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Problem-solving and mathematical competence: A look to the relation during the study of Linear Programming

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ABSTRACT

This study seeks reflection on the approaches of 11th grade students to Linear Programming problems, discussing the approaches taken at different moments of the teaching process. It aims to analyze: How is the students' mathematical competence characterized in relation to problem-solving; What differences can be identified in the resolutions at different moments of the teaching and learning process. We adopt a qualitative and interpretative methodology, analyzing the approaches of two pairs of students with different mathematical backgrounds. The analysis is guided by Pólya's stages of solving a problem and aspects of the understanding of mathematical competence. The results show different approaches to the problems depending on the teaching moment and different competences. The mathematical background impacts the students' success when they implement routine procedures, however it does not seem to determine the students' competence to reason about a problem.

1. Introduction

Enhancing students' curiosity and motivating them to learn is a demanding and complex task (Goldin et al., 2016). Indeed, promoting meaningful learning requires the commitment and determination of those involved. It is necessary to take part in experiences that go beyond listening to the teacher and solving exercises. One possibility, as emphasized by Clarke and Roche (2018), is the use of contextualized tasks. The present perspectives on mathematics education, according to Leavy and Hourigan (2020), highlight conceptual understanding, suggesting higher-level problem-solving as an important activity for students. English and Gainsburg (2016) emphasize this idea, reinforcing the relevance, at the level of citizenship, of the competence to analyze a problem, and make the fundamental decisions to solve it. Goos and Kaya (2020) go further, interconnecting problem-solving and mathematical thinking.

Even so, school mathematics often adopts approaches in which the memorized knowledge of facts and the mechanical repetition of procedures are valued. But, as Pólya (1957) alerts, the teacher who chooses to focus on routine and training tasks, destroys the students' interest and limits their possibilities for intellectual development.

Linear Programming is an area focused on problem-solving, where situations from reality play an important role (Kenney et al., 2020). Its study is important so that students become familiar with day-to-day management situations and so that they can develop skills to make correct decisions in terms of management and planning (English & Gainsburg, 2016).

In this study, we seek to reflect on the approaches that 11th grade students (16 years old) make to Linear Programming problems.

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To this end, we intend to discuss the approaches chosen by students at a time prior to the teaching process and compare them with later approaches. Something that has not been the subject of much attention in research on problem-solving and mathematical competence of students.

For this analysis, we take as a basis Pólya's (1957) four stages of solving a problem and aspects of the understanding of mathematical competence as conceptualized by Niss and Højgaard (2019). More specifically, with this study, we aim to analyze and understand:

i) How is the students' mathematical competence characterized in relation to problem solving (focusing on the students' approaches, mathematical representations, mathematical communication, mathematical formalism, and technology use)?
ii) What differences can be identified in the resolutions made by students at different moments of the teaching and learning process?

In this study, we assume what English and Gainsburg (2016), (p. 314) call the traditional definition of problem: "tasks in which the solver does not know how to arrive at an answer". An understanding inspired by Schoenfeld (1983), (p. 41), who states that if the task "can be solved comfortably by routine or familiar procedures", it is not a problem. We do not address the cognitive level of the problem as commented by Yeo (2017) or by Carlson and Bloom (2005). This means we are not going to discuss the understanding of the term *problem*, assuming it broadly. We are not also going to discuss whether at a certain point the students are already familiar with the kind of task proposed to them, and they are solving an exercise rather than a problem.

2. Problem-solving

It is considered that studying mathematics improves the ability to think, reason and solve problems; justifying the introduction of problem-solving in curricula as a way to promote mathematics and the ability to reason (Stanic & Kilpatrick, 1989). Studying mathematics through problem-solving, making it more real, develops students' creativity, critical spirit, interaction and autonomy (Clarke & Roche, 2018), and promotes the development of citizens able to live in society (English & Gainsburg, 2016). During problem-solving, the students develop reasoning and build new concepts in a process where the teachers assume the role of mediator or advisor of the teaching and learning process, and where they are responsible for the systematization of the new knowledge acquired (Pólya, 1957). The students assume an active participation in their own learning, while they get involved in the development of several reasonings in an attempt to solve the problem, analyzing different hypotheses of approach, comparing the results achieved with those of their colleagues and considering the reasonableness of the entire process developed (Rocha, 2020a). But problem-solving is admittedly not an easy task and aspects such as how it should be approached, the distinction between good and poor problem solvers, the type/difficulty of the problem, and the metacognition in problem-solving, have been the focus of several authors, generating prolonged debate (English & Gainsburg, 2016; Kilpatrick, 1985; Lester, 1994; Schoenfeld, 1992).

Particularly when considering problems that have a real context, as happens in the present study, cases are known in which students do not adequately take into account reality, proposing solutions inconsistent with the situations presented, as described by Palm (2008). Some reasons pointed out by the author are related to the experiences provided by the school, leading students to consciously ignore the aspects of reality referred to in the task, or to assume that all tasks have a single solution. Even so, justifications of another type are mentioned, such as the students lack of knowledge about the context in question or the adoption of an interpretation different from that of the teacher.

Pólya (1957) was concerned about discovering how to solve problems and how to teach strategies to solve problems. The author clearly characterizes the techniques for solving a problem and exposes his method in four steps: the 1st step is to understand the problem, that is, it is necessary to try to understand the problem, define the unknown, define the conditions of the problem and verify whether it is possible to estimate the response; the 2nd step is to elaborate a plan, being necessary to find resolution strategies, organize the data and try to solve the problem in phases; the 3rd stage is the execution of the plan, where it is necessary to verify each step of the resolution, to execute all the calculations and implement all the strategies outlined; the 4th and last step is to verify the solution, where it is necessary to verify that the obtained solution is correct, to examine all the reasoning carried out, to verify if there is another way to solve the problem and if it is possible to apply the resolution made to similar problems. The reflection on the entire resolution of the problem, reconsidering and reexamining the result and the process that led to it, allows to consolidate the knowledge and improve the students' ability to solve problems, since it is always possible to improve the understanding of the resolution (Pólya, 1957).

Schoenfeld (1985, 1992) also considered a set of stages in problem solving, however, his work was based on an empirical level, while the work of Pólya (1957) was based on a theoretical level (Liljedahl et al., 2016). The five stages proposed, which have similarities to the ones proposed by Pólya (1957) are: reading, analysis, exploration, planning and implementation, and verification (Schoenfeld, 1985).

The teachers are responsible for working on the mathematics so that the students build their own knowledge (Rocha, 2022), arousing their curiosity and critical spirit (Clarke & Roche, 2018). In this way, the teachers can encourage students to identify and correct any errors, to generalize strategies for more complex problems and to reflect on their interpretations and needs (Čadež & Kolar, 2015). When interacting with students, the teachers should start with generic and simple suggestions, and gradually move on to more elaborate ones, seeking to support the students, but without reducing the cognitive demand of the task (Estrella et al., 2020).

Problem-solving should offer students opportunities to develop reasons, make sense of relationships and concepts, and search for the development of their mathematical knowledge, while facing challenges suitable for their level (DiNapoli & Miller, 2022). Čadež and Kolar (2015) assume that the students should solve sets of similar problems as a way to improve their ability to solve problems. These experiences will offer the students opportunities to find relations between the problems, allowing them to develop generalizations about how to approach a problem with certain characteristics.

Despite the potential ascribed to problem-solving and the initiatives to foster it in schools, students tend to face difficulties when confronted with problems (Martino, 2019). These difficulties suggested the relevance of factors going beyond the cognitive approach to problem-solving based on heuristics, as proposed by the work of Pólya. Some studies (e.g., Schoenfeld 1985) point to the influence of affective or emotional factors, referring to how they impact the students' "approach to the problem, what they perceive to be important in the problem, which techniques will be used or avoided, how long and how hard they will work on it, and so on" (Martino, 2019, p. 294). That is also the case of Lester (1994), who assumes the competence to solve a problem as the result of several independent factors, such as knowledge, control, beliefs, and sociocultural context. The attention given to these factors has grown over time, with Hannula (2015), (p. 355) stating that a "positive disposition to mathematics has a reciprocal relationship with achievement, both enhancing the other over time". However, problem-solving and what contributes to achieve success on it, is not yet fully understood (Carlson & Bloom, 2005).

3. Mathematical competence

The development of mathematical thinking is, according to Goos and Kaya (2020), assumed as one of the major goals of schooling. This is a concept that puts the focus on the process and not on the content, and is often approached based on problem-solving, although, as recognized by the authors, it is not an easy concept to define. Other authors refer instead to mathematical literacy, as is the case with the OECD (2018), which describes it as the individual ability to reason mathematically and to formulate, employ and interpret mathematics to solve problems in a real context. But for Geraniou and Jankvist (2019), the notion of mathematical literacy has currently shifted to the notion of mathematical competence, which has become a key construct of the educational paradigm.

Defining what mathematical competence is with certainty is difficult. Even so, Niss and Højgaard (2019) seek to clarify the understanding regarding this concept. They assume competence as the willingness to act appropriately in response to all types of mathematical challenges related to certain situations. A competency is always action-oriented (whether physical or mental), based on a certain insight and where meeting the challenge is in some way what determines the existence of competence.

By focusing on mathematical competence and not on mathematical content, we intend, like Niss and Højgaard (2019), to focus on the use of mathematics, on what is possible to do based on it, and not so much on procedures and their mechanic reproduction. That is, we intend to put the focus on mathematical activity. According to the authors, four components are essential to mathematical activity: fundamental mathematical thinking; posing and solving mathematical problems; mathematical modeling and work around mathematical models; and develop mathematical reasoning. For that, it is necessary to master the mathematical language, the mathematical concepts and the available tools, which involves four other components: the mathematical representations; mathematical symbols and associated formalism; mathematical communication; and the materials and resources available, such as technologies.

Considering the specificity of the present study, we will adopt the conceptualization of Niss and Højgaard (2019) but we will focus only on problem solving, assuming it is one of the components of mathematical activity. This competence then involves the ability to solve problems, but also to critically analyze the resolution or attempted resolution made by someone else or by oneself, being able to develop and implement strategies that prove to be adequate to solve the problem. And the performance of this activity requires, as already mentioned, four skills. The competence to deal with different representations of mathematical entities consists of the ability to interpret information made available through a diversity of representations, as well as to know which representation is most appropriate to choose at each moment, being aware of the strengths and weaknesses of each one, as well as the information they convey. The competence to deal with mathematical formalism and symbols consists of the ability to deal with mathematical symbols and expressions, as well as the respective rules for transforming them. The competence of mathematical communication consists of the ability to communicate mathematical ideas in different ways, using appropriate ways. And finally, the competence to deal with different materials and tools that may prove useful in the course of mathematical activity. At this level, in this study we will pay particular attention to the use of technology, and specifically to the graphing calculator, and the ability to take advantage of it for mathematical activity, being aware of its affordances and limitations. This includes not only being aware of the graphing calculator operation, but being aware about how and when to use it, reflecting critically on the information provided and on the contribution to solve the task.

4. Methodology and study context

Considering the characteristics of the present study, we opted for a qualitative and interpretative methodology (Bryman, 2004).

Data collection was based on observation of lessons, collection of resolutions made by the students, and field notes prepared by the class teacher (the second author of this paper). An audio recording of the observed classes was also made.

The study participants were three pairs of students of an 11th grade class, aged 16 years old, from a Portuguese school in the Azores islands, with different levels of performance and different levels of interest in Mathematics. The organization of students in pairs is intended to promote collaboration, once its potential to promote mathematical learning and to improve problem-solving competency is recognized (Saadati & Felmer, 2021). However, this is not a focus of the data analysis. Although, as noticed by the OECD (2017), the outcomes of the pair could be different (and greater) than the outputs of each member, in this study the focus is on the pair of students. The pairs were organized according to their grades in Mathematics, intending to create homogeneous pairs. Besides this criterion, friendship among students was also considered.

The chosen students were designated as AB pair, CD pair and EF pair. According to the grades previously achieved and to the information provided by their teacher, AB pair has the best level of performance and interest in the discipline and EF pair shows some difficulties in understanding and applying mathematical knowledge and less interest in relation to the discipline. The pairs of students also have a different perspective about themselves and the study of mathematics, according to the data collected by their teacher in a small interview. The AB pair assumes Mathematics as a very interesting and relevant subject and expresses confidence about their ability to be successful in its study. The EF pair also recognize the importance of Mathematics, however they perceive it as a difficult subject, requiring a lot of work and dedication. The CD pair was one average pair, achieving grades lower than AB pair and higher than EF pair. Although the data collected about this pair are interesting, they do not add much to the conclusions based on the other two pairs. As so, due to space constraints, here only the AB and EF pairs are analyzed.

The study focused on the detailed analysis of the resolution of three tasks, designated 1A, 1B and 2.

The proposed tasks designed specifically by the teacher for this study, reflected situations linked to the students' reality and were worked in the context of the students' study of Linear Programming. Linear Programming is one of the contents in the mathematics syllabus and it was chosen for this study due to its emphasis on problem solving.

The students had their first contact with Linear Programming problems with task 1A (see Fig. 1). It was intended that they find the solution to the proposed problem based on their own resolution strategies (without any kind of external guidance about how to approach such a problem).

Task 1B is a guideline for solving the problem of task 1A. It was intended to introduce the method of solving Linear Programming problems. The task presupposed that the students would solve the proposed steps to find the optimal solution. After filling in tables identical to the one in Fig. 2 but for specific cases, the task asked the students to fill in the table in the general case. Later, it sought to guide the writing of the conditions to be imposed, then asking for a geometric representation and the calculation of the coordinates of the vertices of the obtained polygon. After that, it requested the expression of the profit function, its value at the vertices of the polygon and then an analysis of all the elements to find the answer to the problem.

Task 2 (see Fig. 3) was intended to confront students with a new Linear Programming problem, to allow the analysis of their resolutions. This was the second problem on Linear Programming proposed to the students. However, they were not given any guidance as to how they should approach the problem.

The data analysis focuses on each of the tasks proposed. It started from the identification of the stages of Pólya (1957) in the resolutions made by the students and from illustrative episodes of the components of the problem-solving competence according to Niss and Højgaard (2019). The following dimensions, based on the referred authors' conceptualization, were used to guide the analysis: problem-solving competency, competence with mathematical representations, competence in relation to symbols and mathematical formalism, mathematical communication competence, and competence with technology. The analysis related to the first dimension – problem-solving competence – used the stages of Pólya and focused on the students' resolution, trying to identify the four stages. The competence with mathematical representations started from an identification of the representations used and then it was based on the analysis of the contribution of those representations. A similar process was used to analyze the competence in relation to symbols and mathematical formalism. The analysis of the mathematical communication competence was based on the identification of moments where the students communicate their ideas and on the analysis of the coherence and reason presented by the students to support their options. The analysis of the competence with technology, started with the identification of moments of technology use, the analysis was then guided by the ease of use (from a technical point of view) and the characteristics of that use (calculations, getting a graph, getting the intended information from the graph, using it in the problem).

5. Results

5.1. AB pair

5.1.1. Task 1A

In the resolution of task 1A, the AB pair started by reading carefully and, after a first analysis, registered and organized the information that they considered important. Thus, the registration in Fig. 4 was made, with the available quantities of each ingredient

Maria produces two types of traditional Azorean sweets: *ear cookies* and *cavacas*. Each kilogram of *cavacas* gives a profit of 5€ and each kilogram of *ear cookies* a profit of 7€. Regarding the products needed to make the sweets, Maria has only two limitations: she only has 10 kg of sugar and 6 kg of flour.

It is known that:

- Each kilogram of *cavacas* takes 0.4 kg of sugar and 0.2 kg of flour.

- Each kilogram of *ear cookies* takes 0.2 kg of sugar and 0.3 kg of flour.

How many kilograms of *cavacas* and how many kilograms of *ear cookies* should Maria produce to make the highest possible profit? Calculate the value of that profit.

Fig. 1. Task 1A.

		Quantity of sugar (kg)	Quantity of flour (kg)	Profit
Quantity of *cavacas* (kg)	x			
Quantity of *ear cookies* (kg)	y			
Total				

Fig. 2. One of the tables on task 1B.

Joseph has a sports store and needs to buy handball balls to sell to the local sports club. In a warehouse selling SOL brand products, he found the following poster:

> On handball balls, ONLY pay:
> *€12 for each official brand ball **
> *€8 for each SOL brand ball*
>
> * promotion valid on purchases of more than 10 handballs of the official brand

Joseph wants to take advantage of this promotion, but only 45 balls fit in the trunk of the car and the number of balls of the SOL brand must be less than or equal to double the number of balls of the official brand.

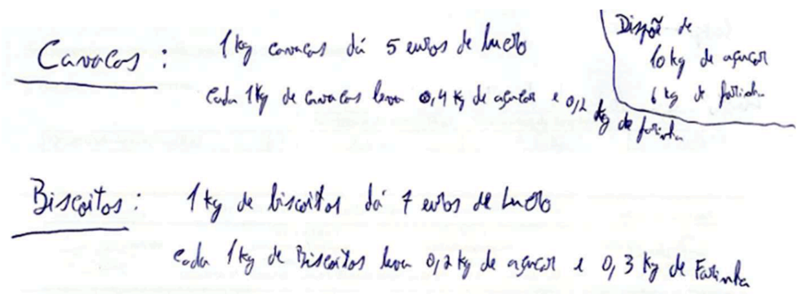
After analyzing the prices applied in competing stores, Joseph intends to sell each ball of the official brand at €14 and each ball of the SOL brand at €11.

Knowing that Joseph intends to maximize the profit, L, in the sale of the balls, determine the number of balls of each brand that Joseph must buy.

Fig. 3. Task 2.

written in the corner of the sheet. Finally, they read the question and expressed what is asked of them in the following words: "We have this to manufacture as much as possible".

The students explored the task by experimenting with various possibilities for the quantities of *cavacas* and *ear cookies* to make, considering the amounts of sugar and flour available. So, they adopted a trial and error strategy. They started by dividing the quantities of sugar and flour available, leaving 5 kg of sugar and 3 kg of flour for making each type of sweets. However, when trying to determine



		Available 10 kg of sugar 6 kg of flour
Cavacas:	1 kg of cavacas gives 5 euros of profit Each 1 kg of cavacas takes 0,4 kg of sugar and 0,2 kg of flour	
Ear cookies:	1 kg of cookies gives 7 euros of profit Each 1 kg of cookies takes 0,2 kg of sugar and 0,3 kg of flour	

Fig. 4. Registration of AB pair after reading task 1A.

the number of *cavacas* they could make with 5 kg of sugar, they did the opposite reason and determined the amounts of sugar and flour necessary to make 5 kg of *cavacas*. After this, they also determined the amounts of sugar and flour needed to make 5 kg of *ear cookies*. As they found that the quantities available of sugar and flour were much higher than the quantities needed for this production, they decided to increase the confection of *ear cookies*, due to the fact that their profit is higher. So, they tried to find out the quantity of *cavacas* and *ear cookies* they had to produce in order to use all the sugar. After several attempts (5 kg of *cavacas* and 8 kg of *ear cookies*, 10 kg of *cavacas* and 15 kg of *ear cookies*, 17 kg of *cavacas* and 15 kg of *ear cookies*, and, finally, 17 kg of *cavacas* and 16 kg of *ear cookies*), they concluded that with the confection of 17 kg of *cavacas* and 16 kg of *ear cookies*, they needed 10 kg of sugar (the available quantity). Then they determined the amount of flour needed for this production. As they found that, with this confection, the amount of flour needed was higher than the amount available, they chose to decrease the amount of *cavacas* to be made and, subsequently, also to decrease the amount of *ear cookies* to make. Thus, they determined the amount of flour needed to make 15 kg of *cavacas* and 16 kg of *ear cookies*, 15 kg of *cavacas* and 14 kg of *ear cookies*, 15 kg of *cavacas* and 13 kg of *ear cookies* and 15 kg of *cavacas* and 10 kg of *ear cookies*. They therefore reduced the quantities of sweets produced until they found a situation where they did not exceed the amount of flour available. They concluded that the best solution would be to make 15 kg of *cavacas* and 10 kg of *ear cookies* and determined the amount of sugar needed for this production. See Table 1 for a summary of AB students' options.

In all cases, to determine the required quantities, students used a mechanical procedure, as evidenced by the record they make and the dialogues they establish. As an example, we present the first of the dialogues, when they determine the amount of sugar to produce 5 kg of *cavacas*:

Student B: 1 kg of *cavacas* takes 0.4 kg of sugar.

Student A: So, 5 kg will take x.

Student B: x is equal to 5 times 0.4 to divide by 1 which gives 2.

Student A: You can make 2 kg. So, for every 5 kg of *cavacas* we use 2 kg of sugar.

At this point, the students reread the initial question of the problem and calculated the profit obtained from this production, using the calculator to carry out the arithmetic part. When this calculation was completed, the problem was assumed to be solved.

5.1.2. Task 2

In task 2, students immediately adopt the method they followed in task 1B (define the variables and the objective function; identify the problem constraints; build the mathematical model; obtain the optimal solution) and, thus, started to define the variables "official brand" and "SOL brand" (strictly it should be "number of balls of the official brand" and "number of balls of the SOL brand"), and organized the information of the problem. They have some doubts about the value corresponding to the profit, and they have a feeling that something is not right, as the following dialogue illustrates:

Student B: The profit is 14.

Student A: 14? It's too much… it can't be. I do not understand… Here the 12… Teacher?… I am not getting…

Teacher: (…) Is the profit 14? You buy at this price [pointing] and sell at this. How much profit will you have?

Student A: 14 - 12 = 2. Then 2x.

They determined the expressions of the total number of balls, $x + y$, and the total profit, $2x + 3y$, and organized the data into a table (Fig. 5):

Table 1
Summary of AB students' options during problem solving.

Strategy	Implementation	Strategy reformulation
Consider half the ingredients for making each type of sweet	By mistake they determine the quantity of ingredients for making 5 kg of *cavacas* and 5 kg of *ear cookies*	With ingredients available to make more sweets, they decide to increase the number *of ear cookies* because they are the most profitable
Find the amount of *ear cookies* and *cavacas* that use all the sugar	Sucessive attempts: 5 kg *cavacas* + 8 kg *ear cookies* 10 kg *cavacas* + 15 kg *ear cookies* 17 kg *cavacas* + 15 kg *ear cookies* 17 kg *cavacas* + 16 kg *ear cookies* Found a production that allows to use the 10 kg of sugar available, they determine the corresponding flour needs	As the quantity of flour is not enough for the intended production, they decide to reduce the amount of *cavacas*
Find the quantity of *ear cookies* and *cavacas* for which the flour is enough	Calculate the amount of flour for the production of: 15 kg *cavacas* + 16 kg *ear cookies*	As the quantity of flour is not enough for the intended production, they decide to also reduce the amount of *ear cookies*
	Calculate the amount of flour for the production of: 15 kg *cavacas* + 14 kg *ear cookies* 15 kg *cavacas* + 13 kg *ear cookies* 15 kg *cavacas* + 10 kg *ear cookies*	Confirm that sugar is also enough. A solution has been found

After organizing the data, they began to write the problem constraints. Among them, they consider $y \leq 12x$ and the teacher makes them rethink the condition. Accuracy is not a characteristic of the students' work, as they refer to condition $x > 10$, but this is not the record they make, and they also end up making a mistake in writing in order to y the first condition considered, as shown in Fig. 6.

The resolution proceeds with the students choosing the graphical resolution of the problem, an option similar to the one in task 1B. Which side of the line to shade to represent each condition, is something where they still need support:

Student A: Teacher, is it like this?

Teacher: Do you have all the conditions?

Student A: One is missing here…

Teacher: Ok. Now paint.

Student B: To this side?…

Teacher: Wrong side.

Student A: Right … it's smaller.

After the representation on the calculator, the students draw the admissible region on paper and determined, with the help of the calculator's commands, the coordinates of the vertices of the obtained polygon. However, they need to confirm with the teacher one of the commands to be used. Then, they quickly defined the objective function, calculated its value at each of the vertices of the polygon and identified the optimal solution, concluding that Joseph should buy 15 balls of the official brand and 30 balls of the SOL brand (Fig. 7).

5.2. *EF pair*

5.2.1. *Task 1A*

The students started by reading the task, but could not understand it, immediately deciding to ask the teacher for guidance.

Student F: Did you understand anything?

Student E: Calm down, I'm reading…

Student F: That kg is killing me… how much is it?

Student E: Each kg of *cavacas* gives a profit of €5.

Student F: And the *cookies* €7.

Student E: She has only 10 kg of sugar and 6 kg of flour.

Student F: Teacher, can you come here?

Student E: Let the teacher explain the problem. We are not understanding this…

It was necessary to read the task together with the students and exemplify, questioning concretely what quantities of flour and sugar are needed to make 3 kg of *cavacas* and 4 kg of *ear cookies*, in order to help clarify the problem and allow the students to approach it:

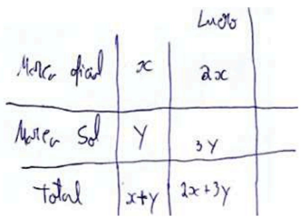
Student F: Do we have to make a comparison between the two or do we do it first for *cavacas* and then for *ear cookies*? Can we do it for both?

Teacher: You can do it separately, but you must consider that, at most, you can use 10 kg of sugar and 6 kg of flour.

Student F: So, we make a simple rule of three.

Teacher: Does Maria with 10 kg of sugar and 6 kg of flour manage to produce 3 kg of *cavacas* and 4 kg of *ear cookies*, for example?

Student F: I don't understand how to calculate the amount of sugar …



		Profit
Official brand	x	$2x$
SOL brand	y	$3y$
Total	$x+y$	$2x+3y$

Fig. 5. Definition of variables and organization of data for task 2, by AB pair.

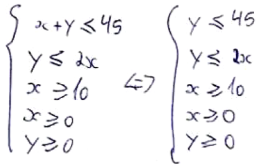


Fig. 6. Identification of the constraints of the problem of task 2, by AB pair.

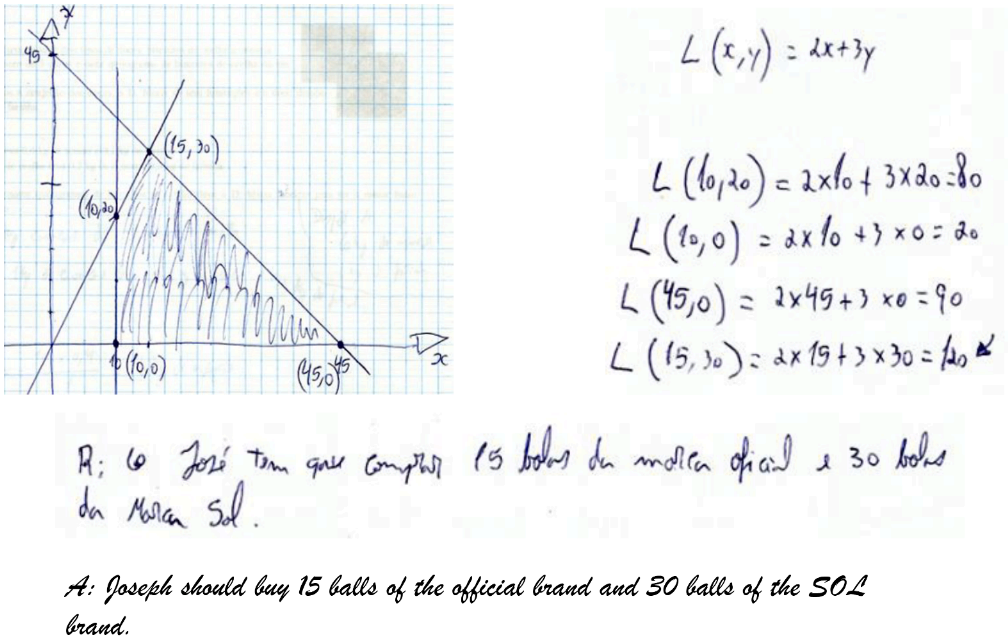


Fig. 7. Final part of the AB pair resolution.

Teacher: If 1 kg of *cavacas* takes 0.4 kg of sugar, 3 kg…

Student F: Ah! Ok…

Teacher: Try 3 kg or other values… We want to know the biggest profit Maria can get. Did you get the idea? Try it!

Student E: The biggest profit without exceeding 10 kg of sugar and 6 kg of flour…

After this, they started calculating the amount of flour needed to make 3 kg of *cavacas* and 3 kg of *ear cookies*, and then, as they verified that the amount of flour used in this production was considerably less than the one available, they decided to increase the quantity of *cavacas* and *ear cookies* to 7 kg and determined the quantities of flour and sugar needed (aiming to use all the flour available). They concluded that it was still too little and decided to increase the quantities of sweets to 12 kg. They found that in this case they used all the available flour and calculated the profit obtained.

It then occurred to them that they did not have to make the same quantity of the two types of sweets and that it would be advantageous to make more *ear cookies* than *cavacas*, as the profit was greater with the first. Although they are interpreting this condition of the problem properly, they felt the need to confirm with the teacher that the quantities of ingredients available were those that they could use in the total production of the two types of sweets and that this meant that they could not use more flour or sugar than the available quantities. Induced by the teacher, the students abandoned the line of reasoning they were trying to follow and then decided to look for another solution in which they would also use all the available sugar. They thus determined the amount of flour to make 10 kg of *cavacas* and 10 kg of *ear cookies*. As the quantity of flour did not total 6 kg, they determined the quantities of flour and sugar to be used in making 5 kg of *cavacas* and 20 kg of *ear cookies* and as these quantities exceeded the total available flour, they determined the amount of flour to be used in making 5 kg of *cavacas* and 19 kg of *ear cookies* and 5 kg of *cavacas* and 17 kg of *ear cookies*. Then, they determined the quantities of flour and sugar needed to make 4 kg of *cavacas* and 20 kg of *ear cookies*, 4 kg of *cavacas* and 19 kg of *ear cookies*, 4 kg of *cavacas* and 17 kg of *ear cookies*. They concluded that the latter was a valid solution, but as the profit was lower than what they had already obtained with 12 kg of each of the sweets, they abandoned it. They then decided to consider producing 1 kg of *cavacas* and 20 kg of *ear cookies*, but in this case too, the quantity of ingredients was not enough. They concluded that the best solution would be to make 12 kg of *cavacas* and 12 kg of *ear cookies*, since their profit is higher. For this production, 7.2 kg of sugar and 6 kg of flour were needed. See Table 2 for a summary of EF students' options.

The calculation of the quantity of ingredients required to produce each of the quantities of the two types of sweets is done quickly based on the proportion, as shown in the following dialogue:

> Student E: Let's try with 3. Imagine that she makes 3 kg of *cavacas* and 3 kg of *ear cookies*. And now to do the math… For 1 kg she needs 0.2 kg, for 3 kg she needs 0.6 kg of flour. Now for the *ear cookies*: 1 kg needs 0.3; 3 kg needs 0.9. So, 0.6 + 0.9 gives 1.5… it cannot be 3, it has to be a higher number.

5.2.2. Task 2

In task 2, students immediately recognize the similarity of this task with the previous one, immediately and tacitly deciding to follow the procedures adopted in solving task 1B (define the variables and the objective function; identify the problem restrictions; build the mathematical model; obtain the optimal solution), but the interpretation of the task proved to be once again a difficult step in solving the problem:

> Student F: This is like the other one we did. Here 45. Then the number of balls from the SOL brand must be less than or equal to the double of the number of balls from the official brand… I think we should have to make the conditions.
>
> Student E: But first we must define x and y.
>
> Student F: Only 45 balls can fit in the trunk.
>
> Student E: Let's make a table. It is the most difficult part, the rest is easy.
>
> Student F: Yes, it is like this: official brand, SOL brand. Teacher, could you come here? Here 10 balls are €12, right?
>
> Teacher: Why 10?
>
> Student F: Here it says 10…
>
> Teacher: No. Please note that what it says is that the promotion is only valid for purchases over 10 balls!

For the writing of the conditions, they felt once again the need to ask for the teacher's support:

> Student F: Teacher, we have doubts here.
>
> Teacher: The number of balls of the SOL brand must be less than or equal to double the number of balls of the official brand. Ok… how do we write this in mathematical language? Have you defined the variables? What does x stand for? And y? (students do not respond) How do you want to write if you have not yet defined the variables?
>
> Student F: Ok… Isn't that so?
>
> Teacher: How much does each ball of the official brand cost?
>
> Student F:…
>
> Teacher: What is the price of each ball?
>
> Student F: Twelve.
>
> Teacher: At what value do you want to sell each ball?
>
> Student F: Fourteen.

Table 2
Summary of EF students' options during problem solving.

Strategy	Implementation	Strategy reformulation
Calculate the quantity of ingredients needed for some productions and check if they have enough	Successive attempts 3 kg *cavacas* + 3 kg *ear cookies* 7 kg *cavacas* + 7 kg *ear cookies* 12 kg *cavacas* + 12 kg *ear cookies* They verify they used all the available flour (it is not possible to continue to increase the quantities produced) and calculate the profit	It is not mandatory to produce the same quantity of the two types of sweets. They decide to produce more *ear cookies* than *cavacas* to obtain a higher profit.
Influenced by the teacher, they decide to change the strategy they were considering and look for a production that uses all the sugar	Successive attempts 10 kg *cavacas* + 10 kg *ear cookies* 5 kg *cavacas* + 20 kg *ear cookies* 5 kg *cavacas* + 19 kg *ear cookies* 5 kg *cavacas* + 17 kg *ear cookies* valid solution, but with lower profit than previously found 1 kg *cavacas* + 20 kg *ear cookies* insufficient ingredients	They decide that they have already found the biggest profit and consider the problem solved.

Teacher: So how much is the profit?

Student E: Two euros.

Student F: Then it is 2x.

They then drew up the table in Fig. 8 (where the number of balls for each brand should appear, not just the brand name) and began to write down the restrictions of the problem. This again requires the interpretation of the task and its translation into mathematical language and, as in previous cases where it was necessary to interpret the task, the students needed support.

Teacher: He can only benefit from this promotion if he buys more than 10 official brand balls. How can we write this now?

Student E: $x>10$.

Teacher: The number of balls of the SOL brand must be less than or equal to the double of the number of balls of the official brand. How can we write this in mathematical language?

Student F:…

Teacher: What is the number of balls from the SOL brand?

Student F: x… no, 45.

Teacher: Number of balls from the SOL brand?

Student F: y.

Teacher: Then write…

Student F: $y \leq$ 2x.

The different answers given by student F to the teacher's question on how to write the number of balls of the SOL brand suggest that the process carried out by the students lacks meaning for them. The way in which they solve in order to y the first of the conditions they consider (leaving a denominator 1) is also elucidative of the way students deal with algebraic manipulations (Fig. 9).

As in task 1B, students choose to solve the problem graphically. The use of the graphing calculator still poses some difficulties to the students, something quite evident at a certain moment in the reaction of student F: "I don't know how this changed by itself… Teacher, my calculator changed on its own!". But, in addition, some aspects are also identifiable that illustrate the students' mastery (or lack of it) in terms of mathematical knowledge that can be considered elementary for this level of education. This is the case of identifying the coordinates of points in a graphic representation:

Teacher: Paint the region and find the vertices of this polygon. Two of them you already have, find the others. What is this?

Student F: (0, 10).

Teacher: In this order?

Student F: (10, 0).

The dependence on the teacher is another aspect that marks the resolution of this pair, notorious for the number of times that her presence is requested, with frequent occasions where she ends up guiding students in what to do next, as illustrated by the previous dialogue.

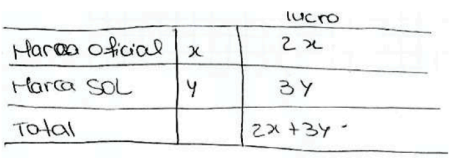
After determining the coordinates of the vertices of the obtained polygon, they register the graphic representation on paper, making the absence of a scale noticeable. They proceed to the calculation of the values of the profit function at these points, however they do not use the function previously determined (Fig. 10).

Thus, their resolution is concluded without presenting an answer to what was requested and identifying the option that allows to obtain the greatest profit.

6. Discussion of the results

6.1. Problem solving competency

All the students adopted similar approaches in the proposed tasks. In the first task (task 1A), they based their work on



		Profit
Official brand	x	$2x$
SOL brand	y	$3y$
Total		$2x+3y$

Fig. 8. Definition of variables and organization of data for task 2, by the EF pair.

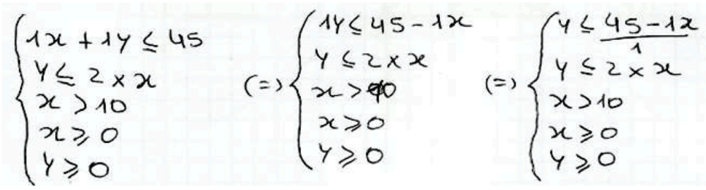


Fig. 9. Identification of the problem restrictions of task 2, by the EF pair.

experimentation and analysis around successively chosen cases. However, later (on task 2) they adopted a different strategy. They recognized the similarity of this task with the previous ones, and they assume that the appropriate approach would be the one developed in task 1B. This option might be based on what Palm (2008) refers to as the socio-mathematical norms or the enculturation of students in the school environment, which makes them consider that the desirable approach is the one they tried in task 1B. This belief leads them to choose this approach, even when they do not understand it, which sometimes takes them to a level of performance lower than they had in task 1A, where they showed they could understand the problem and the solution they want to find (case of the EF pair).

Cirillo and Hummer (2021) paid attention to the difference between experienced and novice problem solvers. According to the authors, novice problem solvers spend more time doing than thinking or planning. This result is based on a study from Cai (1994), valuing the time spent in orientation and organization, and not so much on execution. This conclusion can be related to the stages of problem solving. In the present study, the problem solving experience suggests the 1st stage (understanding the problem) and the 4th stage (verifying the solution), according to the classification developed by Pólya (1957), as the most complex to implement. These are stages related to thinking and not to doing, what suggests an alignment with the conclusions of Cirillo and Hummer (2021), since the participants of this study are novice problem solvers.

Understanding the problem is a particularly delicate step for the EF pair, who always needs support. But the global analysis of the work done (4th stage) is a sensitive part for all the students involved. The AB pair never seems to feel the need to do it. Analyzing the solution in the context of the problem is one important component of this stage. But, according to Martino (2019), the context tends to be ignored by the students. They assume it is something irrelevant. A simple delay in the process of getting to the right answer. However, in what concerns the EF pair, the situation is different from what we can find in several studies (Martino, 2019) and very interesting. The students seem to make a global analysis of their work during task 1A, but this is no longer the case in task 2, where they do not even complete the task since they do not respond to what is asked. According to Carotenuto et al.'s (2021) analysis of several studies, this is related to the students' perception of the reality in the problem. If the students assume the context of the problem as real and relevant, they keep it in mind and try to achieve an answer suitable to the situation. In the other cases, they just ignore the context. And this pair's approach to this task differs significantly from what they did in task 1A. In the first case the students were involved in solving the problem keeping in mind what was intended, in the last case the students are not actually developing a problem solving process. On the contrary, the students are trying to reproduce something that they assume is what is expected of them. So, it seems that the results of this study are aligned with the results of those reported by Martino (2019), i.e., when the students are acting according to what they think is expected from them in a Mathematics class, they actually ignore the context of the problem, not feeling the need to analyze the solution achieved in the context.

It is also interesting to note that the EF pair is composed of students with poor performance in Mathematics, but in task 1A this is the pair that, once the problem is understood, is able to make the most coherent decisions in the choice of the situations they analyze; which finds more cases respecting the restrictions imposed by the problem, not tending to assume problems as a single solution task (one of the common difficulties in problem solving identified by Palm (2008)); and the one that always keeps the focus of what is intended during task 1A, not suspending their sense making and keeping their "intrinsic reason to confront their solution with reality"

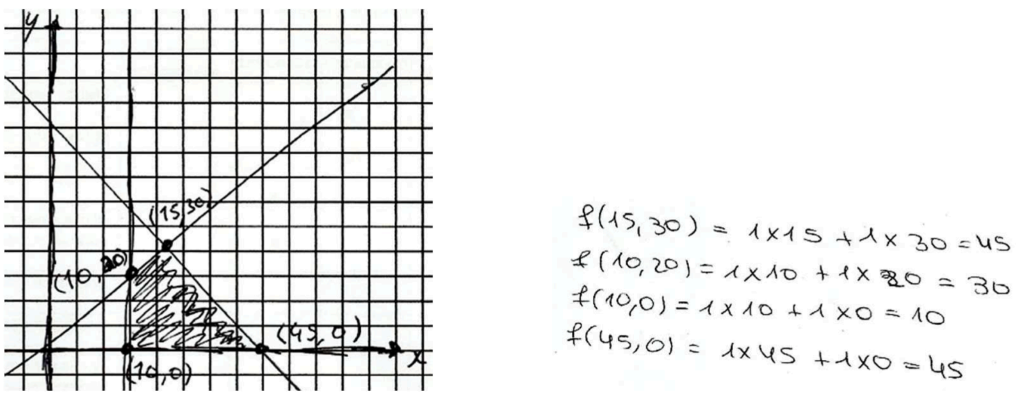


Fig. 10. Conclusion of task 2 resolution by EF pair.

in line with Carotenuto et al. (2021, p. 819).

6.2. Competence with mathematical representations

The interpretation of the information provided is something that is always complex for the EF pair, who is unable to analyze the representation provided in the task (written representation) and proceed to its transition to a representation that is relevant to the students and allows them to make sense of the information. Although all the tasks use verbal representation, students in this pair always need the support of the teacher to make a transition to another representation that makes sense to them. Still, it is interesting to note that once this transition is made to a representation that, being equally verbal, seems to be different from the one initially made available, the students can make a coherent approach to task 1A, maintaining meaning in all the options assumed. Other representations, such as the graphical one they use in task 2, prove to be more difficult to deal with for this pair of students, being namely evident difficulties in reading the coordinates. On the contrary, the AB pair seems to deal better with the different representations, but apparently tends to neglect some aspects linked to the real context of the information made available, in line with what was described by Palm (2008) and Carotenuto et al. (2021).

6.3. Competence in relation to symbols and mathematical formalism

This competence is particularly marked by the students' mathematical background. While the AB pair does not seem to have problems dealing with symbols and notations, the same is no longer the case with the EF pair. This last pair faces difficulties in getting the coordinates of a point from the graph, changing its order, and solves an inequality in order to one of the variables in a way that illustrates the lack of flexibility and familiarity with the process. These difficulties in relation to symbols and mathematical formalism are also identified in other studies, as discussed by Cañadas et al. (2018). As Cañadas et al. (2018) recognize, despite the efforts to promote the students' familiarity with algebraic symbolism, many struggle to use it and to understand it. However, Diego-Mantecón et al. (2021, p. 354) refer "the complexity of adapting mathematics techniques to solve real-life tasks". Cirillo and Hummer (2021) consider that there can be a difference between having mathematical knowledge and knowing how to apply it in problem solving. Or, as Mamona-Downs and Downs (2005) say, one student can achieve high results on a test and do not achieve the same high level in problem solving. In this study, the students achieving higher results are also showing a higher competence in relation to the use of symbols and mathematical formalism.

The approaches chosen by the EF pair, when they are trying to follow a strategy that they consider should be reproduced, also reflect the level of competence they have with mathematical formalism. However, when solving a problem on their own some differences were observed. When calculating the amount of flour and sugar needed to produce a certain amount of cakes, the AB pair uses an algorithm, on the contrary, the EF pair relies on the notion of proportion. As pointed by Norton (2005), one pair of students based the calculation on a mechanical procedure, and the other pair on the understanding of the proportion notion. This fact allows us to raise questions about the characteristics of the difficulties of EF pair. We can question whether the difficulties are related to a mechanical use of procedures disconnected from understanding and not so much with the use of mathematics with understanding. Or, in other words, we can question if experiences with a mechanical approach to mathematics did impact the learning of these students and even their involvement in its learning. In line with this idea, Cañadas et al. (2018) question the understanding achieved by the students during their experience in the learning of algebraic symbolism. However, Mamona-Downs and Downs (2005) emphasize the relation between mathematical knowledge and problem solving competence, recognizing the complexity of the relation.

6.4. Mathematical communication competence

The competence of mathematical communication seems to be closely related to the mathematical competence that students attribute to themselves. The EF pair, being composed of students who are weaker in mathematics, is more insecure and has a greater need for validation by the teacher. However, among themselves, the students show the ability to logically base the options they assume during task 1A, after the initial stage of understanding the problem. According to Bach and Bikner-Ahsbahs (2022, p. 180), based on the understanding of Niss and Højgaard (2019), "mathematical communication competency concerns being able to express oneself mathematically using mathematical terms and words, as well as understanding and interpreting other people's mathematical expressions". In these circumstances, it is not surprising the difference in the performance of AB pair.

The AB pair, formed by students with better performance in the discipline, tends to assume a more confident posture. Still, at times it does not seem to be completely capable of adequately justify the reasons that underpinned the options assumed, once again questioning the competency to apply their mathematical knowledge in a problem solving context, as referred by Mamona-Downs and Downs (2005).

6.5. Competence with technology

In terms of competence with technology, all the students were starting to use the graphing calculator and, therefore, there was little familiarity with the technology, as it should be expected (Rocha, 2022). Even so, the AB pair was more confident in the use of technology, being the EF pair the one that most often needed to ask for support in relation to it. A difference that may be related to the fact that the calculator is used mainly when students are looking to reproduce a technique that was presented to them, and this is something that ends up being related to the confidence that students have in themselves in learning Mathematics and even with their performance in the discipline. However, the idea of competence with technology is related to the characteristics of its use. And Rocha (2020b) speaks about the knowledge a student needs to make a suitable use of the technology, i.e., to develop competence with it. According to Niss and Højgaard (2019, p. 18) competence with technology is the "individual's ability to put such aids and tools to constructive use in mathematical work". This understanding suggests an use of technology to look for mathematical understanding. In the case of this study, technology was used by the students when they were reproducing a process, and this as consequences for the characteristics of its use. As so, none of the students showed competence with technology.

7. Conclusion

The results of this study offer some interesting conclusions, although the reduced number of participants is a limitation, suggesting the need for future studies and for a deeper understanding about the relations between problem solving and competency.

The study suggests, in line with Palm (2008), that the students' enculturation may have led them to value a performance in the mathematics class based on the reproduction of techniques such as procedures and representations. However, as advanced by Mamona-Downs and Downs (2005), a greater mastery of the techniques worked in the mathematics class does not translate into a better performance in a problem solving situation. On the contrary, in this case it seems that students with lower performance in these techniques are able to show some important aspects regarding problem solving, namely some of the aspects associated with the fourth phase proposed by Pólya (1957). Actually, these are the students who, at the end of the calculations, read the question posed again, before completing their work, not showing an action marked by the idea of a single solution discussed by Palm (2008) (as seems to be the case of their colleagues), neither assuming the context as irrelevant, as is often done as described by Martino (2019). Even so, one evidence of this study is that in task 1A the students were faced with a problem situation. However, that is not the case in task 2, where all the students assumed that they should reproduce the process presented in task 1B, instead of looking for the best strategies to find the answer to the problem.

The main conclusion of this study points, therefore, to the lack of a direct link between the mastery of techniques, procedures and mathematical representations, and the performance in a problem solving situation. This somehow contradicts the idea presented by Achmetli et al. (2019) of a relation between previous knowledge and competence, although the authors are considering procedural and conceptual knowledge, and in this study we cannot be sure about the conceptual knowledge of the students. However, the conclusions support the point of view of Mamona-Downs and Downs (2005) of this relation as a complex one. Overall, it seems to be possible to conclude that the mastery of techniques, procedures and mathematical representations does not translate into an improvement in the problem solving capacity, and may even be responsible for the development of conceptualizations that negatively affect this ability. This conclusion poses questions about what is being valued in mathematical learning, emphasizing the relevance of the teachers and of their professional development.

CRediT authorship contribution statement

Helena Rocha: Conceptualization, Formal analysis, Methodology, Writing – original draft, Writing – review & editing. **Ana Babo:** Conceptualization, Formal analysis, Methodology, Writing – original draft.

Data availability

When the confidentiality agreement assumed with the participants of the study allows it, the data will be made available on request by the second author of this article.

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