,\end{aligned}\tag{5.13}$$

for a.e. $t\in[0,\mathscr{T}]$. We denote $[\mathrm{A}(\boldsymbol{z}_\rho)]_{ij}:=b_{ij}^\rho$ which will be used in the sequel and recall that $[\mathrm{A}(\boldsymbol{v})]_{ij}=a_{ij}$. Next, using (2.11), (2.12), (2.13), Hölder's, Agmon's and Young's inequalities, we estimate all the terms in the right hand side of (5.13) as follows:

$$|(\boldsymbol{\psi},\mathscr{A}\boldsymbol{z}_\rho)|\leq\|\boldsymbol{\psi}\|_2\|\mathscr{A}\boldsymbol{z}_\rho\|_2\leq C_\mu\|\boldsymbol{\psi}\|_2^2+\frac{\mu}{6}\|\mathscr{A}\boldsymbol{z}_\rho\|_2^2,\tag{5.14}$$

$$\begin{aligned}|b(\boldsymbol{v},\boldsymbol{z}_\rho,\mathscr{A}\boldsymbol{z}_\rho)|&\leq\|\boldsymbol{v}\|_\infty\|\nabla\boldsymbol{z}_\rho\|_2\|\mathscr{A}\boldsymbol{z}_\rho\|_2\leq C\|\mathscr{A}\boldsymbol{v}\|_2\|\mathrm{A}(\boldsymbol{z}_\rho)\|_2\|\mathscr{A}\boldsymbol{z}_\rho\|_2\\&\leq C_\mu\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2\|\nabla\boldsymbol{z}_\rho\|_2^2+\frac{\mu}{6}\|\mathscr{A}\boldsymbol{z}_\rho\|_2^2,\end{aligned}\tag{5.15}$$

$$\begin{aligned}|b(\boldsymbol{z}_\rho,\boldsymbol{v},\mathscr{A}\boldsymbol{z}_\rho)|&\leq\|\boldsymbol{z}_\rho\|_2\|\nabla\boldsymbol{v}\|_\infty\|\mathscr{A}\boldsymbol{z}_\rho\|_2\leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2\|\nabla\boldsymbol{z}_\rho\|_2\|\mathscr{A}\boldsymbol{z}_\rho\|_2\\&\leq C_\mu\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2\|\nabla\boldsymbol{z}_\rho\|_2^2+\frac{\mu}{6}\|\mathscr{A}\boldsymbol{z}_\rho\|_2^2,\end{aligned}\tag{5.16}$$

$$
\begin{aligned}
&\frac{\beta\rho^2}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho(x))|^2[\mathrm{A}(\boldsymbol{z}_\rho(x)):\Delta\mathrm{A}(\boldsymbol{z}_\rho(x))]\mathrm{d}x\\
&=-\frac{\beta\rho^2}{2}\sum_{m=1}^{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho(x))|^2|\partial_m\mathrm{A}(\boldsymbol{z}_\rho(x))|^2\mathrm{d}x\\
&-\beta\rho^2\sum_m\int_{\mathbb{T}^2}\left[\sum_{i,j}b_{ij}^\rho(x)\partial_m b_{ij}^\rho(x)\right]^2\mathrm{d}x,
\end{aligned}\tag{5.17}
$$

$$
\begin{aligned}
&\frac{\beta\rho}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho(x))|^2[\mathrm{A}(\boldsymbol{v}(x)):\Delta\mathrm{A}(\boldsymbol{z}_\rho(x))]\mathrm{d}x\\
&=\frac{\beta\rho}{2}\sum_m\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho(x))|^2[\mathrm{A}(\boldsymbol{v}(x)):\partial_m^2\mathrm{A}(\boldsymbol{z}_\rho(x))]\mathrm{d}x\\
&=\frac{\beta\rho}{2}\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}[b_{ij}^\rho(x)]^2a_{kl}(x)\partial_m^2b_{kl}^\rho(x)\mathrm{d}x\\
&=-\frac{\beta\rho}{2}\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}[b_{ij}^\rho(x)]^2\partial_m a_{kl}(x)\partial_m b_{kl}^\rho(x)\mathrm{d}x\\
&\quad-\beta\rho\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}b_{ij}^\rho(x)\partial_m b_{ij}^\rho(x)a_{kl}(x)\partial_m b_{kl}^\rho(x)\mathrm{d}x\\
&\leq\frac{\beta\rho^2}{12}\sum_m\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho(x))|^2|\partial_m\mathrm{A}(\boldsymbol{z}_\rho(x))|^2\mathrm{d}x\\
&\quad+C_\beta\|\mathrm{A}(\boldsymbol{z}_\rho)\|_{\frac{2}{\varepsilon}}^2\|\boldsymbol{v}\|^2_{\mathbb{W}^{2,\frac{2}{1-\varepsilon}}}+C_\beta\|\mathrm{A}(\boldsymbol{v})\|_\infty^2\|\boldsymbol{z}_\rho\|^2_{\dot{\mathbb{H}}^2}\\
&\leq\frac{\beta\rho^2}{12}\sum_m\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho(x))|^2|\partial_m\mathrm{A}(\boldsymbol{z}_\rho(x))|^2\mathrm{d}x\\
&\quad+C_\beta\|\boldsymbol{z}_\rho\|^2_{\dot{\mathbb{H}}^2}\|\boldsymbol{v}\|^2_{\dot{\mathbb{H}}^{2+\varepsilon}}\\
&\leq\frac{\beta\rho^2}{12}\sum_m\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho(x))|^2|\partial_m\mathrm{A}(\boldsymbol{z}_\rho(x))|^2\mathrm{d}x+C_\beta\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2\|\mathscr{A}\boldsymbol{z}_\rho\|_2^2,
\end{aligned}\tag{5.18}
$$

$$
\begin{aligned}
&\frac{\beta}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}(x))|^2[\mathrm{A}(\boldsymbol{z}_\rho(x)):\Delta\mathrm{A}(\boldsymbol{z}_\rho(x))]\mathrm{d}x\\
&=\frac{\beta}{2}\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}[a_{ij}(x)]^2b_{kl}^\rho(x)\partial_m^2b_{kl}^\rho(x)\mathrm{d}x\\
&=-\frac{\beta}{2}\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}[a_{ij}(x)]^2[\partial_m b_{kl}^\rho(x)]^2\mathrm{d}x\\
&\quad-\beta\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}a_{ij}(x)\partial_m a_{ij}(x)b_{kl}^\rho(x)\partial_m b_{kl}^\rho(x)\mathrm{d}x\\
&=-\frac{\beta}{2}\sum_m\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}(x))|^2|\partial_m\mathrm{A}(\boldsymbol{z}_\rho(x))|^2\mathrm{d}x\\
&\quad-\beta\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}a_{ij}(x)\partial_m a_{ij}(x)b_{kl}^\rho(x)\partial_m b_{kl}^\rho(x)\mathrm{d}x\\
&\leq-\frac{\beta}{2}\sum_m\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}(x))|^2|\partial_m\mathrm{A}(\boldsymbol{z}_\rho(x))|^2\mathrm{d}x\\
&\quad+\frac{\beta}{12}\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}[a_{ij}(x)]^2[\partial_m b_{kl}^\rho(x)]^2\mathrm{d}x+C_\beta\|\mathrm{A}(\boldsymbol{z}_\rho)\|_{\frac{2}{\varepsilon}}^2\|\boldsymbol{v}\|^2_{\mathbb{W}^{2,\frac{2}{1-\varepsilon}}}\\
&\leq-\frac{\beta}{2}\sum_m\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}(x))|^2|\partial_m\mathrm{A}(\boldsymbol{z}_\rho(x))|^2\mathrm{d}x
\end{aligned}
$$

$$+\frac{\beta}{12}\sum_{m}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}(x))|^2|\partial_m\mathrm{A}(\boldsymbol{z}_\rho(x))|^2\mathrm{d}x+C_\beta\|\mathscr{A}\boldsymbol{z}_\rho\|_2^2\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2, \tag{5.19}$$

$$\begin{aligned}
&\beta\rho\int_{\mathbb{T}^2}[\mathrm{A}(\boldsymbol{z}_\rho(x)):\mathrm{A}(\boldsymbol{v}(x))][\mathrm{A}(\boldsymbol{z}_\rho(x)):\Delta\mathrm{A}(\boldsymbol{z}_\rho(x))]\mathrm{d}x\\
&=\beta\rho\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}b_{ij}^{\rho}(x)a_{ij}(x)b_{kl}^{\rho}(x)\partial_m^2 b_{kl}^{\rho}(x)\mathrm{d}x\\
&=-\beta\rho\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}\partial_m b_{ij}^{\rho}(x)a_{ij}(x)b_{kl}^{\rho}(x)\partial_m b_{kl}^{\rho}(x)\mathrm{d}x\\
&\quad-\beta\rho\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}b_{ij}^{\rho}(x)\partial_m a_{ij}(x)b_{kl}^{\rho}(x)\partial_m b_{kl}^{\rho}(x)\mathrm{d}x\\
&\quad-\beta\rho\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}b_{ij}^{\rho}(x)a_{ij}(x)\partial_m b_{kl}^{\rho}(x)\partial_m b_{kl}^{\rho}(x)\mathrm{d}x\\
&\leq\frac{\beta\rho^2}{12}\sum_{m}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho(x))|^2|\partial_m\mathrm{A}(\boldsymbol{z}_\rho(x))|^2\mathrm{d}x+C_\beta\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2\|\mathscr{A}\boldsymbol{z}_\rho\|_2^2,
\end{aligned}\tag{5.20}$$

$$\begin{aligned}
&\beta\int_{\mathbb{T}^2}[\mathrm{A}(\boldsymbol{z}_\rho(x)):\mathrm{A}(\boldsymbol{v}(x))][\mathrm{A}(\boldsymbol{v}(x)):\Delta\mathrm{A}(\boldsymbol{z}_\rho(x))]\mathrm{d}x\\
&=\beta\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}b_{ij}^{\rho}(x)a_{ij}(x)a_{kl}(x)\partial_m^2 b_{kl}^{\rho}(x)\mathrm{d}x\\
&=-\beta\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}\partial_m b_{ij}^{\rho}(x)a_{ij}(x)a_{kl}(x)\partial_m b_{kl}^{\rho}(x)\mathrm{d}x\\
&\quad-\beta\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}b_{ij}^{\rho}(x)\partial_m a_{ij}(x)a_{kl}(x)\partial_m b_{kl}^{\rho}(x)\mathrm{d}x\\
&\quad-\beta\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}b_{ij}^{\rho}(x)a_{ij}(x)\partial_m a_{kl}(x)\partial_m b_{kl}^{\rho}(x)\mathrm{d}x\\
&\leq\frac{\beta}{12}\sum_{i,j,k,l,m}\int_{\mathbb{T}^2}[a_{ij}(x)]^2[\partial_m b_{kl}^{\rho}(x)]^2\mathrm{d}x+C_\beta\|\mathscr{A}\boldsymbol{z}_\rho\|_2^2\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2\\
&\leq\frac{\beta}{12}\sum_{m}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}(x))|^2|\partial_m\mathrm{A}(\boldsymbol{z}_\rho(x))|^2\mathrm{d}x+C_\beta\|\mathscr{A}\boldsymbol{z}_\rho\|_2^2\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2,
\end{aligned}\tag{5.21}$$

$$\begin{aligned}
&\left|\frac{\alpha\rho}{2}\int_{\mathbb{T}^2}\mathrm{A}(\boldsymbol{z}_\rho(x))\mathrm{A}(\boldsymbol{z}_\rho(x)):\Delta\mathrm{A}(\boldsymbol{z}_\rho(x))\mathrm{d}x\right|\leq\rho|\alpha|\sum_{m}\int_{\mathbb{T}^2}|\partial_m\mathrm{A}(\boldsymbol{z}_\rho(x))|^2|\mathrm{A}(\boldsymbol{z}_\rho(x))|\mathrm{d}x\\
&\leq\frac{\beta\rho^2}{12}\sum_{m}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho(x))|^2|\partial_m\mathrm{A}(\boldsymbol{z}_\rho(x))|^2\mathrm{d}x+C_{\alpha,\beta}\|\mathscr{A}\boldsymbol{z}_\rho\|_2^2,
\end{aligned}\tag{5.22}$$

$$\begin{aligned}
&\left|\frac{\alpha}{2}\int_{\mathbb{T}^2}\mathrm{A}(\boldsymbol{z}_\rho(x))\mathrm{A}(\boldsymbol{v}(x)):\Delta\mathrm{A}(\boldsymbol{z}_\rho(x))\mathrm{d}x+\frac{\alpha}{2}\int_{\mathbb{T}^2}\mathrm{A}(\boldsymbol{v}(x))\mathrm{A}(\boldsymbol{z}_\rho(x)):\Delta\mathrm{A}(\boldsymbol{z}_\rho(x))\mathrm{d}x\right|\\
&\leq|\alpha|\sum_{m}\int_{\mathbb{T}^2}|\partial_m\mathrm{A}(\boldsymbol{z}_\rho(x))|^2|\mathrm{A}(\boldsymbol{v}(x))|\mathrm{d}x\\
&\quad+|\alpha|\sum_{m}\int_{\mathbb{T}^2}|\partial_m\mathrm{A}(\boldsymbol{z}_\rho(x))||\partial_m\mathrm{A}(\boldsymbol{v}(x))||\mathrm{A}(\boldsymbol{z}_\rho(x))|\mathrm{d}x\\
&\leq\frac{\beta}{12}\sum_{m}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}(x))|^2|\partial_m\mathrm{A}(\boldsymbol{z}_\rho(x))|^2\mathrm{d}x\\
&\quad+C_{\alpha,\beta}\|\mathscr{A}\boldsymbol{z}_\rho\|_2^2+C_{\alpha,\beta}\|\mathrm{A}(\boldsymbol{z}_\rho)\|_{\frac{2}{\varepsilon}}\|\mathscr{A}\boldsymbol{z}_\rho\|_2\|\boldsymbol{v}\|_{\mathbb{W}^{2,\frac{2}{1-\varepsilon}}}\\
&\leq\frac{\beta}{12}\sum_{m}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{v}(x))|^2|\partial_m\mathrm{A}(\boldsymbol{z}_\rho(x))|^2\mathrm{d}x
\end{aligned}$$

$$+ C_{\alpha,\beta}\|\mathscr{A}\boldsymbol{z}_{\rho}\|_2^2 + C_{\alpha,\beta}\|\mathscr{A}\boldsymbol{z}_{\rho}\|_2^2\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2. \tag{5.23}$$

Combining (5.13)-(5.23), we find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|\nabla \boldsymbol{z}_{\rho}(t)\|_2^2 + \gamma\|\mathscr{A}\boldsymbol{z}_{\rho}(t)\|_2^2\Big] + \mu\|\mathscr{A}\boldsymbol{z}_{\rho}(t)\|_2^2 \\
&\leq C_{\mu}\|\boldsymbol{\psi}(t)\|_2^2 + C_{\mu}\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}(t)\|_2^2\|\nabla \boldsymbol{z}_{\rho}(t)\|_2^2 + C_{\alpha,\beta}\big[1+\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}(t)\|_2^2\big]\|\mathscr{A}\boldsymbol{z}_{\rho}(t)\|_2^2 \\
&\leq C_{\mu}\|\boldsymbol{\psi}(t)\|_2^2 + C_{\alpha,\beta,\gamma}\big[1+\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}(t)\|_2^2\big]\Big[\|\nabla \boldsymbol{z}_{\rho}(t)\|_2^2 + \gamma\|\mathscr{A}\boldsymbol{z}_{\rho}(t)\|_2^2\Big],
\end{aligned} \tag{5.24}$$

for a.e. $t \in [0, \mathscr{T}]$, which implies by an application of Gronwall's inequality that (5.4) holds. This completes the proof. $\square$

6. Gateaux differentiability of the control-to-state mapping

The differentiality of the control-to-state mapping is examined in this section. More specifically, we will demonstrate that the solution of the linearized equation provides the Gateaux derivative of the control-to-state mapping using the stability property defined in the preceding section. Let us define

$$\chi_{\rho} := \boldsymbol{z}_{\rho} - \boldsymbol{z} = \frac{\boldsymbol{v}_{\rho} - \boldsymbol{v}}{\rho} - \boldsymbol{z}, \tag{6.1}$$

where, $\boldsymbol{v}$ is the solution of system (3.1) in the sense of Definition 3.1, $\boldsymbol{v}_{\rho}$ is the solution of system (3.1) with $\boldsymbol{f}$ replaced by $\boldsymbol{f} + \rho\boldsymbol{\psi}$ in the sense of Definition 3.1 and $\boldsymbol{z}$ is the solution of system (4.2) in the sense of Definition 4.1. Then, χ_{ρ} satisfies the following system:

$$\left\{\begin{aligned}
&\partial_t\left(\chi_{\rho}(t) + \gamma\mathscr{A}\chi_{\rho}(t)\right) + \mu\mathscr{A}\chi_{\rho}(t) + \rho\mathscr{B}(\boldsymbol{z}_{\rho}(t)) + \mathscr{B}(\boldsymbol{v}(t),\chi_{\rho}(t)) + \mathscr{B}(\chi_{\rho}(t),\boldsymbol{v}(t)) \\
&-\alpha\rho\mathbb{P}\mathrm{div}[\mathrm{A}(\boldsymbol{z}_{\rho}(t))\mathrm{A}(\boldsymbol{z}_{\rho}(t))] - \alpha\mathbb{P}\mathrm{div}[\mathrm{A}(\chi_{\rho}(t))\mathrm{A}(\boldsymbol{v}(t))] - \alpha\mathbb{P}\mathrm{div}[\mathrm{A}(\boldsymbol{v}(t))\mathrm{A}(\chi_{\rho}(t))] \\
&\qquad -\beta\rho^2\mathbb{P}\mathrm{div}[|\mathrm{A}(\boldsymbol{z}_{\rho}(t))|^2\mathrm{A}(\boldsymbol{z}_{\rho}(t))] - \beta\rho\mathbb{P}\mathrm{div}[|\mathrm{A}(\boldsymbol{z}_{\rho}(t))|^2\mathrm{A}(\boldsymbol{v}(t))] \\
&-2\beta\rho\mathbb{P}\mathrm{div}[(\mathrm{A}(\boldsymbol{z}_{\rho}(t)) : \mathrm{A}(\boldsymbol{v}(t)))\mathrm{A}(\boldsymbol{z}_{\rho}(t))] - 2\beta\mathbb{P}\mathrm{div}[(\mathrm{A}(\chi_{\rho}(t)) : \mathrm{A}(\boldsymbol{v}(t)))\mathrm{A}(\boldsymbol{v}(t))] \\
&\qquad -\beta\mathbb{P}\mathrm{div}[|\mathrm{A}(\boldsymbol{v}(t))|^2\mathrm{A}(\chi_{\rho}(t))] = \boldsymbol{0}, \\
&\chi_{\rho}(0) = \boldsymbol{0},
\end{aligned}\right. \tag{6.2}$$

for a.e. $t \in [0, \mathscr{T}]$, in $\mathbb{V}'$. Next lemma helps us to obtain the Gateaux differentiability of control-to-state mapping.

Lemma 6.1. *For some $\varepsilon > 0$, let $\boldsymbol{v} \in \mathrm{L}^{\infty}(0, \mathscr{T}; \mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))$. Then, χ_{ρ} satisfies*

$$\lim_{\rho\to 0}\|\chi_{\rho}(t)\|_2^2 + \gamma\lim_{\rho\to 0}\|\nabla\chi_{\rho}(t)\|_2^2 + \mu\lim_{\rho\to 0}\int_0^t\|\nabla\chi_{\rho}(s)\|_2^2 = 0, \quad \text{for all } t \in [0, \mathscr{T}]. \tag{6.3}$$

Proof of Lemma 6.1. Taking the inner product by χ_{ρ} to the equation $(6.2)_1$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|\chi_{\rho}(t)\|_2^2 + \gamma\|\nabla\chi_{\rho}(t)\|_2^2\Big] + \mu\|\nabla\chi_{\rho}(t)\|_2^2 \\
&\quad + \frac{\beta}{2}\||\mathrm{A}(\boldsymbol{v}(t))|\mathrm{A}(\chi_{\rho}(t))\|_2^2 + \beta\int_{\mathbb{T}^2}[\mathrm{A}(\boldsymbol{v}(x,t)) : \mathrm{A}(\chi_{\rho}(x,t))]^2\mathrm{d}x \\
&= -\rho b(\boldsymbol{z}_{\rho}(t), \boldsymbol{z}_{\rho}(t), \chi_{\rho}(t)) - b(\chi_{\rho}(t), \boldsymbol{v}(t), \chi_{\rho}(t))
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\alpha\rho}{2}\int_{\mathbb{T}^2}\mathrm{A}(\boldsymbol{z}_\rho(x,t))\mathrm{A}(\boldsymbol{z}_\rho(x,t)):\mathrm{A}(\chi_\rho(x,t))\mathrm{d}x\\
&\quad-\frac{\alpha}{2}\int_{\mathbb{T}^2}\mathrm{A}(\chi_\rho(x,t))\mathrm{A}(\boldsymbol{v}(x,t)):\mathrm{A}(\chi_\rho(x,t))\mathrm{d}x\\
&-\frac{\alpha}{2}\int_{\mathbb{T}^2}\mathrm{A}(\boldsymbol{v}(x,t))\mathrm{A}(\chi_\rho(x,t)):\mathrm{A}(\chi_\rho(x,t))\mathrm{d}x\\
&\quad-\frac{\beta\rho^2}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho(x,t))|^2[\mathrm{A}(\boldsymbol{z}_\rho(x,t)):\mathrm{A}(\chi_\rho(x,t))]\mathrm{d}x\\
&-\frac{\beta\rho}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho(x,t))|^2[\mathrm{A}(\boldsymbol{v}(x,t)):\mathrm{A}(\chi_\rho(x,t))]\mathrm{d}x\\
&\quad-\beta\rho\int_{\mathbb{T}^2}[\mathrm{A}(\boldsymbol{z}_\rho(x,t)):\mathrm{A}(\boldsymbol{v}(x,t))][\mathrm{A}(\boldsymbol{z}_\rho(x,t)):\mathrm{A}(\chi_\rho(x,t))]\mathrm{d}x, \qquad (6.4)
\end{aligned}
$$

for a.e. $t\in[0,\mathscr{T}]$. Next, using (2.12), Hölder's, Sobolev and Young's inequalities, we estimate all the terms in the right hand side of (6.4) as follows:

$$
\begin{aligned}
|\rho b(\boldsymbol{z}_\rho,\boldsymbol{z}_\rho,\chi_\rho)|&=|\rho b(\boldsymbol{z}_\rho,\chi_\rho,\boldsymbol{z}_\rho)|\leq\rho\|\boldsymbol{z}_\rho\|_4^2\|\chi_\rho\|_2\leq C\rho\|\mathscr{A}\boldsymbol{z}_\rho\|_2^2\|\nabla\chi_\rho\|_2\\
&\leq C\rho^2\|\mathscr{A}\boldsymbol{z}_\rho\|_2^4+\frac{\mu}{16}\|\nabla\chi_\rho\|_2^2, \qquad (6.5)\\
|b(\chi_\rho,\boldsymbol{v},\chi_\rho)|&\leq\|\chi_\rho\|_4^2\|\nabla\boldsymbol{v}\|_2\leq C\|\nabla\chi_\rho\|_2^2\|\nabla\boldsymbol{v}\|_2\\
&\leq C\|\nabla\boldsymbol{v}\|_2^2\|\nabla\chi_\rho\|_2^2+\frac{\mu}{16}\|\nabla\chi_\rho\|_2^2, \qquad (6.6)\\
&\left|\frac{\alpha\rho}{2}\int_{\mathbb{T}^2}\mathrm{A}(\boldsymbol{z}_\rho)\mathrm{A}(\boldsymbol{z}_\rho):\mathrm{A}(\chi_\rho)\mathrm{d}x\right|\\
&\leq\frac{|\alpha|\rho}{2}\|\mathrm{A}(\boldsymbol{z}_\rho)\|_4^2\|\mathrm{A}(\chi_\rho)\|_2\\
&\leq C\rho\|\mathscr{A}\boldsymbol{z}_\rho\|_2^2\|\nabla\chi_\rho\|_2\leq C\rho^2\|\mathscr{A}\boldsymbol{z}_\rho\|_2^4+\frac{\mu}{16}\|\nabla\chi_\rho\|_2^2, \qquad (6.7)\\
&\left|\frac{\alpha}{2}\int_{\mathbb{T}^2}\mathrm{A}(\boldsymbol{v})\mathrm{A}(\chi_\rho):\mathrm{A}(\chi_\rho)\mathrm{d}x\right|\\
&\leq\frac{|\alpha|}{2}\|\mathrm{A}(\boldsymbol{v})\|_\infty\|\mathrm{A}(\chi_\rho)\|_2^2\leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2\|\nabla\chi_\rho\|_2^2\\
&\leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2\|\nabla\chi_\rho\|_2^2+\frac{\mu}{16}\|\nabla\chi_\rho\|_2^2, \qquad (6.8)\\
&\left|\frac{\alpha}{2}\int_{\mathbb{T}^2}\mathrm{A}(\chi_\rho)\mathrm{A}(\boldsymbol{v}):\mathrm{A}(\chi_\rho)\mathrm{d}x\right|\\
&\leq\frac{|\alpha|}{2}\|\mathrm{A}(\boldsymbol{v})\|_\infty\|\mathrm{A}(\chi_\rho)\|_2^2\leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2\|\nabla\chi_\rho\|_2^2\\
&\leq C\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2\|\nabla\chi_\rho\|_2^2+\frac{\mu}{16}\|\nabla\chi_\rho\|_2^2, \qquad (6.9)\\
&\left|\frac{\beta\rho^2}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho)|^2[\mathrm{A}(\boldsymbol{z}_\rho):\mathrm{A}(\chi_\rho)]\mathrm{d}x\right|\\
&\leq\frac{\beta\rho^2}{2}\|\mathrm{A}(\boldsymbol{z}_\rho)\|_6^3\|\mathrm{A}(\chi_\rho)\|_2
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\rho^2\|\mathscr{A}\boldsymbol{z}_\rho\|_2^3\|\mathrm{A}(\chi_\rho)\|_2 \leq C\rho^4\|\mathscr{A}\boldsymbol{z}_\rho\|_2^6 + \frac{\mu}{16}\|\nabla\chi_\rho\|_2^2, \qquad (6.10)\\
&\left|\frac{\beta\rho}{2}\int_{\mathbb{T}^2}|\mathrm{A}(\boldsymbol{z}_\rho)|^2[\mathrm{A}(\boldsymbol{v}):\mathrm{A}(\chi_\rho)]\mathrm{d}x\right|\\
&\leq \frac{\beta\rho}{2}\|\mathrm{A}(\boldsymbol{z}_\rho)\|_4^2\|\mathrm{A}(\boldsymbol{v})\|_\infty\|\mathrm{A}(\chi_\rho)\|_2\\
&\leq C\rho\|\mathscr{A}\boldsymbol{z}_\rho\|_2^2\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2\|\nabla\chi_\rho\|_2\\
&\leq C\rho^2\|\mathscr{A}\boldsymbol{z}_\rho\|_2^4\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2 + \frac{\mu}{16}\|\nabla\chi_\rho\|_2^2, \qquad (6.11)\\
&\left|\beta\rho\int_{\mathbb{T}^2}[\mathrm{A}(\boldsymbol{z}_\rho):\mathrm{A}(\boldsymbol{v})][\mathrm{A}(\boldsymbol{z}_\rho):\mathrm{A}(\chi_\rho)]\mathrm{d}x\right|\\
&\leq C\rho^2\|\mathscr{A}\boldsymbol{z}_\rho\|_2^4\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}\|_2^2 + \frac{\mu}{16}\|\nabla\chi_\rho\|_2^2. \qquad (6.12)
\end{aligned}
$$

Combining (6.4)-(6.12), we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\chi_\rho(t)\|_2^2 + \gamma\|\nabla\chi_\rho(t)\|_2^2\right] + \mu\|\nabla\chi_\rho(t)\|_2^2\\
&\leq C\rho^2[1+\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}(t)\|_2^2]\|\mathscr{A}\boldsymbol{z}_\rho(t)\|_2^4 + C\rho^4\|\mathscr{A}\boldsymbol{z}_\rho(t)\|_2^6\\
&+ C\left[\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}(t)\|_2^2 + \|\nabla\boldsymbol{v}(t)\|_2^2\right]\|\nabla\chi_\rho(t)\|_2^2, \qquad (6.13)
\end{aligned}
$$

for a.e. $t \in [0,\mathscr{T}]$, which implies (6.3) in view of the Gronwall inequality (3.8), (3.15), Theorems 3.2-3.3 and (5.3)-(5.4). This completes the proof. □

6.1. Variation of the cost functional (2.15)

As a consequence of Lemma 6.1, we get the following result on the variation for the cost functional (2.15).

Proposition 6.1. *For some $\varepsilon > 0$, let $\boldsymbol{v} \in \mathrm{L}^\infty(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))$ be the solution of system (3.1) in the sense of Definition 3.1, $\boldsymbol{v}_\rho \in \mathrm{L}^\infty(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))$ be the solution of system (3.1) with $\boldsymbol{f}$ replaced by $\boldsymbol{f}_\rho := \boldsymbol{f} + \rho\boldsymbol{\psi}$ in the sense of Definition 3.1 and $\boldsymbol{z}$ be the solution of system (4.2) in the sense of Definition 4.1. Then, we have*

$$
\begin{aligned}
\mathrm{J}(\boldsymbol{f}_\rho,\boldsymbol{v}_\rho) = \mathrm{J}(\boldsymbol{f},\boldsymbol{v}) + \rho\int_0^{\mathscr{T}}&\{(\nabla_{\boldsymbol{f}}\mathfrak{L}(t,\boldsymbol{f}(t),\boldsymbol{v}(t)),\boldsymbol{\psi}(t))\\
&+(\nabla_{\boldsymbol{v}}\mathfrak{L}(t,\boldsymbol{f}(t),\boldsymbol{v}(t)),\boldsymbol{z}(t))\}\mathrm{d}t + o(\rho).
\end{aligned}
$$

7. Adjoint system

In this section, our goal is to demonstrate that, for $\boldsymbol{g} \in \mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{L}}^2(\mathbb{T}^2))$ and $\boldsymbol{v} \in \mathrm{L}^\infty(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))$ (for some $\varepsilon > 0$), the following adjoint system is well-posed:

$$\begin{cases} -\partial_t(\boldsymbol{p} - \gamma\Delta\boldsymbol{p}) = \boldsymbol{g} - \nabla\pi^* + \mu\Delta\boldsymbol{p} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{p} - \boldsymbol{p}\cdot\nabla\boldsymbol{v} & \\ \qquad + \alpha\mathrm{div}\big(\mathrm{A}(\boldsymbol{v})\mathrm{A}(\boldsymbol{p}) + \mathrm{A}(\boldsymbol{p})\mathrm{A}(\boldsymbol{v})\big) & \\ \qquad + \beta\big[\mathrm{div}\big(|\mathrm{A}(\boldsymbol{v})|^2\mathrm{A}(\boldsymbol{p})\big)\big] + 2\beta\mathrm{div}\big((\mathrm{A}(\boldsymbol{p}):\mathrm{A}(\boldsymbol{v}))\mathrm{A}(\boldsymbol{v})\big), & \text{in } \mathbb{T}^2\times(0,\mathscr{T}], \\ \mathrm{div}\,\boldsymbol{p} = 0, & \text{in } \mathbb{T}^2\times[0,\mathscr{T}], \\ \boldsymbol{p}(x,\mathscr{T}) = \boldsymbol{0}, & \text{in } \mathbb{T}^2, \end{cases} \tag{7.1}$$

with periodic boundary conditions, where $\boldsymbol{p} = (p^1,p^2)$ and π^* are unknown vector and scalar fields, receptively, and $\mathrm{A}(\boldsymbol{p}) = \nabla\boldsymbol{p} + (\nabla\boldsymbol{p})^T$. Taking the projection $\mathbb{P}$ to the system (7.1), we obtain

$$\begin{cases} -\partial_t(\boldsymbol{p} + \gamma\mathscr{A}\boldsymbol{p}) = \mathbb{P}\boldsymbol{g} - \mu\mathscr{A}\boldsymbol{p} + \mathscr{B}(\boldsymbol{v},\boldsymbol{p}) - \mathbb{P}[\boldsymbol{p}\cdot\nabla\boldsymbol{v}] + \alpha\mathbb{P}\big[\mathrm{div}\big(\mathrm{A}(\boldsymbol{v})\mathrm{A}(\boldsymbol{p}) + \mathrm{A}(\boldsymbol{p})\mathrm{A}(\boldsymbol{v})\big)\big] \\ \qquad + \beta\mathbb{P}\big[\mathrm{div}\big(|\mathrm{A}(\boldsymbol{v})|^2\mathrm{A}(\boldsymbol{p})\big)\big] + 2\beta\mathbb{P}\big[\mathrm{div}\big((\mathrm{A}(\boldsymbol{p}):\mathrm{A}(\boldsymbol{v}))\mathrm{A}(\boldsymbol{v})\big)\big], \\ \boldsymbol{p}(\mathscr{T}) = \boldsymbol{0}, \end{cases} \tag{7.2}$$

in $\mathbb{V}'$. We obtain the well-posedness of system (7.2) in the following sense:

Definition 7.1. A function $\boldsymbol{p} \in \mathrm{C}([0,\mathscr{T}];\mathbb{V}) \cap \mathrm{L}^\infty(0,\mathscr{T};\mathrm{D}(\mathscr{A}))$ with $\partial_t\boldsymbol{p} \in \mathrm{L}^2(0,\mathscr{T};\mathbb{V})$, is called a *strong solution* to system (7.2), if for $\boldsymbol{g} \in \mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{L}}^2(\mathbb{T}^2))$, it satisfies

(i) for any $\boldsymbol{\phi} \in \mathbb{V}$,

$$\begin{aligned} &-\langle\partial_t[\mathrm{I} + \gamma\mathscr{A}]\boldsymbol{p}(t),\boldsymbol{\phi}\rangle \\ &= \big\langle\boldsymbol{g}(t) - \mu\mathscr{A}\boldsymbol{p}(t) + \mathscr{B}(\boldsymbol{v}(t),\boldsymbol{p}(t)) - \boldsymbol{p}(t)\cdot\nabla\boldsymbol{v}(t) \\ &\quad + \alpha\mathrm{div}\big(\mathrm{A}(\boldsymbol{v}(t))\mathrm{A}(\boldsymbol{p}(t)) + \mathrm{A}(\boldsymbol{p}(t))\mathrm{A}(\boldsymbol{v}(t))\big) \\ &\quad + \beta\mathrm{div}\big(|\mathrm{A}(\boldsymbol{v}(t))|^2\mathrm{A}(\boldsymbol{p}(t))\big) + 2\beta\mathrm{div}\big((\mathrm{A}(\boldsymbol{p}(t)):\mathrm{A}(\boldsymbol{v}(t)))\mathrm{A}(\boldsymbol{v}(t))\big),\boldsymbol{\phi}\big\rangle, \end{aligned} \tag{7.3}$$

for a.e. $t \in [0,\mathscr{T}]$;

(ii) the terminal data:

$$\boldsymbol{z}(\mathscr{T}) = \boldsymbol{0}.$$

Next theorem is the main result of this section which provides the existence and uniqueness of global solutions to system (7.2) in the sense of Definition 7.1 satisfying energy equality.

Theorem 7.1. *For $\boldsymbol{g} \in \mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{L}}^2(\mathbb{T}^2))$, there exists a unique solution $\boldsymbol{p}$ to system (7.2) in the sense of Definition 7.1.*

Proof of Theorem 7.1. Notice that $\boldsymbol{p}$ is the solution of (7.1) if and only if $\boldsymbol{q}(t) = \boldsymbol{p}(\mathscr{T} - t)$ is the solution of the following initial value problem with $\bar{\boldsymbol{v}}(t) = \boldsymbol{v}(\mathscr{T} - t), \bar{\boldsymbol{g}}(t) = \boldsymbol{g}(\mathscr{T} - t)$ and $\bar{\pi}^*(t) = \pi^*(\mathscr{T} - t)$

$$\begin{cases} \partial_t(\boldsymbol{q} - \gamma\Delta\boldsymbol{q}) = \mathbb{P}\bar{\boldsymbol{g}} - \nabla\bar{\pi}^* + \mu\Delta\boldsymbol{q} + (\bar{\boldsymbol{v}}\cdot\nabla)\boldsymbol{q} - \boldsymbol{q}\cdot\nabla\bar{\boldsymbol{v}} & \\ \qquad + \alpha\mathrm{div}\big(\mathrm{A}(\bar{\boldsymbol{v}})\mathrm{A}(\boldsymbol{q}) + \mathrm{A}(\boldsymbol{q})\mathrm{A}(\bar{\boldsymbol{v}})\big) & \\ \qquad + \beta\big[\mathrm{div}\big(|\mathrm{A}(\bar{\boldsymbol{v}})|^2\mathrm{A}(\boldsymbol{q})\big)\big] + 2\beta\mathrm{div}\big((\mathrm{A}(\boldsymbol{q}) : \mathrm{A}(\bar{\boldsymbol{v}}))\mathrm{A}(\bar{\boldsymbol{v}})\big), & \text{in } \mathbb{T}^2 \times (0, \mathscr{T}], \\ \mathrm{div}\,\boldsymbol{q} = 0, & \text{in } \mathbb{T}^2 \times [0, \mathscr{T}], \\ \boldsymbol{q}(x, 0) = \mathbf{0}, & \text{in } \mathbb{T}^2, \end{cases} \tag{7.4}$$

with periodic boundary conditions. According to the Definition 7.1, $\boldsymbol{q}$ is the solution of system (7.4) if $\boldsymbol{q} \in \mathrm{C}([0, \mathscr{T}]; \mathbb{V}) \cap \mathrm{L}^\infty(0, \mathscr{T}; \mathrm{D}(\mathscr{A}))$ with $\partial_t\boldsymbol{q} \in \mathrm{L}^2(0, \mathscr{T}; \mathbb{V}')$, $\boldsymbol{q}(0) = \mathbf{0}$ and , in addition, the following equality holds, for all $\boldsymbol{\phi} \in \mathbb{V}$,

$$\begin{aligned} &\langle\partial_t[\mathrm{I} + \gamma\mathscr{A}]\boldsymbol{q}(t), \boldsymbol{\phi}\rangle \\ &= \big\langle\bar{\boldsymbol{g}}(t) - \mu\mathscr{A}\boldsymbol{q}(t) + (\bar{\boldsymbol{v}}(t)\cdot\nabla)\boldsymbol{q}(t) \\ &\quad - \boldsymbol{q}(t)\cdot\nabla\bar{\boldsymbol{v}}(t) \\ &+ \alpha\mathrm{div}\big(\mathrm{A}(\bar{\boldsymbol{v}}(t))\mathrm{A}(\boldsymbol{q}(t)) + \mathrm{A}(\boldsymbol{q}(t))\mathrm{A}(\bar{\boldsymbol{v}}(t))\big) \\ &\quad + \beta\mathrm{div}\big(|\mathrm{A}(\bar{\boldsymbol{v}}(t))|^2\mathrm{A}(\boldsymbol{q}(t))\big) + 2\beta\mathrm{div}\big((\mathrm{A}(\boldsymbol{q}(t)) : \mathrm{A}(\bar{\boldsymbol{v}}(t)))\mathrm{A}(\bar{\boldsymbol{v}}(t))\big), \boldsymbol{\phi}\big\rangle, \end{aligned} \tag{7.5}$$

for a.e. $t \in [0, \mathscr{T}]$.

Let us consider the following approximate equation for system (7.5) on the finite-dimensional space $\mathbb{H}_n$:

$$\begin{cases} \partial_t(\boldsymbol{q}^n(t) + \gamma\mathscr{A}\boldsymbol{q}^n(t)) = \mathrm{P}_n\mathbb{P}\bar{\boldsymbol{g}}(t) - \mu\mathrm{P}_n\mathscr{A}\boldsymbol{q}^n(t) - \mathrm{P}_n\mathscr{B}(\bar{\boldsymbol{v}}(t), \boldsymbol{q}^n(t)) \\ \qquad - \mathrm{P}_n\mathbb{P}\boldsymbol{q}^n(t)\cdot\nabla\bar{\boldsymbol{v}}(t) \\ \qquad + \alpha\mathrm{P}_n\mathbb{P}\big[\mathrm{div}\big(\mathrm{A}(\bar{\boldsymbol{v}}(t))\mathrm{A}(\boldsymbol{q}^n(t)) + \mathrm{A}(\boldsymbol{q}^n(t))\mathrm{A}(\bar{\boldsymbol{v}}(t))\big)\big] \\ \qquad + \beta\mathrm{P}_n\mathbb{P}\big[\mathrm{div}\big(|\mathrm{A}(\bar{\boldsymbol{v}}(t))|^2\mathrm{A}(\boldsymbol{q}^n(t))\big)\big] \\ \qquad + 2\beta\mathrm{P}_n\mathbb{P}\big[\mathrm{div}\big((\mathrm{A}(\boldsymbol{q}^n(t)) : \mathrm{A}(\bar{\boldsymbol{v}}(t)))\mathrm{A}(\bar{\boldsymbol{v}}(t))\big)\big], \\ \boldsymbol{q}^n(0) = \mathbf{0}, \end{cases} \tag{7.6}$$

Since system (7.4) is very similar to system (4.1), by following the same arguments as in the proof of Theorem 4.1, one can obtain the existence and uniqueness of solution of system (7.4). Consequently, this will imply the existence and uniqueness of solution of system (7.1) with the claimed regularity. □

8. Existence of optimal control and optimality condition

This section provides a duality relation between the solutions of the adjoint equation and the linearized equation, and establishes the existence of a solution to the control problem. Then, taking into account the duality relation, we show that the solution to the control problem satisfies the first order optimality criterion.

8.1. Duality property

Proposition 8.1. *Let* $\boldsymbol{v} \in \mathrm{L}^{\infty}(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))$ *(for some* $\varepsilon > 0$*) and* $\boldsymbol{g},\boldsymbol{\psi} \in \mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{L}}^2(\mathbb{T}^2))$*. Then we have*

$$\int_0^{\mathscr{T}} (\boldsymbol{\psi}(t),\boldsymbol{p}(t))\mathrm{d}t = \int_0^{\mathscr{T}} (\boldsymbol{g}(t),\boldsymbol{z}(t))dt,$$

where $\boldsymbol{p}$ *is the solution of (7.1) and* $\boldsymbol{z}$ *is the solution of (4.1).*

Proof. Let us consider the following approximate equations corresponding to systems (4.3) and (7.3) on the finite-dimensional space $\mathbb{H}_n$, respectively:

$$\begin{cases} (\partial_t(\boldsymbol{z}^n(t)+\gamma\mathscr{A}\boldsymbol{z}^n(t)),\boldsymbol{w}^n) = (\mathrm{P}_n\mathbb{P}\boldsymbol{\psi}(t) - \mu\mathrm{P}_n\mathscr{A}\boldsymbol{z}^n(t) - \mathrm{P}_n\mathscr{B}(\boldsymbol{v}(t),\boldsymbol{z}^n(t)) \\ \qquad - \mathrm{P}_n\mathscr{B}(\boldsymbol{z}^n(t),\boldsymbol{v}(t)),\boldsymbol{w}^n) \\ \qquad + (\alpha\mathrm{P}_n\mathbb{P}\big[\mathrm{div}\big(\mathrm{A}(\boldsymbol{v}(t))\mathrm{A}(\boldsymbol{z}^n(t)) + \mathrm{A}(\boldsymbol{z}^n(t))\mathrm{A}(\boldsymbol{v}(t))\big)\big],\boldsymbol{w}^n) \\ \qquad + (\beta\mathrm{P}_n\mathbb{P}\big[\mathrm{div}\big(|\mathrm{A}(\boldsymbol{v}(t))|^2\mathrm{A}(\boldsymbol{z}^n(t))\big)\big],\boldsymbol{w}^n) \\ \qquad + (2\beta\mathrm{P}_n\mathbb{P}\big[\mathrm{div}\big((\mathrm{A}(\boldsymbol{z}^n(t)):\mathrm{A}(\boldsymbol{v}(t)))\mathrm{A}(\boldsymbol{v}(t))\big)\big],\boldsymbol{w}^n), \\ \boldsymbol{z}^n(0) = \boldsymbol{0}, \end{cases} \tag{8.1}$$

for a.e. $t \in [0,\mathscr{T}]$ and for all $\boldsymbol{w}^n \in \mathbb{H}_n$, and

$$\begin{cases} -(\partial_t(\boldsymbol{p}^n(t)+\gamma\mathscr{A}\boldsymbol{p}^n(t)),\boldsymbol{w}^n) = (\mathrm{P}_n\mathbb{P}\boldsymbol{g}(t) - \mu\mathrm{P}_n\mathscr{A}\boldsymbol{p}^n(t) + \mathrm{P}_n\mathscr{B}(\boldsymbol{v}(t),\boldsymbol{p}^n(t)) \\ \qquad - \mathrm{P}_n\mathbb{P}[\boldsymbol{p}^n\cdot\nabla\boldsymbol{v}],\boldsymbol{w}^n) \\ \qquad + (\alpha\mathrm{P}_n\mathbb{P}\big[\mathrm{div}\big(\mathrm{A}(\boldsymbol{v}(t))\mathrm{A}(\boldsymbol{p}^n(t)) + \mathrm{A}(\boldsymbol{p}^n(t))\mathrm{A}(\boldsymbol{v}(t))\big)\big],\boldsymbol{w}^n) \\ \qquad + (\beta\mathrm{P}_n\mathbb{P}\big[\mathrm{div}\big(|\mathrm{A}(\boldsymbol{v}(t))|^2\mathrm{A}(\boldsymbol{p}^n(t))\big)\big],\boldsymbol{w}^n) \\ \qquad + (2\beta\mathrm{P}_n\mathbb{P}\big[\mathrm{div}\big((\mathrm{A}(\boldsymbol{p}^n(t)):\mathrm{A}(\boldsymbol{v}(t)))\mathrm{A}(\boldsymbol{v}(t))\big)\big],\boldsymbol{w}^n), \\ \boldsymbol{p}^n(\mathscr{T}) = \boldsymbol{0}, \end{cases} \tag{8.2}$$

for a.e. $t \in [0,\mathscr{T}]$ and for all $\boldsymbol{w}^n \in \mathbb{H}_n$.

Setting $\boldsymbol{w}^n = \boldsymbol{p}^n$ in $(8.1)_1$, we get

$$\begin{aligned} &(\partial_t(\boldsymbol{z}^n(t)+\gamma\mathscr{A}\boldsymbol{z}^n(t)),\boldsymbol{p}^n(t)) \\ &= (\mathrm{P}_n\mathbb{P}\boldsymbol{\psi}(t) - \mu\mathrm{P}_n\mathscr{A}\boldsymbol{z}^n(t) - \mathrm{P}_n\mathscr{B}(\boldsymbol{v}(t),\boldsymbol{z}^n(t)) - \mathrm{P}_n\mathscr{B}(\boldsymbol{z}^n(t),\boldsymbol{v}(t)),\boldsymbol{p}^n(t)) \\ &\quad + (\alpha\mathrm{P}_n\mathbb{P}\big[\mathrm{div}\big(\mathrm{A}(\boldsymbol{v}(t))\mathrm{A}(\boldsymbol{z}^n(t)) + \mathrm{A}(\boldsymbol{z}^n(t))\mathrm{A}(\boldsymbol{v}(t))\big)\big] \\ &\quad + \beta\mathrm{P}_n\mathbb{P}\big[\mathrm{div}\big(|\mathrm{A}(\boldsymbol{v}(t))|^2\mathrm{A}(\boldsymbol{z}^n(t))\big)\big],\boldsymbol{p}^n(t)) \\ &\quad + (2\beta\mathrm{P}_n\mathbb{P}\big[\mathrm{div}\big((\mathrm{A}(\boldsymbol{z}^n(t)):\mathrm{A}(\boldsymbol{v}(t)))\mathrm{A}(\boldsymbol{v}(t))\big)\big],\boldsymbol{p}^n(t)), \end{aligned} \tag{8.3}$$

for a.e. $t \in [0,\mathscr{T}]$. Also, setting $\boldsymbol{w}^n = \boldsymbol{z}^n(t)$ in $(8.2)_1$, we get

$$\begin{aligned} &-(\partial_t(\boldsymbol{p}^n(t)+\gamma\mathscr{A}\boldsymbol{p}^n(t)),\boldsymbol{z}^n(t)) \\ &= (\mathrm{P}_n\mathbb{P}\boldsymbol{g}(t) - \mu\mathrm{P}_n\mathscr{A}\boldsymbol{p}^n(t) + \mathrm{P}_n\mathscr{B}(\boldsymbol{v}(t),\boldsymbol{p}^n(t)) - \mathrm{P}_n\mathbb{P}[\boldsymbol{p}^n\cdot\nabla\boldsymbol{v}],\boldsymbol{z}^n(t)) \\ &\quad + (\alpha\mathrm{P}_n\mathbb{P}\big[\mathrm{div}\big(\mathrm{A}(\boldsymbol{v}(t))\mathrm{A}(\boldsymbol{p}^n(t)) + \mathrm{A}(\boldsymbol{p}^n(t))\mathrm{A}(\boldsymbol{v}(t))\big)\big] \\ &\quad + \beta\mathrm{P}_n\mathbb{P}\big[\mathrm{div}\big(|\mathrm{A}(\boldsymbol{v}(t))|^2\mathrm{A}(\boldsymbol{p}^n(t))\big)\big],\boldsymbol{z}^n(t)) \\ &\quad + (2\beta\mathrm{P}_n\mathbb{P}\big[\mathrm{div}\big((\mathrm{A}(\boldsymbol{p}^n(t)):\mathrm{A}(\boldsymbol{v}(t)))\mathrm{A}(\boldsymbol{v}(t))\big)\big],\boldsymbol{z}^n(t)), \end{aligned} \tag{8.4}$$

for a.e. $t \in [0,\mathscr{T}]$. The differentiation rules give

$$\begin{aligned} &-(\partial_t(\boldsymbol{p}^n(t)+\gamma\mathscr{A}\boldsymbol{p}^n(t)),\boldsymbol{z}^n(t)) = (\partial_t(\boldsymbol{z}^n(t)+\gamma\mathscr{A}\boldsymbol{z}^n(t)),\boldsymbol{p}^n(t)) \\ &- \partial_t\big[(\boldsymbol{p}^n(t),\boldsymbol{z}^n(t)) + \gamma(\nabla\boldsymbol{p}^n(t),\nabla\boldsymbol{z}^n(t))\big]. \end{aligned} \tag{8.5}$$

Integrating (8.5) with respect to t over $[0,\mathscr{T}]$, and using that $\boldsymbol{z}^n(0)=\boldsymbol{p}^n(\mathscr{T})=0$, we obtain

$$-\int_0^{\mathscr{T}}(\partial_t(\boldsymbol{p}^n(t)+\gamma\mathscr{A}\boldsymbol{p}^n(t)),\boldsymbol{z}^n(t))\mathrm{d}t=\int_0^{\mathscr{T}}(\partial_t(\boldsymbol{z}^n(t)+\gamma\mathscr{A}\boldsymbol{z}^n(t)),\boldsymbol{p}^n(t))\mathrm{d}t. \tag{8.6}$$

Substituting (8.4) and (8.3) in (8.6), integrating by parts and using the fact that $(A,B)=(A^T,B^T)$, for any $A,B\in\mathscr{M}_{2\times2}(\mathbb{R})$, we obtain

$$\int_0^{\mathscr{T}}(\boldsymbol{\psi}(t),\boldsymbol{p}^n(t))\mathrm{d}t=\int_0^{\mathscr{T}}(\boldsymbol{g}(t),\boldsymbol{z}^n(t))\mathrm{d}t.$$

Therefore, taking the limit as $n\to\infty$, we complete the proof. □

Considering $\boldsymbol{g}=\nabla_{\boldsymbol{v}}\mathfrak{L}(\cdot,\boldsymbol{f},\boldsymbol{v})\in\mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{L}}^2(\mathbb{T}^2))$ in Proposition 8.1, we obtain

Corollary 8.1. *Under the assumtions of Proposition 8.1, the following duality relation holds*

$$\int_0^{\mathscr{T}}(\boldsymbol{\psi}(t),\boldsymbol{p}(t))\mathrm{d}t=\int_0^{\mathscr{T}}(\nabla_{\boldsymbol{v}}\mathfrak{L}(t,\boldsymbol{f},\boldsymbol{v}),\boldsymbol{z}(t))\mathrm{d}t.$$

8.2. Existence of an optimal control for (2.16)

Suppose that $\{\boldsymbol{f}_n,\boldsymbol{v}_n\}_{n\in\mathbb{N}}$ is a minimizing sequence, remark that $\{\boldsymbol{f}_n\}_{n\in\mathbb{N}}$ is uniformly bounded in the closed convex set $\mathscr{F}_{ad}\subset\mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{H}}^\varepsilon(\mathbb{T}^2))$ for some $\varepsilon>0$. On the other hand, let us denote by $\boldsymbol{v}_n$ the solution of (1.4), where $\boldsymbol{f}$ is replaced by $\boldsymbol{f}_n$, Theorems 3.1, 3.2 and 3.3 ensure that $\{\boldsymbol{v}_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $\mathrm{C}([0,\mathscr{T}];\mathbb{V})\cap\mathrm{L}^\infty(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))\cap\mathrm{H}^1(0,\mathscr{T};\mathbb{V})$.

By Banach-Alaoglu theorem, there exists

$$(\widetilde{\boldsymbol{f}},\widetilde{\boldsymbol{v}})\in\mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{H}}^\varepsilon(\mathbb{T}^2))\times\left(\mathrm{C}([0,\mathscr{T}];\mathbb{V})\cap\mathrm{L}^\infty(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))\cap\mathrm{H}^1(0,\mathscr{T};\mathbb{V})\right)$$

such that the following convergences hold, up to a subsequence (denoting by the same)

$$\begin{array}{lll}\boldsymbol{f}_n & \rightharpoonup\widetilde{\boldsymbol{f}} & \text{in } \mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{H}}^\varepsilon(\mathbb{T}^2)),\\ \boldsymbol{v}_n & \overset{*}{\rightharpoonup}\widetilde{\boldsymbol{v}} & \text{in } \mathrm{L}^\infty(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}})),\\ \boldsymbol{v}_n & \rightharpoonup\widetilde{\boldsymbol{v}} & \text{in } \mathrm{L}^2(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}})),\\ \partial_t\boldsymbol{v}_n & \rightharpoonup\partial_t\widetilde{\boldsymbol{v}} & \text{in } \mathrm{L}^2(0,\mathscr{T};\mathbb{V}).\end{array} \tag{8.7}$$

By (8.7) and [41, Lemma 3.1, p. 404], we observe that $\widetilde{\boldsymbol{v}}\in\mathrm{C}([0,\mathscr{T}],\mathbb{V})$ and therefore $\boldsymbol{v}_n(0)$ converges to $\widetilde{\boldsymbol{v}}(0)$ in $\mathbb{V}$, which gives $\widetilde{\boldsymbol{v}}(0)=\boldsymbol{v}_0$. Now, a similar arguments as in the proof of Theorem 3.1 provide that $(\widetilde{\boldsymbol{f}},\widetilde{\boldsymbol{v}})$ solves (1.4).

Recall that $\mathrm{J}:\mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{H}}^\varepsilon(\mathbb{T}^2))\times\mathrm{L}^\infty(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}}))\to\mathbb{R}^+$ given by (2.15) is convex and continuous. From (8.7) we have

$$\boldsymbol{f}_n\rightharpoonup\widetilde{\boldsymbol{f}}\quad\text{in } \mathrm{L}^2(0,\mathscr{T};\dot{\mathbb{H}}^\varepsilon(\mathbb{T}^2))\text{ and }\boldsymbol{v}_n\rightharpoonup\widetilde{\boldsymbol{v}}\quad\text{in } \mathrm{L}^2(0,\mathscr{T};\mathrm{D}(\mathscr{A}^{1+\frac{\varepsilon}{2}})).$$

The weak lower semicontinuity of J ensures

$$\mathrm{J}(\widetilde{\boldsymbol{f}},\widetilde{\boldsymbol{v}})\le\liminf_n\mathrm{J}(\boldsymbol{f}_n,\boldsymbol{v}_n),$$

which gives that $(\widetilde{\boldsymbol{f}},\widetilde{\boldsymbol{v}})$ is an optimal pair.

8.3. A necessary optimality condition for (2.16)

Suppose that $(\widetilde{\boldsymbol{f}},\widetilde{\boldsymbol{v}})$ is an optimal control pair. Consider $\boldsymbol{\psi}\in\mathscr{F}_{ad}$ and define $\boldsymbol{f}_\rho:=\widetilde{\boldsymbol{f}}+\rho(\boldsymbol{\psi}-\widetilde{\boldsymbol{f}})$. Thanks to Lemma 6.1 and Proposition 6.1, we have

$$\frac{\mathrm{J}(\boldsymbol{f}_\rho,\boldsymbol{v}_\rho)-\mathrm{J}(\widetilde{\boldsymbol{f}},\widetilde{\boldsymbol{v}})}{\rho}$$
$$=\int_0^{\mathscr{T}}\{(\nabla_{\boldsymbol{f}}\mathfrak{L}(t,\widetilde{\boldsymbol{f}}(t),\widetilde{\boldsymbol{v}}(t)),\boldsymbol{\psi}(t)-\widetilde{\boldsymbol{f}}(t))+(\nabla_{\boldsymbol{v}}\mathfrak{L}(t,\widetilde{\boldsymbol{f}}(t),\widetilde{\boldsymbol{v}}(t)),\boldsymbol{z}(t))\}\mathrm{d}t+\frac{o(\rho)}{\rho}.$$

Then, the Gateaux derivative of the cost functional J is given by

$$\lim_{\rho\to0}\frac{\mathrm{J}(\boldsymbol{f}_\rho,\boldsymbol{v}_\rho)-\mathrm{J}(\widetilde{\boldsymbol{f}},\widetilde{\boldsymbol{v}})}{\rho}$$
$$=\int_0^{\mathscr{T}}\{(\nabla_{\boldsymbol{f}}\mathfrak{L}(t,\widetilde{\boldsymbol{f}}(t),\widetilde{\boldsymbol{v}}(t)),\boldsymbol{\psi}(t)-\widetilde{\boldsymbol{f}}(t))+(\nabla_{\boldsymbol{v}}\mathfrak{L}(t,\widetilde{\boldsymbol{f}}(t),\widetilde{\boldsymbol{v}}(t)),\boldsymbol{z}(t))\}\mathrm{d}t.$$

Therefore, we have

$$\int_0^{\mathscr{T}}\{(\nabla_{\boldsymbol{f}}\mathfrak{L}(t,\widetilde{\boldsymbol{f}}(t),\widetilde{\boldsymbol{v}}(t)),\boldsymbol{\psi}(t)-\widetilde{\boldsymbol{f}}(t))+(\nabla_{\boldsymbol{v}}\mathfrak{L}(t,\widetilde{\boldsymbol{f}}(t),\widetilde{\boldsymbol{v}}(t)),\boldsymbol{z}(t))\}\mathrm{d}t\geq0, \tag{8.8}$$

where $\boldsymbol{z}$ is the unique solution to the linearized problem (4.1) with $\boldsymbol{\psi}$ replaced by $\boldsymbol{\psi}-\widetilde{\boldsymbol{f}}$.

Let $\tilde{\boldsymbol{p}}$ be the unique solution of (7.1). The application of Proposition 8.1 yields

$$\int_0^{\mathscr{T}}(\boldsymbol{\psi}(t)-\widetilde{\boldsymbol{f}}(t),\tilde{\boldsymbol{p}}(t))dt=\int_0^{\mathscr{T}}(\nabla_{\boldsymbol{v}}\mathfrak{L}(t,\widetilde{\boldsymbol{f}}(t),\widetilde{\boldsymbol{v}}(t)),\boldsymbol{z}(t))\mathrm{d}t. \tag{8.9}$$

Finally, we obtain the following optimality condition, for any $\boldsymbol{\psi}\in\mathscr{F}_{ad}$

$$\int_0^{\mathscr{T}}(\boldsymbol{\psi}(t)-\widetilde{\boldsymbol{f}}(t),\tilde{\boldsymbol{p}}(t)+\nabla_{\boldsymbol{f}}\mathfrak{L}(t,\widetilde{\boldsymbol{f}}(t),\widetilde{\boldsymbol{v}}(t)))\mathrm{d}t$$
$$=\int_0^{\mathscr{T}}(\nabla_{\boldsymbol{v}}\mathfrak{L}(t,\widetilde{\boldsymbol{f}}(t),\widetilde{\boldsymbol{v}}(t)),\boldsymbol{z}(t))\mathrm{d}t+\int_0^{T}(\nabla_{\boldsymbol{f}}\mathfrak{L}(t,\widetilde{\boldsymbol{f}}(t),\widetilde{\boldsymbol{v}}(t)),\boldsymbol{\psi}(t)-\widetilde{\boldsymbol{f}}(t))dt\geq0, \tag{8.10}$$

where we have used (8.8) and (8.9). The combination of the preceding sections leads to the proof of Theorem 2.1.

9. Uniqueness of the optimal solution

The coupled system composed of the so-called first-order optimality conditions for the control problem was deduced in the previous section. Accordingly, a solution of the coupled system is currently only a candidate for the optimal solution. Thus, a priori knowledge about the the uniqueness of the solution to the coupled system is a key factor in determining the optimal solution. This section focuses on the proof of Theorem 2.2, which establishes an uniqueness result for the coupled system by using the methods in [2].

For the reader's convenience, we first restate the coupled optimality system below:

$$\left.\begin{aligned} \partial_t\left(\boldsymbol{v}+\gamma\mathscr{A}\boldsymbol{v}\right) &= -\mu\mathscr{A}\boldsymbol{v}-\mathscr{B}(\boldsymbol{v})-\alpha\mathscr{J}(\boldsymbol{v})-\beta\mathscr{K}(\boldsymbol{v})+\mathbb{P}\boldsymbol{f},\\ \boldsymbol{v}(0) &= \boldsymbol{v}_0, \end{aligned}\right\}$$

(State equation (Forward))

$$\left.\begin{aligned} -\partial_t(\boldsymbol{p}+\gamma\mathscr{A}\boldsymbol{p}) &= \mathbb{P}(\boldsymbol{v}-\boldsymbol{v}_d)-\mu\mathscr{A}\boldsymbol{p}+\mathscr{B}(\boldsymbol{v},\boldsymbol{p})-\mathbb{P}[\boldsymbol{p}\cdot\nabla\boldsymbol{v}]\\ &\quad+\alpha\mathbb{P}\big[\mathrm{div}\big(\mathrm{A}(\boldsymbol{v})\mathrm{A}(\boldsymbol{p})+\mathrm{A}(\boldsymbol{p})\mathrm{A}(\boldsymbol{v})\big)\big]\\ &\quad+\beta\mathbb{P}\big[\mathrm{div}\big(|\mathrm{A}(\boldsymbol{v})|^2\mathrm{A}(\boldsymbol{p})\big)\big]\\ &\quad+2\beta\mathbb{P}\big[\mathrm{div}\big((\mathrm{A}(\boldsymbol{p}):\mathrm{A}(\boldsymbol{v}))\mathrm{A}(\boldsymbol{v})\big)\big],\\ \boldsymbol{p}(\mathscr{T}) &= \boldsymbol{0}, \end{aligned}\right\}$$

(Adjoint equation (Backward))

$$\int_0^{\mathscr{T}}(\boldsymbol{\psi}(t)-\boldsymbol{f}(t),\boldsymbol{p}(t)+\lambda\boldsymbol{f}(t))\mathrm{d}t\geq 0,\quad \text{for all}\quad \boldsymbol{\psi}\in\mathscr{F}_{ad}.$$

(Optimality condition)

9.1. Proof of Theorem 2.2

Suppose that $\boldsymbol{f}_1,\boldsymbol{f}_2$ are two optimal control variables for (2.16) and $\boldsymbol{v}_1,\boldsymbol{v}_2$ are the corresponding optimal states with the adjoint states $\boldsymbol{p}_1,\boldsymbol{p}_2$, respectively.

Now, let us consider $\mathfrak{v}=\boldsymbol{v}_1-\boldsymbol{v}_2$, $\mathfrak{f}=\boldsymbol{f}_1-\boldsymbol{f}_2$ and notice that $\mathfrak{v}$ solves the equation

$$\partial_t(\mathrm{I}+\gamma\mathscr{A})\mathfrak{v}(t)=-\left[\mathscr{G}(\boldsymbol{v}_1(t))-\mathscr{G}(\boldsymbol{v}_2(t))\right]+\mathbb{P}\mathfrak{f}(t),\tag{9.1}$$

for a.e. $t\in[0,\mathscr{T}]$, in $\mathbb{V}'$ with $\mathfrak{v}(0)=\boldsymbol{0}$.

Corollary 9.1. *For some $\varepsilon>0$, there exists a constant $\kappa>0$ such that the solution $\boldsymbol{v}_i$ of system (1.4) corresponding to $\boldsymbol{f}_i$ satisfies*

$$\sup_{t\in[0,\mathscr{T}]}\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}_i(t)\|_2^2\leq\kappa^2.\tag{9.2}$$

Corollary 9.2. *Let us consider the problem (2.16) and $\boldsymbol{g}=\nabla_{\boldsymbol{v}}\mathfrak{L}(\cdot,\boldsymbol{f}_i,\boldsymbol{v})$ in system (7.1). Then, there exists a constant $\widetilde{\lambda}>0$ such that*

$$\sup_{t\in[0,\mathscr{T}]}\left[\|\nabla\boldsymbol{p}_i(t)\|_2^2+\|\mathscr{A}\boldsymbol{p}_i(t)\|_2^2\right]\leq C(\mathscr{T})\int_0^{\mathscr{T}}\|\boldsymbol{v}_i(s)-\boldsymbol{v}_d(s)\|_2^2ds\leq\widetilde{\lambda}^2.\tag{9.3}$$

Note that existence of $\widetilde{\lambda}$ is confirmed due to (3.8).

Corollary 9.3. *There exists a constant $\Gamma>0$ such that*

$$|b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w})|\leq\Gamma\|\nabla\boldsymbol{w}\|_2\|\nabla\boldsymbol{u}\|_2^2,\qquad \textit{for all}\ \ \boldsymbol{u},\boldsymbol{w}\in\mathbb{V}.\tag{9.4}$$

Let us now take the inner product (9.1) by $\boldsymbol{p}_2$ to obtain

$$\begin{aligned}
&\langle \partial_t \mathfrak{v}, \boldsymbol{p}_2\rangle + \gamma \langle \partial_t \nabla \mathfrak{v}, \nabla \boldsymbol{p}_2\rangle \\
&= (\mathfrak{f}, \boldsymbol{p}_2) - \mu(\nabla \mathfrak{v}, \nabla \boldsymbol{p}_2) - b(\mathfrak{v}, \mathfrak{v}, \boldsymbol{p}_2) - b(\mathfrak{v}, \boldsymbol{v}_2, \boldsymbol{p}_2) - b(\boldsymbol{v}_2, \mathfrak{v}, \boldsymbol{p}_2) \\
&- \frac{\alpha}{2}\big((\mathrm{A}(\mathfrak{v}))^2 + \mathrm{A}(\mathfrak{v})\mathrm{A}(\boldsymbol{v}_2) + \mathrm{A}(\boldsymbol{v}_2)\mathrm{A}(\mathfrak{v}), \mathrm{A}(\boldsymbol{p}_2)\big) \\
&\quad - \frac{\beta}{2}\big(|\mathrm{A}(\mathfrak{v})|^2\mathrm{A}(\mathfrak{v}) + |\mathrm{A}(\mathfrak{v})|^2\mathrm{A}(\boldsymbol{v}_2) + (\mathrm{A}(\mathfrak{v}) : \mathrm{A}(\boldsymbol{v}_2))\,\mathrm{A}(\mathfrak{v}), \mathrm{A}(\boldsymbol{p}_2)\big) \\
&\quad - \frac{\beta}{2}\big(\,(\mathrm{A}(\mathfrak{v}) : \mathrm{A}(\boldsymbol{v}_2))\,\mathrm{A}(\boldsymbol{v}_2) + (\mathrm{A}(\boldsymbol{v}_2) : \mathrm{A}(\mathfrak{v}))\,\mathrm{A}(\mathfrak{v}) \\
&+ (\mathrm{A}(\boldsymbol{v}_2) : \mathrm{A}(\mathfrak{v}))\,\mathrm{A}(\boldsymbol{v}_2) + |\mathrm{A}(\boldsymbol{v}_2)|^2\mathrm{A}(\mathfrak{v}), \mathrm{A}(\boldsymbol{p}_2)\big).
\end{aligned} \tag{9.5}$$

Considering the adjoint equation $(7.1)_1$ for $\boldsymbol{p}_2$ (corresponding to $\boldsymbol{v}_2$ and $\boldsymbol{g} = \boldsymbol{v}_2 - \boldsymbol{v}_d$) and taking the inner product with $\mathfrak{v}$, we write

$$\begin{aligned}
&\langle \partial_t \boldsymbol{p}_2, \mathfrak{v}\rangle + \gamma \langle \partial_t \nabla \boldsymbol{p}_2, \nabla \mathfrak{v}\rangle \\
&= -(\boldsymbol{v}_2 - \boldsymbol{v}_d, \mathfrak{v}) + \mu(\nabla \boldsymbol{p}_2, \nabla \mathfrak{v}) - b(\boldsymbol{v}_2, \boldsymbol{p}_2, \mathfrak{v}) + b(\mathfrak{v}, \boldsymbol{v}_2, \boldsymbol{p}_2) \\
&\quad + \frac{\alpha}{2}\big(\mathrm{A}(\boldsymbol{v}_2)\mathrm{A}(\boldsymbol{p}_2) + \mathrm{A}(\boldsymbol{p}_2)\mathrm{A}(\boldsymbol{v}_2), \mathrm{A}(\mathfrak{v})\big) \\
&\quad + \frac{\beta}{2}\big(|\mathrm{A}(\boldsymbol{v}_2)|^2\mathrm{A}(\boldsymbol{p}_2), \mathrm{A}(\mathfrak{v})\big) + \beta\big((\mathrm{A}(\boldsymbol{p}_2) : \mathrm{A}(\boldsymbol{v}_2))\mathrm{A}(\boldsymbol{v}_2), \mathrm{A}(\mathfrak{v})\big).
\end{aligned} \tag{9.6}$$

Now summing the equalities (9.5) and (9.6), we get

$$\begin{aligned}
&\partial_t \langle \mathfrak{v}, \boldsymbol{p}_2\rangle + \gamma \partial_t \langle \nabla \mathfrak{v}, \nabla \boldsymbol{p}_2\rangle \\
&= -(\boldsymbol{v}_2 - \boldsymbol{v}_d, \mathfrak{v}) + (\mathfrak{f}, \boldsymbol{p}_2) - b(\mathfrak{v}, \mathfrak{v}, \boldsymbol{p}_2) - \alpha\big((\mathrm{A}(\mathfrak{v}))^2, \nabla \boldsymbol{p}_2\big) \\
&\quad - \beta\big(|\mathrm{A}(\mathfrak{v})|^2\mathrm{A}(\mathfrak{v}) + |\mathrm{A}(\mathfrak{v})|^2\mathrm{A}(\boldsymbol{v}_2), \nabla \boldsymbol{p}_2\big) - 4\beta\big(\,(\mathrm{A}(\boldsymbol{v}_2) : \mathrm{A}(\mathfrak{v}))\,\mathrm{A}(\mathfrak{v}), \nabla \boldsymbol{p}_2\big),
\end{aligned} \tag{9.7}$$

where we used the symmetry of $\mathrm{A}(\cdot)$ and the the property (2.6) of trilinear form $b(\cdot,\cdot,\cdot)$. By integrating (9.7) from $t = 0$ to $t = \mathscr{T}$ and taking into account the initial and terminal conditions for $\mathfrak{v}$ and $\boldsymbol{p}_2$, we have

$$\begin{aligned}
0 &= -\int_0^{\mathscr{T}} (\boldsymbol{v}_2(t) - \boldsymbol{v}_d(t), \mathfrak{v}(t))\mathrm{d}t + \int_0^{\mathscr{T}} (\mathfrak{f}(t), \boldsymbol{p}_2(t))\mathrm{d}t - \int_0^{\mathscr{T}} b(\mathfrak{v}(t), \mathfrak{v}(t), \boldsymbol{p}_2(t))\mathrm{d}t \\
&\quad - \alpha \int_0^{\mathscr{T}} \big((\mathrm{A}(\mathfrak{v}(t)))^2, \nabla \boldsymbol{p}_2(t)\big)\mathrm{d}t \\
&\quad\quad - \beta \int_0^{\mathscr{T}} \big(|\mathrm{A}(\mathfrak{v}(t))|^2\mathrm{A}(\mathfrak{v}(t)) + |\mathrm{A}(\mathfrak{v}(t))|^2\mathrm{A}(\boldsymbol{v}_2(t)), \nabla \boldsymbol{p}_2(t)\big)\mathrm{d}t \\
&\quad - 4\beta \int_0^{\mathscr{T}} \big(\,(\mathrm{A}(\boldsymbol{v}_2(t)) : \mathrm{A}(\mathfrak{v}(t)))\,\mathrm{A}(\mathfrak{v}(t)), \nabla \boldsymbol{p}_2(t)\big)\mathrm{d}t.
\end{aligned} \tag{9.8}$$

Analogously, we can show that $\bar{\mathfrak{v}} = -\mathfrak{v}$ verifies the relation

$$\begin{aligned}
&\partial_t \langle \bar{\mathfrak{v}}, \boldsymbol{p}_1\rangle + \gamma \partial_t \langle \nabla \bar{\mathfrak{v}}, \nabla \boldsymbol{p}_1\rangle \\
&= -(\boldsymbol{v}_1 - \boldsymbol{v}_d, \bar{\mathfrak{v}}) - (\mathfrak{f}, \boldsymbol{p}_1) - b(\bar{\mathfrak{v}}, \bar{\mathfrak{v}}, \boldsymbol{p}_1) - \alpha\big((\mathrm{A}(\bar{\mathfrak{v}}))^2, \nabla \boldsymbol{p}_1\big) \\
&\quad - \beta\big(|\mathrm{A}(\bar{\mathfrak{v}})|^2\mathrm{A}(\bar{\mathfrak{v}}) + |\mathrm{A}(\bar{\mathfrak{v}})|^2\mathrm{A}(\boldsymbol{v}_1), \nabla \boldsymbol{p}_1\big) - 4\beta\big(\,(\mathrm{A}(\boldsymbol{v}_1) : \mathrm{A}(\bar{\mathfrak{v}}))\,\mathrm{A}(\bar{\mathfrak{v}}), \nabla \boldsymbol{p}_1\big),
\end{aligned}$$

Integrating from $t = 0$ to $t = \mathscr{T}$, making use of the fact that $\bar{\mathfrak{v}} = -\mathfrak{v}$ and using the initial and terminal conditions for $\mathfrak{v}$ and $\boldsymbol{p}_2$, we have

$$\begin{aligned} 0 &= \int_0^{\mathscr{T}} (\boldsymbol{v}_1(t) - \boldsymbol{v}_d(t), \mathfrak{v}(t))\mathrm{d}t \\ &- \int_0^{\mathscr{T}} (\mathfrak{f}(t), \boldsymbol{p}_1(t))\mathrm{d}t - \int_0^{\mathscr{T}} b(\mathfrak{v}(t), \mathfrak{v}(t), \boldsymbol{p}_1(t))\mathrm{d}t - \alpha \int_0^{\mathscr{T}} \big((\mathrm{A}(\mathfrak{v}(t)))^2, \nabla \boldsymbol{p}_1(t)\big)\mathrm{d}t \\ &+ \beta \int_0^{\mathscr{T}} \big(|\mathrm{A}(\mathfrak{v}(t))|^2 \mathrm{A}(\mathfrak{v}(t)) - |\mathrm{A}(\mathfrak{v}(t))|^2 \mathrm{A}(\boldsymbol{v}_1(t)), \nabla \boldsymbol{p}_1(t)\big)\mathrm{d}t \\ &- 4\beta \int_0^{\mathscr{T}} \big((\mathrm{A}(\boldsymbol{v}_1(t)) : \mathrm{A}(\mathfrak{v}(t)))\, \mathrm{A}(\mathfrak{v}(t)), \nabla \boldsymbol{p}_1(t)\big)\mathrm{d}t. \end{aligned} \tag{9.9}$$

By summing (9.8) and (9.9), we infer

$$\begin{aligned} &\int_0^{\mathscr{T}} \|\mathfrak{v}(t)\|_2^2 \mathrm{d}t - \int_0^{\mathscr{T}} (\mathfrak{f}(t), \boldsymbol{p}_1(t) - \boldsymbol{p}_2(t))\mathrm{d}t \\ &= \underbrace{\int_0^{\mathscr{T}} b(\mathfrak{v}(t), \mathfrak{v}(t), \boldsymbol{p}_1(t) + \boldsymbol{p}_2(t))\mathrm{d}t}_{I_1} \\ &+ \underbrace{\alpha \int_0^{\mathscr{T}} \big((\mathrm{A}(\mathfrak{v}(t)))^2, \nabla \boldsymbol{p}_1(t) + \nabla \boldsymbol{p}_2(t)\big)\mathrm{d}t}_{I_2} \\ &\underbrace{-\beta \int_0^{\mathscr{T}} \big(|\mathrm{A}(\mathfrak{v}(t))|^2 \mathrm{A}(\mathfrak{v}(t)) - |\mathrm{A}(\mathfrak{v}(t))|^2 \mathrm{A}(\boldsymbol{v}_1(t)), \nabla \boldsymbol{p}_1(t)\big)\mathrm{d}t}_{I_3} \\ &+ \underbrace{4\beta \int_0^{\mathscr{T}} \big((\mathrm{A}(\boldsymbol{v}_1(t)) : \mathrm{A}(\mathfrak{v}(t)))\, \mathrm{A}(\mathfrak{v}(t)), \nabla \boldsymbol{p}_1(t)\big)\mathrm{d}t}_{I_4} \\ &+ \beta \underbrace{\int_0^{\mathscr{T}} \big(|\mathrm{A}(\mathfrak{v}(t))|^2 \mathrm{A}(\mathfrak{v}(t)) + |\mathrm{A}(\mathfrak{v}(t))|^2 \mathrm{A}(\boldsymbol{v}_2(t)), \nabla \boldsymbol{p}_2(t)\big)\mathrm{d}t}_{I_5} \\ &+ \underbrace{4\beta \int_0^{\mathscr{T}} \big((\mathrm{A}(\boldsymbol{v}_2(t)) : \mathrm{A}(\mathfrak{v}(t)))\, \mathrm{A}(\mathfrak{v}(t)), \nabla \boldsymbol{p}_2(t)\big)\mathrm{d}t}_{I_6}. \end{aligned} \tag{9.10}$$

From (8.10), the following optimality conditions hold

$$\int_0^{\mathscr{T}} (\boldsymbol{\psi}(t) - \boldsymbol{f}_1(t), \boldsymbol{p}_1(t) + \lambda \boldsymbol{f}_1(t))\mathrm{d}t \geq 0, \quad \text{for all } \boldsymbol{\psi} \in \mathscr{F}_{ad}, \tag{9.11}$$

and

$$\int_0^{\mathscr{T}} (\boldsymbol{\psi}(t) - \boldsymbol{f}_2(t), \boldsymbol{p}_2(t) + \lambda \boldsymbol{f}_2(t))\mathrm{d}t \geq 0, \quad \text{for all } \boldsymbol{\psi} \in \mathscr{F}_{ad}. \tag{9.12}$$

Setting $\boldsymbol{\psi}=\boldsymbol{f}_2$ and $\boldsymbol{\psi}=\boldsymbol{f}_1$ in (9.11) and (9.12), respectively, we achieve

$$\lambda\int_0^{\mathscr{T}}\|\mathfrak{f}(t)\|_2^2\mathrm{d}t\leq-\int_0^{\mathscr{T}}(\mathfrak{f}(t),\boldsymbol{p}_1(t)-\boldsymbol{p}_2(t))\mathrm{d}t. \tag{9.13}$$

Using (2.12), (2.13), (5.3), (5.4), (9.2), (9.3) and (9.4), we obtain

$$|I_1|\leq\Gamma\int_0^{\mathscr{T}}(\|\nabla\boldsymbol{p}_1(t)\|_2+\|\nabla\boldsymbol{p}_2(t)\|_2)\|\nabla\mathfrak{v}(t)\|_2^2\mathrm{d}t\leq 2\Gamma\Gamma_2\widetilde{\lambda}\int_0^{\mathscr{T}}\|\mathfrak{f}(t)\|_2^2\mathrm{d}t, \tag{9.14}$$

$$\begin{aligned}|I_2|&\leq|\alpha|\int_0^{\mathscr{T}}(\|\nabla\boldsymbol{p}_1(t)\|_4+\|\nabla\boldsymbol{p}_2(t)\|_4)\|\mathrm{A}(\mathfrak{v}(t))\|_4\|\mathrm{A}(\mathfrak{v}(t))\|_2\mathrm{d}t\\&\leq 4|\alpha|[S_4]^2\int_0^{\mathscr{T}}(\|\mathscr{A}\boldsymbol{p}_1(t)\|_2+\|\mathscr{A}\boldsymbol{p}_2(t)\|_2)\|\mathscr{A}\mathfrak{v}(t)\|_2\|\nabla\mathfrak{v}(t)\|_2\mathrm{d}t\\&\leq\frac{8|\alpha|}{\mu}[S_4]^2\Gamma_1\Gamma_2\widetilde{\lambda}\int_0^{\mathscr{T}}\|\mathfrak{f}(t)\|_2^2\mathrm{d}t,\end{aligned} \tag{9.15}$$

$$\begin{aligned}\sum_{i=3}^{6}|I_i|&\leq 7\beta\int_0^{\mathscr{T}}(\|\nabla\boldsymbol{p}_1(t)\|_4+\|\nabla\boldsymbol{p}_2(t)\|_4)(\|\mathrm{A}(\boldsymbol{v}_1(t))\|_\infty\\&\quad+\|\mathrm{A}(\boldsymbol{v}_2(t))\|_\infty)\|\mathrm{A}(\mathfrak{v}(t))\|_4\|\mathrm{A}(\mathfrak{v}(t))\|_2\mathrm{d}t\\&\leq 56\beta[S_4]^2S^\varepsilon\int_0^{\mathscr{T}}(\|\mathscr{A}\boldsymbol{p}_1(t)\|_2\\&\quad+\|\mathscr{A}\boldsymbol{p}_2(t)\|_2)(\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}_1(t)\|_2+\|\mathscr{A}^{1+\frac{\varepsilon}{2}}\boldsymbol{v}_2(t)\|_2)\|\mathscr{A}\mathfrak{v}(t)\|_2\|\nabla\mathfrak{v}(t)\|_2\mathrm{d}t\\&\leq\frac{224\beta}{\mu}[S_4]^2S^\varepsilon\kappa\Gamma_1\Gamma_2\widetilde{\lambda}\int_0^{\mathscr{T}}\|\mathfrak{f}(t)\|_2^2\mathrm{d}t.\end{aligned} \tag{9.16}$$

Combining (9.10), (9.13)-(9.16), we conclude

$$\begin{aligned}&\int_0^{\mathscr{T}}\|\mathfrak{v}(t)\|_2^2\mathrm{d}t+\lambda\int_0^{\mathscr{T}}\|\mathfrak{f}(t)\|_2^2\mathrm{d}t\\&\leq 2\Gamma_2\widetilde{\lambda}\left[\Gamma+\frac{4}{\mu}|\alpha|[S_4]^2\Gamma_1+\frac{112\beta}{\mu}[S_4]^2S^\varepsilon\kappa\Gamma_1\right]\int_0^{\mathscr{T}}\|\mathfrak{f}(t)\|_2^2\mathrm{d}t,\end{aligned}$$

which provides the required result.

10. Conclusion and future directions

This work aims to advance the understanding of the optimal control problem for the class of incompressible third-grade fluid equations (1.2) with $\alpha_1=0$ (see (1.3)). The optimal control of flows governed by incompressible third-grade fluid equations with $\alpha_1>0$ on a bounded domain D, subject to Navier-slip boundary conditions, was previously investigated in [38]. As discussed in the introduction, the term $\mathrm{div}(|\mathrm{A}(\boldsymbol{v})|^2\mathrm{A}(\boldsymbol{v}))$ introduces significant challenges, requiring higher regularity of the state solution to estimate nonlinear terms and to ensure the well-posedness of the linearized system corresponding to (1.3). To address this, we study the optimal control problem for a regularized version of (1.3), referred to as the third-grade-Voigt equations (see (1.4)). First, we establish the well-posedness of system (1.4) using the Faedo-Galerkin approximation method and derive the solution regularity necessary for our analysis. Next, we prove the existence of an optimal pair and derive the first-order necessary optimality condition. Finally, we show that for sufficiently large $\lambda > 0$, the optimal pair is unique.

Next, we outline potential future directions that could pave the way for numerous interesting research objectives:

- One can think to propose an approximation scheme to the velocity tracking problem for two-dimensional third-grade flows (1.3), relying on the optimal solution to the two-dimensional third-grade-Voigt equations.
- One can think to extend the results of this work to three-dimensional domains.
- A numerical analysis would be interesting to obtain the optimal pair.

Acknowledgements

We express our deep gratitude to editor and the anonymous referee for the careful reading and valuable suggestions that have improved this manuscript. We express our deep gratitude to editor and the anonymous referee for the careful reading and valuable suggestions that have improved this manuscript. “This work is funded by national funds through the FCT – Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 (https://doi.org/10.54499/UIDB/00297/2020) and UIDP/00297/2020 (https://doi.org/10.54499/UIDP/00297/2020) (Center for Mathematics and Applications)”.

Author contributions All authors contributed equally.

Funding Open access funding provided by FCT—FCCN (b-on).

Data Availability Statement No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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Received: 8 June 2025.
Revised: 21 August 2025.
Accepted: 22 August 2025.