

A Work Project, presented as part of the requirements for the Award of a Master's degree
in Finance from the Nova School of Business and Economics.

Pricing and Analysis of the structure of Barrier Options based on Geometric
Brownian motion using Monte Carlo Simulation

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Abstract

The objective of this work is to analyze the pricing behavior of barrier options considering different scenarios. The pricing structure is done considering that the underlying follows a geometric Brownian motion, while the simulations are based on the construction of possible underlying trajectories through Monte Carlo Simulation. The interest and focus of this work are to expose the big difference between the payoffs of vanilla and exotic options even when based on one of the most popular and simple exotic options traded in the market.

Keywords (Exotic Options, Monte Carlo Simulation, Brownian Motion, Pricing)

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1 Introduction

The derivatives market has grown impressively over the last twenty years, both in terms of volume and in terms of financial products available. Banks and hedge funds enter the market offering a large number of risk protection products, usually options, which allow the search for protection of the portfolio or a certain position against the strong volatility of exchange rates, interest rates or even the underlying itself.

As increasingly complex contracts are negotiated, the greater the problems to price them in a minimally correct way, which provides the market with opportunities for arbitrageurs, who seek arbitrage opportunities, since these mispricing situations occur more frequently in complex products, for this purpose, mathematical and computational tools are used.

Options, whether traditional or exotic, are a type of derivative contract, whose main objective is to have their value derived according to the price of an underlying asset. In the case of traditional options, defined as vanilla options, we have a purchase option (Call) and a sell option (Put), the buyer has a right upon payment of an amount (premium) that he may or may not exercise in a period or date specific. On the other hand, options are said to be exotic when they deviate from the standards of vanilla contracts and present flexibility and the possibility of customizing the characteristics of the contract.

In other words, exotic options can be seen as a series of options very different from traditional call and put options, whose payoff structure is personalized and non-trivial. Exotic options have unique characteristics that make them an excellent option for risk management of a portfolio or a given position. With this, exotic options expand the range of possibilities for investors, as it enables the creation of products capable of better adapting to their speculation or hedge demands.

One of the most well-known and popular exotic options is the Barrier options, which is precisely the exotic option model that this work will be based on. The payoff of this type of option is similar to the payoff of vanilla options, but there are restrictions on the exercise of options, which leads to large differences in option pricing. Barrier options are based on the market reaching a negotiated (predetermined) threshold where the option will either be activated (knocked-in) or deactivated (knocked-out).

The objective of this work is to build and simulate barrier option pricing assuming different scenarios. Pricing construction is done considering that the underlying follows a geometric Brownian motion, while the simulations are based on the construction of possible underlying trajectories through Monte Carlo Simulation. Before presenting the results obtained by the simulations, we are going to expose the theoretical background on which we based ourselves.

2 Concepts

2.1 Options

An option contract is a futures contract in which the right to buy or sell a defined amount of an asset traded in the market is established, at a predetermined price, defined as the strike price, on a certain date or future period and can be classified as according to its class (call and put option), style (European and American), marketplace (over the counter and exchange-traded), etc.

Options contracts can belong to two classes: Call and Put options. The counterparties of this contract are defined as buyer, who buys the option, and issuer, who sells it. A call option is defined where the option buyer has the right to buy the underlying asset, while the issuer has the obligation to sell it as negotiated in the contract. On the contrary, the put option is defined where the buyer has the right to sell the underlying asset and the issuer has the obligation to buy it if the buyer decides to exercise the option.

We can also distinguish the options as American and European. The main difference between the two is with regard to expiration. A European option can be defined as an option that can be redeemed only at its maturity date. Whereas, an American option can be defined as an option that can be redeemed anytime during the period lying between the strike date (first valid date) and maturity date. We can also highlight Bermuda options. This style option indicates specific days before expiration on which the trader can execute the option, so it can be seen as a restricted form of the American option that allows for early exercise but only at set dates.

A call and put option can be defined numerically as the maximum between zero and the value that the option would have if it were exercised immediately, that is, the value of a

call option would be the maximum between zero and the difference between the spot price of the underlying asset (S) and the strike price of the option (K): $\max(S - K; 0)$. In the case of a put option, it would be the maximum value between the difference between the strike price and the spot price of the underlying asset and zero: $\max(K - S; 0)$, according to Hull (2022).

2.2 Vanilla x Exotic

A vanilla option can be defined as a financial instrument that gives the holder the right, but not the obligation, to buy or sell an underlying asset at a predetermined price (Strike price) within a negotiated period or a single date. About the payoff structure, a vanilla option is a call or put option that has no flexible and customizable features.

On the other hand, exotic options are options contracts that differ from vanilla options in their maturity period, payoff structures, etc. One of the most valued features of the exotic options is the fact that it can be customized to meet the counterparties' desire, such as the hedge for some position or portfolio, manage the risk tolerance or generate profit for the investor. Therefore, to achieve these goals the option must have more complex features and be traded over the counter. There are many types of exotic options such as Barrier option, Autocallables, Capped Bonus, Twin-Win, etc. As regards valuation, given their complexity, exotic derivatives are usually modeled using specialized simulation techniques.

2.3 Volatility

The volatility of the underlying asset price means the uncertainty related to the market price of this asset at a certain future date, that means, it corresponds to the fluctuations that may occur in the price of the underlying asset during a particular period. The greatest probability of the option being exercised is when large movements in the price of the asset are expected, this statement is valid for most options, however, due to the flexibility of exotic options, it is possible to have options that do not validate this statement. When we talk about high or low volatility, we are not determining a specific direction for the trajectory of the asset.

There are several methods to estimate the volatility of the price of an asset and the attempt to develop new models for this calculation is recurrent. One of the most used methods is the calculation of historical volatility. According to the definition suggested by Silva (2008), historical volatility is normally calculated as a percentage by the standard deviation of movements in the price of the underlying asset.

2.4 Barrier Options

Zhang (1998) indicates that barrier options are the oldest type of exotic option and were already available on the American over-the-counter market in the late 1960s. Even though there is a theoretical background on pricing exotic contracts as early as the 1970s and 1980s, the term exotic is attributed to Mark Rubinstein (1990).

Barrier options are path-dependent that are similar in some ways to ordinary options. We can define a barrier event as when the underlying crosses the barrier level. Barrier event

verification is usually done in three ways, only at maturity (in fine), considering the daily closing price (discrete) or at any time during the considered period (continuous).

The valuation of a barrier options can be tricky, because unlike other simpler options as described above, they are path-dependent. The valuation of this type of option is totally linked to the type of barrier negotiated in the contract.

The four main types of barrier options are:

- I) Up-and-in: spot price starts below the barrier level and has to hit this level for the option to become activated.
- II) Down-and-in: spot price starts above the barrier level and has to hit this level for the option to become activated.
- III) Up-and-out: spot price starts below the barrier level and has to hit this level for the option to become null or assume a rebate level.
- IV) Down-and-out: spot price starts above the barrier level and has to hit this level for the option to become null or assume a rebate level.

2.5 Brownian motion

As defined by Silva and Lima (2007), Brownian motion is a random movement where the position at a given instant of time only depends on the position at the previous instant, and not on the complete path. When we analyze the graph of the price of a stock or index, if we try to zoom any position, it is not possible to reach a point in which the derivative is zero. This is due to the fact that the stock and index prices follow a Brownian motion. It can be explained because there is randomness within financial markets.

Brownian motion is a type of stochastic process that is similar to a random walk. It follows a normal distribution with mean of zero and variance equal to the time period t . The random walk theory is used in the field of finance, where it describes that the processes in the stock market evolve as a random walk and that it is not possible to determine exactly its next step.

2.6 Geometric Brownian motion

A geometric Brownian motion (GBM) is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion (also called a Wiener process) with drift.

To model the behavior of stock prices over time, geometric Brownian motion can be used. A stochastic process S_t is said to follow a geometric Brownian motion if it satisfies the following stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (2.1)$$

where S_t is the percentage change in the share price over time, μ and σ are the mean (the percentage drift) and standard deviation of the returns (the percentage volatility) respectively, and W_t is a Wiener process.

The percentage drift is used to describe deterministic trends, while the percentage volatility term is used to model unpredictable events occurring during this process. The key distinguishing point among different Brownian motions as defined by Wu and Buyya (2015) is the different types of drift. If the drift is equal to 0, it is standard BM. If the drift is constant, it is BM with constant drift. If the drift is linear, it is geometric BM.

3 Monte Carlo Simulation

In developing financial research, the valuation of more sophisticated options is a major focus. The solutions proposed by Black and Scholes (1973) were based on European options on non-dividend-paying stocks. The method of Monte Carlo Simulation, developed by Metropolis and Ulam in 1949, has a wide range of applications in the world of finance, specifically for option contracts, and emerged as an alternative to meet this demand for new numerical procedures in the pricing of options more complex or with peculiar characteristics.

Briefly, the Monte Carlo method consists of generating a certain number of random future scenarios and analyzing the probabilistic distribution of each one of them. Monte Carlo Simulation is one of the most used tools by analysts and financial specialists across the market. In 1977, Boyle introduced Monte Carlo Simulations to the field of option pricing with the publication of his article “Options: A Monte-Carlo Approach” in the Journal of Financial Economics.

The famous Black-Scholes model presents difficulties when dealing with exotic options, which encourages and values the Monte Carlo Simulation because, with the evolution of computational calculation, these simulations are more applicable in the pricing of options of different complexities and other topics involving decision-making under uncertainty. With this, the simulation collaborates in order that the investor or analyst can anticipate situations.

3.1 Pricing

Determining the pricing of options through Monte Carlo Simulations boils down to simulating the values assumed by the variables of the model in question, generating random numbers and normal probability distribution for each one of them, and thus pricing the option through the averages of the simulations.

Considering the hypothesis of risk-neutral valuation, the rate of return μ in the above equation can be replaced by the risk-free interest rate:

$$dS_t = rS_t dt + \sigma S_t dW_t \quad (3.1)$$

The option value would be the maximum between zero and the difference between the spot price of the underlying asset (S) and the strike price of the option (K), for call option we have $\max(S - K; 0)$ and for put option we have $\max(K - S; 0)$.

Then, each return obtained would be discounted based on the risk-free interest rate and, finally, the arithmetic mean of these would be calculated.

We will use Itô's Rule to define the function that represents the evolution of the share price. We will not delve into the hypotheses and everything that is behind this definition.

For this I recommend reading Shreve (2004). Below is Itô's Rule:

$$df(S_t) = \frac{\partial f}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (dS_t)^2 \quad (3.2)$$

The next step is apply Itô's Rule for $\ln(S_t)$, so, we have:

$$f(S_t) = \ln(S_t), \quad f'(S_t) = 1/S_t, \quad f''(S_t) = -1/S_t^2$$

Thus,

$$d\ln(S_t) = \frac{1}{S_t} dS_t - \frac{1}{2} \frac{1}{S_t^2} (dS_t)^2 \quad (3.3)$$

We already know that $dS_t = \mu S_t dt + \sigma S_t dW_t$.

Therefore, with algebraic manipulations and stochastic calculus properties, we have the function that represents the evolution of the share price defined as:

$$S_t = S_0 e^{(\mu - 0.5\sigma^2)t + \sigma W_t} \quad (3.4)$$

Now, in order to take the Monte Carlo Simulations, we still need to define the volatility, which will be explained in the next section.

4 Data and Results

Initially, we will analyze the pricing of a Barrier option being defined as a call option with an Up and Out barrier of 140% of the initial value of the share. It is common for the barrier to be agreed in percentage based on the initial value of the share, since the path followed by the share can be seen as performance. The stock chosen for this example was Apple Inc. (AAPL) and all the information used to calculate the volatility and also the initial price was based on the Yahoo Finance API. It is also important to note that all values are defined in US Dollar.

We must define some parameters to calculate the option value. We consider the initial value of the share and the Strike price of the option as the closing price on April 17, 2023 (165.23). As explained, the barrier will be Up and Out at the level of 140% of the initial value of the share (231.32). We will run 100 simulations of the possible course of action and the number of time steps considered was 251, as between April 18, 2022 and April 17, 2023 we have 251 days in which we had some update on the closing price.

Two other variables to consider are the risk-free annual rate and the stock's volatility. We are considering the Risk Neutral Pricing hypothesis, so we will use the U.S. 1 Year Treasury Rate on April 17, 2023 (4.8%). Stock volatility was calculated based on historical volatility over a 1-year period. For this, we apply the logarithm of the daily variation of the closing price in relation to the previous price and then calculate the standard deviation of the log returns, defining a volatility of 34.32%.

We will use the Monte Carlo method, which, as explained earlier, is based on the random generation of possible asset price trajectories and the calculation of the option value for each simulation concluded. With that being said, we are able to calculate the average of

the option price. We will run 100 simulations of the possible course of the stock and, as explained above, the number of time steps considered was 251.

First, we will display a graph showing the trajectory of each simulation. It is worth mentioning that the option is a call option with a barrier of 140% of the initial value of the asset (B).

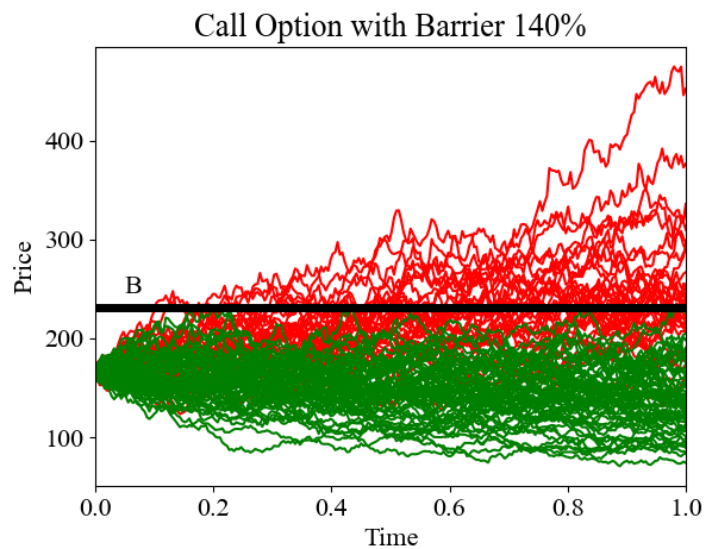


Figure 1

The barrier was fixed at 231.32 (initial value multiplied by 1.4) and with that we have two ways to distinguish a simulation. We can have a trajectory in which, at some point, the price of the asset reaches the value defined as the barrier (red) or we have another case in which the stock does not reach this value at any point (green).

We can also view the trajectories separately.

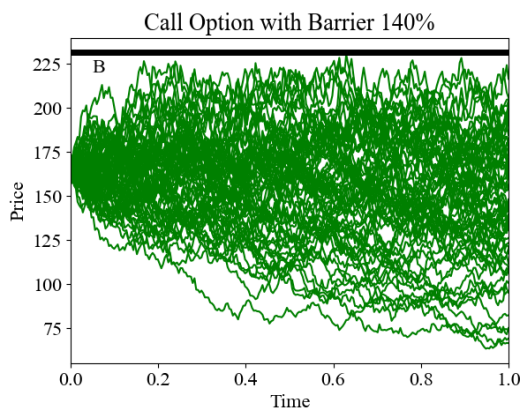


Figure 2

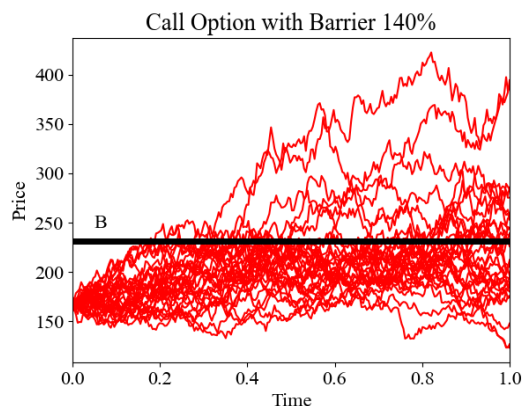


Figure 3

As we define this barrier as Up and Out, the action is interrupted and has a value of 0 (zero), just as an example, we could have the same barrier defined as Up and In, that is, we have the opposite case, while the price of the action does not reach the barrier, we have that the value of the option is 0 (zero).

An important note is that this value does not have to be equal to zero. This value is defined as rebate. Rebate is defined as the value assigned to the option when the barrier interrupts the option's path. In the considered case where the rebate is equal to zero, strictly speaking, when the Up and Out barrier is reached, the option has a value equal to zero.

In this scenario defined above, the option value was 5.72 and we had 29 simulations that reached the barrier and had its value equal to zero (rebate equals zero). This value, as explained above, is the average of the values of the 100 simulations of the stock trajectory.

Still in this scenario, it is reasonable that if agreed that the rebate was equal to 5, for example, in other words, when the barrier is reached, the value of the call option will be equal to 5, the average price of the option is greater than the calculated previously.

Redoing the simulation, but now with the rebate equal to 5, we found that the option value is 7.35 with 31 simulations surpassing the established barrier. The option value is natural

and obvious to be greater than previously calculated, and not having the same number of options crossing the barrier (29 and 31) is due to the randomness of the Monte Carlo Simulation.

Exotic options are customizable and their characteristics are established according to the will of the counterparties involved in the contract. With this in mind, we are able to, for example, have a call option with more than one barrier.

We will establish the same characteristics as in the first case, adding another Down and Out barrier of 80% of the initial share value (132.18), that is, if the option price reaches one of the barriers (140% and 80%), the option is closed with zero rebate.

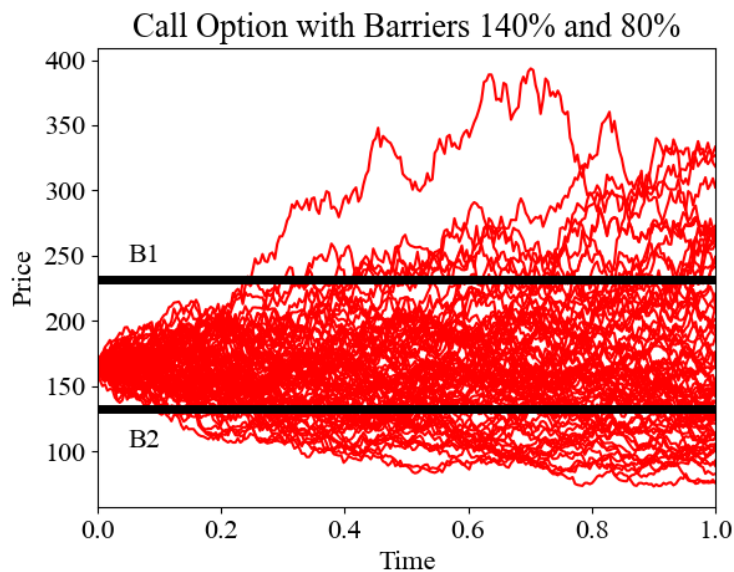


Figure 4

The graph above shows the trajectories that at some point reached one of the barriers, in this scenario, we had 74 simulations out of a total of 100 that meet this requirement. Also, the option value was 3.67. This value is considerably smaller when compared to our first case, in which the only difference is the existence of another barrier being defined as Down and Out at the level of 80% of the initial value of the share. This result is expected

and makes perfect sense, because as we have zero rebate, the option now has three possibilities to expire without value: if the stock reaches the upper barrier; if it reaches the lower barrier, or if it does not reach any of the barriers but ending below its initial value (as we are considering a call option with a strike price equal to its initial value, by definition, the option is only exercised if the value at maturity is greater than the Strike price). As a result, the scenarios in which the share's maturity value is greater than its initial value and less than the upper barrier are reduced.

Option pricing is based on the likelihood that the final price of the underlying asset will end up in-the-money. Generally, a higher historical volatility percentage means to a higher option value. As volatility increases, the price of a vanilla option on that underlying tends to rise. This is because the chances of the final price of the underlying asset finishing in the money likewise increase. This thought and idea is widespread and accepted when considering vanilla options. When we consider exotic options, this will not always be true. Such thinking will depend on the terms agreed between the counterparties.

Now we will consider a call option with a strike price equal to the share's initial value and with two barriers (Up and Out at the 140% level and Down and Out at the 80% level). Let's keep everything the same except volatility. For this case, let's assume a volatility of 45% instead of the calculated historical volatility of 34.32%. By pricing this scenario, we reached a value of 1.18 and obtained 92 simulations that reached one of the two barriers. That means, with the increase in volatility, the price of the option decreased, and as already explained in this work, this is easy to understand due to the peculiarities of the barrier option when compared to the vanilla option.

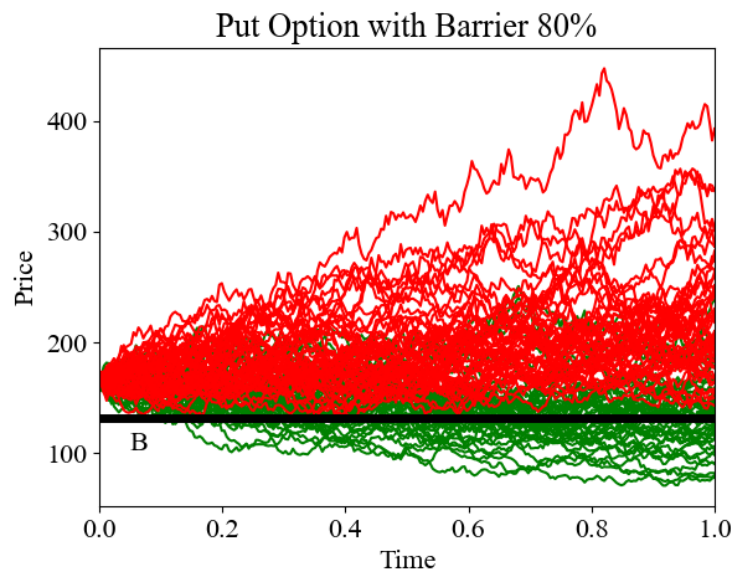
With increasing volatility, the value assigned to the rebate becomes even more important, as the number of simulations that hit one of the barriers increases significantly. So, we

can imagine that in this case, with increasing volatility, the option price tends to be close to the rebate value. We will simulate the same case, only now with a rebate equal to 5 and we will consider a higher volatility equal to 55% to illustrate this point more incisively. We get the option value equal to 4.88, which is quite natural, since 98 simulations reached one of the barriers and assumed the value of the rebate (5).

Our last case will be to explore and expose the inverse of what we have presented so far. For this, we will consider a put option with a Down and In barrier at the level of 80% of the initial value of the share. All the remaining variables, we will keep as exposed in the first case, that is, the stock considered is Apple Inc. on April 17, 2023, with a strike price equal to the initial value of the share and with the risk-free interest rate and historical volatility as already presented in the first case.

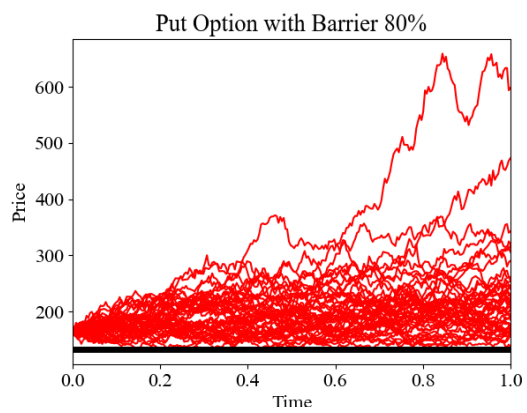
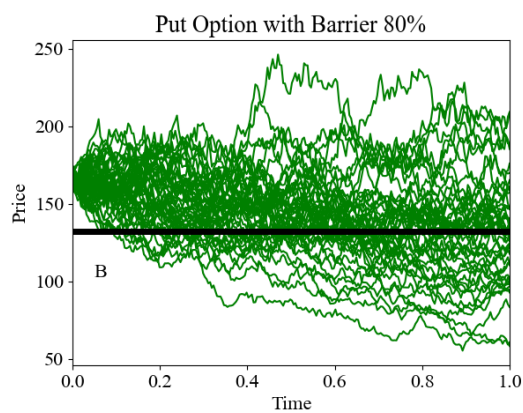
The use of the Down and In barrier changes the dynamics of the option, once the share price reaches the barrier level, the option value is the put option value, otherwise it is equal to the rebate, which in this case we will consider equal to zero.

First, we will present a graph showing the trajectory of each simulation. It is worth mentioning that the option is a put option with a barrier of 80% of the initial value of the asset (B).



The barrier was set at 132.18 (initial value multiplied by 0.8) and with that we have two ways to distinguish a simulation. We can have a trajectory in which the price of the asset reaches the value defined as the barrier (green) at some point, and we have the other case in which the stock does not reach this value at any time (red).

We can also view the trajectories separately.



In this scenario defined above, the option value was 17.48 and we had 47 simulations that did not reach the barrier and had their value equal to zero (rebate equals zero). This value as explained above is the average of the values of the 100 simulations of the stock trajectory.

In the previous case, we noticed that the increase in volatility did not lead to an increase in the value of the option, what was observed was exactly the opposite. However, in this case, the influence of increased volatility on the option price follows the perceived behavior of vanilla options. We will simulate the following scenario and keep everything the same as shown in the last case, except for the volatility level of the stock, assuming again that the calculated volatility was 45%. With that, the new option price becomes equal to 22.21 and we had 34 simulations that did not reach the barrier and had their value equal to zero (rebate equals zero). In this case, the increase in volatility generated an increase in the option price, and this is explained by the type of barrier that was considered. When we consider a Down and In barrier, the increase in volatility generates more trajectories that reach the level of the barrier and start to have some value.

Finally, we will expose the big difference on pricing between the exotic options presented and the vanilla options. For this, we simulate a standard call option (without barriers) to compare with the call option with Up and Out barrier at the level of 140% already exposed and also, a standard put option (without barriers) to compare with the put option with Down and In barrier at the level of 80% previously exposed.

Table 1: Comparison between vanilla and exotic options.

Option	Type	Barrier	Barrier Type	Barrier Level	Price
Call	Exotic	Yes	Up and Out	140%	\$ 5.72
Call	Vanilla	No Barrier	No Barrier	No Barrier	\$ 18.65
Put	Exotic	Yes	Down and In	80%	\$ 17.48
Put	Vanilla	No Barrier	No Barrier	No Barrier	\$ 20.40

As expected, vanilla options have a higher price precisely because they have no barrier that defines their entry or exit. We can notice a huge difference between the exposed call options, this is due to the fact that 29 simulations reached the barrier of 140% and had their value equal to zero. The biggest point in this fact is that these options are options that would add a great value to the option price, as at some point it would be generating a positive payoff of \$40.

Compared to put options, the difference is significantly smaller by two points. The barrier level (80%) is closer to Strike price (100%) than call option barrier (140%). Another point is the type of barrier, in this case it is Down and In, that is, the option only has some value when it reaches the barrier, so the simulations that would have the most positive payoffs are not cancelled.

5 Conclusion

The popularization and importance of exotic options is due to the fact that these strategies allow considerable returns in times with low market interest rates, expressive returns in very volatile markets and, above all, the flexibility and customization of products to meet the desire and needs of counterparties.

This flexibility generates more complex payoffs than those observed in vanilla options, when we insert a certain level of the underlying price for the option to start or end, the payoff undergoes significant changes. Furthermore, the interpretation of the product is changed, the importance and effect of volatility for example, is changed only by the type of barrier to be traded.

When working and pricing exotic options, the analyst must be aware that the customization of this contract has severe consequences on the payoff of the option, and some statements that are considered mainstream do not remain true, largely because these statements are based on the payoffs of vanilla options. As presented, volatility is not always proportional to the option price, which may mean that greater volatility is not always a benefit for the option buyer, as a result, the option price becomes inversely proportional to volatility.

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