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Tail Index Estimation:
A Study on the Evolution of Heavy Tails of Regional Bank Indices and its Impact on Value-at-Risk

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Abstract

This study peruses the leptokurtic evolution of regional banking indices belonging to the Americas, Asia, Australasia and Europe from 1973-2016 as measured by the tail index ζ in order to assess the impact of tail index variation on VaR. Breaks in ζ are detected under the testing framework proposed by Quintos *et al.* (2001) combined with both the originally suggested Hill estimator and, as an innovation, the newly proposed rank-size statistic. It was concluded that changes in ζ led to material differences in VaR and the new test statistic showed superior statistical power over the Hill statistic.

Keywords: Hill statistic, Rank-size statistic, Value-at-Risk, Tail index variation

1 – Introduction

The anatomical scrutiny of extreme value geometries in statistics has an indisputable relevance in a wide array of quantitative subjects, especially in the field of finance. Financial asset returns are notorious for straying away from the mesokurtic characteristic of Gaussian curves, but tend to be rather symmetrical. As a matter of fact, stock returns have a strong tendency to display “heavy tails” or, under more precise terminology, leptokurtosis. This refers to distributions with positive excess kurtosis (Lux, 2008). Visually, this translates into more amassed tails compared to a distribution with null excess kurtosis, as is the case of the normal curve. Failing to factor this into one´s analysis could lead to large sums of money being lost as a result of underestimating the potential extreme losses of a given event. As tails gain more mass, extreme events become more likely. In other words, the existence of heavy tails and their variation in time must be properly accounted for in order to avoid a misleading depiction of the financial risk contained in VaR figures. The impact of heavy-tailedness and its variation in time on this risk metric will therefore be one of the focal points of the current study. The VaR measure is vastly employed throughout the financial industry from financial regulators and large asset managers to small-scale individual investors as a fundamental risk gauge.

The first central purpose of this study is therefore to investigate how ζ, which is a measure of the fatness of tails and ergo a financial indicator of extreme risk profiles, evolves throughout turbulent periods across several regions in order to assess its impact on risk analysis. Daily returns from banking indices extracted from Thomson Reuters for the following areas will be considered: Americas, Asia, Australasia, and Europe. Following Werner and Upper (2004), the effect of tail variation on risk analysis is pondered in terms of a comparative evolutionary study between the tail index VaR and the volatility of the indices throughout the entire range of the data set from January 1973 to January 2016. In other words, the question of whether or not the tail index itself adds additional value to the risk measure of volatility is addressed. A normal distribution based VaR is also included as an illustrative example of how it may underestimate tail risk. The approach adopted here is analogous to the one used by Werner and Upper (2004) in their perusal of bond future returns where three main points are discussed: the confirmation of heavy tails in returns, the verification of tail changes over time and lastly the consideration of their impact on VaR. The statistical testing framework that is employed to identify changes in ζ is the same as the one detailed by Quintos *et al.* (2001). The second objective of this paper is to assess the comparative statistical power of a newly proposed test statistic that links the OLS rank-size tail estimator with their structural break methodology relative to the originally employed Hill estimator.

This paper brings two innovations to the branch of extreme value theory. The first is the investigation of regional bank stock indices. Most sources have concentrated their efforts towards evaluating returns and VaRs from equities, as Lux (2008) did, stock futures like Cotter (2001), bond futures by Werner and Upper (2004) or exchange rates by Hartmann, Straetmans and De Vries (2010). The second novelty consists in not restricting the structural break analysis and computation of ζ to only the volatility-adjusted version of Hill´s estimator as specified by Quintos *et al.* (2001), but in also incorporating a more robust method of tail calculation as specified by Gabaix and Ibragimov (2011). The latter pertains to the OLS log-log rank-size regression, and is used here as a new way of linking the rank-size regression with the structural break testing methodology proposed by Quintos *et al.* (2001).

The content of this report is organized as follows: part 2 expands upon essential theoretical concepts underlying the current study and discusses the overall methodology that will be put into practice, part 3 includes descriptive statistics of the data along with the tail index and VaR results from the two methods of calculation incorporated in this paper. It will also include the outcome of statistical tests used to identify structural breaks and aim to map lower ζ values with regional crises. Analysis of a Monte Carlo simulation used to study the power of the newly proposed rank-size test statistic is also included. Finally, section 4 summarizes key points and provides concluding remarks.

2 – Literature Review and Methodology

The present section provides an overview of the theory that currently exists within the topic of heavy tails and thoroughly explains how this investigation will contribute to its field of study. Additionally, it aims to clarify the methodologies implemented in this research.

2.1 – Measurement of Heavy Tails

Studies in extreme value theory widely define the fatness of tails in terms of both kurtosis, the fourth order central moment of a distribution, and the tail index parameter ζ, which is a scalar measure for the thickness of tails that was first theorized by Hill (1975).

The presence of fatter tails is identified when excess kurtosis is positive, which defines the existence of leptokurtosis. Lux (2008) claims that financial returns "always" exhibit positive excess kurtosis and supports this by citing the work of Fergusson and Platen (2006) that showcases the excess kurtosis for the following stock indices: FTSE100, NASDAQ, CAC 40, MSCI Australia. On a similar note, Bradley and Taqqu (2001) verify that the daily returns of the S&P500, the USD/GBP exchange rate, the Baht/USD exchange rate and finally the NASDAQ index all exhibit positive excess kurtoses.

Section 3.1 will characterize the regional bank indices in terms of their sample excess kurtosis and skewness. This will then allow for a preliminary diagnosis of the type of kurtosis involved. Consequently, the Jarque-Bera statistic may be computed and compared with $\chi^2_{(2)} = 9.2103$ for a 1% significance level so as to verify the non-normal nature of bank stock samples.

The tail index ζ is another measure of fat tails that quantifies their thickness in the form of a decay exponent belonging to a given power law function that is presumed to model a given univariate random variable (Ibragimov *et al.*, 2015). Ibragimov and Kattuman (2013) define this type of power law, for both left and right tails, as:

$$P(|X| > x) \sim Cx^{-\zeta}, \quad C, x, \zeta > 0 \tag{2.1}$$

with C being a positive constant.

Specific cases which can be obtained from (2.1) are the Cauchy, Pareto, Levy and other stable distributions with $\zeta < 2$. Gourieroux and Jasiak (2001) state that a distribution may be classified as "Pareto type" or approximately Pareto if the following is true about the survival function $1 - F(x)$:

$$P(X > x) \equiv 1 - F(x) \sim x^{-\zeta} G(x) \tag{2.2}$$

where $G(x)$ is a function that must be "slowly varying" at infinity and $F(x)$ is the CDF of a distribution such that $F(x) = P(X \leq x)$ with X being a random variable. Straetmans and Chaudhry (2012) as well as Gourieroux and Jasiak (2001) describe $G(x)$ in the following way:

$$\lim_{x \to \infty} \frac{G(\lambda x)}{G(x)} = 1, \quad \forall \lambda > 0. \tag{2.3}$$

Analogous to (2.2) and (2.3), Werner and Upper (2004) write the necessary condition as:

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\zeta}, \zeta > 0. \tag{2.4}$$

Danielsson, Hartmann and de Vries (1998) ensure that heavy-tailed distributions, at least through a first-order approximation of its tails, may be precisely modeled by a Pareto distribution. Leptokurtic functions must therefore satisfy conditions (2.1) to (2.4).

Intuitively speaking, the fatness of tails will increase with decreasing values of ζ, implying also that extreme events are more likely with lower tail indices. Hartmann, Straetmans and De Vries (2010) mention the exponent in (2.1) as also being indicative of the number of finite distributional moments. Ibragimov and Kattuman (2013) add that current studies associate heavy-tailed models for stock indices with $\zeta \in (2, 5)$.

2.2 – Tail Index Estimators

One of the most well-known methods of calculation of the tail index ζ, is Hill´s (1975) conditional maximum likelihood estimator, which can be given as:

$$\hat{\zeta}_{Hill} = \left(\frac{1}{m_T} \sum_{i=1}^{m_T} \left(\ln X_{(T-i+1)}^{T} - \ln X_{(T-m_T+1)}^{T} \right) \right)^{-1} \quad (2.5)$$

where m_T is the largest m data points or tail values within a sample of size T. They define $X_{(i)}^{J}$ as the i-th value within a sample of size J in ascending order such that $X_{(1)}^{J} \leq X_{(2)}^{J} \leq \cdots \leq X_{(J)}^{J}$. In order to calculate $\hat{\zeta}$ associated with the outer portions of the distribution, considering the right tail as they did, m_T must be replaced with m_{w_t} and T with w_t. m_{w_t} would now represent the subsample with an endpoint at t and size of w_t. The endpoints t and m_{w_t} are designated in the following manner via the floor function $t = \lfloor rT \rfloor, r \in (0,1)$ and according to DuMouchel (1983) the number of observations m_{w_t} from the right tail (region where the distribution is power-like) to be included in the estimation of ζ should be defined as $m_{w_t} = \lfloor k w_t \rfloor$ where $k = 0.1$ is suggested[1]. Werner and Upper (2004) stress the importance of choosing an appropriate value for m_{w_t}, claiming that as one moves farther along the tails, a better computation of ζ is

[1] Note that for the rank-size regression method, a value of $k = 0.15$ is used.

generally possible given that the Pareto approximation improves. Nuyts (2010) notes the exceptions in which certain functions do not behave as power laws even at extreme regions of the tail. These give rise to the “Hill horror plots”. Equation (2.5) represents one of the fundamental methods with which ζ will be estimated in section 3.

Knowing that the Hill estimator may produce inaccurate results under significant deviations from power laws, this study will also include an OLS log-log rank-size regression for ζ as a comparative verification of consistency between both methods. Gabaix and Ibragimov (2011) explain how this regression grew popular mainly due to its simplicity and robustness when compared to other techniques, including the Hill estimator. The regression is as follows:

$$\log(t-\gamma) = a - b\log Z_{(t)} \tag{2.6}$$

where $Z_{(t)}$ refers to an observation from the data set arranged in descending order: $Z_{(1)} \geq Z_{(2)} \geq \cdots \geq Z_{(n)}$ from the n largest values. t alludes to the relative rank of a data point and $Z_{(t)}$ is also commonly called its size. To correct for small sample bias, Gabaix and Ibragimov (2011) were the first to propose that the optimal shift be characterized by $\gamma = \frac{1}{2}$. The estimator $\hat{b}_n^{\gamma}$ that they illustrate for the slope coefficient b is also the tail index estimate $\hat{\zeta}_{RS}$:

$$\hat{\zeta}_{RS} \equiv \hat{b}_n^{\gamma} = -\frac{\sum_{t=1}^{n}\left(\log\left(Z_{(t)}\right) - \frac{1}{n}\sum_{t=1}^{n}\log Z_{(t)}\right)\left(\log(t-\gamma) - \frac{1}{n}\sum_{t=1}^{n}\log(t-\gamma)\right)}{\sum_{t=1}^{n}\left(\log\left(Z_{(t)}\right) - \frac{1}{n}\sum_{t=1}^{n}\log Z_{(t)}\right)^2} \tag{2.7}$$

with an associated robust standard error specified as $SE(\hat{\zeta}_{RS}) = \hat{\zeta}_{RS}\sqrt{\frac{2}{n}}$. The corresponding 95% confidence interval is thus: $\hat{\zeta}_{RS} \pm 1.96\hat{\zeta}_{RS}\sqrt{\frac{2}{n}}$.

Equation (2.6) with $\gamma = \frac{1}{2}$ represents the second technique that is employed in section 3 to calculate the tail index. It is important to stress that this optimal shift with the correct standard errors, $SE(\hat{\zeta}_{RS})$, provides adequate results for leptokurtic GARCH - dependent processes (Ibragimov and Walden, 2015).

2.3 – Statistical Tests for Structural Breaks in the Tails

Phillips and Loretan (1990) and Koedijk *et al.* (1990) centered their work on structural breaks tests with a known break date. Through Hill´s estimator and a null hypothesis positing a constant ζ throughout all subsamples, they discovered that ζ was variable for exchange rate returns in the US, Japan and part of Western Europe. By noting w_i as the length of subsample i such that $T = \sum_{i=1}^{g} w_i$ for g partitions of the full sample, Koedijk *et al.* (1990) indicate the following statistic that is comparable with a Chi-squared value with g degrees of freedom:

$$Q = m_1 \left(\frac{\hat{\zeta}_1}{\zeta_1} - 1 \right)^2 + \cdots + m_g \left(\frac{\hat{\zeta}_g}{\zeta_g} - 1 \right)^2 \sim \chi_g^2 \qquad (2.8)$$

Quintos *et al.* (2001) were the first to propose a recursive method to test for tail variation given that the breakpoint was unknown. Their test statistic is $Y_T(t) = \left(\frac{t m_t}{T}\right)^{1/2} \left(\frac{\hat{\zeta}_t}{\hat{\zeta}_T} - 1\right)$ where t denotes a certain time and $m_t = 0.1 \cdot T$, $\hat{\zeta}_T$ is the Hill estimator for the entire sample and $\hat{\zeta}_t$ for a subsample up to t. The corresponding null hypothesis of tail constancy is $H_0\colon \zeta_{\lfloor rT \rfloor} = \zeta$ where r belongs to a subset of $(0,1)$ such that $r \in R_\pi = [\pi, 1 - \pi]$ with $\pi > 0$. The unadjusted test statistic mentioned by Quintos *et al.* (2001) is:

$$Q = \sup_{r \in R_\pi} Y_T(\lfloor rT \rfloor)^2 = \sup_{r \in R_\pi} \left\{ \left(\frac{\lfloor rT \rfloor m_{\lfloor rT \rfloor}}{T} \right)^{1/2} \left(\frac{\hat{\zeta}_{\lfloor rT \rfloor}}{\hat{\zeta}_T} - 1 \right) \right\}^2 \qquad (2.9)$$

Quintos *et al.* (2001) propose a correction that accounts for $\mathrm{GARCH}(1,1)$ dynamics by incorporating a scaling factor $\hat{\eta}_t$ in (2.9):

$$Q_{Hill}^{*} = \sup_{r \in R_\pi} \hat{\eta}_t^{-1} Y_T(\lfloor rT \rfloor)^2 \qquad (2.10)$$

This scaling factor $\hat{\eta}_t$ serves the purpose of correcting for GARCH effects (volatility variation) that are commonly encountered in financial returns data (Straetmans and Candelon, 2008). $\hat{\eta}_t$ is given as a linear combination of $\hat{\chi}_t$, $\hat{\psi}_t$ and $\hat{\omega}_t$, i.e.,

$$\hat{\eta}_t = 1 + \hat{\chi}_t + \hat{\omega}_t - 2\hat{\psi}_t \qquad (2.11)$$

where $\hat{\chi}_t = \frac{2\bar{\zeta}_t^2}{m_t}\sum_{j=1}^{t-1} c_{tj}c_{tj+1}$, $\widehat{\omega}_t = \frac{2}{m_t}\sum_{j=1}^{t-1} d_{tj}d_{tj+1}$ and $\hat{\psi}_t = \frac{\bar{\zeta}_t}{m_t}\sum_{j=1}^{t-1}\left(c_{tj}d_{tj+1} + c_{tj+1}c_{tj}\right)$. Note that c_{tj} may be defined in terms of the maximum function $X_+ = \max(X, 0)$ and d_{tj} relative to the indicator function $I(.)$ as seen in (2.12):

$$\begin{cases} c_{tj} = \left(\log X_i^{2t} - \log X_{(t-m_t+1)}^{2t}\right)_+ \\ d_{tj} = I\left(\log X_i^{2t} > \log X_{(t-m_t+1)}^{2t}\right) \end{cases} \tag{2.12}$$

Through the methodology that ties in the GARCH-adapted statistic (2.10), Werner and Upper (2004) concluded that there were significant breaks in the tail index of Bund futures. Quintos *et al.* (2001) were previously able to show that the market returns from the Asian stock market, composed of data from Thailand (SET Index), Malaysia (KLCI Index) and Indonesia (JCI Index), prove that the tails of returns also present a time-varying nature.

The outcomes from (2.10) computed, with the aid of MATLAB, are shown in section 3.3. The decision mechanism used to map structural breaks for any region is centered around the 95% critical value of $Q_{crit} = 1.78$. Two test statistics will be computed for each time period: the forward and the reverse. The forward recursive test detects a decrease in the tail index ($\zeta_1 > \zeta_2$), or when tails vary from thinner to thicker. The reverse recursive test identifies a significant increase in the tail index ($\zeta_1 < \zeta_2$) or when tails change from thicker to thinner (Quintos *et al.*, 2001). The Hill estimator suffers from small sample bias so the search for breaks ceases once a sample of less than 500 observations is observed.

The current paper introduces a new method for evaluating the test statistic using a rank-size regression. It is analogous to (2.10) with a scaling factor based on the corresponding standard error. Now $m_t = 0.15 \cdot T$, as suggested by Nicolau and Rodrigues (2016). Thus,

$$Q_{RS}^* = \sup_{r \in R_\pi} \sqrt{\frac{2}{m_t}} Y_T(\lfloor rT \rfloor)^2. \tag{2.13}$$

The power of the test statistic in (2.13) relative to that in (2.10) is assessed in terms of a Monte Carlo simulation emanating from the Pareto distribution, which is analyzed in section 3.3.3.

2.4 – Measuring the Impact on Value-at-Risk

VaR may simultaneously be the greatest possible associated loss or the smallest potential extreme loss. $\mathrm{VaR}_{95\%} = 1M€$ may be interpreted in two equivalent ways: either there exists a 95% chance that $1M€$ will be the maximum loss or that there is a 5% probability that this value will be the smallest overall cost. It is a fairly recent concept that was developed in the 1990s by leading investment banks including J.P. Morgan to quantify a company´s overall exposure to market risk by a single figure (Bradley and Taqqu, 2001). If the periodicity of this VaR estimate is daily, then on any given trading day, the worst loss can amount to $1M€$.

Bradley *et al.* (2001) quantify this small probability associated with the scenario in which the loss exceeds the VaR in the following manner:

$$P(\mathrm{Loss} > \mathrm{VaR}) \leq 1 - \alpha \tag{2.14}$$

where α is a confidence level such that $\alpha \in]0,1[$ but more commonly between 0.95 and 1. By letting F_X denote the CDF of random variable X such that it represents the negative profit and loss behavior of a financial asset at a specific time period t, the VaR of X or $\mathrm{VaR}_\alpha(X)$ is:

$$\mathrm{VaR}_\alpha(X) = \inf\{x | F_X(x) \geq \alpha\}. \tag{2.15}$$

By further developing (2.15), the parametric VaR according to Bradley *et al.* (2001) for a generic probability distribution is obtained, i.e.,

$$\mathrm{VaR}_\alpha(X) = \mu_t + \sigma_t q_\alpha \tag{2.16}$$

with $q_\alpha = F_{\tilde{X}}^{-1}(\alpha)$ being the α-quantile of a standardized random variable $\tilde{X} = \sigma_t^{-1}(X - \mu_t)$. In the context of a normal distribution, i.e. $F_X \sim N(\mu_t, \sigma_t^2)$, (2.16) may be written as follows:

$$\mathrm{VaR}_\alpha(X) = \mu_t + F_{\tilde{X}}^{-1}(\alpha) \cdot \sigma_t = \mu_t + \Phi^{-1}(\alpha) \cdot \sigma_t. \tag{2.17}$$

Note that the periodicity of this VaR estimate depends on the frequency of the return data from which the volatility is calculated.

Werner and Upper (2004) verified that the VaRs calculated from assuming that Bund future returns derived from a normal distribution were significantly lower than those calculated from the tail index VaR, as is expected from the effect of heavy tails. Bradley and Taqqu (2001) justify this difference by simply noting that $F_{normal}^{-1}(\alpha) \leq F_{t-dist}^{-1}(\alpha)$, in the case that heavy tails were to be modeled by the t-distribution. The more important discovery, however, concerns the peril of assessing market risk solely based on volatility, which they measured through the standard deviation. Results showed that the 99% tail index VaR (see (2.19) or (2.20)) did not always accompany the movements of volatility, as did the 99% VaR from a normal distribution (see (2.17)). This shows that volatility does not provide a complete picture of the market risk as it ignores the evolution of the tails of a distribution and as such, the VaR must be adapted to contemplate ζ variation, at least in the context of future Bund returns.

Several solutions have been proposed to deal with heavy-tailedness. In 1995, the Basel Committee on Banking Supervision (BCBS) proposed the use of a 10-day VaR at a 99% confidence level scaled by a safety factor of at least 3 as an adjusted figure to reliably help establish the minimum amount of capital reserves needed by financial institutions during more turbulent times to guarantee solvency (Borak *et al.*, 2011). The safety factor is a response to the leptokurtic irregularities of the distribution from which the returns emanate.

Gourieroux and Jasiak (2001) claim that if the condition in (2.3) is true, then the following may be said regarding the VaR for a small probability $1-\alpha$:

$$-\mathrm{VaR}(\boldsymbol{w}, 1-\alpha) \sim (1-\alpha)^{-1/\zeta} \tag{2.18}$$

However, when $1-\alpha_0 < 10\%$ the VaR estimation involving the empirical $(1-\alpha_0)$ - quantile will not have enough extreme data points to guarantee an accurate figure. As a result, they propose that extreme $(1-\alpha)$ - quantiles such that $1-\alpha < 1-\alpha_0 \leq 10\%$ be estimated by applying a Pareto type model of the tails to the initial empirical quantile, $\widehat{\mathrm{VaR}}_{1-\alpha_0}$:

$$\widehat{\mathrm{VaR}}_{1-\alpha} = \widehat{\mathrm{VaR}}_{1-\alpha_0}\left(\frac{1-\alpha_0}{1-\alpha}\right)^{1/\hat{\zeta}} \tag{2.19}$$

Quintos *et al.* (2001) define the VaR in terms of the tail index ζ in a way that can be obtained directly from (2.19) if one considers that $\widehat{\text{VaR}}_{1-\alpha_0} = X_{(T-m+1)}$ and $1 - \alpha_0 = \frac{m}{T}$:

$$\text{VaR}_{\alpha} = \left[\frac{T}{m}(1-\alpha)\right]^{-1/\zeta} X_{(T-m+1)} \qquad (2.20)$$

Equations (2.17) and (2.20) will be used in the VaR analysis in section 3. Since the return data is daily, both equations will be used to compute daily or 1-day 99% VaR estimates.

2.5 – Contribution of Current Research within Theoretical Framework

Unlike most studies conducted in this field, which have focused on exchange rate returns, Bund future returns or indices of the likes of the S&P500, the current paper adds to the understanding of heavy tails by analyzing regional bank stock indices. Additionally, it presents results not only from the Hill estimator but also from the OLS rank-size regression and applies both to the structural break testing framework proposed by Quintos *et al.* (2001). The novel contribution is in the form of (2.13), whose power is tested via a Monte Carlo simulation. Lastly, it will emulate Werner and Upper´s (2004) approach by studying the evolution of the tail index VaR with the volatility to determine whether or not the tail index brings more information about market risk than the latter within the context of regional bank indices. It will also contrast the VaRs from the normal distribution with those from the tail index.

3 – Empirical Analysis

3.1 – Data and Descriptive Statistics

This section summarizes the descriptive statistics or "pseudo-moments" of daily returns of bank stock indices from 1973 to 2016 belonging to the following regions: the Americas, Asia, Australasia, and Europe.

Table 1: Pseudo-moments of Regional Bank Stock Indices (01/1973 – 01/2016)

Region	Mean	Std. Deviation	Minimum	Maximum	Skewness	Excess Kurtosis	Jarque-Bera
Americas	0.0209%	1.2694%	-14.68%	14.60%	-0.1491	19.59	179700.04
Asia	0.0216%	1.2467%	-17.09%	12.07%	0.3649	12.16	69425.42
Australasia	0.0224%	1.5291%	-21.76%	12.89%	-0.6183	11.26	60089.48
Europe	0.0134%	1.3429%	-11.47%	15.03%	-0.0275	12.00	67367.67

Note: the number of observations for the Americas is 11233 and 11234 for the remaining regions.

The statistics in table 1 show how the daily average returns for most bank indices hover around the 0.02% mark with Europe exceptionally showing a lower value of 0.01%. Volatility averages at about 1.3%. The high absolute value of the minimum and maximum returns compared to the mean points to the presence of recessional occurrences within the sampling period, these crises will be covered in more detail in section 3.4.

The most notable feature overall is the seemingly leptokurtic and highly non-normal nature of these bank index returns, but of course the tail index must be considered in order to confirm this as both the kurtosis and skewness may be infinite for $0 < \zeta < 3$.

Figures 1-4 confirm the stationary behavior in the form of mean reversion that is characteristic to most financial returns. Additionally, volatility clustering is also observed, with

Figure 1: Regional Bank Stock Index Returns from the Americas

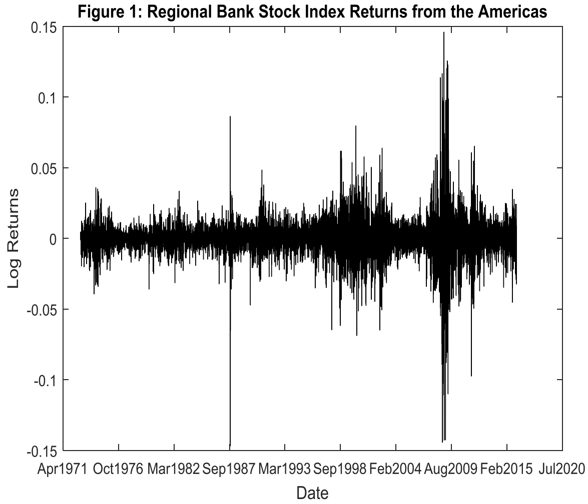


Figure 2: Regional Bank Stock Index Returns from Asia

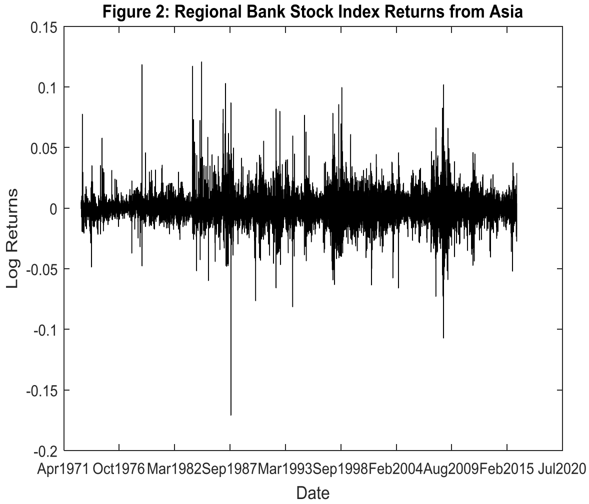


Figure 3: Regional Bank Stock Index Returns from Australasia

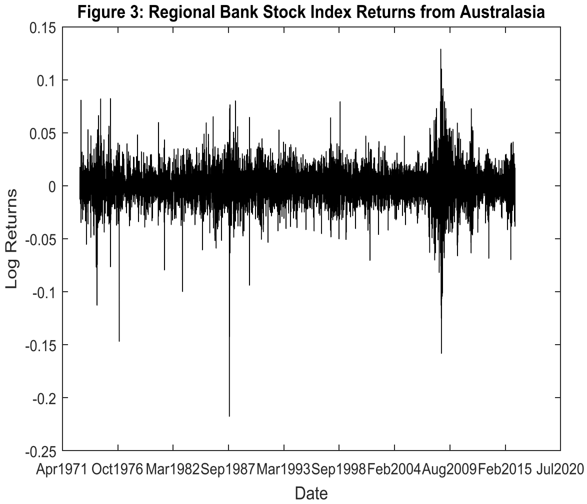
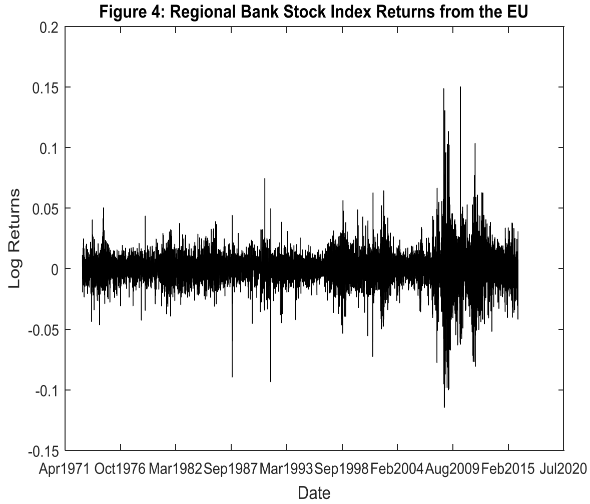


Figure 4: Regional Bank Stock Index Returns from the EU



the period between 2007 and 2009 globally showcasing the highest volatilities, this is precisely why the recursive test in (2.10) has the time-varying factor $\hat{\eta}_t$ (Straetmans and Candelon, 2008).

3.2 – Full Sample Tail Properties

This section provides an overview of results obtained for ζ using the entire sample (01/1973 – 01/2016) for each region based on the Hill estimator and rank-size regression. Table 2 shows a regional breakdown of ζ calculated according to both of these methods. The 95% confidence interval for $\hat{\zeta}_{Hill}$ is constructed according to Quintos *et al.* (2001).

Table 2: Full Sample Tail Indices (01/1973 – 01/2016)

Region	Hill Estimator (Left Tail)	Hill Estimator (Right Tail)	Rank-Size OLS (Left Tail)	Rank-Size OLS (Right Tail)
Americas	1.9263 [1.8654, 1.9873]	1.9801 [1.9183, 2.0419]	2.0258 [1.8890, 2.1626]	2.0468 [1.9086, 2.1851]
Asia	2.1654 [2.0997, 2.2310]	2.0154 [1.9520, 2.0787]	2.3058 [2.1501, 2.4616]	2.1491 [2.0039, 2.2942]
Australasia	2.2705 [2.2025, 2.3385]	2.3160 [2.2473, 2.3847]	2.4065 [2.2439, 2.5690]	2.4947 [2.3262, 2.6632]
Europe	2.0794 [2.0154, 2.1434]	2.2037 [2.1378, 2.2695]	2.1937 [2.0455, 2.3419]	2.2853 [2.1309, 2.4396]

Table 2 shows that, if breaks were ignored, all regions would possess a finite first distributional moment since the tail index is always above 1. It is also true that they would be defined by infinitely large values for skewness (third moment) and kurtosis (fourth moment) since the tail exponent would not only be universally below 3 but also below 4. As for the variance, Asia, Australasia, and Europe would likely exhibit a finite value. By considering the 95% confidence interval for the right tail of the Hill estimator, there would be a possibility of an infinite second moment based on the lower limit of 1.9520 for Asia. The rank-size estimator would essentially guarantee the existence of the first two moments for all regions except for the Americas. It is evident from table 2 that the Hill estimator always calculates a slightly lower value of ζ. It is worth noting that both estimators show that, for all regions except Asia, the left tail has a lower ζ than the right. This means that the left tail, which represents negative returns, is predominantly heavier than the right, which is representative of positive returns. This characteristic was already adumbrated by table 1 since Asia was the only region with a positive skewness. Since the tail exponent of a Gaussian distribution is infinite, the data set of the regional bank indices is indeed leptokurtic.

3.3 – Structural Tail Breaks

The purpose of this section is to peruse the evolution of the test statistic calculated by (2.10) for the case of the Hill estimator and (2.13) for the rank-size estimator, in order to identify any statistically significant structural breaks in the tail index ζ. It will then consider the results of the Monte Carlo exercise to compare the power of each test statistic.

3.3.1 – Hill Estimator

As an illustrative example on structural break analysis, consider the Americas. According to Figures 5 and 6, there exists a significant break in the forward statistic taking place on June 17, 1998 since $\max(Q_{Forward}) > 1.78$ using a 95% confidence level.

Figure 5: Structural Change Test in Right Tail Index for the Americas (Hill Estimator)

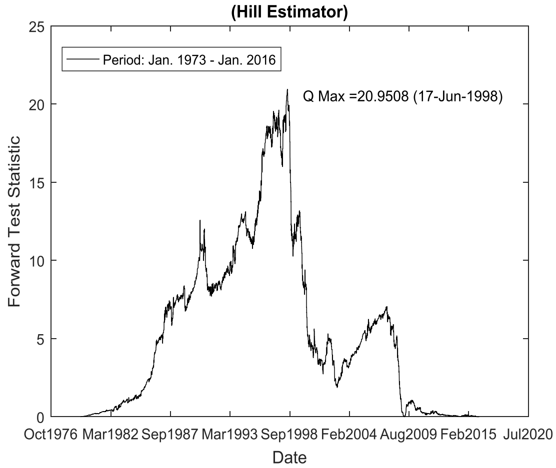
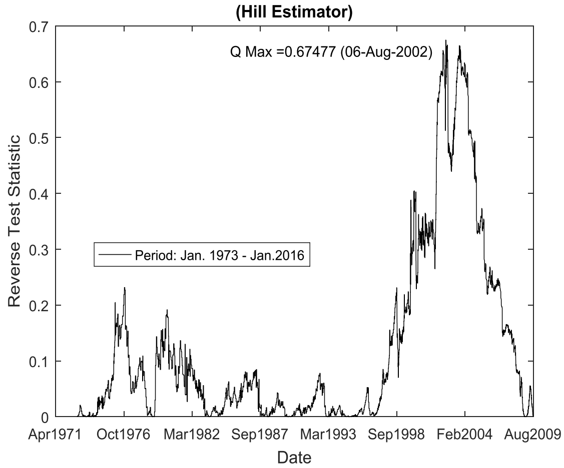


Figure 6: Structural Change Test in Right Tail Index for the Americas (Hill Estimator)



Upon splitting the original data into two quadrants, the test is run again from Jan. 1973 to June 1998 and then from June 1998 to Jan. 2016 in search of significant discontinuities in the tail index. This procedure continues until either the sample size is less than 500 or no breakpoints are found in the remaining intervals. See "Work Project Supplement" for a compilation of test statistic graphs for both tails using the Hill and rank-size estimators.

Table 3: Tail Index Evolution using Hill´s Estimator: Americas

Period (Right Tail)	Volatility (Right Tail)	Right Tail Index	VaR 99% (normal)	VaR 99% (Right-Tail, Hill)	Period (Left Tail)	Volatility (Left Tail)	Left Tail Index	VaR 99% (normal)	VaR 99% (Left-Tail, Hill)
Jan. 3, 1973 - Feb. 8, 1991	0.0075	2.4438 [2.3342, 2.5534]	0.0175	0.0203	Jan. 3, 1973 - Jun. 17, 1998	0.0078	2.3865 [2.2944, 2.4786]	0.0182	0.0224
Feb. 9, 1991 - Jan. 17, 1997	0.0074	2.5472 [2.3431, 2.7512]	0.0172	0.0221	Jun. 18, 1998 - Sep. 9, 2003	0.0171	2.7381 [2.5109, 2.9652]	0.0398	0.0470
Jan. 18, 1997 - Jun. 17, 1998	0.0117	3.4815 [2.9892, 3.9737]	0.0273	0.0266	Sep. 10, 2003 - Feb. 27, 2007	0.0065	2.9106 [2.6123, 3.2090]	0.0150	0.0176
Jun. 18, 1998 - Oct. 14, 1998	0.0236	2.0299 [1.1741, 2.8857]	0.0548	0.0782	Feb. 28, 2007 - Sep. 10, 2008	0.0192	3.1433 [2.6876, 3.5989]	0.0446	0.0490
Oct. 15, 1998 - Jul. 12, 2001	0.0175	3.5000 [3.1293, 3.8707]	0.0407	0.0459	Sep. 11, 2008 - Dec. 9, 2011	0.0285	1.8409 [1.6209, 2.0609]	0.0662	0.0988
Jul. 13, 2001 - Jun. 14, 2002	0.0126	4.4052 [3.7221, 5.0883]	0.0292	0.0271	Dec. 10, 2011 - Jul. 20, 2015	0.0125	2.6655 [1.9014, 3.4295]	0.0292	0.0348
Jun. 15, 2002 - Oct. 16, 2003	0.0165	2.5014 [2.0809, 2.9219]	0.0383	0.0488	Jul. 21, 2015 - Aug. 20, 2015	0.0082	2.4617 [2.1829, 2.7406]	0.0191	0.0253
Oct. 17, 2003 - May 13, 2005	0.0066	4.1274 [3.5288, 4.7260]	0.0154	0.0153	Aug. 21, 2015 - Jan. 22, 2016	0.0140	3.4373 [1.6837, 5.1910]	0.0326	0.0370
May 14, 2005 - Apr. 27, 2007	0.0067	2.5123 [2.1627, 2.8618]	0.0156	0.0201	-	-	-	-	-
Apr. 28, 2007 - May 25, 2009	0.0344	2.1213 [1.8148, 2.4278]	0.0801	0.1133	-	-	-	-	-
May 26, 2009 - Jan. 22, 2016	0.0129	2.4236 [2.2406, 2.6066]	0.0300	0.0367	-	-	-	-	-

Table 3 denotes that the right tail index practically guarantees the existence of a finite mean and variance in all periods except from June 1998 to Oct. 1998 and from April 2007 to May 2009, where the lower limits of the 95% confidence interval are less than 2. As for the left tail, there are more instances where the tail exponent drops below 2 as well. It is clear that the tail index does not remain constant over time for the Americas, varying from a point estimate as low as 1.8409 in the left tail all the way up to 4.4052 in the right tail. Globally, the lowest value of $\hat{\zeta}$ in both tails and hence the heaviest tail, occurs during the 2007-2008 financial crisis.

Table 4: Tail Index Evolution using Hill´s Estimator: Asia

Period (Right Tail)	Volatility (Right Tail)	Right Tail Index	VaR 99% (normal)	VaR 99% (Right-Tail, Hill)	Period (Left Tail)	Volatility (Left Tail)	Left Tail Index	VaR 99% (normal)	VaR 99% (Left-Tail, Hill)
Jan. 2, 1973 - Jun. 1, 1979	0.0078	1.7550 [1.6041, 1.9059]	0.0181	0.0247	Jan. 2, 1973 - Jan. 22, 2016	0.0124	2.1654 [2.0997, 2.2310]	0.0288	0.0356
Jun. 2, 1979 - Jan. 9, 1984	0.0072	2.4327 [2.2042, 2.6612]	0.0167	0.0224	-	-	-	-	-
Jan. 10, 1984 - Oct. 20, 1987	0.0154	1.8650 [1.6585, 2.0715]	0.0357	0.0560	-	-	-	-	-
Oct. 21, 1987 - Jan. 22, 2016	0.0131	2.2372 [2.1538, 2.3206]	0.0304	0.0394	-	-	-	-	-

Table 4 shows that the right tail index cannot be described continuously across the entire data set. Tail exponents ensure the existence of both the mean and variance from June 1979 to Jan. 1984 and from Oct. 1987 to Jan. 2016. A finite mean is only secured from Jan. 1973 to June 1979 and from Jan. 1984 to Oct. 1987. The left tail, per contra, reveals that tail thickness remains constant throughout the full sample, showing that the Asian bank index returns can be characterized by a finite mean and variance with a 95% confidence level.

Table 5: Tail Index Evolution using Hill´s Estimator: Australasia

Period (Right Tail)	Volatility (Right Tail)	Right Tail Index	VaR 99% (normal)	VaR 99% (Right-Tail, Hill)	Period (Left Tail)	Volatility (Left Tail)	Left Tail Index	VaR 99% (normal)	VaR 99% (Left-Tail, Hill)
Jan. 2, 1973 - Dec. 22, 1982	0.0135	2.0331 [1.8993, 2.1670	0.0314	0.0420	Jan. 2, 1973 - Mar. 25, 1988	0.0148	2.2816 [2.1668, 2.3964]	0.0344	0.0420
Dec. 23, 1982 - Aug. 27, 1987	0.0142	2.9459 [2.6954, 3.1964]	0.0331	0.0378	Mar. 26, 1988 - Mar. 27, 2007	0.0126	2.7025 [2.5865, 2.8185]	0.0292	0.0351
Aug. 28, 1987 - Sep. 14, 1988	0.0259	2.2535 [1.8168, 2.6901]	0.0603	0.0706	Mar. 28, 2007 - Nov. 29, 2011	0.0253	2.6128 [2.3866, 2.8390]	0.0588	0.0711
Sep. 15, 1988 - Jan. 24, 2008	0.0126	2.6885 [2.5741, 2.8029]	0.0292	0.0339	Nov. 30, 2011 - Jan. 22, 2016	0.0126	2.8774 [2.6156, 3.1392]	0.0294	0.0347
Jan. 25, 2008 - Jun. 2, 2010	0.0308	2.5099 [2.1957, 2.8241]	0.0716	0.0878	-	-	-	-	-
Jun. 3, 2010 - Jan. 22, 2016	0.0144	2.8786 [2.6576, 3.0995]	0.0335	0.0385	-	-	-	-	-

Table 5 further enforces the predominant tendency that the rate of decay $\hat{\zeta}$ is variable throughout 1973 to 2016 for both tails. All point estimates support the existence of the first and second distributional moments. During the period from Aug. 1987 to Sep. 1988 it is possible to see a lower limit on the right tail index of 1.8168, visibly indicative of a return distribution that only possesses a finite mean. On the other hand, there are a few ranges whose upper limit suggests the boundedness of the third moment, or skewness. Such is the case between June 2010 and Jan. 2016 on the right tail with an upper limit of 3.0995 and between Nov. 2011 to Jan. 2016 on the left tail with a limit of 3.1392.

Table 6: Tail Index Evolution using Hill´s Estimator: Europe

Period (Right Tail)	Volatility (Right Tail)	Right Tail Index	VaR 99% (normal)	VaR 99% (Right-Tail, Hill)	Period (Left Tail)	Volatility (Left Tail)	Left Tail Index	VaR 99% (normal)	VaR 99% (Left-Tail, Hill)
Jan. 2, 1973 - Jun. 16, 1998	0.0093	2.5938 [2.4963, 2.6913]	0.0216	0.0256	Jan. 2, 1973 - Jul. 30, 1974	0.0097	3.7438 [3.2625, 4.2250]	0.0227	0.0231
Jun. 17, 1998 - Mar. 24, 2003	0.0145	2.5471 [2.3255, 2.7688]	0.0336	0.0413	Jul. 31, 1974 - Jan. 23, 1979	0.0084	2.2193 [2.0037, 2.4348]	0.0196	0.0257
Mar. 25, 2003 - Dec. 25, 2007	0.0094	2.8751 [2.6356, 3.1147]	0.0220	0.0240	Jan. 24, 1979 - Jul. 8, 1981	0.0089	2.6549 [2.3281, 2.9818]	0.0206	0.0247
Dec. 26, 2007 - May 22, 2009	0.0356	2.0168 [1.6375, 2.3961]	0.0829	0.1219	Jul. 9, 1981 - Sep. 22, 1987	0.0098	3.4729 [3.2323, 3.7135]	0.0228	0.0240
May 23, 2009 - Jul. 1, 2010	0.0232	3.4225 [2.8688, 3.9762]	0.0539	0.0503	Sep. 23, 1987 - Aug. 13, 2008	0.0112	2.2475 [2.1503, 2.3448]	0.0261	0.0341
Jul. 2, 2010 - Jan. 8, 2013	0.0222	2.8051 [2.4691, 3.1410]	0.0516	0.0620	Aug. 14, 2008 - Jul. 1, 2009	0.0422	3.7140 [3.0396, 4.3884]	0.0981	0.1082
Jan. 9, 2013 - Aug. 6, 2014	0.0121	4.0461 [3.5426, 4.5497]	0.0281	0.0276	Jul. 2, 2009 - Jan. 18, 2011	0.0216	2.8113 [2.3826, 3.2401]	0.0501	0.0574
Aug. 7, 2014 - Jan. 22, 2016	0.0129	2.6681 [2.2361, 3.1002]	0.0299	0.0357	Jan. 19, 2011 - Aug. 3, 2011	0.0175	5.2153 [4.0100, 6.4205]	0.0407	0.0418
-	-	-	-	-	Aug. 4, 2011 - Sep. 25, 2012	0.0269	3.5142 [2.9478, 4.0805]	0.0626	0.0695
-	-	-	-	-	Sep. 26, 2012 - Jan. 22, 2016	0.0126	2.7219 [2.4364, 3.0073]	0.0293	0.0355

Table 6 confirms a vast number of structural breaks for Europe in both tails. There are periods where the upper limit of the tail index rises to a value of 6.4205, which safeguards the existence of the fourth distributional moment, or kurtosis.

Overall, even though Asia´s left tail showed no breakpoints, it can be said that on a worldwide scale, the tail index suffered multiple breaks and therefore the corresponding impact on the VaR metric is worth studying, at least in the realm of the Hill statistic given by (2.10).

3.3.2 – OLS Log-log Rank-size Regression

Table 7: Tail Index Evolution using a Rank-Size Regression: Americas

Period (Right Tail)	Volatility (Right Tail)	Right Tail Index	VaR 99% (normal)	VaR 99% (Right-Tail, Rank-Size)	Period (Left Tail)	Volatility (Left Tail)	Left Tail Index	VaR 99% (normal)	VaR 99% (Left-Tail, Rank-Size)
Jan. 3, 1973 - Aug. 7, 1998	0.0079	2.6109 [2.3822, 2.8397]	0.0183	0.0207	Jan. 3, 1973 - Mar. 13, 1997	0.0075	2.5915 [2.3580, 2.8250]	0.0175	0.0200
Aug. 8, 1998 - Jan. 22, 2016	0.0173	2.0616 [1.8430, 2.2803]	0.0403	0.0499	Mar. 14, 1997 - Jan. 22, 2016	0.0170	2.1074 [1.8924, 2.3224]	0.0396	0.0497

Table 7 shows that the first half is associated with lower tail risk, where the distribution during this time possesses both a finite mean and variance. The second half, however, is characterized by a lower tail exponent on both tails where the lower limit of the confidence interval drops below 2, thereby ensuring the existence of only the mean. Unlike table 3, table 7 shows a reduced number of breaks when shifting methods to a rank-size regression.

Table 8: Tail Index Evolution using a Rank-Size Regression: Asia

Period (Right Tail)	Volatility (Right Tail)	Right Tail Index	VaR 99% (normal)	VaR 99% (Right-Tail, Rank-Size)	Period (Left Tail)	Volatility (Left Tail)	Left Tail Index	VaR 99% (normal)	VaR 99% (Left-Tail, Rank-Size)
Jan. 2, 1973 - Jan. 22, 2016	0.0124	2.1491 [2.0039, 2.2942]	0.0288	0.0368	Jan. 2, 1973 - Jan. 22, 2016	0.0124	2.3058 [2.1501, 2.4616]	0.0288	0.0333

Table 8 reveals no significant discontinuities for either tail. This implies that a single distribution, with a finite mean and variance, is capable of modeling the Asian bank returns. The Hill and rank-size methods both concur that the left tail can indeed be described by a single tail exponent, however, there are many more breaks in the right tail for the Hill method.

Table 9: Tail Index Evolution using a Rank-Size Regression: Australasia

Period (Right Tail)	Volatility (Right Tail)	Right Tail Index	VaR 99% (normal)	VaR 99% (Right-Tail, Rank-Size)	Period (Left Tail)	Volatility (Left Tail)	Left Tail Index	VaR 99% (normal)	VaR 99% (Left-Tail, Rank-Size)
Jan. 2, 1973 - Jan. 22, 2016	0.0153	2.4947 [2.3262, 2.6632]	0.0355	0.0405	Jan. 2, 1973 - Jan. 22, 2016	0.0153	2.4065 [2.2439, 2.5690]	0.0355	0.0421

Table 9 demonstrates that banking returns for Australasia do not have significant breakpoints. Again, this equates to the possibility of using only one distribution with bounded values for both of its first two distributional moments to model each of the tails.

Table 10: Tail Index Evolution using a Rank-Size Regression: Europe

Period (Right Tail)	Volatility (Right Tail)	Right Tail Index	VaR 99% (normal)	VaR 99% (Right-Tail, Rank-Size)	Period (Left Tail)	Volatility (Left Tail)	Left Tail Index	VaR 99% (normal)	VaR 99% (Left-Tail, Rank-Size)
Jan. 3, 1973 - Jan. 16, 2008	0.0102	2.6835 [2.4826, 2.8844]	0.0237	0.0265	Jan. 3, 1973 - Dec. 13, 2007	0.0102	2.6541 [2.4552, 2.8531]	0.0237	0.0277
Jan. 17, 2008 - Apr. 8, 2009	0.0367	1.9541 [1.1641, 2.7442]	0.0854	0.1208	Dec. 14, 2007 - Jan. 22, 2016	0.0225	2.2156 [1.8706, 2.5605]	0.0523	0.0681
Apr. 9, 2009 - Sep. 13, 2012	0.0234	3.0191 [2.2962, 3.7420]	0.0544	0.0600	-	-	-	-	-
Sep. 14, 2012 - Jan. 22, 2016	0.0127	3.2040 [2.4281, 3.9799]	0.0295	0.0316	-	-	-	-	-

Table 10 confirms that Europe´s bank returns cannot be modeled by a single distribution for either tail. The right tail shows a clear distinction during the crisis of 2007-2008. The period from Jan. 2008 to April 2009 displays the global minimum rate of decay of 1.9541, with a lower limit of 1.1641. As is discussed in section 3.4.1, this has a significant impact on VaR. Overall, table 10 suggests that three statistical distributions are needed: one with a finite mean and variance, another with a finite mean and a third with a bounded mean, variance and skewness.

3.3.3 – Monte Carlo Simulation

The Monte Carlo simulation (Appendix B) was generated based on the Pareto distribution (see (3.1)). The results are empirical rejection frequencies wherein a significance level of 5% is used. The breakpoints were imposed manually and are represented by $\tau = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$. The null hypothesis states that the entire sample belongs to a Pareto distribution with no breaks and $\zeta = 5$. The sample size is 1000 with 1000 iterations. The data generating process consists in generating returns with pre-selected tail indices and then in obtaining the Hill statistic according to (2.10) and the rank-size statistic according to (2.13). The number of rejections for each statistic based on the 5% significance level is then noted together with the entire sample size to construct a rejection frequency table. The CDF of the Pareto distribution follows,

$$\mathrm{F}_X(x) = 1 - x^{-\zeta}, \qquad x \geq 1. \tag{3.1}$$

Comparing results from both the Hill and the rank-size estimators, it is promising to see that the rank-size has greater power overall in being able to detect structural breaks in the tail index. Consider for instance, when a break occurs at a third of the sample, $\tau = \frac{1}{3}$. If the first quadrant is modeled by a Pareto with $\zeta_1 = 5$ and the second with a $\zeta_2 = 4$, then the forward rank-size statistic will reject the null more often (25.20%) than the Hill estimator (15.60%). Globally, the newly proposed rank-size test statistic (2.13) shows superior performance.

The difference in the number of breakpoints seen in the previous two subsections between both estimators may be due to the fact that the Hill estimator has less power than the new rank-size estimator. This suggests that the true number of breaks could be the ones indicated by the rank-size method, although a more in-depth analysis is required. In both cases, it is still evident that Europe and the Americas cannot be described by a single tail index.

3.4 – Discussion

This fourth section aims to analyze the impact of the tail index variation on VaR. It will also contrast the VaR metric calculated according to a normal distribution (2.17) and one which accounts for leptokurtosis (2.20). Lastly, the limitations of the volatility as a measure of risk are considered relative to the tail index.

3.4.1 – The Impact on VaR and Volatility as a Standalone Measure of Risk

Table 11 shows the daily 99% VaR figures computed from point estimates of the tail indices obtained from table 2. It is clear that the full sample VaR computed according to the Gaussian curve mostly underestimates extreme financial risk.

Considering the Americas under a normal curve, an investor would lose, in a day, in the worst of cases 2.95% of his initial investment. Since the return distribution is heavy-tailed, the extreme loss actually climbs to 3.87%. Contextualizing with a $1M investment, the unforeseen loss can be estimated to nearly $10,000 in any given day. The use of the normal approximation is justified only in periods with higher tail exponents, like the one from Oct. 2003 to May 2005 in table 3, where the normal VaR appears slightly larger than the right tail VaR.

Table 11: Full Sample Value-at-Risk Measures

Region	VaR 99% (normal)	VaR 99% (Left-Tail, Hill)	VaR 99% (Right-Tail, Hill)	VaR 99% (Left-Tail, Rank-Size)	VaR 99% (Right-Tail, Rank-Size)
Americas	0.0295	0.0387	0.0369	0.0365	0.0355
Asia	0.0288	0.0356	0.0395	0.0333	0.0368
Australasia	0.0355	0.0446	0.0435	0.0421	0.0405
Europe	0.0311	0.0404	0.0377	0.0381	0.0364

If the tail index is presumed constant for an entire region as a result of ignoring structural break analysis, then there could be severe unaccounted losses originating from higher VaR values that are associated with turbulent periods. Table C.1 in Appendix C shows that within the Hill method, Europe´s full sample VaR is underestimated by 8.42% compared to its right tail VaR from the turbulent period of the 2007-2008 crisis. For the left tail, the underestimation would be 6.78%. The same occurs for Europe within the rank-size method in C.2, where there is a divergence of 8.44% between the full sample VaR from table 11 and the right tail VaR from table 10 belonging to the same period of the global crisis. Therefore, in the presence of tail breaks, a dissection of ζ with its corresponding subsample VaR estimates is essential.

The measure of financial risk in the form of volatility clearly increases during crises. It fails, however, in accurately informing about extreme losses of a given return distribution. It is not clear from results whether or not volatility can predict the direction of variation of the tail index. According to table 12, volatility seems to behave inversely with $\hat{\zeta}$, with a low correlation coefficient of around 30%. Europe´s left tail index, however, shows a positive correlation. In table 3 for example, a decrease in volatility from 0.0075 to 0.0074 is accompanied by an increase in the right tail VaR from 0.0203 to 0.0221, instead of increasing as one might think. This erratic behavior solidifies the conclusion that volatility is not a standalone measure of risk.

Table 12: Correlation[2] between Tail Index $\hat{\zeta}$ and Volatility

Region	Right Tail Index vs. Volatility (Hill)	Left Tail Index vs. Volatility (Hill)	Right Tail Index vs. Volatility (Rank-Size)	Left Tail Index vs. Volatility (Rank-Size)
Americas	-0.3877	-0.3270	-	-
Asia	-0.2411	-	-	-
Australasia	-0.2859	-0.1788	-	-
Europe	-0.3807	0.3401	-	-0.7478

3.4.2 – Crisis Mapping

The current section is devoted to historically making sense of the breakpoints that were found. The goal is to try and understand the intervals with elevated tail risk and to match that with a crisis that may have caused it.

[2] Unavailable fields are due to insufficient breakpoint data for a given tail type in a given region.

In the Americas, both the Hill and rank-size estimators show a low tail exponent starting from the second half of 1998. This in fact is attributed to the 1998 Russian financial crisis linked to the devaluation of the ruble (Werner and Upper, 2004). The right tail from table 3 also indicates a lower tail index that could have been caused by the 2002 stock market downturn, which was catalyzed by terror attacks that took place in September 2001. The financial crisis of 2007-2008 associated with the housing market bubble is seen again in the right tail of the Hill estimator. The rank-size estimate (table 7) does not detect this as a significant breakpoint.

In Asia, the most notable drop in the tail index takes place in the right tail (table 4) from Jan. 1973 to Jun. 1979. This could have been due to the 1973-1974 stock market crash that resulted from the fall of the Bretton Woods system, affecting stock markets around the world. It is, however, hard to state whether or not this was the true cause since the left tail of the same data and both tails from the rank-size estimator (table 8) do not detect any significant discontinuities. It is interesting to note the detection of the Black Monday event that took place in October 19, 1987 and impacted financial markets all around. The right tail in table 4 is accurately able to detect this structural break.

With regard to Australasia, the period with the highest risk profile occurs from Jan. 1973 to Dec. 1982 (table 5). As was the case with Asia, this may have been due to the 1973-1974 stock market crash since most parts of the globe were affected. The other high risk profile taking place from Aug. 1987 to Sep. 1988 (table 5) may again have been influenced by the Black Monday stock market plummet. It is interesting to note that the rank-size regression did not detect any significant breaks (table 9).

The lowest ζ in Europe is related to the 2007-2008 global crisis emanating from North America and spreading to many other parts of the world, as is confirmed through this data. This is seen by both the Hill and the rank-size estimators (see tables 6 and 10 respectively).

4 – Conclusion

The first conclusion that this paper draws concerns the heavy-tailedness of the regional bank indices. Both the Hill and rank-size estimators confirm that the majority of calculated tail exponents hovers around the range of Ibragimov and Kattuman (2013) for stock returns: $\zeta \in (2, 5)$. The pseudo-moments in table 1 also support the existence of leptokurtic tails.

Secondly, the Hill estimator showed several breaks throughout all regions. This indicates that the heaviness of tails does not stay constant throughout time. This means that a single distribution cannot be used to model the entire time span of 1973-2016. The rank-size estimator, on the other hand, shows significantly less breakpoints occurring only for the Americas and Europe. Based on the Monte Carlo exercise, the latter test statistic has more statistical power, which not only shows a promising future for the new test statistic (2.13) proposed by this paper but also suggests that only the Americas and Europe have statistically significant breaks in both of its tails.

Regarding the study on the impact of VaR, it can be seen that due to changes in the tail index, the VaR too suffers considerable changes. This would indeed translate to additional material losses if structural break analysis is overlooked. As the tails become thicker, the more inaccurate the parametric VaR estimate from the normal distribution becomes compared to the more realistic leptokurtic VaR. It was also seen that volatility alone is not a complete indicator of financial risk since it cannot quantify extreme risk. The correlation between ζ and volatility suggests that the latter cannot be used to predict the direction of movement of the former.

As a next step in this research, it would be worthwhile to delve deeper into the Monte Carlo simulation by considering other types of fat-tailed distributions like the Levy, the Burr, the Student´s t-distribution and the Cauchy with sample sizes larger than 1000 in order to better test the comparative power of the new test statistic (2.13) relative to the volatility-adjusted test statistic of the Hill estimator (2.10).

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Appendix A: Statistical Table

Critical values for recursive, rolling and sequential tests

Tests	0·50	0·60	0·70	0·80	0·90	0·95	0·975	0·99
$Q(r)$	0·67	0·79	0·94	1·14	1·46	1·78	2·11	2·54

Source: Quintos *et al*. (2001, pg.662)

Appendix B: Monte Carlo Simulation

	Hill Estimator Statistic		Log-log Rank-Size Statistic	
	Forward	Reverse	Forward	Reverse
$H_0: \zeta = 5$	0.0500	0.0500	0.0500	0.0500
τ = 1/2				
$\zeta_1 = 5; \zeta_2 = 4$	0.1860	0.0010	0.3060	0.0060
$\zeta_1 = 4; \zeta_2 = 5$	0.0100	0.1620	0.0200	0.2340
$\zeta_1 = 3; \zeta_2 = 5$	0.0040	0.6400	0.0120	0.7020
$\zeta_1 = 5; \zeta_2 = 3$	0.6640	0.0020	0.7720	0.0020
τ = 2/3				
$\zeta_1 = 5; \zeta_2 = 4$	0.1860	0.0000	0.3480	0.0040
$\zeta_1 = 4; \zeta_2 = 5$	0.0120	0.0940	0.0280	0.1240
$\zeta_1 = 3; \zeta_2 = 5$	0.0020	0.3600	0.0220	0.4340
$\zeta_1 = 5; \zeta_2 = 3$	0.7440	0.0000	0.8620	0.0000
τ = 1/3				
$\zeta_1 = 5; \zeta_2 = 4$	0.1560	0.0040	0.2520	0.0140
$\zeta_1 = 4; \zeta_2 = 5$	0.0000	0.1080	0.0120	0.1780
$\zeta_1 = 3; \zeta_2 = 5$	0.0000	0.6180	0.0040	0.7080
$\zeta_1 = 5; \zeta_2 = 3$	0.5100	0.0020	0.6380	0.0120

Appendix C: Full-sample VaR (VaR_{FS}) versus Sub-sample VaR (VaR_{SS})

Table C.1: VaRs using the Hill Estimator

Americas		Asia		Australasia		Europe	
LT	RT	LT	RT	LT	RT	LT	RT
1.63%	1.66%	0.00%	1.48%	0.26%	0.15%	1.73%	1.21%
-0.83%	1.48%	-	1.71%	0.95%	0.57%	1.47%	-0.36%
2.11%	1.03%	-	-1.65%	-2.65%	-2.71%	1.57%	1.37%
-1.03%	-4.13%	-	0.01%	0.99%	0.96%	1.64%	-8.42%
-6.01%	-0.90%	-	-	-	-4.43%	0.63%	-1.26%
0.39%	0.98%	-	-	-	0.50%	-6.78%	-2.43%
1.34%	-1.19%	-	-	-	-	-1.70%	1.01%
0.17%	2.16%	-	-	-	-	-0.14%	0.20%
-	1.68%	-	-	-	-	-2.91%	-
-	-7.64%	-	-	-	-	0.49%	-
-	0.02%	-	-	-	-	-	-

Table C.2: VaRs using the OLS Rank-size Regression

Americas		Asia		Australasia		Europe	
LT	RT	LT	RT	LT	RT	LT	RT
1.65%	1.48%	0.00%	0.00%	0.00%	0.00%	1.04%	0.99%
-1.32%	-1.44%	-	-	-	-	-3.00%	-8.44%
-	-	-	-	-	-	-	-2.36%
-	-	-	-	-	-	-	0.48%

Values in C.1 and C.2 calculated using $VaR_{FS} - VaR_{SS}$. VaR_{FS} is from table 11, VaR_{SS} is from tables 3-10. LT=Left Tail, RT=Right Tail.