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Journal of Approximation Theory 314 (2026) 106223

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full Length Article

On multiplier analogues of the algebra $C + H^{\infty}$ on weighted rearrangement-invariant sequence spaces

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Received 20 July 2024; received in revised form 27 December 2024; accepted 13 August 2025
Available online 19 August 2025

Communicated by Natan Kruglyak

To Professor Yuri Brudnyi on the occasion of his 90th birthday

Abstract

Let $X(\mathbb{Z})$ be a reflexive rearrangement-invariant Banach sequence space with nontrivial Boyd indices α_X, β_X and let w be a symmetric weight in the intersection of the Muckenhoupt classes $A_{1/\alpha_X}(\mathbb{Z})$ and $A_{1/\beta_X}(\mathbb{Z})$. Let $M_{X(\mathbb{Z},w)}$ denote the collection of all periodic distributions a generating bounded Laurent operators $L(a)$ on the space $X(\mathbb{Z}, w) = \{\varphi : \mathbb{Z} \to \mathbb{C} : \varphi w \in X(\mathbb{Z})\}$. We show that $M_{X(\mathbb{Z},w)}$ is a Banach algebra. Further, we consider the closure of trigonometric polynomials in $M_{X(\mathbb{Z},w)}$ denoted by $C_{X(\mathbb{Z},w)}$ and $H^{\infty,\pm}_{X(\mathbb{Z},w)} = \{a \in M_{X(\mathbb{Z},w)} : \widehat{a}(\pm n) = 0 \text{ for } n < 0\}$. We prove that $C_{X(\mathbb{Z},w)} + H^{\infty,\pm}_{X(\mathbb{Z},w)}$ are closed subalgebras of $M_{X(\mathbb{Z},w)}$.

MSC: primary 47B35; secondary 46E30

Keywords: Laurent operator; Rearrangement-invariant Banach sequence space; Boyd indices; Muckenhoupt weights

1. Introduction

Let C and L^{∞} be the Banach algebras of continuous and essentially bounded functions on the unit circle, respectively, and

$$H^{\infty} := \left\{ f \in L^{\infty} \ : \ \widehat{f}(n) = 0 \text{ for } n < 0 \right\}.$$

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https://doi.org/10.1016/j.jat.2025.106223

In 1967, Sarason observed that $C + H^\infty$ is a closed subalgebra of L^∞ (see [26, p. 191] and also [27, p. 290]). An elegant alternative proof of this fact was given by Zalcman [30, Theorem 6.1]. A few years later, Rudin [25, Theorem 1.2] presented the following abstract version of Zalcman's argument (see also [20, Ch. VII, Section 3, p. 149] and [8, Lemma 2.52]).

In the theorem below and in what follows, for a Banach space $\mathcal{X}$, let $\mathcal{B}(\mathcal{X})$ denote the Banach algebra of all bounded linear operators on $\mathcal{X}$ equipped with the operator norm.

Theorem 1.1. *Suppose $\mathcal{Y}$ and $\mathcal{Z}$ are closed subspaces of a Banach space $\mathcal{X}$, and suppose there is a collection $\Phi \subset \mathcal{B}(\mathcal{X})$ with the following properties:*

(i) Every $\Lambda \in \Phi$ maps $\mathcal{X}$ into $\mathcal{Y}$.
(ii) Every $\Lambda \in \Phi$ maps $\mathcal{Z}$ into $\mathcal{Z}$.
(iii) $\sup\limits_{\Lambda\in\Phi} \|\Lambda\|_{\mathcal{B}(\mathcal{X})} < \infty$.
(iv) To every $y \in \mathcal{Y}$ and to every $\varepsilon > 0$ corresponds a $\Lambda \in \Phi$ such that $\|y - \Lambda y\|_{\mathcal{X}} < \varepsilon$.

Then $\mathcal{Y} + \mathcal{Z}$ is closed in $\mathcal{X}$.

Let $1 < p < \infty$ and M_p be the Banach algebra of symbols a generating bounded Laurent operators $L(a)$ on $\ell^p(\mathbb{Z})$, let C_p be the closure of trigonometric polynomials in M_p and $H_p^\infty = H^\infty \cap M_p$. In 1987, Böttcher and Silbermann proved that $C_p + H_p^\infty$ is a closed subalgebra of M_p (see [6, Theorem 1] and also [8, Theorem 2.53]). Their proof is based on Theorem 1.1 with $\mathcal{X} = M_p$, $\mathcal{Y} = C_p$, $\mathcal{Z} = H_p^\infty$, and Φ being the collection of all convolution operators with the Fejér kernels.

The aim of this paper is to extend the result by Böttcher and Silbermann mentioned above to the setting of reflexive rearrangement-invariant Banach sequence spaces (see [2, Ch. 2] and Section 3.1) with nontrivial Boyd indices α_X, β_X (see [9] and Section 3.2) and symmetric weights belonging to the intersection of the Muckenhoupt classes $A_{1/\alpha_X}(\mathbb{Z})$ and $A_{1/\beta_X}(\mathbb{Z})$. Note that the class of rearrangement-invariant Banach sequence spaces is very wide. The most interesting examples are Lebesgue sequence spaces $\ell^p(\mathbb{Z})$ with $1 \le p \le \infty$, Lorentz sequence spaces $\ell^{p,q}(\mathbb{Z})$ with $1 < p < \infty$ and $1 \le q \le \infty$, Orlicz sequence spaces $\ell^\Phi(\mathbb{Z})$ with suitable convex functions $\Phi : [0,\infty) \to [0,\infty]$. Boyd indices α_X and β_X satisfy $0 \le \alpha_X \le \beta_X \le 1$ and, roughly speaking, play a role of $1/p$ for the spaces $\ell^p(\mathbb{Z})$. More precisely, $\alpha_{\ell^p} = \beta_{\ell^p} = 1/p$ for all $1 \le p \le \infty$, however there are reflexive Orlicz spaces $\ell^\Phi(\mathbb{Z})$ such that $0 < \alpha_{\ell^\Phi} < \beta_{\ell^\Phi} < 1$ (see [22, Corollary 11.12, Theorem 11.5] and [10]). It is customary to say that Boyd indices of a rearrangement-invariant space $X(\mathbb{Z})$ are nontrivial if

$$0 < \alpha_X, \quad \beta_X < 1.$$

Without going into technical details, we mention that our main result (Theorem 1.3) seems to be new already in the setting of weighted Lebesgue sequence spaces $\ell^p(\mathbb{Z}, w)$ with $1 < p < \infty$ and symmetric Muckenhoupt weights $w \in A_p(\mathbb{Z})$.

Now we give some important definitions. Let $X(\mathbb{Z})$ be a Banach sequence space (not necessarily rearrangement-invariant) (see [2, Ch. 1] and Section 2.1). Let $\mathcal{P}'$ be the space of periodic distributions (see, e.g., [1, Ch. 3 and 5] and Section 5.2) and let $S_0(\mathbb{Z})$ denote the set of all finitely supported sequences. For $a \in \mathcal{P}'$ and $\varphi \in S_0(\mathbb{Z})$, we define the convolution $a * \varphi$ as the sequence

$$(a * \varphi)_j = \sum_{k\in\mathbb{Z}} \widehat{a}(j-k)\varphi_k, \quad j \in \mathbb{Z},$$

where $\{\widehat{a}(j)\}_{j\in\mathbb{Z}}$ is the sequence of Fourier coefficients of the distribution a. By $M_{X(\mathbb{Z})}$ we denote the collection of all distributions $a \in \mathcal{P}'$ for which $a * \varphi \in X(\mathbb{Z})$ whenever $\varphi \in S_0(\mathbb{Z})$ and

$$\|a\|_{M_{X(\mathbb{Z})}} := \sup\left\{ \frac{\|a * \varphi\|_{X(\mathbb{Z})}}{\|\varphi\|_{X(\mathbb{Z})}} \ : \ \varphi \in S_0(\mathbb{Z}),\ \varphi \neq 0 \right\} < \infty. \tag{1.1}$$

If the space $X(\mathbb{Z})$ is separable, then $S_0(\mathbb{Z})$ is dense in $X(\mathbb{Z})$ (see Lemma 2.1(a) below). In this case, for $a \in M_{X(\mathbb{Z})}$, the operator from $S_0(\mathbb{Z})$ to $X(\mathbb{Z})$ defined by $\varphi \mapsto a * \varphi$ extends to a bounded operator

$$L(a) : X(\mathbb{Z}) \to X(\mathbb{Z}), \quad \varphi \mapsto a * \varphi,$$

which is referred to as the Laurent operator with symbol a.

Let $\mathbb{Z}_+ := \{0, 1, 2, \dots\} = \{0\} \cup \mathbb{N}$. We will equip $\mathbb{S} \in \{\mathbb{N}, \mathbb{Z}_+, \mathbb{Z}\}$ with the counting measure m, that is, m is a purely atomic measure and $m(\{k\}) = 1$ for every $k \in \mathbb{S}$.

A weight on $\mathbb{S}$ is a sequence $w = \{w_k\}_{k\in\mathbb{S}}$ of positive numbers. Let $l, n \in \mathbb{S}$ satisfy $l \leq n$. We call the set of the form $J = \{l, \dots, n\}$ an interval of $\mathbb{S}$. Let $1 < p < \infty$ and $1/p + 1/q = 1$. A weight $w = \{w_k\}_{k\in\mathbb{S}}$ is said to belong to the Muckenhoupt class $A_p(\mathbb{S})$ if

$$[w]_{A_p(\mathbb{S})} := \sup_{J\subset\mathbb{S}} \frac{1}{m(J)} \left(\sum_{k\in J} w_k^p \right)^{1/p} \left(\sum_{k\in J} w_k^{-q} \right)^{1/q} < \infty,$$

where the supremum is taken over all intervals $J \subset \mathbb{S}$.

A weight $w = \{w_k\}_{k\in\mathbb{Z}}$ is said to be symmetric if $w_{-k} = w_k$ for all $k \in \mathbb{Z}$. The collection of all symmetric weights in the Muckenhoupt class $A_p(\mathbb{Z})$ will be denoted by $A_p^{\mathrm{sym}}(\mathbb{Z})$.

For a Banach sequence space $X(\mathbb{Z})$ and a weight $w = \{w_k\}_{k\in\mathbb{Z}}$, the weighted Banach sequence space $X(\mathbb{Z}, w)$ consists of all sequences $f = \{f_k\}_{k\in\mathbb{Z}}$ such that $fw := \{f_k w_k\}_{k\in\mathbb{Z}}$ belongs to $X(\mathbb{Z})$. It is easy to see that $X(\mathbb{Z}, w)$ is itself a Banach sequence space with respect to the norm

$$\|f\|_{X(\mathbb{Z},w)} := \|fw\|_{X(\mathbb{Z})}.$$

For $1 \leq p \leq \infty$, let $L^p(-\pi, \pi)$ denote the space of all 2π-periodic measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\|f\|_{L^p(-\pi,\pi)} := \begin{cases} \left(\dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} |f(\theta)|^p \, d\theta \right)^{1/p}, & 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} |f(\theta)|, & p = \infty, \end{cases}$$

is finite. A 2π-periodic function $a : \mathbb{R} \to \mathbb{C}$ is said to be of bounded variation if

$$V(a) := \sup \sum_{j=1}^{n} |a(\theta_{j+1}) - a(\theta_j)| < \infty,$$

where the supremum is taken over all partitions $-\pi \leq \theta_1 < \theta_2 < \dots < \theta_{n+1} = \pi$. It is well known that the collection of all functions of bounded variation $BV[-\pi, \pi]$ is a Banach algebra with respect to the norm

$$\|a\|_{BV[-\pi,\pi]} := \|a\|_{L^\infty(-\pi,\pi)} + V(a).$$

Our first result is a generalization of properties of $M_{X(\mathbb{Z},w)}$ listed for $\ell^p(\mathbb{Z})$ and $w = 1$ in [8, Section 2.5(d), (f), (g)].

Theorem 1.2. *Let $X(\mathbb{Z})$ be a reflexive rearrangement-invariant Banach sequence space with nontrivial Boyd indices α_X, β_X and let $w \in A^{\rm sym}_{1/\alpha_X}(\mathbb{Z}) \cap A^{\rm sym}_{1/\beta_X}(\mathbb{Z})$.*

(a) If $a \in M_{X(\mathbb{Z},w)}$, then $a \in L^\infty(-\pi,\pi)$ and

$$\|a\|_{L^\infty(-\pi,\pi)} \le \|a\|_{M_{X(\mathbb{Z},w)}}.$$

(b) $M_{X(\mathbb{Z},w)}$ is a Banach algebra under pointwise multiplication and the norm

$$\|a\|_{M_{X(\mathbb{Z},w)}} = \|L(a)\|_{\mathcal{B}(X(\mathbb{Z},w))}.$$

(c) There is a constant $C_{X(\mathbb{Z},w)} \in (0,\infty)$ depending only on $X(\mathbb{Z})$ and w such that

$$\|a\|_{M_{X(\mathbb{Z},w)}} \le C_{X(\mathbb{Z},w)} \left(\|a\|_{L^\infty(-\pi,\pi)} + V(a)\right)$$

for all $a \in BV[-\pi,\pi]$. In particular, $BV[-\pi,\pi] \hookrightarrow M_{X(\mathbb{Z},w)}$.

It would be interesting to clarify whether the above result remains true if one drops the assumption that the weight w is symmetric.

Let $\mathcal{TP}$ denote the set of all trigonometric polynomials, that is, functions of the form

$$p(\theta) = \sum_{|k|\le n} c_k e^{ik\theta}, \quad \theta \in \mathbb{R},$$

where $n \in \mathbb{Z}_+$ and $c_k \in \mathbb{C}$ for all $k \in \{-n,\dots,n\}$. Since each function in $\mathcal{TP}$ is of bounded variation, it follows from Theorem 1.2(c) that $\mathcal{TP} \subset M_{X(\mathbb{Z},w)}$. We denote by $C_{X(\mathbb{Z},w)}$ the closure of $\mathcal{TP}$ with respect to the norm of the Banach algebra $M_{X(\mathbb{Z},w)}$ and put

$$H^{\infty,\pm}_{X(\mathbb{Z},w)} := \left\{a \in M_{X(\mathbb{Z},w)} : \widehat{a}(\pm n) = 0 \text{ for } n < 0\right\}.$$

The following extension of [6, Theorem 1] and [8, Theorem 2.53] is the main result of this paper.

Theorem 1.3. *Let $X(\mathbb{Z})$ be a reflexive rearrangement-invariant Banach sequence space with nontrivial Boyd indices α_X, β_X and let $w \in A^{\rm sym}_{1/\alpha_X}(\mathbb{Z}) \cap A^{\rm sym}_{1/\beta_X}(\mathbb{Z})$. Then the sets $C_{X(\mathbb{Z},w)} + H^{\infty,\pm}_{X(\mathbb{Z},w)}$ are closed subalgebras of the Banach algebra $M_{X(\mathbb{Z},w)}$.*

The paper is organized as follows. The first two sections cover the preliminaries. In Section 2, we collect properties of Banach sequence spaces culminating with the fact that if an operator is bounded on a Banach sequence space $X(\mathbb{Z})$ and on its associate space $X'(\mathbb{Z})$, then it is bounded on the space $\ell^2(\mathbb{Z})$. Section 3 is devoted to rearrangement-invariant Banach sequence spaces, their Boyd indices, and the Boyd interpolation theorem [9].

In Section 4, we prove that if $w \in A^{\rm sym}_{p_0}(\mathbb{Z})$, then $w^{1+\varepsilon} \in A^{\rm sym}_p(\mathbb{Z})$ for all ε sufficiently close to zero and all p sufficiently close to p_0. This result extends the previous stability result for discrete Muckenhoupt weights by Böttcher and Seybold [4, Theorem 1.1], which did not allow to "shake" p_0. We follow closely the proof given by Böttcher and Karlovich [3, Theorem 2.31] for continuous Muckenhoupt weights.

Section 5 is dedicated to the study of properties of $M_{X(\mathbb{Z})}$. We show that if $X(\mathbb{Z})$ is a reflection-invariant Banach sequence space, then $M_{X(\mathbb{Z})} = M_{X'(\mathbb{Z})}$ with equal norms. If, in addition, $X(\mathbb{Z})$ is reflexive, then the interpolation theorem from Section 2 allows us to prove that $M_{X(\mathbb{Z})} \hookrightarrow L^\infty(-\pi,\pi)$ and that $M_{X(\mathbb{Z})}$ is a Banach algebra. We conclude this section with the proof of the uniform boundedness of Fejér means of $a \in M_{X(\mathbb{Z})}$ if $X(\mathbb{Z})$ is a reflexive reflection-invariant Banach sequence space.

We start Section 6 with the observation that if $X(\mathbb{Z})$ is a reflexive rearrangement-invariant Banach sequence space and a weight w is symmetric, then $X(\mathbb{Z},w)$ is a reflexive reflection-invariant Banach sequence space. So, Theorem 1.2(a)–(b) is a consequence of results of Section 5. Further, combining the stability result for Muckenhoupt weights of Section 4, the Boyd interpolation theorem (Theorem 3.1), and Stechkin's inequality for $\ell^p(\mathbb{Z},w)$ with $w \in A_p(\mathbb{Z})$ proved by Böttcher and Seybold [4, Theorem 1.2], we prove Theorem 1.2(c).

The proof of Theorem 1.3 essentially consists of verification of properties (i)–(iv) of Theorem 1.1 for $\mathcal{X} = M_{X(\mathbb{Z},w)}$, $\mathcal{Y} = C_{X(\mathbb{Z},w)}$, $\mathcal{Z} := H^{\infty,\pm}_{X(\mathbb{Z},w)}$ and Φ being the collection of all convolution operators with the Fejér kernels $a \mapsto \sigma_n(a)$. The most difficult is the verification of property (iv), where we have to prove that $\sigma_n(a) \to a$ as $n \to \infty$ in the norm of $M_{X(\mathbb{Z},w)}$ for every $a \in C_{X(\mathbb{Z},w)}$. It follows from the definition of $C_{X(\mathbb{Z},w)}$ and the last result of Section 5 that it is enough to consider $a \in \mathcal{TP} \subset C[-\pi,\pi] \cap BV[-\pi,\pi]$. On the other hand, the combination of the Boyd and Stein–Weiss interpolation theorems with the Stechkin inequality allows us to reduce the study of convergence $\sigma_n(a) \to a$ in the norm of $M_{X(\mathbb{Z},w)}$ for $a \in C[-\pi,\pi] \cap BV[-\pi,\pi]$ to the study of convergence $\sigma_n(a) \to a$ in the norm of $M_{\ell^2(\mathbb{Z})} = L^\infty(-\pi,\pi)$, which is a classical result. Once again, our ability to perform this interpolation argument is based on the stability result for symmetric Muckenhoupt weights obtained in Section 4. This will complete the proof of Theorem 1.3.

2. Preliminaries on Banach sequence spaces

2.1. Banach sequence spaces

Let $\mathbb{S} \in \{\mathbb{N},\mathbb{Z}\}$, let $\ell^0(\mathbb{S})$ be the linear space of all sequences $f : \mathbb{S} \to \mathbb{C}$, and let $\ell^0_+(\mathbb{S})$ be the cone of nonnegative sequences in $\ell^0(\mathbb{S})$. According to [2, Ch. 1, Definition 1.1], a Banach function norm $\varrho : \ell^0_+(\mathbb{S}) \to [0,\infty]$ is a mapping which satisfies the following axioms for all $f, g \in \ell^0_+(\mathbb{S})$, for all sequences $\{f^{(n)}\}_{n\in\mathbb{N}}$ in $\ell^0_+(\mathbb{S})$, for all finite subsets $E \subset \mathbb{S}$, and all constants $\alpha \geq 0$:

(A1) $\varrho(f) = 0 \Leftrightarrow f = 0,\ \varrho(\alpha f) = \alpha\varrho(f),\ \varrho(f+g) \leq \varrho(f)+\varrho(g)$,

(A2) $0 \leq g \leq f \quad \Rightarrow \quad \varrho(g) \leq \varrho(f)$ (the lattice property),

(A3) $0 \leq f^{(n)} \uparrow f \quad \Rightarrow \quad \varrho(f^{(n)}) \uparrow \varrho(f)$ (the Fatou property),

(A4) $\varrho(\chi_E) < \infty$,

(A5) $\displaystyle\sum_{k\in E} f_k \leq C_E\varrho(f)$,

where χ_E is the characteristic (indicator) function of E, and the constant $C_E \in (0,\infty)$ may depend on ϱ and E, but is independent of f. The set $X(\mathbb{S})$ of all sequences $f \in \ell^0(\mathbb{S})$ for which $\varrho(|f|) < \infty$ is called a Banach sequence space. For each $f \in X(\mathbb{S})$, the norm of f is defined by

$$\|f\|_{X(\mathbb{S})} := \varrho(|f|).$$

The set $X(\mathbb{S})$ equipped with the natural linear space operations and this norm becomes a Banach space (see [2, Ch. 1, Theorems 1.4 and 1.6]). If ϱ is a Banach function norm, its associate norm ϱ' is defined on $\ell^0_+(\mathbb{S})$ by

$$\varrho'(g) := \sup\left\{\sum_{k\in\mathbb{S}} f_k g_k \ :\ f = \{f_k\}_{k\in\mathbb{S}} \in \ell^0_+(\mathbb{S}),\ \varrho(f) \leq 1\right\}, \quad g \in \ell^0_+(\mathbb{S}).$$

It is a Banach function norm itself [2, Ch. 1, Theorem 2.2]. The Banach sequence space $X'(\mathbb{S})$ determined by the Banach function norm ϱ' is called the associate space (Köthe dual) of $X(\mathbb{S})$. The associate space $X'(\mathbb{S})$ can be viewed as a subspace of the Banach dual space $X^*(\mathbb{S})$.

Let us conclude these subsections with two results highlighting the idea that in many cases arbitrary sequences can be approximated by finitely supported sequences.

Lemma 2.1. *Let $X(\mathbb{Z})$ be a Banach sequence space and $X'(\mathbb{Z})$ be its associate space.*

(a) If $X(\mathbb{Z})$ is separable, then the set $S_0(\mathbb{Z})$ is dense in the space $X(\mathbb{Z})$.
(b) If $X(\mathbb{Z})$ is reflexive, then the set $S_0(\mathbb{Z})$ is dense in the space $X(\mathbb{Z})$ and in the space $X'(\mathbb{Z})$.

Part (a) follows from [2, Ch. 1, Proposition 3.10, Theorem 3.11, and Corollary 5.6]. Part (b) follows from part (a) and [2, Ch. 1, Corollary 4.4].

Lemma 2.2 (*See [17, Lemma 2.1] and also [16, Lemma 2.10]*)**.** *Let $X(\mathbb{Z})$ be a Banach sequence space and $X'(\mathbb{Z})$ be its associate space. For every sequence $f = \{f_k\}_{k\in\mathbb{Z}} \in X(\mathbb{Z})$,*

$$\|f\|_{X(\mathbb{Z})} = \sup\left\{\left|\sum_{k\in\mathbb{Z}} f_k s_k\right| \;:\; s = \{s_k\}_{k\in\mathbb{Z}} \in S_0(\mathbb{Z}),\ \|s\|_{X'(\mathbb{Z})} \le 1\right\}.$$

2.2. Minkowski integral inequality for Banach sequence spaces

We will need the following Minkowski integral inequality for Banach sequence spaces, which is a special case of [28, Proposition 2.1].

Lemma 2.3. *Let $\mathbb{Z}$ be equipped with the counting measure m and $[-\pi, \pi]$ be equipped with a σ-finite probability measure μ. Suppose $X(\mathbb{Z})$ is a Banach sequence space. If f is $m \times \mu$-measurable, then*

$$\left\|\left\{\int_{-\pi}^{\pi} |f(j,x)|\, d\mu(x)\right\}_{j\in\mathbb{Z}}\right\|_{X(\mathbb{Z})} \le \int_{-\pi}^{\pi} \left\|\{f(j,x)\}_{j\in\mathbb{Z}}\right\|_{X(\mathbb{Z})} d\mu(x).$$

2.3. Reflection-invariant Banach sequence spaces

By analogy with [16, Subsection 7.2], we say that a Banach sequence space $X(\mathbb{Z})$ is reflection-invariant if $\|\varphi\|_{X(\mathbb{Z})} = \|\widetilde{\varphi}\|_{X(\mathbb{Z})}$ for every $\varphi \in X(\mathbb{Z})$, where $\widetilde{\varphi}$ denotes the reflection of a sequence $\varphi = \{\varphi_k\}_{k\in\mathbb{Z}}$ defined by

$$\widetilde{\varphi}_k := \varphi_{-k}, \quad k \in \mathbb{Z}.$$

Lemma 2.4. *A Banach sequence space $X(\mathbb{Z})$ is reflection-invariant if and only if its associate space $X'(\mathbb{Z})$ is reflection-invariant.*

Proof. Fix $\psi = \{\psi_k\}_{k\in\mathbb{Z}} \in X'(\mathbb{Z})$. If $X(\mathbb{Z})$ is reflection-invariant, then it follows from [2, Ch. 1, Lemma 2.8] that

$$\|\widetilde{\psi}\|_{X'(\mathbb{Z})} = \sup\left\{\left|\sum_{k\in\mathbb{Z}} \widetilde{\psi}_k \varphi_k\right| \;:\; \varphi = \{\varphi_k\}_{k\in\mathbb{Z}} \in X(\mathbb{Z}),\ \|\varphi\|_{X(\mathbb{Z})} \le 1\right\}$$

$$= \sup\left\{\left|\sum_{k\in\mathbb{Z}} \psi_k\widetilde{\varphi}_k\right| \ :\ \varphi = \{\varphi_k\}_{k\in\mathbb{Z}} \in X(\mathbb{Z}),\ \|\varphi\|_{X(\mathbb{Z})} \le 1\right\}$$

$$= \sup\left\{\left|\sum_{k\in\mathbb{Z}} \psi_k\varphi_k\right| \ :\ \varphi = \{\varphi_k\}_{k\in\mathbb{Z}} \in X(\mathbb{Z}),\ \|\varphi\|_{X(\mathbb{Z})} \le 1\right\} = \|\psi\|_{X'(\mathbb{Z})}$$

because $\|\widetilde{\varphi}\|_{X(\mathbb{Z})} = \|\varphi\|_{X(\mathbb{Z})}$ for all $\varphi = \{\varphi_k\}_{k\in\mathbb{Z}} \in X(\mathbb{Z})$. Hence $X'(\mathbb{Z})$ is reflection-invariant.

If $X'(\mathbb{Z})$ is reflection-invariant, then, by what has already been proved, $X''(\mathbb{Z})$ is reflection-invariant. It remains to recall that, by the Lorentz–Luxemburg theorem (see [2, Ch. 1, Theorem 2.7]), the space $X''(\mathbb{Z})$ coincides with the space $X(\mathbb{Z})$ and $\|\varphi\|_{X(\mathbb{Z})} = \|\varphi\|_{X''(\mathbb{Z})}$ for all $\varphi = \{\varphi_k\}_{k\in\mathbb{Z}} \in X(\mathbb{Z})$. Thus, $X(\mathbb{Z})$ is reflection-invariant. □

2.4. Interpolation of operators between a Banach sequence space and its associate space

Let $X_0(\mathbb{Z})$ and $X_1(\mathbb{Z})$ be Banach sequence spaces. Suppose $0 < \theta < 1$. The Calderón product $(X_0^{1-\theta}X_1^{\theta})(\mathbb{Z})$ of $X_0(\mathbb{Z})$ and $X_1(\mathbb{Z})$ consists of all sequences $x = \{x_k\}_{k\in\mathbb{Z}} \in \ell^0(\mathbb{Z})$ such that the inequality

$$|x_k| \le \lambda|y_k|^{1-\theta}|z_k|^{\theta}, \quad k \in \mathbb{Z}, \tag{2.1}$$

holds for some $\lambda > 0$ and elements $y = \{y_k\}_{k\in\mathbb{Z}} \in X_0(\mathbb{Z})$ and $z = \{z_k\}_{k\in\mathbb{Z}} \in X_1(\mathbb{Z})$ with $\|y\|_{X_0(\mathbb{Z})} \le 1$ and $\|z\|_{X_1(\mathbb{Z})} \le 1$ (see [11]). The norm of x in $(X_0^{1-\theta}X_1^{\theta})(\mathbb{Z})$ is defined to be the infimum of all values of λ in (2.1).

Theorem 2.5. *Let $X(\mathbb{Z})$ be a Banach sequence space and $X'(\mathbb{Z})$ be its associate space. If $A : \ell^0(\mathbb{Z}) \to \ell^0(\mathbb{Z})$ is a linear operator bounded on $X(\mathbb{Z})$ and on $X'(\mathbb{Z})$, then A is bounded on $\ell^2(\mathbb{Z})$ and*

$$\|A\|_{\mathcal{B}(\ell^2(\mathbb{Z}))} \le \|A\|^{1/2}_{\mathcal{B}(X(\mathbb{Z}))}\|A\|^{1/2}_{\mathcal{B}(X'(\mathbb{Z}))}.$$

Proof. It follows from the Lozanovskii interpolation formula that

$$(X^{1/2}(X')^{1/2})(\mathbb{Z}) = \ell^2(\mathbb{Z}) \tag{2.2}$$

with equality of the norms (see, e.g., [22, Ch. 15, Example 6, p. 185]). The desired result is a consequence of (2.2) and an abstract version of the Riesz–Thorin interpolation theorem given in [23, Theorem 3.11]. □

3. Rearrangement-invariant Banach sequence spaces and their Boyd indices

3.1. Rearrangement-invariant Banach sequence spaces

Let $\mathbb{S} \in \{\mathbb{N}, \mathbb{Z}\}$. The distribution function of a sequence $f = \{f_k\}_{k\in\mathbb{S}} \in \ell^0(\mathbb{S})$ is defined by

$$d_f(\lambda) := m\{k \in \mathbb{S} : |f_k| > \lambda\}, \quad \lambda \ge 0,$$

where $m(S)$ denotes the measure (cardinality) of a set $S \subset \mathbb{S}$. For any sequence $f \in \ell^0(\mathbb{S})$, its decreasing rearrangement is given by

$$f^*(n) := \inf\{\lambda \ge 0 : d_f(\lambda) \le n - 1\}, \quad n \in \mathbb{N}.$$

One says that sequences $f = \{f_k\}_{k\in\mathbb{S}_1} \in \ell^0(\mathbb{S}_1)$, $g = \{g_k\}_{k\in\mathbb{S}_2} \in \ell^0(\mathbb{S}_2)$ with $\mathbb{S}_1, \mathbb{S}_2 \in \{\mathbb{N}, \mathbb{Z}\}$ are equimeasurable if $d_f = d_g$.

A Banach function norm $\varrho : \ell_+^0(\mathbb{S}) \to [0,\infty]$ is said to be rearrangement-invariant if $\varrho(f) = \varrho(g)$ for every pair of equimeasurable sequences $f = \{f_k\}_{k\in\mathbb{S}}$, $g = \{g_k\}_{k\in\mathbb{S}} \in \ell_+^0(\mathbb{S})$. In that case, the Banach sequence space $X(\mathbb{S})$ generated by ϱ is said to be a rearrangement-invariant Banach sequence space (cf. [2, Ch. 2, Definition 4.1]). It follows from [2, Ch. 2, Proposition 4.2] that if a Banach sequence space $X(\mathbb{S})$ is rearrangement-invariant, then its associate space $X'(\mathbb{S})$ is also a rearrangement-invariant Banach sequence space.

3.2. Boyd indices

Let $X(\mathbb{Z})$ be a rearrangement-invariant Banach sequence space generated by a rearrangement-invariant Banach function norm $\varrho : \ell_+^0(\mathbb{Z}) \to [0,\infty]$. By the Luxemburg representation theorem (see [2, Ch. 2, Theorem 4.10]), there exists a unique rearrangement-invariant Banach function norm $\overline{\varrho} : \ell_+^0(\mathbb{N}) \to [0,\infty]$ such that

$$\varrho(f) = \overline{\varrho}(f^*), \quad f \in \ell_+^0(\mathbb{Z}).$$

The rearrangement-invariant Banach sequence space generated by $\overline{\varrho}$ is denoted by $\overline{X}(\mathbb{N})$ and is called the Luxemburg representation of $X(\mathbb{Z})$.

Let $t \ge 0$. We denote by $\lfloor t \rfloor$ the greatest integer less than or equal to t. For $f = \{f_k\}_{k\in\mathbb{N}} : \mathbb{N} \to [0,\infty)$, consider the following operators:

$$(E_j f)_k := f_{jk}, \quad (F_j f)_k := f_{\lfloor (k-1)/j \rfloor + 1}, \quad j,k \in \mathbb{N}.$$

For $j \in \mathbb{N}$, let

$$H(j,X) := \sup\left\{\overline{\varrho}(E_j f^*) \ : \ f \in \overline{X}(\mathbb{N}),\ \overline{\varrho}(f) \le 1\right\},$$
$$K(j,X) := \sup\left\{\overline{\varrho}(F_j f^*) \ : \ f \in \overline{X}(\mathbb{N}),\ \overline{\varrho}(f) \le 1\right\}.$$

It follows from [9, Lemmas 3–5] that the following limits

$$\alpha_X := \lim_{j\to\infty} \frac{-\log H(j,X)}{\log j}, \quad \beta_X := \lim_{j\to\infty} \frac{\log K(j,X)}{\log j}$$

exist and satisfy

$$0 \le \alpha_X \le \beta_X \le 1, \quad \alpha_{X'} = 1 - \beta_X, \quad \beta_{X'} = 1 - \alpha_X.$$

The numbers α_X and β_X are called the lower and the upper Boyd indices, respectively.

3.3. Boyd's interpolation theorem

The next theorem follows from the Boyd interpolation theorem [9, Theorem 1] for quasi-linear operators of weak types (p,p) and (q,q). Its proof for nonatomic measure spaces can also be found in [2, Ch. 3, Theorem 5.16] and [21, Theorem 2.b.11].

Theorem 3.1. *Let $1 \le q < p \le \infty$ and $X(\mathbb{Z})$ be a rearrangement-invariant Banach sequence space with the Boyd indices α_X, β_X satisfying*

$$1/p < \alpha_X \le \beta_X < 1/q.$$

Then there exists a constant $C_{p,q} \in (0,\infty)$ with the following property. If a linear operator $T : \ell^0(\mathbb{Z}) \to \ell^0(\mathbb{Z})$ is bounded on the spaces $\ell^p(\mathbb{Z})$ and $\ell^q(\mathbb{Z})$, then it is also bounded on the rearrangement-invariant Banach sequence space $X(\mathbb{Z})$ and

$$\|T\|_{\mathcal{B}(X(\mathbb{Z}))} \le C_{p,q} \max\left\{\|T\|_{\mathcal{B}(\ell^p(\mathbb{Z}))}, \|T\|_{\mathcal{B}(\ell^q(\mathbb{Z}))}\right\}. \tag{3.1}$$

Notice that estimate (3.1) is not stated explicitly in [2,9,21]. However, it can be extracted from the proof of the Boyd interpolation theorem.

4. Discrete Muckenhoupt weights

4.1. Symmetric reproduction of Muckenhoupt weights

Recall that $A_p^{\mathrm{sym}}(\mathbb{Z})$ denotes the collection of weights $v \in A_p(\mathbb{Z})$ satisfying $v_{-k} = v_k$ for all $k \in \mathbb{Z}$. The following lemma allows us to extend a weight $w \in A_p(\mathbb{Z}_+)$ to a weight $v \in A_p^{\mathrm{sym}}(\mathbb{Z})$ (see [18, Lemma 3.2] and also [3, Section 2.4], where similar results in the continuous case are considered).

Lemma 4.1. *Let $1 < p < \infty$, $w : \mathbb{Z}_+ \to (0, \infty)$ and $v_n := w_{|n|}$, $n \in \mathbb{Z}$. Then*

$$w \in A_p(\mathbb{Z}_+) \quad \Longleftrightarrow \quad v \in A_p^{\mathrm{sym}}(\mathbb{Z}).$$

4.2. Reverse Hölder inequality

We will need the following result contained in [4, Lemma 2.3].

Theorem 4.2. *Let $1 < p < \infty$ and $w \in A_p(\mathbb{Z}_+)$. Then there exist $\delta > 0$ and $C > 0$ depending only on p and w such that*

$$\left(\frac{1}{m(R)} \sum_{k\in R} w_k^{p(1+\delta)}\right)^{1/(1+\delta)} \le \frac{C}{m(R)} \sum_{k\in R} w_k^p$$

for all intervals $R \subset \mathbb{Z}_+$ such that $m(R) = 2^r$ with $r \in \mathbb{N}$.

4.3. "Convexity" property of discrete Muckenhoupt weights

The following lemma is analogous to [3, Proposition 2.30]. We give its proof for the sake of completeness of presentation.

Lemma 4.3. *Let $w : \mathbb{Z}_+ \to (0, \infty)$ be a weight. Then the set*

$$\Gamma := \left\{(p, \delta) \in (1, \infty) \times \mathbb{R} \ : \ w^{\delta/p} \in A_p(\mathbb{Z}_+)\right\}$$

is convex.

Proof. Let $(p_1, \delta_1) \in \Gamma$ and $(p_2, \delta_2) \in \Gamma$ and put

$$p(\theta) := (1-\theta)p_1 + \theta p_2, \quad \delta(\theta) := (1-\theta)\delta_1 + \theta\delta_2, \quad \theta \in (0,1).$$

By definition of Γ, we have $w^{\delta_1/p_1} \in A_{p_1}(\mathbb{Z}_+)$ and $w^{\delta_2/p_2} \in A_{p_2}(\mathbb{Z}_+)$. We have to show that $w^{\delta(\theta)/p(\theta)} \in A_{p(\theta)}(\mathbb{Z}_+)$ for all $\theta \in (0, 1)$.

Let $J \subset \mathbb{Z}_+$ be any interval. Consider

$$\alpha := 1/(1-\theta), \quad \beta := 1/\theta. \tag{4.1}$$

Then for all $k \in J$,

$$\left(w_k^{\delta(\theta)/p(\theta)}\right)^{p(\theta)} = w_k^{(1-\theta)\delta_1+\theta\delta_2} = \left(w_k^{\delta_1}\right)^{1/\alpha} \left(w_k^{\delta_2}\right)^{1/\beta}.$$

Hölder's inequality with α, β given by (4.1) implies that

$$\begin{aligned}
\left(\frac{1}{m(J)}\sum_{k\in J}\left(w_k^{\delta(\theta)/p(\theta)}\right)^{p(\theta)}\right)^{1/p(\theta)} &= \left(\frac{1}{m(J)}\sum_{k\in J}\left(w_k^{\delta_1}\right)^{1/\alpha}\left(w_k^{\delta_2}\right)^{1/\beta}\right)^{1/p(\theta)}\\
&\le \left(\left(\frac{1}{m(J)}\sum_{k\in J} w_k^{\delta_1}\right)^{1/\alpha}\left(\frac{1}{m(J)}\sum_{k\in J} w_k^{\delta_2}\right)^{1/\beta}\right)^{1/p(\theta)}\\
&= \left(\frac{1}{m(J)}\sum_{k\in J}\left(w_k^{\delta_1/p_1}\right)^{p_1}\right)^{\frac{1}{p_1}\frac{(1-\theta)p_1}{p(\theta)}}\left(\frac{1}{m(J)}\sum_{k\in J}\left(w_k^{\delta_2/p_2}\right)^{p_2}\right)^{\frac{1}{p_2}\frac{\theta p_2}{p(\theta)}}. \qquad (4.2)
\end{aligned}$$

Define q_j and $q(\theta)$ by

$$\frac{1}{p_j}+\frac{1}{q_j}=1, \quad j=1,2, \qquad \frac{1}{p(\theta)}+\frac{1}{q(\theta)}=1, \quad \theta\in(0,1).$$

Let

$$\gamma := \frac{p(\theta)}{q(\theta)}\frac{q_1}{p_1}\frac{1}{1-\theta}, \quad \mu := \frac{p(\theta)}{q(\theta)}\frac{q_2}{p_2}\frac{1}{\theta}. \qquad (4.3)$$

Then

$$\begin{aligned}
\frac{1}{\gamma}+\frac{1}{\mu} &= \frac{q(\theta)}{p(\theta)}\left(\frac{p_1}{q_1}(1-\theta)+\frac{p_2}{q_2}\theta\right)\\
&= \frac{1}{p(\theta)-1}((p_1-1)(1-\theta)+(p_2-1)\theta) = \frac{p(\theta)-1}{p(\theta)-1} = 1
\end{aligned}$$

and

$$\begin{aligned}
-\frac{\delta(\theta)q(\theta)}{p(\theta)} &= -\frac{q(\theta)}{p(\theta)}((1-\theta)\delta_1+\theta\delta_2)\\
&= -\frac{q(\theta)}{p(\theta)}\frac{p_1}{q_1}(1-\theta)\delta_1\frac{q_1}{p_1} - \frac{q(\theta)}{p(\theta)}\frac{p_2}{q_2}\theta\delta_2\frac{q_2}{p_2}\\
&= \left(-\frac{\delta_1 q_1}{p_1}\right)\frac{1}{\gamma}+\left(-\frac{\delta_2 q_2}{p_2}\right)\frac{1}{\mu}.
\end{aligned}$$

Hence, for all $k\in J$,

$$\left(w_k^{-\delta(\theta)/p(\theta)}\right)^{q(\theta)} = \left(w_k^{-\delta_1 q_1/p_1}\right)^{1/\gamma}\left(w_k^{-\delta_2 q_2/p_2}\right)^{1/\mu}.$$

Applying Hölder's inequality with γ and μ given by (4.3), we get

$$\begin{aligned}
&\left(\frac{1}{m(J)}\sum_{k\in J}\left(w_k^{-\delta(\theta)/p(\theta)}\right)^{q(\theta)}\right)^{1/q(\theta)}\\
&= \left(\frac{1}{m(J)}\sum_{k\in J}\left(w_k^{-\delta_1 q_1/p_1}\right)^{1/\gamma}\left(w_k^{-\delta_2 q_2/p_2}\right)^{1/\mu}\right)^{1/q(\theta)}\\
&\le \left(\left(\frac{1}{m(J)}\sum_{k\in J} w_k^{-\delta_1 q_1/p_1}\right)^{1/\gamma}\left(\frac{1}{m(J)}\sum_{k\in J} w_k^{-\delta_2 q_2/p_2}\right)^{1/\mu}\right)^{1/q(\theta)}
\end{aligned}$$

$$= \left(\frac{1}{m(J)} \sum_{k \in J} \left(w_k^{-\delta_1/p_1} \right)^{q_1} \right)^{1/(\gamma q(\theta))} \left(\frac{1}{m(J)} \sum_{k \in J} \left(w_k^{-\delta_2/p_2} \right)^{q_2} \right)^{1/(\mu q(\theta))}$$

$$= \left(\frac{1}{m(J)} \sum_{k \in J} \left(w_k^{-\delta_1/p_1} \right)^{q_1} \right)^{\frac{1}{q_1} \frac{(1-\theta)p_1}{p(\theta)}} \left(\frac{1}{m(J)} \sum_{k \in J} \left(w_k^{-\delta_2/p_2} \right)^{q_2} \right)^{\frac{1}{q_2} \frac{\theta p_2}{p(\theta)}}. \tag{4.4}$$

Multiplication of (4.2) and (4.4) shows that

$$\left(\frac{1}{m(J)} \sum_{k \in J} \left(w_k^{\delta(\theta)/p(\theta)} \right)^{p(\theta)} \right)^{1/p(\theta)} \left(\frac{1}{m(J)} \sum_{k \in J} \left(w_k^{-\delta(\theta)/p(\theta)} \right)^{q(\theta)} \right)^{1/q(\theta)}$$

$$\leq \left(\frac{1}{m(J)} \sum_{k \in J} \left(w_k^{\delta_1/p_1} \right)^{p_1} \right)^{\frac{1}{p_1} \frac{(1-\theta)p_1}{p(\theta)}} \left(\frac{1}{m(J)} \sum_{k \in J} \left(w_k^{-\delta_1/p_1} \right)^{q_1} \right)^{\frac{1}{q_1} \frac{(1-\theta)p_1}{p(\theta)}}$$

$$\times \left(\frac{1}{m(J)} \sum_{k \in J} \left(w_k^{\delta_2/p_2} \right)^{p_2} \right)^{\frac{1}{p_2} \frac{\theta p_2}{p(\theta)}} \left(\frac{1}{m(J)} \sum_{k \in J} \left(w_k^{-\delta_2/p_2} \right)^{q_2} \right)^{\frac{1}{q_2} \frac{\theta p_2}{p(\theta)}}$$

$$\leq \left(\left[w^{\delta_1/p_1} \right]_{A_{p_1}(\mathbb{Z}_+)} \right)^{\frac{(1-\theta)p_1}{p(\theta)}} \left(\left[w^{\delta_2/p_2} \right]_{A_{p_2}(\mathbb{Z}_+)} \right)^{\frac{\theta p_2}{p(\theta)}}.$$

Passing to the supremum over all intervals $J \subset \mathbb{Z}_+$, we obtain

$$\left[w^{\delta(\theta)/p(\theta)} \right]_{A_{p(\theta)}(\mathbb{Z}_+)} \leq \left(\left[w^{\delta_1/p_1} \right]_{A_{p_1}(\mathbb{Z}_+)} \right)^{\frac{(1-\theta)p_1}{p(\theta)}} \left(\left[w^{\delta_2/p_2} \right]_{A_{p_2}(\mathbb{Z}_+)} \right)^{\frac{\theta p_2}{p(\theta)}}$$

and $w^{\delta(\theta)/p(\theta)} \in A_{p(\theta)}(\mathbb{Z}_+)$. $\square$

4.4. Stability property of discrete Muckenhoupt weights

The stability property of weights in $A_p(\mathbb{Z}_+)$ available in [4, Theorem 1.1] is not sufficient for our purposes. We will need a more general version of it, which is analogous to [3, Theorem 2.31].

Theorem 4.4. *Let $1 < p_0 < \infty$. If $w \in A_{p_0}(\mathbb{Z}_+)$, then there exists $\varepsilon_0 > 0$ such that $w^{1+\varepsilon} \in A_p(\mathbb{Z}_+)$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and all $p \in (p_0 - \varepsilon_0, p_0 + \varepsilon_0)$.*

Proof. The proof given below is an adaptation of the proof of [3, Theorem 2.31] to the discrete setting. If $w \in A_{p_0}(\mathbb{Z}_+)$, then $w^{-1} \in A_{q_0}(\mathbb{Z}_+)$, where $1/p_0 + 1/q_0 = 1$. By Theorem 4.2, there are constants $\delta, \eta \in (0, \infty)$ and $C, D \in (0, \infty)$ such that

$$\left(\frac{1}{m(R)} \sum_{k \in R} w_k^{p_0(1+\delta)} \right)^{1/(1+\delta)} \leq \frac{C}{m(R)} \sum_{k \in R} w_k^{p_0}, \tag{4.5}$$

$$\left(\frac{1}{m(R)} \sum_{k \in R} w_k^{-q_0(1+\eta)} \right)^{1/(1+\eta)} \leq \frac{D}{m(R)} \sum_{k \in R} w_k^{-q_0} \tag{4.6}$$

for all intervals $R \subset \mathbb{Z}_+$ such that $m(R) = 2^r$ with $r \in \mathbb{N}$.

If $-1 < \lambda_1 \le \lambda_2$, then $\Phi(x) = x^{(1+\lambda_2)/(1+\lambda_1)}$ is convex on $[0, \infty)$ and we may apply Jensen's inequality to deduce that

$$\left(\frac{1}{m(R)}\sum_{k\in R} v_k^{1+\lambda_1}\right)^{1/(1+\lambda_1)} \le \left(\frac{1}{m(R)}\sum_{k\in R} v_k^{1+\lambda_2}\right)^{1/(1+\lambda_2)}$$

for any weight $v = \{v_k\}_{k\in\mathbb{Z}_+}$. Thus, letting $\lambda := \min\{\delta, \eta\}$, we obtain from (4.5) and (4.6) that

$$\begin{aligned}\left(\frac{1}{m(R)}\sum_{k\in R} w_k^{p_0(1+\lambda)}\right)^{1/(1+\lambda)} &\le \left(\frac{1}{m(R)}\sum_{k\in R} w_k^{p_0(1+\delta)}\right)^{1/(1+\delta)}\\ &\le \frac{C}{m(R)}\sum_{k\in R} w_k^{p_0}, \qquad (4.7)\\ \left(\frac{1}{m(R)}\sum_{k\in R} w_k^{-q_0(1+\lambda)}\right)^{1/(1+\lambda)} &\le \left(\frac{1}{m(R)}\sum_{k\in R} w_k^{-q_0(1+\eta)}\right)^{1/(1+\eta)}\\ &\le \frac{D}{m(R)}\sum_{k\in R} w_k^{-q_0} \qquad (4.8)\end{aligned}$$

for all intervals $R \subset \mathbb{Z}_+$ such that $m(R) = 2^r$ with $r \in \mathbb{N}$. It follows from (4.7) and (4.8) that

$$\begin{aligned}&\left(\frac{1}{m(R)}\sum_{k\in R} w_k^{p_0(1+\lambda)}\right)^{1/p_0}\left(\frac{1}{m(R)}\sum_{k\in R} w_k^{-q_0(1+\lambda)}\right)^{1/q_0}\\ &\le \left[\left(\frac{C}{m(R)}\sum_{k\in R} w_k^{p_0}\right)^{1/p_0}\left(\frac{D}{m(R)}\sum_{k\in R} w_k^{-q_0}\right)^{1/q_0}\right]^{1+\lambda}\\ &\le \left(C^{1/p_0}D^{1/q_0}\right)^{1+\lambda}[w]_{A_{p_0}(\mathbb{Z}_+)}^{1+\lambda}\end{aligned}$$

for all intervals $R \subset \mathbb{Z}_+$ such that $m(R) = 2^r$ with $r \in \mathbb{N}$.

Now let $J \subset \mathbb{Z}_+$ be an arbitrary interval with $m(J) \ge 2$. Choose any interval $R \subset \mathbb{Z}_+$ such that $m(R) = 2^r$ for some $r \in \mathbb{N}$, $J \subset R$ and $m(R) \le 2m(J)$. Then

$$\begin{aligned}&\frac{1}{m(J)}\left(\sum_{k\in J}\left(w_k^{1+\lambda}\right)^{p_0}\right)^{1/p_0}\left(\sum_{k\in J}\left(w_k^{1+\lambda}\right)^{-q_0}\right)^{1/q_0}\\ &\le \frac{2}{m(R)}\left(\sum_{k\in R}\left(w_k^{1+\lambda}\right)^{p_0}\right)^{1/p_0}\left(\sum_{k\in R}\left(w_k^{1+\lambda}\right)^{-q_0}\right)^{1/q_0}\\ &\le 2\left(C^{1/p_0}D^{1/q_0}\right)^{1+\lambda}[w]_{A_{p_0}(\mathbb{Z}_+)}^{1+\lambda}.\end{aligned}$$

Passing to the supremum over all intervals $J \subset \mathbb{Z}_+$, we obtain

$$\left[w^{1+\lambda}\right]_{A_{p_0}(\mathbb{Z}_+)} \le 2\left(C^{1/p_0}D^{1/q_0}\right)^{1+\lambda}[w]_{A_{p_0}(\mathbb{Z}_+)}^{1+\lambda}.$$

Consequently, $(p_0, (1+\lambda)p_0) \in \Gamma$. Because, by Lemma 4.3, the set Γ is convex and contains the half-line $(1, \infty) \times \{0\}$, it follows that (p_0, p_0) is an inner point of Γ (in the usual topology of $\mathbb{R}^2$), which is equivalent to the assertion of the theorem. □

Lemma 4.1 and Theorem 4.4 immediately imply the following.

Corollary 4.5. *Let $1 < p_0 < \infty$. If $w \in A_{p_0}^{\mathrm{sym}}(\mathbb{Z})$, then there exists $\varepsilon_0 > 0$ such that $w^{1+\varepsilon} \in A_p^{\mathrm{sym}}(\mathbb{Z})$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and all $p \in (p_0 - \varepsilon_0, p_0 + \varepsilon_0)$.*

4.5. Stechkin's inequality

In 1950, Stechkin [29] proved that if a 2π-periodic function $a : \mathbb{R} \to \mathbb{Z}$ is of bounded variation, then $a \in M_{\ell^p(\mathbb{Z})}$. Böttcher and Seybold [4, Theorem 1.2] proved the following weighted extension of Stechkin's inequality.

Theorem 4.6. *Let $1 < p < \infty$. If $w \in A_p(\mathbb{Z})$, then there is a constant $c_{p,w} \in (0, \infty)$ depending only on p and w such that*

$$\|a\|_{M_{\ell^p(\mathbb{Z},w)}} \leq c_{p,w} \left(\|a\|_{L^\infty(-\pi,\pi)} + V(a)\right)$$

for all $a \in BV[-\pi, \pi]$. In particular, $BV[-\pi, \pi] \hookrightarrow M_{\ell^p(\mathbb{Z},w)}$.

5. Banach algebra $M_{X(\mathbb{Z})}$

5.1. Laurent operators on the space $\ell^2(\mathbb{Z})$

We start this section with the following well-known result (see, e.g., [7, Theorem 1.1 and equalities (1.5)–(1.6)] and also [14, Section XXIII.2], [15, Section 3.1]).

Theorem 5.1. *Let $\{a_n\}_{n\in\mathbb{Z}}$ be a sequence of complex numbers. The Laurent matrix $(a_{j-k})_{j,k=-\infty}^{\infty}$ generates a bounded operator A on the space $\ell^2(\mathbb{Z})$ if and only if there exists a function $a \in L^\infty(-\pi, \pi)$ such that $a_n = \widehat{a}(n)$ for all $n \in \mathbb{Z}$. In this case $A = L(a)$ and*

$$\|a\|_{M_{\ell^2(\mathbb{Z})}} = \|L(a)\|_{\mathcal{B}(\ell^2(\mathbb{Z}))} = \|a\|_{L^\infty(-\pi,\pi)}.$$

Moreover, if $a, b \in L^\infty(-\pi, \pi)$, then $L(a)L(b) = L(ab)$ on the space $\ell^2(\mathbb{Z})$.

5.2. Periodic distributions and their Fourier coefficients

Let $\mathcal{P}$ be the set of all infinitely differentiable 2π-periodic functions from $\mathbb{R}$ to $\mathbb{C}$. Elements of $\mathcal{P}$ are called periodic test functions. One can equip $\mathcal{P}$ with the countable family of seminorms

$$\|u\|_{k,\mathcal{P}} := \sup_{x\in[-\pi,\pi]} |D^{k-1}u(x)|, \quad k \in \mathbb{N},$$

where $D^k u$ denotes the kth derivative of u and $D^0 u = u$, and the metric

$$d(u, v) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|u - v\|_{k,\mathcal{P}}}{1 + \|u - v\|_{k,\mathcal{P}}}.$$

Then the set $\mathcal{P}$ endowed with the metric d is a complete linear metric space (see [1, Ch. 3, Theorems 2.1–2.2]).

A periodic distribution is a continuous linear functional on the complete linear metric space $(\mathcal{P}, d)$. The set of all periodic distributions is denoted by $\mathcal{P}'$. The functions

$$E_n(\theta) := e^{in\theta}, \quad \theta \in \mathbb{R},$$

belong to $\mathcal{TP} \subset \mathcal{P}$ for all $n \in \mathbb{Z}$. The Fourier coefficients of a periodic distribution $a \in \mathcal{P}'$ are defined by

$$\widehat{a}(n) := a(E_{-n}), \quad n \in \mathbb{Z}.$$

The complex conjugate $\overline{a}$ of a periodic distribution $a \in \mathcal{P}'$ is defined by

$$\overline{a}(u) := \overline{a(\overline{u})}, \quad u \in \mathcal{P}.$$

It is easy to see that

$$\widehat{\overline{a}}(n) = \overline{a}(E_{-n}) = \overline{a(\overline{E_{-n}})} = \overline{a(E_n)} = \overline{\widehat{a}(-n)}, \quad n \in \mathbb{N}. \tag{5.1}$$

5.3. Complex conjugate of an element of $M_{X(\mathbb{Z})}$ belongs to $M_{X'(\mathbb{Z})}$

For $\varphi = \{\varphi_k\}_{k\in\mathbb{Z}} \in X(\mathbb{Z})$ and $\psi = \{\psi_k\}_{k\in\mathbb{Z}} \in X'(\mathbb{Z})$, put

$$(\varphi, \psi) := \sum_{k\in\mathbb{Z}} \varphi_k \overline{\psi_k}.$$

For $j \in \mathbb{Z}$, let $e_j := \{\delta_{jk}\}_{k\in\mathbb{Z}}$, where δ_{jk} is the Kronecker delta. It is clear that $e_k \in S_0(\mathbb{Z})$ for every $k \in \mathbb{Z}$.

The following lemma resembles [8, Section 2.5(a)].

Lemma 5.2. *Let $X(\mathbb{Z})$ be a Banach sequence space and $X'(\mathbb{Z})$ be its associate space. If $a \in M_{X(\mathbb{Z})}$, then $\overline{a} \in M_{X'(\mathbb{Z})}$ and $\|a\|_{M_{X(\mathbb{Z})}} = \|\overline{a}\|_{M_{X'(\mathbb{Z})}}$.*

Proof. If $i, j \in \mathbb{Z}$, then taking into account (5.1), we get

$$\begin{aligned}(a * e_i, e_j) &= \sum_{m\in\mathbb{Z}} (a * e_i)_m \overline{(e_j)_m} = (a * e_i)_j = \sum_{k\in\mathbb{Z}} \widehat{a}(j-k)(e_i)_k = \widehat{a}(j-i) \\ &= \overline{\widehat{\overline{a}}(i-j)} = \overline{\sum_{k\in\mathbb{Z}} \widehat{\overline{a}}(i-k)(e_j)_k} = \overline{(\overline{a} * e_j)_i} = \sum_{m\in\mathbb{Z}} (e_i)_m \overline{(\overline{a} * e_j)_m} \\ &= (e_i, \overline{a} * e_j).\end{aligned}$$

Hence, for all $\varphi, \psi \in S_0(\mathbb{Z})$,

$$(a * \varphi, \psi) = (\varphi, \overline{a} * \psi).$$

Fix $\psi \in S_0(\mathbb{Z})$ and recall that $X(\mathbb{Z}) = X''(\mathbb{Z})$ with equal norms (see [2, Ch. 1, Theorem 2.7]). It follows from the above observations and Lemma 2.2 that

$$\begin{aligned}\|\overline{a} * \psi\|_{X'(\mathbb{Z})} &= \sup\{|(\varphi, \overline{a} * \psi)| \ : \ \varphi \in S_0(\mathbb{Z}), \ \|\varphi\|_{X''(\mathbb{Z})} \le 1\} \\ &= \sup\{|(a * \varphi, \psi)| \ : \ \varphi \in S_0(\mathbb{Z}), \ \|\varphi\|_{X(\mathbb{Z})} \le 1\}.\end{aligned} \tag{5.2}$$

On the other hand, Hölder's inequality for Banach sequence spaces (see [2, Ch. 1, Theorem 2.4]) and (1.1) imply that for all $\varphi \in S_0(\mathbb{Z})$,

$$|(a * \varphi, \psi)| \le \|a * \varphi\|_{X(\mathbb{Z})} \|\psi\|_{X'(\mathbb{Z})} \le \|a\|_{M_{X(\mathbb{Z})}} \|\varphi\|_{X(\mathbb{Z})} \|\psi\|_{X'(\mathbb{Z})}. \tag{5.3}$$

Combining (5.2) and (5.3), we arrive at

$$\|\overline{a}\|_{M_{X'(\mathbb{Z})}} = \sup\left\{\frac{\|\overline{a} * \psi\|_{X'(\mathbb{Z})}}{\|\psi\|_{X'(\mathbb{Z})}} \ : \ \psi \in S_0(\mathbb{Z}), \ \|\psi\|_{X'(\mathbb{Z})} \le 1\right\} \le \|a\|_{M_{X(\mathbb{Z})}}. \tag{5.4}$$

Applying inequality (5.4) to $\overline{a}$ and $X'(\mathbb{Z})$ in place of a and $X(\mathbb{Z})$, respectively, we get

$$\|a\|_{M_{X(\mathbb{Z})}} = \left\|\overline{\overline{a}}\right\|_{M_{X''(\mathbb{Z})}} \le \|\overline{a}\|_{M_{X'(\mathbb{Z})}}. \tag{5.5}$$

Inequalities (5.4) and (5.5) imply that $\|a\|_{M_{X(\mathbb{Z})}} = \|\overline{a}\|_{M_{X'(\mathbb{Z})}}$. $\square$

5.4. *The sets $M_{X(\mathbb{Z})}$ and $M_{X'(\mathbb{Z})}$ coincide if $X(\mathbb{Z})$ is reflection-invariant*

The next lemma extends [8, Section 2.5(b)].

Lemma 5.3. *If $X(\mathbb{Z})$ is a reflection-invariant Banach sequence space, then $a \in M_{X(\mathbb{Z})}$ if and only if $\overline{a} \in M_{X(\mathbb{Z})}$ and $\|a\|_{M_{X(\mathbb{Z})}} = \|\overline{a}\|_{M_{X(\mathbb{Z})}}$.*

Proof. If $X(\mathbb{Z})$ is reflection-invariant, then the anti-linear operator R defined by $R\varphi = \overline{\widetilde{\varphi}}$, that is,

$$(R\varphi)_j = \overline{\varphi}_{-j}, \quad j \in \mathbb{Z},$$

is an isometry on $X(\mathbb{Z})$ and $R^2 = I$.

If $\varphi \in S_0(\mathbb{Z})$, then taking into account (5.1), we get for all $j \in \mathbb{Z}$,

$$\begin{aligned}(\overline{a} * \varphi)_j &= \sum_{k\in\mathbb{Z}} \widehat{\overline{a}}(j-k)\varphi_k = \sum_{k\in\mathbb{Z}} \overline{\widehat{a}(k-j)}\varphi_k = \sum_{k\in\mathbb{Z}} \overline{\widehat{a}(k-j)}\widetilde{\varphi}_{-k} \\ &= \overline{\sum_{k\in\mathbb{Z}} \widehat{a}(k-j)\overline{\widetilde{\varphi}}_{-k}} = \overline{\sum_{k\in\mathbb{Z}} \widehat{a}(-j-k)\overline{\widetilde{\varphi}}_{k}} = \overline{\sum_{k\in\mathbb{Z}} \widehat{a}(-j-k)(R\varphi)_k} \\ &= \overline{(a * R\varphi)_{-j}} = (R(a * R\varphi))_j.\end{aligned}$$

Hence, taking into account that $\varphi \in S_0(\mathbb{Z})$ if and only if $\psi = R\varphi \in S_0(\mathbb{Z})$, we get

$$\begin{aligned}\|\overline{a}\|_{M_{X(\mathbb{Z})}} &= \sup\left\{\frac{\|\overline{a} * \varphi\|_{X(\mathbb{Z})}}{\|\varphi\|_{X(\mathbb{Z})}} \;:\; \varphi \in S_0(\mathbb{Z}),\ \varphi \ne 0\right\} \\ &= \sup\left\{\frac{\|R(a * R\varphi)\|_{X(\mathbb{Z})}}{\|\varphi\|_{X(\mathbb{Z})}} \;:\; \varphi \in S_0(\mathbb{Z}),\ \varphi \ne 0\right\} \\ &= \sup\left\{\frac{\|a * R\varphi\|_{X(\mathbb{Z})}}{\|R(R\varphi)\|_{X(\mathbb{Z})}} \;:\; \varphi \in S_0(\mathbb{Z}),\ \varphi \ne 0\right\} \\ &= \sup\left\{\frac{\|a * \psi\|_{X(\mathbb{Z})}}{\|R\psi\|_{X(\mathbb{Z})}} \;:\; \psi \in S_0(\mathbb{Z}),\ \psi \ne 0\right\} \\ &= \sup\left\{\frac{\|a * \psi\|_{X(\mathbb{Z})}}{\|\psi\|_{X(\mathbb{Z})}} \;:\; \psi \in S_0(\mathbb{Z}),\ \psi \ne 0\right\} = \|a\|_{M_{X(\mathbb{Z})}},\end{aligned}$$

which completes the proof. $\square$

The previous two lemmas lead to the following.

Corollary 5.4. *If $X(\mathbb{Z})$ is a reflection-invariant Banach sequence space and $X'(\mathbb{Z})$ is its associate space, then $M_{X(\mathbb{Z})} = M_{X'(\mathbb{Z})}$ and $\|a\|_{M_{X(\mathbb{Z})}} = \|a\|_{M_{X'(\mathbb{Z})}}$ for all $a \in M_{X(\mathbb{Z})}$.*

Proof. By Lemma 2.4, $X'(\mathbb{Z})$ is a reflection-invariant Banach sequence space. If $a \in M_{X(\mathbb{Z})}$, then $\overline{a} \in M_{X'(\mathbb{Z})}$ and $\|a\|_{M_{X(\mathbb{Z})}} = \|\overline{a}\|_{M_{X'(\mathbb{Z})}}$ in view of Lemma 5.2. On the other hand, Lemma

5.3 applied to $M_{X'(\mathbb{Z})}$ implies that $a \in M_{X'(\mathbb{Z})}$ and $\|\overline{a}\|_{M_{X'(\mathbb{Z})}} = \|a\|_{M_{X'(\mathbb{Z})}}$. So,

$$M_{X(\mathbb{Z})} \hookrightarrow M_{X'(\mathbb{Z})} \tag{5.6}$$

and

$$\|a\|_{M_{X(\mathbb{Z})}} = \|a\|_{M_{X'(\mathbb{Z})}} \quad \text{for} \quad a \in M_{X(\mathbb{Z})}. \tag{5.7}$$

It follows from what has already been proved with $X'(\mathbb{Z})$ in place of $X(\mathbb{Z})$ and the equality $X(\mathbb{Z}) = X''(\mathbb{Z})$ with equal norms (see [2, Ch. 1, Theorem 2.7]) that

$$M_{X'(\mathbb{Z})} \hookrightarrow M_{X''(\mathbb{Z})} = M_{X(\mathbb{Z})} \tag{5.8}$$

and

$$\|a\|_{M_{X'(\mathbb{Z})}} = \|a\|_{M_{X''(\mathbb{Z})}} = \|a\|_{M_{X(\mathbb{Z})}} \quad \text{for} \quad a \in M_{X'(\mathbb{Z})}. \tag{5.9}$$

Combining (5.6)–(5.9), we arrive at the desired conclusion. □

5.5. *Continuous embedding of $M_{X(\mathbb{Z})}$ into $L^\infty(-\pi, \pi)$*

The following result is the key ingredient in the proof of the main result of this section. It extends one of the assertions in [8, Section 2.5(d)].

Theorem 5.5. *Let $X(\mathbb{Z})$ be a reflexive reflection-invariant Banach sequence space. If $a \in M_{X(\mathbb{Z})}$, then $a \in L^\infty(-\pi, \pi)$ and*

$$\|a\|_{L^\infty(-\pi,\pi)} \le \|a\|_{M_{X(\mathbb{Z})}}.$$

Proof. If $a \in M_{X(\mathbb{Z})}$, then in view of Corollary 5.4, $a \in M_{X'(\mathbb{Z})}$ and $\|a\|_{M_{X(\mathbb{Z})}} = \|a\|_{M_{X'(\mathbb{Z})}}$. Since the space $X(\mathbb{Z})$ is reflexive, Lemma 2.1(b) implies that $S_0(\mathbb{Z})$ is dense in $X(\mathbb{Z})$ and in $X'(\mathbb{Z})$. Hence $L(a) \in \mathcal{B}(X(\mathbb{Z}))$ and $L(a) \in \mathcal{B}(X'(\mathbb{Z}))$ with

$$\|L(a)\|_{\mathcal{B}(X(\mathbb{Z}))} = \|L(a)\|_{\mathcal{B}(X'(\mathbb{Z}))} = \|a\|_{M_{X(\mathbb{Z})}} = \|a\|_{M_{X'(\mathbb{Z})}}.$$

It follows from Theorem 2.5 that

$$\|a\|_{M_{\ell^2(\mathbb{Z})}} = \|L(a)\|_{\mathcal{B}(\ell^2(\mathbb{Z}))} \le \|L(a)\|^{1/2}_{\mathcal{B}(X(\mathbb{Z}))} \|L(a)\|^{1/2}_{\mathcal{B}(X'(\mathbb{Z}))} = \|a\|_{M_{X(\mathbb{Z})}}.$$

It remains to recall that $M_{\ell^2(\mathbb{Z})} = L^\infty(-\pi, \pi)$ and $\|a\|_{M_{\ell^2(\mathbb{Z})}} = \|a\|_{L^\infty(-\pi,\pi)}$ (see Theorem 5.1). □

5.6. *$M_{X(\mathbb{Z})}$ is a Banach algebra if $X(\mathbb{Z})$ is reflexive and reflection-invariant*

Now we are in a position to prove the main result of this section. It generalizes [8, Section 2.5(g)].

Theorem 5.6. *Let $X(\mathbb{Z})$ be a reflexive reflection-invariant Banach sequence space. Then $M_{X(\mathbb{Z})}$ is a Banach algebra under pointwise multiplication and the norm*

$$\|a\|_{M_{X(\mathbb{Z})}} = \|L(a)\|_{\mathcal{B}(X(\mathbb{Z}))}. \tag{5.10}$$

Proof. If $a, b \in M_{X(\mathbb{Z})}$, then by Theorem 5.5, $a, b \in L^\infty(-\pi, \pi)$. Hence, Theorem 5.1 implies that

$$L(ab)\varphi = L(a)L(b)\varphi$$

for all $\varphi \in S_0(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$. Since $S_0(\mathbb{Z})$ is dense in $X(\mathbb{Z})$ (see Lemma 2.1(b)), we conclude that $L(ab) = L(a)L(b)$ on $X(\mathbb{Z})$. Hence $ab \in M_{X(\mathbb{Z})}$ and

$$\|ab\|_{M_{X(\mathbb{Z})}} = \|L(ab)\|_{\mathcal{B}(X(\mathbb{Z}))} \le \|L(a)\|_{\mathcal{B}(X(\mathbb{Z}))}\|L(b)\|_{\mathcal{B}(X(\mathbb{Z}))} = \|a\|_{M_{X(\mathbb{Z})}}\|b\|_{M_{X(\mathbb{Z})}}.$$

So, $M_{X(\mathbb{Z})}$ is a normed algebra under pointwise multiplication and the norm (5.10).

It remains to show that $M_{X(\mathbb{Z})}$ is complete. Let $\{a^{(n)}\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $M_{X(\mathbb{Z})}$. Then Theorem 5.5 implies that $\{a^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(-\pi, \pi)$. Since the latter space is complete, there exists $a \in L^\infty(-\pi, \pi)$ such that $\|a^{(n)} - a\|_{L^\infty(-\pi,\pi)} \to 0$ as $n \to \infty$. Theorem 5.1 implies that

$$\|L(a^{(n)}) - L(a)\|_{\mathcal{B}(\ell^2(\mathbb{Z}))} \to 0 \quad \text{as} \quad n \to \infty. \tag{5.11}$$

Since $\{L(a^{(n)})\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}(X(\mathbb{Z}))$ and $\mathcal{B}(X(\mathbb{Z}))$ is complete, there exists an operator $A \in \mathcal{B}(X(\mathbb{Z}))$ such that

$$\|L(a^{(n)}) - A\|_{\mathcal{B}(X(\mathbb{Z}))} \to 0 \quad \text{as} \quad n \to \infty. \tag{5.12}$$

Let $\varphi = \{\varphi_k\}_{k\in\mathbb{Z}} \in S_0(\mathbb{Z}) \subset \ell^2(\mathbb{Z}) \cap X(\mathbb{Z})$. It follows from (5.11) and (5.12) that

$$\|L(a^{(n)})\varphi - L(a)\varphi\|_{\ell^2(\mathbb{Z})} \to 0, \quad \|L(a^{(n)})\varphi - A\varphi\|_{X(\mathbb{Z})} \to 0 \quad \text{as} \quad n \to \infty.$$

In view of [2, Ch. 1, Theorem 1.4], these relations imply that for every finite sets $S \subset \mathbb{Z}$ and every $\varepsilon > 0$,

$$\begin{aligned} &m\Big\{k \in S \ : \ \big|(L(a^{(n)})\varphi)_k - (L(a)\varphi)_k\big| \ge \varepsilon\Big\} \to 0, \\ &m\Big\{k \in S \ : \ \big|(L(a^{(n)})\varphi)_k - (A\varphi)_k\big| \ge \varepsilon\Big\} \to 0 \end{aligned}$$

as $n \to \infty$. Since $m(B) < 1$ if and only if $B \subset \mathbb{Z}$ is empty, we conclude that for every finite set $S \subset \mathbb{Z}$ and every $\varepsilon > 0$, there exists $n_0 = n_0(S, \varepsilon) \in \mathbb{N}$ such that for all $k \in S$ and all $n > n_0$,

$$\big|(L(a^{(n)})\varphi)_k - (L(a)\varphi)_k\big| < \varepsilon, \quad \big|(L(a^{(n)})\varphi)_k - (A\varphi)_k\big| < \varepsilon.$$

Hence, for all $k \in \mathbb{Z}$,

$$(L(a)\varphi)_k = \lim_{n\to\infty}(L(a^{(n)}\varphi)_k = (A\varphi)_k.$$

Thus $L(a)\varphi = A\varphi$ for all $\varphi \in S_0(\mathbb{Z})$. Lemma 2.1(b) implies that $A = L(a)$ on $X(\mathbb{Z})$. Since $L(a) \in \mathcal{B}(X(\mathbb{Z}))$, we conclude that $a \in M_{X(\mathbb{Z})}$. So, the Cauchy sequence $\{a^{(n)}\}_{n\in\mathbb{Z}}$ in $M_{X(\mathbb{Z})}$ converges to $a \in M_{X(\mathbb{Z})}$. Thus $M_{X(\mathbb{Z})}$ is complete. □

5.7. Uniform boundedness of the Fejér means in the norm of $M_{X(\mathbb{Z})}$

Let $a \in \mathcal{P}'$ and let $\{\widehat{a}(n)\}_{n\in\mathbb{Z}}$ be the sequence of Fourier coefficients of a. The partial sums of the Fourier series of a are defined by

$$[s_n(a)](\theta) := \sum_{k=-n}^{n} \widehat{a}(k)e^{ik\theta}, \quad \theta \in \mathbb{R}, \quad n \in \mathbb{Z}_+,$$

and the Fejér (or Fejér–Cesàro) means of a are defined by

$$\sigma_n(a) := \frac{1}{n+1}\sum_{k=0}^{n} s_k(a), \quad n \in \mathbb{Z}_+.$$

For $f, g \in L^1(-\pi,\pi)$, define the convolution of f and g by

$$(f*g)(\theta) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta-\phi)g(\phi)\,d\phi.$$

For $n \in \mathbb{Z}_+$, let

$$K_n(\theta) := \sum_{|k|\le n}\left(1-\frac{|k|}{n+1}\right)e^{i\theta k} = \frac{1}{n+1}\left(\frac{\sin\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}}\right)^2, \quad \theta \in \mathbb{R},$$

be the nth Fejér kernel. If $a \in L^1(-\pi,\pi)$, then

$$\sigma_n(a) = K_n * a \tag{5.13}$$

(see, e.g., [19, Ch. I, Section 2]).

The following result for spaces $\ell^p(\mathbb{Z})$ is essentially due to Nikol'skiĭ (see corollary to [24, Theorem 5]). Its proof is given in [8, Lemma 2.44]. Our proof is analogous to the latter one.

Lemma 5.7. *Let $X(\mathbb{Z})$ be a reflexive reflection-invariant Banach sequence space. If $a \in M_{X(\mathbb{Z})}$, then*

$$\|\sigma_n(a)\|_{M_{X(\mathbb{Z})}} \le \|a\|_{M_{X(\mathbb{Z})}}$$

for all $n \in \mathbb{Z}_+$.

Proof. Fix $n \in \mathbb{Z}_+$. It follows from Theorem 5.5 that $a \in L^\infty(-\pi,\pi)$. For $x \in \mathbb{R}$, define the function

$$a_x(\theta) := a(\theta - x), \quad \theta \in \mathbb{R}.$$

Then, for all $j \in \mathbb{Z}$,

$$\begin{aligned}\widehat{a}_x(j) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} a(\theta-x)e^{-ij\theta}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} a(\theta)e^{-ij(\theta+x)}\,d\theta \\ &= e^{-ijx}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} a(\theta)e^{-ij\theta}\,d\theta = e^{-ijx}\widehat{a}(j).\end{aligned} \tag{5.14}$$

It follows from (5.13) that

$$[\sigma_n(a)](\theta) = (K_n * a)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} a_x(\theta)K_n(x)dx. \tag{5.15}$$

For $x \in \mathbb{R}$, define the operator $D_x : X(\mathbb{Z}) \to X(\mathbb{Z})$ by

$$\{\varphi_j\}_{j\in\mathbb{Z}} \mapsto \{e^{ijx}\varphi_j\}_{j\in\mathbb{Z}}.$$

It follows from the definition of the norm in $X(\mathbb{Z})$ that D_x is an isometry on $X(\mathbb{Z})$.

If $\varphi = \{\varphi_j\}_{j\in\mathbb{Z}} \in S_0(\mathbb{Z})$, then taking into account (5.14), we get

$$(D_{-x}L(a)D_x)\varphi = D_{-x}L(a)\left(\left\{e^{ijx}\varphi_j\right\}_{j\in\mathbb{Z}}\right) = D_{-x}\left(\left\{\sum_{k\in\mathbb{Z}}\widehat{a}(j-k)e^{ikx}\varphi_k\right\}_{j\in\mathbb{Z}}\right)$$

$$= \left\{ e^{-ijx} \sum_{k\in\mathbb{Z}} \widehat{a}(j-k) e^{ikx} \varphi_k \right\}_{j\in\mathbb{Z}} = \left\{ \sum_{k\in\mathbb{Z}} \widehat{a}(j-k) e^{i(k-j)x} \varphi_k \right\}_{j\in\mathbb{Z}}$$
$$= \left\{ \sum_{k\in\mathbb{Z}} \widehat{a}_x(j-k) \varphi_k \right\}_{j\in\mathbb{Z}} = L(a_x)\varphi. \tag{5.16}$$

It follows that for all $\varphi = \{\varphi_j\}_{j\in\mathbb{Z}} \in S_0(\mathbb{Z})$, all $x, y \in \mathbb{R}$, and all $j \in \mathbb{Z}$,

$$|(D_x\varphi)_j - (D_y\varphi)_j| = |(e^{ijx} - e^{ijy})\varphi_j| \le |j|\,|x-y|\,|\varphi_j| \le \left(\max_{j\in\operatorname{supp}\varphi} |j| \right) |x-y|\,|\varphi_j|.$$

The above inequality and Axioms (A1)–(A2) in the definition of a Banach sequence space imply that

$$\|D_x\varphi - D_y\varphi\|_{X(\mathbb{Z})} \le \left(\max_{j\in\operatorname{supp}\varphi} |j| \right) |x-y|\,\|\varphi\|_{X(\mathbb{Z})}. \tag{5.17}$$

Equality (5.16) and inequality (5.17) imply that for every $\varphi \in S_0(\mathbb{Z})$, the function

$$[-\pi,\pi] \to X(\mathbb{Z}), \quad x \mapsto L(a_x)\varphi \tag{5.18}$$

is continuous. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \|L(a_x)\varphi\|_{X(\mathbb{Z})} K_n(x)\,dx \le \max_{x\in[-\pi,\pi]} \|L(a_x)\varphi\|_{X(\mathbb{Z})} < \infty.$$

So, by [12, Ch. 2. Section 2, Theorem 2], the function given in (5.18) is Bochner integrable on $[-\pi,\pi]$ with respect to the probability measure $d\mu_n(x) = K_n(x)\,dx/(2\pi)$. Similarly, the constant function

$$[-\pi,\pi] \to X(\mathbb{Z}), \quad x \mapsto \varphi \tag{5.19}$$

is continuous and, hence, is Bochner integrable.

Equality (5.15) and Fubini's theorem imply that for all $j \in \mathbb{Z}$,

$$\begin{aligned}
(\sigma_n(a))\widehat{\ }(j) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\sigma_n(a)](\theta) e^{-ij\theta}\,d\theta \\
&= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} a_x(\theta) K_n(x)\,dx \right) e^{-ij\theta}\,d\theta \\
&= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} a_x(\theta) e^{-ij\theta}\,d\theta \right) K_n(x)\,dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{a}_x(j) K_n(x)\,dx.
\end{aligned}$$

Hence, for all $\varphi \in S_0(\mathbb{Z})$ and all $j \in \mathbb{Z}$,

$$\begin{aligned}
(L(\sigma_n(a))\varphi)_j &= \sum_{k\in\mathbb{Z}} (\sigma_n(a))\widehat{\ }(j-k)\varphi_k \\
&= \sum_{k\in\mathbb{Z}} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{a}_x(j-k) K_n(x)\,dx \right) \varphi_k \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k\in\mathbb{Z}} \widehat{a}_x(j-k)\varphi_k \right) K_n(x)\,dx
\end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}(L(a_x)\varphi)_j K_n(x)\,dx. \tag{5.20}$$

Since the functions given by (5.18) and (5.19) are Bochner integrable, it follows from Hille's theorem (see [12, Ch. 2, Section 2, Theorem 6]) with $T = T_j : X(\mathbb{Z}) \to \mathbb{C}$, sending a sequence $\psi = \{\psi_i\}_{i\in\mathbb{Z}} \in X(\mathbb{Z})$ to its jth coordinate, that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}(L(a_x)\varphi)_j K_n(x)\,dx &= \int_{-\pi}^{\pi} T_j(L(a_x)\varphi)\,d\mu_n(x) = T_j\left(\int_{-\pi}^{\pi} L(a_x)\varphi\,d\mu_n(x)\right)\\ &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} L(a_x)\varphi K_n(x)\,dx\right)_j.\end{aligned}$$

Combining this equality with (5.20), we get for all $\varphi \in S_0(\mathbb{Z})$,

$$L(\sigma_n(a))\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi} L(a_x)\varphi K_n(x)\,dx. \tag{5.21}$$

Let

$$f(j,x) := (L(a_x)\varphi)_j, \quad j \in \mathbb{Z}, \quad x \in [-\pi,\pi].$$

It follows from (5.21), the Minkowski integral inequality (see Lemma 2.3 with the above function $f(j,x)$ and the probability measure $d\mu_n(x) = K_n(x)\,dx/(2\pi)$ on $[-\pi,\pi]$) that

$$\begin{aligned}\|L(\sigma_n(a))\varphi\|_{X(\mathbb{Z})} &\le \left\|\frac{1}{2\pi}\int_{-\pi}^{\pi}|L(a_x)\varphi| K_n(x)\,dx\right\|_{X(\mathbb{Z})}\\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi}\|L(a_x)\varphi\|_{X(\mathbb{Z})} K_n(x)\,dx.\end{aligned}$$

Then applying (5.16) and taking into account that D_{-x} and D_x are isometries on $X(\mathbb{Z})$, we get for all $\varphi \in S_0(\mathbb{Z})$,

$$\begin{aligned}\|L(\sigma_n(a))\varphi\|_{X(\mathbb{Z})} &\le \frac{1}{2\pi}\int_{-\pi}^{\pi}\|D_{-x}L(a)D_x\varphi\|_{X(\mathbb{Z})} K_n(x)\,dx\\ &\le \|L(a)\|_{\mathcal{B}(X(\mathbb{Z}))}\|\varphi\|_{X(\mathbb{Z})}\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx\\ &= \|a\|_{M_{X(\mathbb{Z})}}\|\varphi\|_{X(\mathbb{Z})}.\end{aligned}$$

Thus,

$$\|\sigma_n(a)\|_{M_{X(\mathbb{Z})}} = \sup\left\{\frac{\|L(\sigma_n(a))\varphi\|_{X(\mathbb{Z})}}{\|\varphi\|_{X(\mathbb{Z})}} : \varphi \in S_0(\mathbb{Z}), \varphi \ne 0\right\} \le \|a\|_{M_{X(\mathbb{Z})}},$$

which completes the proof. □

6. Proof of the main results

6.1. *Rearrangement-invariant Banach sequence spaces with symmetric weights*

Recall that a sequence $f = \{f_k\}_{k\in\mathbb{Z}}$ in a Banach sequence space $X(\mathbb{Z})$ is said to have absolutely continuous norm in $X(\mathbb{Z})$ if $\|f\chi_{E_n}\|_{X(\mathbb{Z})} \to 0$ as $n \to \infty$ for every sequence $\{E_n\}_{n\in\mathbb{N}}$ of sets in $\mathbb{Z}$ satisfying $(\chi_{E_n})_k \to 0$ as $n \to \infty$ for all $k \in \mathbb{Z}$. If all sequences $f = \{f_k\}_{k\in\mathbb{Z}} \in X(\mathbb{Z})$ have this property, then the space itself is said to have absolutely continuous norm (see [2, Ch. 1, Definition 3.1]).

Lemma 6.1. *If $X(\mathbb{Z})$ is a reflexive rearrangement-invariant Banach sequence space and w is a symmetric weight, then $X(\mathbb{Z}, w)$ is a reflexive reflection-invariant Banach sequence space.*

Proof. Since $X(\mathbb{Z})$ is a reflexive Banach sequence, by [2, Ch. 1, Corollary 4.4], $X(\mathbb{Z})$ and $X'(\mathbb{Z})$ have absolutely continuous norms. It is easy to see that then the weighted Banach sequence spaces $X(\mathbb{Z}, w)$ and $X'(\mathbb{Z}, w^{-1})$ have absolutely continuous norms. Since $X'(\mathbb{Z}, w^{-1})$ is the associate space of $X(\mathbb{Z}, w)$, applying [2, Ch. 1, Corollary 4.4] once again, we conclude that $X(\mathbb{Z}, w)$ is reflexive.

Let $f = \{f_k\}_{k\in\mathbb{Z}} \in X(\mathbb{Z})$. It is clear that the counting measure m on $\mathbb{Z}$ is reflection-invariant, that is, $m(-S) = m(S)$ for every $S \subset \mathbb{Z}$. Since the weight w is symmetric, we get for all $\lambda > 0$,

$$\begin{aligned} d_{\widetilde{f}w}(\lambda) &= m\{k \in \mathbb{Z} : |\widetilde{f}_k w_k| > \lambda\} = m\{k \in \mathbb{Z} : |f_{-k} w_{-k}| > \lambda\} \\ &= m\{-k \in \mathbb{Z} : |f_k w_k| > \lambda\} = m\{k \in \mathbb{Z} : |f_k w_k| > \lambda\} = d_{fw}(\lambda). \end{aligned}$$

Since $X(\mathbb{Z})$ is rearrangement-invariant, we obtain

$$\|\widetilde{f}\|_{X(\mathbb{Z},w)} = \|\widetilde{f}w\|_{X(\mathbb{Z})} = \|fw\|_{X(\mathbb{Z})} = \|f\|_{X(\mathbb{Z},w)},$$

that is, $X(\mathbb{Z}, w)$ is reflection-invariant. $\square$

6.2. Proof of Theorem 1.2

Part (a) follows from Lemma 6.1 and Theorem 5.5. Similarly, Lemma 6.1 and Theorem 5.6 yield part (b).

Let us prove part (c). Since $\alpha_X, \beta_X \in (0, 1)$ and $w \in A^{\rm sym}_{1/\alpha_X}(\mathbb{Z}) \cap A^{\rm sym}_{1/\beta_X}(\mathbb{Z})$, Corollary 4.5 yield that there exist p and q such that

$$1 < q < 1/\beta_X \le 1/\alpha_X < p < \infty, \quad w \in A^{\rm sym}_p(\mathbb{Z}) \cap A^{\rm sym}_q(\mathbb{Z}).$$

By Theorem 4.6, there exist $c_{p,w}, c_{q,w} \in (0, \infty)$ depending only on p, q and w such that for all $a \in BV[-\pi, \pi]$,

$$\|L(a)\|_{\mathcal{B}(\ell^p(\mathbb{Z},w))} = \|a\|_{M_{\ell^p(\mathbb{Z},w)}} \le c_{p,w} \left(\|a\|_{L^\infty(-\pi,\pi)} + V(a)\right), \tag{6.1}$$

$$\|L(a)\|_{\mathcal{B}(\ell^q(\mathbb{Z},w))} = \|a\|_{M_{\ell^q(\mathbb{Z},w)}} \le c_{q,w} \left(\|a\|_{L^\infty(-\pi,\pi)} + V(a)\right). \tag{6.2}$$

On the other hand, Boyd's interpolation theorem (Theorem 3.1) implies that

$$\begin{aligned} \|a\|_{M_{X(\mathbb{Z},w)}} &= \|L(a)\|_{\mathcal{B}(X(\mathbb{Z},w))} = \|wL(a)w^{-1}\|_{\mathcal{B}(X(\mathbb{Z}))} \\ &\le C_{p,q} \max\left\{\|wL(a)w^{-1}\|_{\mathcal{B}(\ell^p(\mathbb{Z}))}, \|wL(a)w^{-1}\|_{\mathcal{B}(\ell^q(\mathbb{Z}))}\right\} \\ &= C_{p,q} \max\left\{\|L(a)\|_{\mathcal{B}(\ell^p(\mathbb{Z},w))}, \|L(a)\|_{\mathcal{B}(\ell^q(\mathbb{Z},w))}\right\}. \end{aligned} \tag{6.3}$$

Combining (6.1)–(6.3), we arrive at

$$\|a\|_{M_{X(\mathbb{Z},w)}} \le C_{X(\mathbb{Z},w)} \left(\|a\|_{L^\infty(-\pi,\pi)} + V(a)\right)$$

with $C_{X(\mathbb{Z},w)} := C_{p,q} c_{p,w} c_{q,w}$ independent of $a \in BV[-\pi, \pi]$. $\square$

6.3. Convergence of Fejér means of continuous functions of bounded variation in the multiplier norm

Using interpolation techniques, we prove the following approximation result, which plays a crucial role in the verification of property (iv) in Theorem 1.1 for $\mathcal{Y} = C_{X(\mathbb{Z},w)}$, and, thus, in the proof of Theorem 1.3.

Let $C[-\pi,\pi]$ be the Banach space of all continuous 2π-periodic functions $f:\mathbb{R}\to\mathbb{C}$ with the norm

$$\|f\|_{C[-\pi,\pi]} := \sup_{\theta\in[-\pi,\pi]} |f(\theta)|.$$

Lemma 6.2. *Let $X(\mathbb{Z})$ be a reflexive rearrangement-invariant Banach sequence space with nontrivial Boyd indices α_X,β_X and let $w\in A^{\rm sym}_{1/\alpha_X}(\mathbb{Z})\cap A^{\rm sym}_{1/\beta_X}(\mathbb{Z})$. If $a\in C[-\pi,\pi]\cap BV[-\pi,\pi]$, then*

$$\|\sigma_n(a)-a\|_{M_{X(\mathbb{Z},w)}}\to 0 \quad as \quad n\to\infty.$$

Proof. The idea of the proof is borrowed from [5, Lemma 6.2] and [13, Theorems 3.2–3.3]. Fix $n\in\mathbb{Z}_+$. Since $\alpha_X,\beta_X\in(0,1)$ and $w\in A^{\rm sym}_{1/\alpha_X}(\mathbb{Z})\cap A^{\rm sym}_{1/\beta_X}(\mathbb{Z})$, it follows from Corollary 4.5 that there exist p and q such that

$$1<q<1/\beta_X\le 1/\alpha_X<p<\infty, \quad w\in A^{\rm sym}_p(\mathbb{Z})\cap A^{\rm sym}_q(\mathbb{Z}).$$

It follows from Boyd's interpolation theorem (see Theorem 3.1) that

$$\begin{aligned}\|\sigma_n(a)-a\|_{M_{X(\mathbb{Z},w)}} &= \|L(\sigma_n(a)-a)\|_{\mathcal{B}(X(\mathbb{Z},w))} = \|wL(\sigma_n(a)-a)w^{-1}\|_{\mathcal{B}(X(\mathbb{Z}))}\\ &\le C_{p,q}\max\left\{\|wL(\sigma_n(a)-a)w^{-1}\|_{\mathcal{B}(\ell^p(\mathbb{Z}))}, \|wL(\sigma_n(a)-a)w^{-1}\|_{\mathcal{B}(\ell^q(\mathbb{Z}))}\right\}\\ &= C_{p,q}\max\left\{\|L(\sigma_n(a)-a)\|_{\mathcal{B}(\ell^p(\mathbb{Z},w))}, \|L(\sigma_n(a)-a)\|_{\mathcal{B}(\ell^q(\mathbb{Z},w))}\right\}. \end{aligned} \tag{6.4}$$

Since $w\in A^{\rm sym}_p(\mathbb{Z})$, applying Corollary 4.5 once again, we conclude that $w^{1+\delta_2}\in A^{\rm sym}_{p(1+\delta_1)}(\mathbb{Z})$ whenever $|\delta_1|$ and $|\delta_2|$ are sufficiently small. If $p\ge 2$, then one can find sufficiently small δ_1,δ_2 and a number $\theta_p\in(0,1)$ such that

$$\frac{1}{p}=\frac{1-\theta_p}{2}+\frac{\theta_p}{p(1+\delta_1)}, \quad w=1^{1-\theta_p}w^{(1+\delta_2)\theta_p}, \quad w^{1+\delta_2}\in A^{\rm sym}_{p(1+\delta_1)}(\mathbb{Z}). \tag{6.5}$$

If $1<p<2$, then one can find a sufficiently small number $\delta_2>0$, a number $\delta_1<0$ with a sufficiently small $|\delta_1|$, and a number $\theta_p\in(0,1)$ such that all conditions in (6.5) are fulfilled. Taking into account (6.5), we obtain from the Stein–Weiss interpolation theorem (see, e.g., [2, Ch. 4, Theorem 3.6]) that

$$\begin{aligned}&\|L(\sigma_n(a)-a)\|_{\mathcal{B}(\ell^p(\mathbb{Z},w))}\\ &\quad\le \|L(\sigma_n(a)-a)\|^{1-\theta_p}_{\mathcal{B}(\ell^2(\mathbb{Z}))}\|L(\sigma_n(a)-a)\|^{\theta_p}_{\mathcal{B}(\ell^{p(1+\delta_1)}(\mathbb{Z},w^{1+\delta_2}))}.\end{aligned} \tag{6.6}$$

It follows from Theorem 5.1 that

$$\|L(\sigma_n(a)-a)\|_{\mathcal{B}(\ell^2(\mathbb{Z}))}=\|\sigma_n(a)-a\|_{L^\infty(-\pi,\pi)}\le\|\sigma_n(a)-a\|_{C[-\pi,\pi]}. \tag{6.7}$$

On the other hand, Lemmas 6.1 and 5.7 imply that

$$\|L(\sigma_n(a)-a)\|_{\mathcal{B}(\ell^{p(1+\delta_1)}(\mathbb{Z},w^{1+\delta_2}))} = \|\sigma_n(a)-a\|_{M_{\ell^{p(1+\delta_1)}(\mathbb{Z},w^{1+\delta_2})}} \leq 2\|a\|_{M_{\ell^{p(1+\delta_1)}(\mathbb{Z},w^{1+\delta_2})}}. \tag{6.8}$$

Theorem 4.6 yields that there exists a constant $C(p,w,\delta_1,\delta_2)>0$ such that

$$\|a\|_{M_{\ell^{p(1+\delta_1)}(\mathbb{Z},w^{1+\delta_2})}} \leq C(p,w,\delta_1,\delta_2)\left(\|a\|_{L^\infty(-\pi,\pi)}+V(a)\right). \tag{6.9}$$

Combining (6.6)–(6.9), we get

$$\|L(\sigma_n(a)-a)\|_{\mathcal{B}(\ell^p(\mathbb{Z},w))} \leq (2C(p,w,\delta_1,\delta_2))^{\theta_p}\|\sigma_n(a)-a\|_{C[-\pi,\pi]}^{1-\theta_p}\left(\|a\|_{L^\infty(-\pi,\pi)}+V(a)\right)^{\theta_p}. \tag{6.10}$$

Analogously, one can show that there exist $\delta_3,\delta_4\in\mathbb{R}$ and $\theta_q\in(0,1)$ such that $|\delta_3|,|\delta_4|$ are sufficiently small and

$$\frac{1}{q}=\frac{1-\theta_q}{2}+\frac{\theta_q}{q(1+\delta_3)},\quad w=1^{1-\theta_q}w^{(1+\delta_4)\theta_q},\quad w^{1+\delta_4}\in A^{\mathrm{sym}}_{q(1+\delta_3)}(\mathbb{Z}).$$

Hence

$$\|L(\sigma_n(a)-a)\|_{\mathcal{B}(\ell^q(\mathbb{Z},w))} \leq (2C(q,w,\delta_3,\delta_4))^{\theta_q}\|\sigma_n(a)-a\|_{C[-\pi,\pi]}^{1-\theta_q}\left(\|a\|_{L^\infty(-\pi,\pi)}+V(a)\right)^{\theta_q}, \tag{6.11}$$

where $C(q,w,\delta_3,\delta_4)>0$ is some constant depending only on q, w and δ_3,δ_4. It follows from (6.4) and (6.10)–(6.11) that for all $n\in\mathbb{Z}_+$,

$$\begin{aligned}\|\sigma_n(a)-a\|_{M_{X(\mathbb{Z},w)}} &\leq C\max\left\{\left(\|a\|_{L^\infty(-\pi,\pi)}+V(a)\right)^{\theta_p},\left(\|a\|_{L^\infty(-\pi,\pi)}+V(a)\right)^{\theta_q}\right\}\\ &\quad\times\max\left\{\|\sigma_n(a)-a\|_{C[-\pi,\pi]}^{1-\theta_p},\|\sigma_n(a)-a\|_{C[-\pi,\pi]}^{1-\theta_q}\right\},\end{aligned} \tag{6.12}$$

where

$$C:=C_{p,q}\max\left\{(2C(p,w,\delta_1,\delta_2))^{\theta_p},(2C(q,w,\delta_3,\delta_4))^{\theta_q}\right\}.$$

It is well known that for $a\in C[-\pi,\pi]$,

$$\|\sigma_n(a)-a\|_{C[-\pi,\pi]}\to 0\quad\text{as}\quad n\to\infty \tag{6.13}$$

(see, e.g., [19, Ch. I, Subsections 2.5, 2.10, 2.11]). Combining (6.12) and (6.13), we arrive at the desired conclusion. □

6.4. *Proof of Theorem 1.3*

The idea of the proof is borrowed from [8, Theorem 2.53]. It follows from Theorem 1.2 that $M_{X(\mathbb{Z},w)}$ is a Banach algebra. We apply the Zalcman–Rudin theorem (see Theorem 1.1) to

$$\mathcal{X}=M_{X(\mathbb{Z},w)},\quad \mathcal{Y}=C_{X(\mathbb{Z},w)},\quad \mathcal{Z}=H^{\infty,\pm}_{X(\mathbb{Z},w)},$$

and

$$\Phi=\left\{S_n\in\mathcal{B}(M_{X(\mathbb{Z},w)})\ :\ S_na:=\sigma_n(a),\ n\in\mathbb{Z}_+\right\},$$

where $\sigma_n(a)$ denotes the nth Fejér mean of a. It is clear that properties (i) and (ii) in Theorem 1.1 are satisfied. Property (iii) follows from Lemmas 6.1 and 5.7. It remains to verify property (iv). It follows from the definition of $C_{X(\mathbb{Z},w)}$ that given $a \in C_{X(\mathbb{Z},w)}$ and $\varepsilon > 0$, there exists $f \in \mathcal{TP}$ such that $\|a - f\|_{M_{X(\mathbb{Z},w)}} < \varepsilon/3$. Then Lemmas 6.1 and 5.7 imply that

$$\begin{aligned}\|\sigma_n(a) - a\|_{M_{X(\mathbb{Z},w)}} &\le \|\sigma_n(a - f)\|_{M_{X(\mathbb{Z},w)}} + \|\sigma_n(f) - f\|_{M_{X(\mathbb{Z},w)}} + \|f - a\|_{M_{X(\mathbb{Z},w)}} \\ &\le 2\|a - f\|_{M_{X(\mathbb{Z},w)}} + \|\sigma_n(f) - f\|_{M_{X(\mathbb{Z},w)}} \\ &\le \frac{2\varepsilon}{3} + \|\sigma_n(f) - f\|_{M_{X(\mathbb{Z},w)}}. \end{aligned} \tag{6.14}$$

On the other hand, since $f \in \mathcal{TP} \subset C[-\pi,\pi] \cap BV[-\pi,\pi]$, it follows from Lemma 6.2 that

$$\|\sigma_n(f) - f\|_{M_{X(\mathbb{Z},w)}} < \frac{\varepsilon}{3} \tag{6.15}$$

whenever n is large enough. Combining (6.14) and (6.15), we see that

$$\|S_n a - a\|_{M_{X(\mathbb{Z},w)}} = \|\sigma_n(a) - a\|_{M_{X(\mathbb{Z},w)}} < \varepsilon$$

whenever n is large enough. So, property (iv) in Theorem 1.1 is satisfied. Thus, that theorem implies that $C_{X(\mathbb{Z},w)} + H^{\infty,\pm}_{X(\mathbb{Z},w)}$ are closed subspaces of the Banach algebra $M_{X(\mathbb{Z},w)}$.

Now let $a, b \in C_{X(\mathbb{Z},w)} + H^{\infty,\pm}_{X(\mathbb{Z},w)}$. Then there are $a_k, b_k \in \mathcal{TP} + H^{\infty,\pm}_{X(\mathbb{Z},\pm)}$ for $k \in \mathbb{N}$ such that

$$\|a - a_k\|_{M_{X(\mathbb{Z},w)}} \to 0, \quad \|b - b_k\|_{M_{X(\mathbb{Z},w)}} \to 0 \quad \text{as} \quad k \to \infty. \tag{6.16}$$

It follows from Lemma 6.1 and Theorem 5.5 that $a_k, b_k \in L^\infty(-\pi,\pi) \subset L^2(-\pi,\pi)$. So a_k and b_k can be represented by their Fourier series convergent in $L^2(-\pi,\pi)$. Looking at the Fourier series representation of $a_k b_k \in L^\infty(-\pi,\pi)$, we conclude that $a_k b_k \in \mathcal{TP} + H^{\infty,\pm}_{X(\mathbb{Z},w)}$ for all $k \in \mathbb{N}$. It follows from (6.16) that

$$\|a_k b_k - ab\|_{M_{X(\mathbb{Z},w)}} \to 0 \quad \text{as} \quad k \to \infty.$$

Since $C_{X(\mathbb{Z},w)} + H^{\infty,\pm}_{X(\mathbb{Z},w)}$ is a closed subspace of the Banach algebra $M_{X(\mathbb{Z},w)}$, we conclude that $ab \in C_{X(\mathbb{Z},w)} + H^{\infty,\pm}_{X(\mathbb{Z},w)}$. Thus, $C_{X(\mathbb{Z},w)} + H^{\infty,\pm}_{X(\mathbb{Z},w)}$ is an algebra. $\square$

Funding

This work is funded by national funds through the FCT — Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 (https://doi.org/10.54499/UIDB/00297/2020) and UIDP/ 00297/2020 (https://doi.org/10.54499/UIDP/00297/2020) (Center for Mathematics and Applications). The second author is funded by national funds through the FCT — Fundação para a Ciência e a Tecnologia, I.P., under the scope of the PhD scholarship UI/BD/ 152570/2022.

Acknowledgments

We would like to thank the anonymous referees for useful remarks.

Data availability

No data was used for the research described in the article.

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