A maximal inequality for dependent random variables

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Abstract

For a sequence $\{X_n,\, n \geqslant 1\}$ of random variables satisfying $\mathbb{E}|X_n| < \infty$ for all $n \geqslant 1$, a maximal inequality is established, and used to obtain strong law of large numbers for dependent random variables.

Key words and phrases: Maximal inequality, dependent random variables, strong law of large numbers

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1 Introduction

The most renowned maximal inequality in Probability Theory is, perhaps, the Kolmogorov inequality [12] published in 1928: if $\{X_n,\, n \geqslant 1\}$ is a sequence of (mutually) independent random variables with mean values $\mathbb{E}\, X_n = 0$ and variances $\mathbb{V}(X_n) < \infty$ for each $n \geqslant 1$ then, for any $\varepsilon > 0$,

$$\mathbb{P}\left\{\max_{1\leqslant k\leqslant n} |X_1 + X_2 + \ldots + X_k| > \varepsilon\right\} \leqslant \frac{1}{\varepsilon^2}\sum_{k=1}^{n} \mathbb{V}(X_k).$$

Since then, several authors have extended Kolmogorov's inequality in many ways, namely by weakening the (mutual) independence assumption of the random variables (see [2], [4], [6], [7], [9], [13], [19], [20], [21], [23], [25] among others).

It is commonly known that the significant role of maximal inequalities stems from the fact that these are a cornerstone in the obtention of sharp limit theorems. Recently, it was proved in [27] the Baum-Katz theorem [1] for sequences of pairwise independent and identically distributed random variables under optimal moment conditions, by using a technique developed in [22]. The key ingredient for this achievement was an asymptotic maximal inequality which takes a particular format on the scenario of pairwise independence and identical distribution

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admitted for the random sequence. In this paper, the aforementioned asymptotic maximal inequality is brought to light in a general setting, i.e. the random variables $X_1, X_2, \ldots$ are not assumed to be independent, nor having any particular dependence structure. As consequence, Strong Law of Large Numbers (SLLN) with respect to optimal norming constants can be obtained for sequences of dependent random variables. Over the last decades, many stochastic models involving dependent random variables have emerged in mathematical statistics, statistical physics or reliability theory (see [3] and [11]), whence the importance of having statements (particularly, SLLN) that allow us to deal with it.

We shall need to introduce some relevant notations which will be employed along this paper. Associated to a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we shall consider the space $\mathscr{L}_p$ $(p > 0)$ of all measurable functions X (necessarily random variables) for which $\mathbb{E}\,|X|^p < \infty$. We shall write $x \wedge y$ and $x \vee y$ for $\min\{x, y\}$ and $\max\{x, y\}$, respectively. For any measurable function X, we shall define its positive and negative parts by $X^+ = 0 \vee X$ and $X^- = 0 \vee (-X)$, respectively. Given an event A, we shall denote the indicator random variable of the event A by I_A. Throughout, $\lfloor x \rfloor$ shall be used to represent the largest integer not greater than x. To make the computations be simpler looking, we shall employ the letter C to denote any positive constant that can be explicitly computed and whose value may change between appearances; the symbol $C(r)$ has identical meaning with the additional information that the constant depends on r.

2 Maximal inequality

Let L, K be real numbers such that $0 < L \leqslant K$. We shall introduce the function $g_L(t) = (t \wedge L) \vee (-L)$, which describes the truncation at level L, and

$$h_{L,K}(t) = |g_K(t) - g_L(t)| = 0 \vee [(K - L) \wedge (t - L)] - 0 \wedge [(L - K) \vee (t + L)].$$

Considering two random variables X, Y we shall put

$$\Delta_{X,Y}(x, y) := \mathbb{P}\{X \leqslant x, Y \leqslant y\} - \mathbb{P}\{X \leqslant x\}\,\mathbb{P}\{Y \leqslant y\}$$

and define the following covariance quantities

$$G_{X,Y}(L) := \operatorname{Cov}\big(g_L(X), g_L(Y)\big) = \int_{-L}^{L}\int_{-L}^{L} \Delta_{X,Y}(u, v)\,\mathrm{d}u\,\mathrm{d}v, \tag{2.1}$$

$$\begin{aligned} H_{X,Y}(L, K) &:= \operatorname{Cov}\big(h_{L,K}(X), h_{L,K}(Y)\big) \\ &= \int_{L}^{K}\int_{L}^{K} \Delta_{X,Y}(u, v)\,\mathrm{d}u\,\mathrm{d}v + \int_{-K}^{-L}\int_{-K}^{-L} \Delta_{X,Y}(u, v)\,\mathrm{d}u\,\mathrm{d}v \\ &\quad - \int_{L}^{K}\int_{-K}^{-L} \Delta_{X,Y}(u, v)\,\mathrm{d}u\,\mathrm{d}v - \int_{-K}^{-L}\int_{L}^{K} \Delta_{X,Y}(u, v)\,\mathrm{d}u\,\mathrm{d}v \end{aligned} \tag{2.2}$$

where the last identities in (2.1) and (2.2) are consequences of Theorem 2.3 in [29]. Let us notice that if X, Y are independent random variables then $\Delta_{X,Y}(u, v) = 0$ for any reals u, v.

The main result is given in the following:

Theorem 1 *Let $r > 1$ be an integer and $\{X_n,\, n \geqslant 1\}$ a sequence of random variables verifying $\mathbb{E}|X_n| < \infty$ for all n. If $\{b_n\}$ is nondecreasing sequence of positive constants, and $\{a_{n,k},\, 1 \leqslant k \leqslant n+1, n \geqslant 1\}$ is an array of positive constants satisfying*

(a) $\displaystyle\sum_{k=1}^{n+1} a_{n,k} = O(b_{r^n})$ *as* $n \to \infty$,

(b) $\displaystyle\sum_{k=1}^{n+1} \max_{0\leqslant h<r^{n-k+1}} \sum_{j=1+hr^k}^{hr^k+r^k} \mathbb{E}|X_j| I_{\{|X_j|>b_{r^{k-1}}\}} = o(b_{r^n})$ *as* $n \to \infty$,

then, for any $\varepsilon > 0$, there is a positive integer n_0 and a constant $C(r) > 0$ such that

$$\begin{aligned}
&\mathbb{P}\left\{\max_{1\leqslant m<r^{n+1}} \left|\sum_{k=1}^{m}(X_k - \mathbb{E}\,X_k)\right| > \varepsilon b_{r^n}\right\} \\
&\quad\leqslant \sum_{k=1}^{r^{n+1}-1} \mathbb{P}\left\{|X_k| > b_{r^{n+1}}\right\} + \frac{C(r)}{\varepsilon^2}\sum_{k=1}^{n+1}\sum_{j=1}^{r^{n+1}} a_{n,k}^{-2}\left(\mathbb{E}X_j^2 I_{\left\{|X_j|\leqslant b_{r^k}\right\}} + b_{r^k}^2 \mathbb{P}\left\{|X_j| > b_{r^{k-1}}\right\}\right) \\
&\qquad + \frac{C(r)}{\varepsilon^2}\sum_{k=1}^{n+1} a_{n,k}^{-2} \max_{1\leqslant \ell\leqslant r-1} \sum_{h=0}^{r^{n-k+1}-1} \left[\sum_{1+hr^k\leqslant i<j\leqslant hr^k+\ell r^{k-1}} G_{X_i,X_j}(b_{r^{k-1}})\right]^+ \\
&\qquad + \frac{C(r)}{\varepsilon^2}\sum_{k=1}^{n+1} a_{n,k}^{-2} \sum_{h=0}^{r^{n-k+1}-1} \left[\sum_{1+hr^k\leqslant i<j\leqslant hr^k+r^k} H_{X_i,X_j}(b_{r^{k-1}}, b_{r^k})\right]^+
\end{aligned}$$

for all $n \geqslant n_0$.

Proof. Fix $\varepsilon > 0$ and an integer $r > 1$. From assumption (b), there is a positive integer n_0 such that

$$\sum_{k=1}^{n+1} \max_{0\leqslant h<r^{n-k+1}} \sum_{j=1+hr^k}^{hr^k+r^k} \mathbb{E}|X_j| I_{\{|X_j|>b_{r^{k-1}}\}} < \frac{\varepsilon b_{r^n}}{8} \tag{2.3}$$

for all $n \geqslant n_0$. Setting

$$X'_{j,b_{r^{n+1}}} := g_{b_{r^{n+1}}}(X_j) = X_j I_{\{|X_j|\leqslant b_{r^{n+1}}\}} + b_{r^{n+1}} I_{\{X_j > b_{r^{n+1}}\}} - b_{r^{n+1}} I_{\{X_j < -b_{r^{n+1}}\}},$$

$$\Gamma_n := \bigcap_{j=1}^{r^{n+1}-1} \{X'_{j,b_{r^{n+1}}} = X_j\},$$

$$S_{m,b_{r^{n+1}}} := \sum_{j=1}^{m}(X'_{j,b_{r^{n+1}}} - \mathbb{E}X'_{j,b_{r^{n+1}}}), \qquad n, m \geqslant 0$$

we obtain

$$\begin{aligned}
&\mathbb{P}\left[\left\{\max_{1\leqslant m<r^{n+1}} \left|\sum_{j=1}^{m}(X_j - \mathbb{E}X_j)\right| > \varepsilon b_{r^n}\right\} \cap \Gamma_n\right] \\
&\quad = \mathbb{P}\left\{\max_{1\leqslant m<r^{n+1}} \left|\sum_{j=1}^{m}(X'_{j,b_{r^{n+1}}} - \mathbb{E}X_j)\right| > \varepsilon b_{r^n}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{P}\left\{ \max_{1\leqslant m<r^{n+1}} \left| \sum_{j=1}^{m} [X'_{j,b_{r^{n+1}}} - \mathbb{E}X'_{j,b_{r^{n+1}}} + \mathbb{E}(X'_{j,b_{r^{n+1}}} - X_j)] \right| > \varepsilon b_{r^n} \right\} \\
&\leqslant \mathbb{P}\left\{ \max_{1\leqslant m<r^{n+1}} \left| \sum_{j=1}^{m} (X'_{j,b_{r^{n+1}}} - \mathbb{E}X'_{j,b_{r^{n+1}}}) \right| + \max_{1\leqslant m<r^{n+1}} \left| \sum_{j=1}^{m} \mathbb{E}(X'_{j,b_{r^{n+1}}} - X_j) \right| > \varepsilon b_{r^n} \right\} \\
&= \mathbb{P}\left\{ \max_{1\leqslant m<r^{n+1}} |S_{m,b_{r^{n+1}}}| + \max_{1\leqslant m<r^{n+1}} \left| \sum_{j=1}^{m} \mathbb{E}(X'_{j,b_{r^{n+1}}} - X_j) \right| > \varepsilon b_{r^n} \right\}
\end{aligned}
$$

and, for any $n \geqslant n_0$,

$$
\begin{aligned}
&\max_{1\leqslant m<r^{n+1}} \left| \sum_{j=1}^{m} \mathbb{E}(X'_{j,b_{r^{n+1}}} - X_j) \right| \\
&\quad = \max_{1\leqslant m<r^{n+1}} \left| \sum_{j=1}^{m} \mathbb{E}(b_{r^{n+1}} I_{\{X_j > b_{r^{n+1}}\}} - b_{r^{n+1}} I_{\{X_j < -b_{r^{n+1}}\}} - X_j I_{\{|X_j| > b_{r^{n+1}}\}}) \right| \\
&\quad \leqslant \max_{1\leqslant m<r^{n+1}} \sum_{j=1}^{m} \mathbb{E}|X_j I_{\{|X_j| > b_{r^{n+1}}\}} + b_{r^{n+1}} I_{\{X_j < -b_{r^{n+1}}\}} - b_{r^{n+1}} I_{\{X_j > b_{r^{n+1}}\}}| \\
&\quad = \max_{1\leqslant m<r^{n+1}} \sum_{j=1}^{m} \mathbb{E}(|X_j| - b_{r^{n+1}}) I_{\{|X_j| > b_{r^{n+1}}\}} \\
&\quad \leqslant \sum_{j=1}^{r^{n+1}} \mathbb{E}|X_j| I_{\{|X_j| > b_{r^n}\}} \\
&\quad \leqslant \sum_{k=1}^{n+1} \sum_{j=1}^{r^k} \mathbb{E}|X_j| I_{\left\{|X_j| > b_{r^{k-1}}\right\}} \\
&\quad \leqslant \sum_{k=1}^{n+1} \max_{0\leqslant h<r^{n-k+1}} \sum_{j=1+hr^k}^{hr^k+r^k} \mathbb{E}|X_j| I_{\left\{|X_j| > b_{r^{k-1}}\right\}} \\
&\quad \underset{(2.3)}{\leqslant} \frac{\varepsilon b_{r^n}}{8}.
\end{aligned}
$$

Hence, for every $n \geqslant n_0$,

$$
\begin{aligned}
&\mathbb{P}\left\{ \max_{1\leqslant m<r^{n+1}} \left| \sum_{j=1}^{m} (X_j - \mathbb{E}X_j) \right| > \varepsilon b_{r^n} \right\} \\
&\quad = \mathbb{P}\left[\left\{ \max_{1\leqslant m<r^{n+1}} \left| \sum_{j=1}^{m} (X_j - \mathbb{E}X_j) \right| > \varepsilon b_{r^n} \right\} \cap \Gamma_n \right] \\
&\qquad + \mathbb{P}\left[\left\{ \max_{1\leqslant m<r^{n+1}} \left| \sum_{j=1}^{m} (X_j - \mathbb{E}X_j) \right| > \varepsilon b_{r^n} \right\} \cap \Gamma_n^{\complement} \right]
\end{aligned}
$$

$$\leqslant \mathbb{P}\left\{\max_{1\leqslant m<r^{n+1}}|S_{m,b_{r^{n+1}}}|>\frac{\varepsilon b_{r^n}}{2}\right\}+\sum_{k=1}^{r^{n+1}-1}\mathbb{P}\left\{|X_k|>b_{r^{n+1}}\right\}.$$

Moreover, for each $1\leqslant m<r^{n+1}$, we have

$$\begin{aligned}
S_{m,b_{r^{n+1}}} &= S_{m,b_1}+\sum_{k=1}^{n+1}(S_{m,b_{r^k}}-S_{m,b_{r^{k-1}}}) \\
&= (\overbrace{S_{r^0\lfloor\frac{m}{r^0}\rfloor,b_{r^0}}}^{=S_{m,b_1}}-\underbrace{S_{r^{n+1}\lfloor\frac{m}{r^{n+1}}\rfloor,b_{r^{n+1}}}}_{=S_{0,b_{r^{n+1}}}=0})+\sum_{k=1}^{n+1}(S_{m,b_{r^k}}-S_{m,b_{r^{k-1}}}) \\
&= \sum_{k=1}^{n+1}(S_{r^{k-1}\lfloor\frac{m}{r^{k-1}}\rfloor,b_{r^{k-1}}}-S_{r^k\lfloor\frac{m}{r^k}\rfloor,b_{r^k}})+\sum_{k=1}^{n+1}(S_{m,b_{r^k}}-S_{m,b_{r^{k-1}}}) \\
&= \sum_{k=1}^{n+1}(S_{r^{k-1}\lfloor\frac{m}{r^{k-1}}\rfloor,b_{r^{k-1}}}-S_{r^k\lfloor\frac{m}{r^k}\rfloor,b_{r^{k-1}}}) \\
&\quad +\sum_{k=1}^{n+1}(S_{m,b_{r^k}}-S_{m,b_{r^{k-1}}}-S_{r^k\lfloor\frac{m}{r^k}\rfloor,b_{r^k}}+S_{r^k\lfloor\frac{m}{r^k}\rfloor,b_{r^{k-1}}}).
\end{aligned} \tag{2.4}$$

Further,

$$S_{m,b_{r^k}}-S_{m,b_{r^{k-1}}}=\sum_{j=1}^{m}[(X'_{j,b_{r^k}}-X'_{j,b_{r^{k-1}}})-\mathbb{E}(X'_{j,b_{r^k}}-X'_{j,b_{r^{k-1}}})] \tag{2.5}$$

and

$$S_{r^k\lfloor\frac{m}{r^k}\rfloor,b_{r^k}}-S_{r^k\lfloor\frac{m}{r^k}\rfloor,b_{r^{k-1}}}=\sum_{j=1}^{r^k\lfloor\frac{m}{r^k}\rfloor}[(X'_{j,b_{r^k}}-X'_{j,b_{r^{k-1}}})-\mathbb{E}(X'_{j,b_{r^k}}-X'_{j,b_{r^{k-1}}})]. \tag{2.6}$$

Recall that

$$r^k\left\lfloor\frac{m}{r^k}\right\rfloor\leqslant m<r^k\left\lfloor\frac{m}{r^k}\right\rfloor+r^k,\qquad 1\leqslant k\leqslant n+1 \tag{2.7}$$

because $\lfloor t\rfloor\leqslant t<\lfloor t\rfloor+1$ for any real number t (see [8]). Thereby, (2.5), (2.6) and (2.7) entail

$$\begin{aligned}
&|S_{m,b_{r^k}}-S_{m,b_{r^{k-1}}}-S_{r^k\lfloor\frac{m}{r^k}\rfloor,b_{r^k}}+S_{r^k\lfloor\frac{m}{r^k}\rfloor,b_{r^{k-1}}}| \\
&\quad=\left|\sum_{j=1+r^k\lfloor\frac{m}{r^k}\rfloor}^{m}[(X'_{j,b_{r^k}}-X'_{j,b_{r^{k-1}}})-\mathbb{E}(X'_{j,b_{r^k}}-X'_{j,b_{r^{k-1}}})]\right| \\
&\quad\leqslant\sum_{j=1+r^k\lfloor\frac{m}{r^k}\rfloor}^{m}|(X'_{j,b_{r^k}}-X'_{j,b_{r^{k-1}}})-\mathbb{E}(X'_{j,b_{r^k}}-X'_{j,b_{r^{k-1}}})| \\
&\quad\leqslant\sum_{j=1+r^k\lfloor\frac{m}{r^k}\rfloor}^{r^k\lfloor\frac{m}{r^k}\rfloor+r^k}|(X'_{j,b_{r^k}}-X'_{j,b_{r^{k-1}}})-\mathbb{E}(X'_{j,b_{r^k}}-X'_{j,b_{r^{k-1}}})|
\end{aligned} \tag{2.8}$$

$$
\begin{aligned}
&\leqslant \sum_{j=1+r^k\lfloor\frac{m}{r^k}\rfloor}^{r^k\lfloor\frac{m}{r^k}\rfloor+r^k} (|X'_{j,b_{r^k}} - X'_{j,b_{r^{k-1}}}| + \mathbb{E}|X'_{j,b_{r^k}} - X'_{j,b_{r^{k-1}}}|) \\
&= \sum_{j=1+r^k\lfloor\frac{m}{r^k}\rfloor}^{r^k\lfloor\frac{m}{r^k}\rfloor+r^k} (|X'_{j,b_{r^k}} - X'_{j,b_{r^{k-1}}}| - \mathbb{E}|X'_{j,b_{r^k}} - X'_{j,b_{r^{k-1}}}|) + 2\sum_{j=1+r^k\lfloor\frac{m}{r^k}\rfloor}^{r^k\lfloor\frac{m}{r^k}\rfloor+r^k} \mathbb{E}|X'_{j,b_{r^k}} - X'_{j,b_{r^{k-1}}}| \\
&\leqslant \left| \sum_{j=1+r^k\lfloor\frac{m}{r^k}\rfloor}^{r^k\lfloor\frac{m}{r^k}\rfloor+r^k} (X''_{j,b_{r^{k-1}},b_{r^k}} - \mathbb{E}X''_{j,b_{r^{k-1}},b_{r^k}}) \right| + 2\sum_{j=1+r^k\lfloor\frac{m}{r^k}\rfloor}^{r^k\lfloor\frac{m}{r^k}\rfloor+r^k} \mathbb{E}|X_j| I_{\{|X_j|>b_{r^{k-1}}\}}
\end{aligned}
$$

where

$$X''_{j,b_{r^{k-1}},b_{r^k}} := |X'_{j,b_{r^k}} - X'_{j,b_{r^{k-1}}}| = h_{b_{r^{k-1}},b_{r^k}}(X_j).$$

From (2.7) we still get

$$r\left\lfloor\frac{m}{r^k}\right\rfloor \leqslant \frac{m}{r^{k-1}} < r\left\lfloor\frac{m}{r^k}\right\rfloor + r, \qquad 1 \leqslant k \leqslant n+1$$

whence

$$r\left\lfloor\frac{m}{r^k}\right\rfloor \leqslant \left\lfloor\frac{m}{r^{k-1}}\right\rfloor \leqslant r\left\lfloor\frac{m}{r^k}\right\rfloor + (r-1),$$

i.e.

$$r^{k-1}\left\lfloor\frac{m}{r^{k-1}}\right\rfloor \in \left\{r^k\left\lfloor\frac{m}{r^k}\right\rfloor, r^k\left\lfloor\frac{m}{r^k}\right\rfloor + r^{k-1}, r^k\left\lfloor\frac{m}{r^k}\right\rfloor + 2r^{k-1}, \ldots, r^k\left\lfloor\frac{m}{r^k}\right\rfloor + (r-1)r^{k-1}\right\}$$

for every $1 \leqslant k \leqslant n+1$, $1 \leqslant m < r^{n+1}$ $(n \geqslant 0)$. Thus,

$$
\begin{aligned}
|S_{r^{k-1}\lfloor\frac{m}{r^{k-1}}\rfloor,b_{r^{k-1}}} - S_{r^k\lfloor\frac{m}{r^k}\rfloor,b_{r^{k-1}}}| &= \left| \sum_{j=1+r^k\lfloor\frac{m}{r^k}\rfloor}^{r^{k-1}\lfloor\frac{m}{r^{k-1}}\rfloor} (X'_{j,b_{r^{k-1}}} - \mathbb{E}X'_{j,b_{r^{k-1}}}) \right| \\
&\leqslant \max_{0\leqslant\ell\leqslant r-1} \left| \sum_{j=1+r^k\lfloor\frac{m}{r^k}\rfloor}^{r^k\lfloor\frac{m}{r^k}\rfloor+\ell r^{k-1}} (X'_{j,b_{r^{k-1}}} - \mathbb{E}X'_{j,b_{r^{k-1}}}) \right| \qquad (2.9) \\
&= \max_{1\leqslant\ell\leqslant r-1} \left| \sum_{j=1+r^k\lfloor\frac{m}{r^k}\rfloor}^{r^k\lfloor\frac{m}{r^k}\rfloor+\ell r^{k-1}} (X'_{j,b_{r^{k-1}}} - \mathbb{E}X'_{j,b_{r^{k-1}}}) \right|.
\end{aligned}
$$

By (2.4), (2.8) and (2.9), it follows

$$
\begin{aligned}
|S_{m,b_{r^{n+1}}}| \leqslant &\sum_{k=1}^{n+1} \max_{1\leqslant\ell\leqslant r-1} \left| \sum_{j=1+r^k\lfloor\frac{m}{r^k}\rfloor}^{r^k\lfloor\frac{m}{r^k}\rfloor+\ell r^{k-1}} (X'_{j,b_{r^{k-1}}} - \mathbb{E}X'_{j,b_{r^{k-1}}}) \right| \\
&+ \sum_{k=1}^{n+1} \left| \sum_{j=1+r^k\lfloor\frac{m}{r^k}\rfloor}^{r^k\lfloor\frac{m}{r^k}\rfloor+r^k} (X''_{j,b_{r^{k-1}},b_{r^k}} - \mathbb{E}X''_{j,b_{r^{k-1}},b_{r^k}}) \right| + 2\sum_{k=1}^{n+1} \sum_{j=1+r^k\lfloor\frac{m}{r^k}\rfloor}^{r^k\lfloor\frac{m}{r^k}\rfloor+r^k} \mathbb{E}|X_j| I_{\{|X_j|>b_{r^{k-1}}\}}
\end{aligned}
$$

so that

$$\begin{aligned}
\max_{1\leqslant m<r^{n+1}}|S_{m,b_{r^{n+1}}}| &\leqslant \sum_{k=1}^{n+1}\max_{1\leqslant m<r^{n+1}}\left[\max_{1\leqslant \ell\leqslant r-1}\left|\sum_{j=1+r^k\lfloor\frac{m}{r^k}\rfloor+\ell r^{k-1}}^{r^k\lfloor\frac{m}{r^k}\rfloor+\ell r^{k-1}}(X'_{j,b_{r^{k-1}}}-\mathbb{E}X'_{j,b_{r^{k-1}}})\right|\right]\\
&\quad+\sum_{k=1}^{n+1}\max_{1\leqslant m<r^{n+1}}\left|\sum_{j=1+r^k\lfloor\frac{m}{r^k}\rfloor}^{r^k\lfloor\frac{m}{r^k}\rfloor+r^k}(X''_{j,b_{r^{k-1}},b_{r^k}}-\mathbb{E}X''_{j,b_{r^{k-1}},b_{r^k}})\right|\\
&\quad+2\sum_{k=1}^{n+1}\max_{1\leqslant m<r^{n+1}}\sum_{j=1+r^k\lfloor\frac{m}{r^k}\rfloor}^{r^k\lfloor\frac{m}{r^k}\rfloor+r^k}\mathbb{E}|X_j|I_{\{|X_j|>b_{r^{k-1}}\}} \qquad (2.10)\\
&\leqslant \sum_{k=1}^{n+1}\max_{1\leqslant \ell\leqslant r-1}\left[\max_{0\leqslant h<r^{n-k+1}}\left|\sum_{j=1+hr^k}^{hr^k+\ell r^{k-1}}(X'_{j,b_{r^{k-1}}}-\mathbb{E}X'_{j,b_{r^{k-1}}})\right|\right]\\
&\quad+\sum_{k=1}^{n+1}\max_{0\leqslant h<r^{n-k+1}}\left|\sum_{j=1+hr^k}^{hr^k+r^k}(X''_{j,b_{r^{k-1}},b_{r^k}}-\mathbb{E}X''_{j,b_{r^{k-1}},b_{r^k}})\right|\\
&\quad+2\sum_{k=1}^{n+1}\max_{0\leqslant h<r^{n-k+1}}\sum_{j=1+hr^k}^{hr^k+r^k}\mathbb{E}|X_j|I_{\{|X_j|>b_{r^{k-1}}\}}
\end{aligned}$$

because

$$0=\left\lfloor\frac{1}{r^k}\right\rfloor\leqslant\left\lfloor\frac{m}{r^k}\right\rfloor\leqslant\frac{m}{r^k}<r^{n-k+1}\quad\Longrightarrow\quad 0\leqslant\left\lfloor\frac{m}{r^k}\right\rfloor\leqslant r^{n-k+1}-1$$

for all $1\leqslant k\leqslant n+1$, $1\leqslant m<r^{n+1}$ ($n\geqslant 0$). Combining (2.3) and (2.10) we obtain, for every $n\geqslant n_0$,

$$\begin{aligned}
&\mathbb{P}\left\{\max_{1\leqslant m<r^{n+1}}|S_{m,b_{r^{n+1}}}|>\frac{\varepsilon b_{r^n}}{2}\right\}\\
&\leqslant\mathbb{P}\left\{\sum_{k=1}^{n+1}\max_{1\leqslant \ell\leqslant r-1}\left[\max_{0\leqslant h<r^{n-k+1}}\left|\sum_{j=1+hr^k}^{hr^k+\ell r^{k-1}}(X'_{j,b_{r^{k-1}}}-\mathbb{E}X'_{j,b_{r^{k-1}}})\right|\right]\right.\\
&\quad\left.+\sum_{k=1}^{n+1}\max_{0\leqslant h<r^{n-k+1}}\left|\sum_{j=1+hr^k}^{hr^k+r^k}(X''_{j,b_{r^{k-1}},b_{r^k}}-\mathbb{E}X''_{j,b_{r^{k-1}},b_{r^k}})\right|>\frac{\varepsilon b_{r^n}}{4}\right\}\\
&\leqslant\mathbb{P}\left\{\sum_{k=1}^{n+1}\max_{1\leqslant \ell\leqslant r-1}\left[\max_{0\leqslant h<r^{n-k+1}}\left|\sum_{j=1+hr^k}^{hr^k+\ell r^{k-1}}(X'_{j,b_{r^{k-1}}}-\mathbb{E}X'_{j,b_{r^{k-1}}})\right|\right]>\frac{\varepsilon b_{r^n}}{8}\right\}\\
&\quad+\mathbb{P}\left\{\sum_{k=1}^{n+1}\max_{0\leqslant h<r^{n-k+1}}\left|\sum_{j=1+hr^k}^{hr^k+r^k}(X''_{j,b_{r^{k-1}},b_{r^k}}-\mathbb{E}X''_{j,b_{r^{k-1}},b_{r^k}})\right|>\frac{\varepsilon b_{r^n}}{8}\right\}.
\end{aligned}$$

The thesis holds by noting that

$$\mathbb{P}\left\{\sum_{k=1}^{n+1}\max_{1\leqslant \ell\leqslant r-1}\left[\max_{0\leqslant h<r^{n-k+1}}\left|\sum_{j=1+hr^k}^{hr^k+\ell r^{k-1}}(X'_{j,b_{r^{k-1}}}-\mathbb{E}X'_{j,b_{r^{k-1}}})\right|\right]>\frac{\varepsilon b_{r^n}}{8}\right\}$$

$$
\begin{aligned}
&\leqslant \mathbb{P}\left\{\sum_{k=1}^{n+1}\max_{1\leqslant\ell\leqslant r-1}\left[\max_{0\leqslant h<r^{n-k+1}}\left|\sum_{j=1+hr^k}^{hr^k+\ell r^{k-1}}(X'_{j,b_{r^{k-1}}}-\mathbb{E}X'_{j,b_{r^{k-1}}})\right|\right]>\frac{\varepsilon}{C}\sum_{k=1}^{n+1}a_{n,k}\right\}\\
&\leqslant \sum_{k=1}^{n+1}\mathbb{P}\left\{\max_{1\leqslant\ell\leqslant r-1}\left[\max_{0\leqslant h<r^{n-k+1}}\left|\sum_{j=1+hr^k}^{hr^k+\ell r^{k-1}}(X'_{j,b_{r^{k-1}}}-\mathbb{E}X'_{j,b_{r^{k-1}}})\right|\right]>\frac{\varepsilon a_{n,k}}{C}\right\}\\
&\leqslant \frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\mathbb{E}\left\{\max_{1\leqslant\ell\leqslant r-1}\left[\max_{0\leqslant h<r^{n-k+1}}\left|\sum_{j=1+hr^k}^{hr^k+\ell r^{k-1}}(X'_{j,b_{r^{k-1}}}-\mathbb{E}X'_{j,b_{r^{k-1}}})\right|\right]\right\}^2\\
&= \frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\mathbb{E}\max_{1\leqslant\ell\leqslant r-1}\left\{\max_{0\leqslant h<r^{n-k+1}}\left[\sum_{j=1+hr^k}^{hr^k+\ell r^{k-1}}(X'_{j,b_{r^{k-1}}}-\mathbb{E}X'_{j,b_{r^{k-1}}})\right]^2\right\}\\
&\leqslant \frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\mathbb{E}\sum_{\ell=1}^{r-1}\left\{\sum_{h=0}^{r^{n-k+1}-1}\left[\sum_{j=1+hr^k}^{hr^k+\ell r^{k-1}}(X'_{j,b_{r^{k-1}}}-\mathbb{E}X'_{j,b_{r^{k-1}}})\right]^2\right\}\\
&\leqslant \frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\sum_{\ell=1}^{r-1}\sum_{h=0}^{r^{n-k+1}-1}\sum_{j=1+hr^k}^{hr^k+\ell r^{k-1}}\mathbb{E}|X'_{j,b_{r^{k-1}}}|^2\\
&\quad+\frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\sum_{\ell=1}^{r-1}\sum_{h=0}^{r^{n-k+1}-1}\sum_{1+hr^k\leqslant i<j\leqslant hr^k+\ell r^{k-1}}\mathrm{Cov}(X'_{i,b_{r^{k-1}}};X'_{j,b_{r^{k-1}}})\\
&= \frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\sum_{\ell=1}^{r-1}\sum_{h=0}^{r^{n-k+1}-1}\sum_{j=1+hr^k}^{hr^k+\ell r^{k-1}}\left(\mathbb{E}X_j^2I_{\left\{|X_j|\leqslant b_{r^{k-1}}\right\}}+b_{r^{k-1}}^2\mathbb{P}\left\{|X_j|>b_{r^{k-1}}\right\}\right)\\
&\quad+\frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\sum_{\ell=1}^{r-1}\sum_{h=0}^{r^{n-k+1}-1}\sum_{1+hr^k\leqslant i<j\leqslant hr^k+\ell r^{k-1}}G_{X_i,X_j}(b_{r^{k-1}})\\
&\leqslant \frac{C(r)}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\sum_{h=0}^{r^{n-k+1}-1}\sum_{j=1+hr^k}^{hr^k+r^k}\left(\mathbb{E}X_j^2I_{\left\{|X_j|\leqslant b_{r^{k-1}}\right\}}+b_{r^{k-1}}^2\mathbb{P}\left\{|X_j|>b_{r^{k-1}}\right\}\right)\\
&\quad+\frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\sum_{\ell=1}^{r-1}\sum_{h=0}^{r^{n-k+1}-1}\left[\sum_{1+hr^k\leqslant i<j\leqslant hr^k+\ell r^{k-1}}G_{X_i,X_j}(b_{r^{k-1}})\right]^+\\
&\leqslant \frac{C(r)}{\varepsilon^2}\sum_{k=1}^{n+1}\sum_{j=1}^{r^{n+1}}\frac{1}{a_{n,k}^2}\mathbb{E}X_j^2I_{\left\{|X_j|\leqslant b_{r^{k-1}}\right\}}+\frac{C(r)}{\varepsilon^2}\sum_{k=1}^{n+1}\sum_{j=1}^{r^{n+1}}\frac{b_{r^{k-1}}^2}{a_{n,k}^2}\mathbb{P}\left\{|X_j|>b_{r^{k-1}}\right\}\\
&\quad+\frac{C(r)}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\max_{1\leqslant\ell\leqslant r-1}\sum_{h=0}^{r^{n-k+1}-1}\left[\sum_{1+hr^k\leqslant i<j\leqslant hr^k+\ell r^{k-1}}G_{X_i,X_j}(b_{r^{k-1}})\right]^+
\end{aligned}
$$

and

$$
\mathbb{P}\left\{\sum_{k=1}^{n+1}\max_{0\leqslant h<r^{n-k+1}}\left|\sum_{j=1+hr^k}^{hr^k+r^k}(X''_{j,b_{r^{k-1}},b_{r^k}}-\mathbb{E}X''_{j,b_{r^{k-1}},b_{r^k}})\right|>\frac{\varepsilon b_{r^n}}{8}\right\}
$$

$$
\begin{aligned}
&\leqslant \mathbb{P}\left\{\sum_{k=1}^{n+1}\max_{0\leqslant h<r^{n-k+1}}\left|\sum_{j=1+hr^k}^{hr^k+r^k}(X''_{j,b_{r^{k-1}},b_{r^k}}-\mathbb{E}X''_{j,b_{r^{k-1}},b_{r^k}})\right|>\frac{\varepsilon}{C}\sum_{k=1}^{n+1}a_{n,k}\right\}\\
&\leqslant \sum_{k=1}^{n+1}\mathbb{P}\left\{\max_{0\leqslant h<r^{n-k+1}}\left|\sum_{j=1+hr^k}^{hr^k+r^k}(X''_{j,b_{r^{k-1}},b_{r^k}}-\mathbb{E}X''_{j,b_{r^{k-1}},b_{r^k}})\right|>\frac{\varepsilon a_{n,k}}{C}\right\}\\
&\leqslant \frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\mathbb{E}\left[\max_{0\leqslant h<r^{n-k+1}}\left|\sum_{j=1+hr^k}^{hr^k+r^k}(X''_{j,b_{r^{k-1}},b_{r^k}}-\mathbb{E}X''_{j,b_{r^{k-1}},b_{r^k}})\right|\right]^2\\
&= \frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\mathbb{E}\max_{0\leqslant h<r^{n-k+1}}\left[\sum_{j=1+hr^k}^{hr^k+r^k}(X''_{j,b_{r^{k-1}},b_{r^k}}-\mathbb{E}X''_{j,b_{r^{k-1}},b_{r^k}})\right]^2\\
&\leqslant \frac{C}{\varepsilon^2}\sum_{j=1}^{k+1}\frac{1}{a_{j,r^k}^2}\sum_{h=0}^{r^{n-k+1}-1}\mathbb{E}\left[\sum_{j=1+hr^k}^{hr^k+r^k}(X''_{j,b_{r^{k-1}},b_{r^k}}-\mathbb{E}X''_{j,b_{r^{k-1}},b_{r^k}})\right]^2\\
&\leqslant \frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\sum_{h=0}^{r^{n-k+1}-1}\sum_{j=1+hr^k}^{hr^k+r^k}\mathbb{E}(X''_{j,b_{r^{k-1}},b_{r^k}}-\mathbb{E}X''_{j,b_{r^{k-1}},b_{r^k}})^2\\
&\quad+\frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\sum_{h=0}^{r^{n-k+1}-1}\sum_{1+hr^k\leqslant i<j\leqslant hr^k+r^k}\mathrm{Cov}(X''_{i,b_{r^{k-1}},b_{r^k}};X''_{j,b_{r^{k-1}},b_{r^k}})\\
&\leqslant \frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\sum_{h=0}^{r^{n-k+1}-1}\sum_{j=1+hr^k}^{hr^k+r^k}\mathbb{E}(X''_{j,b_{r^{k-1}},b_{r^k}})^2\\
&\quad+\frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\sum_{h=0}^{r^{n-k+1}-1}\sum_{1+hr^k\leqslant i<j\leqslant hr^k+r^k}H_{X_i,X_j}(b_{r^{k-1}},b_{r^k})\\
&\leqslant \frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\sum_{h=0}^{r^{n-k+1}-1}\sum_{j=1+hr^k}^{hr^k+r^k}\Big(\mathbb{E}X_j^2I_{\{b_{r^{k-1}}<|X_j|\leqslant b_{r^k}\}}+b_{r^k}^2\mathbb{P}\{|X_j|>b_{r^k}\}\Big)\\
&\quad+\frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\sum_{h=0}^{r^{n-k+1}-1}\left[\sum_{1+hr^k\leqslant i<j\leqslant hr^k+r^k}H_{X_i,X_j}(b_{r^{k-1}},b_{r^k})\right]^+\\
&\leqslant \frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\sum_{j=1}^{r^{n+1}}\frac{1}{a_{n,k}^2}\mathbb{E}X_j^2I_{\{b_{r^{k-1}}<|X_j|\leqslant b_{r^k}\}}+\frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\sum_{j=1}^{r^{n+1}}\frac{b_{r^k}^2}{a_{n,k}^2}\mathbb{P}\{|X_j|>b_{r^k}\}\\
&\quad+\frac{C}{\varepsilon^2}\sum_{k=1}^{n+1}\frac{1}{a_{n,k}^2}\sum_{h=0}^{r^{n-k+1}-1}\left[\sum_{1+hr^k\leqslant i<j\leqslant hr^k+r^k}H_{X_i,X_j}(b_{r^{k-1}},b_{r^k})\right]^+
\end{aligned}
$$

for each $n \geqslant n_0$. □

A sequence $\{X_n,\, n \geqslant 1\}$ of random variables is *stochastically dominated* by a random variable X if there exists a constant $C > 0$ such that $\sup_{n\geqslant 1}\mathbb{P}\{|X_n| > t\} \leqslant C\,\mathbb{P}\{|X| > t\}$ for all $t > 0$; the constant C can be taken as having value one according to Theorem 2.4 of [24].

Under this extra assumption on the random sequence $\{X_n,\, n \geqslant 1\}$ and employing Lemma 1 of [15], Theorem 1 can be restated as follows:

Corollary 1 *Let $r > 1$ be an integer and $\{X_n,\, n \geqslant 1\}$ a sequence of random variables stochastically dominated by a random variable $X \in \mathscr{L}_1$. If $\{b_n\}$ is nondecreasing sequence of positive constants, and $\{a_{n,k},\, 1 \leqslant k \leqslant n+1, n \geqslant 1\}$ is an array of positive constants satisfying condition* (a) *of Theorem 1 and*

$$\text{(b')} \sum_{k=1}^{n+1} r^k \mathbb{E}|X| I_{\left\{|X| > b_{r^{k-1}}\right\}} = o(b_{r^n}) \text{ as } n \to \infty$$

then, for any $\varepsilon > 0$, there is a positive integer n_0 and a constant $C(r) > 0$ such that

$$\begin{aligned}
&\mathbb{P}\left\{\max_{1 \leqslant m < r^{n+1}} \left|\sum_{k=1}^{m} (X_k - \mathbb{E}\, X_k)\right| > \varepsilon b_{r^n}\right\} \\
&\quad \leqslant C(r) r^n \mathbb{P}\left\{|X| > b_{r^{n+1}}\right\} + \frac{C(r) r^n}{\varepsilon^2} \sum_{k=1}^{n+1} a_{n,k}^{-2} \mathbb{E} X^2 I_{\left\{|X| \leqslant b_{r^k}\right\}} + \frac{C(r) r^n}{\varepsilon^2} \sum_{k=1}^{n+1} \frac{b_{r^k}^2}{a_{n,k}^2} \mathbb{P}\left\{|X| > b_{r^{k-1}}\right\} \\
&\qquad + \frac{C(r)}{\varepsilon^2} \sum_{k=1}^{n+1} a_{n,k}^{-2} \max_{1 \leqslant \ell \leqslant r-1} \sum_{h=0}^{r^{n-k+1}-1} \left[\sum_{1+hr^k \leqslant i < j \leqslant hr^k + \ell r^{k-1}} G_{X_i, X_j}(b_{r^{k-1}})\right]^+ \\
&\qquad + \frac{C(r)}{\varepsilon^2} \sum_{k=1}^{n+1} a_{n,k}^{-2} \sum_{h=0}^{r^{n-k+1}-1} \left[\sum_{1+hr^k \leqslant i < j \leqslant hr^k + r^k} H_{X_i, X_j}(b_{r^{k-1}}, b_{r^k})\right]^+
\end{aligned}$$

for all $n \geqslant n_0$.

3 SLLN for dependent random variables

The next result allows us to obtain SLLN for sequences $\{X_n,\, n \geqslant 1\}$ of random variables stochastically dominated by a random variable $X \in \mathscr{L}_1$. Let us stress out that no assumption concerning to dependence is assumed for $\{X_n,\, n \geqslant 1\}$.

Theorem 2 *Let $r > 1$ be an integer and $\{X_n,\, n \geqslant 1\}$ be a sequence of random variables stochastically dominated by a random variable $X \in \mathscr{L}_1$. If $\{b_n\}$ is a nondecreasing sequence of positive constants, $\{c_n\}$ a nonincreasing sequence of nonnegative constants, $\{a_{n,k},\, 1 \leqslant k \leqslant n+1, n \geqslant 1\}$ is an array of positive constants satisfying assumptions* (a), (b′), *and*

$$\text{(c)} \sum_{m=0}^{\infty} \sum_{n=r^m}^{r^{m+1}-1} r^m c_n \mathbb{P}\left\{|X| > b_{r^{m+1}}\right\} < \infty,$$

$$\text{(d)} \sum_{m=0}^{\infty} \sum_{n=r^m}^{r^{m+1}-1} \sum_{k=1}^{m+1} \frac{c_n r^m}{a_{m,k}^2} \mathbb{E} X^2 I_{\left\{|X| \leqslant b_{r^k}\right\}} < \infty,$$

(e) $\displaystyle\sum_{m=0}^{\infty}\sum_{n=r^m}^{r^{m+1}-1}\sum_{k=1}^{m+1}\frac{c_n r^m b_{r^k}^2}{a_{m,k}^2}\mathbb{P}\big\{|X|>b_{r^{k-1}}\big\}<\infty,$

(f) $\displaystyle\sum_{m=0}^{\infty}\sum_{n=r^m}^{r^{m+1}-1}\sum_{k=1}^{m+1}\frac{c_n}{a_{m,k}^2}\max_{1\leqslant\ell\leqslant r-1}\sum_{h=0}^{r^{m-k+1}-1}\left[\sum_{1+hr^k\leqslant i<j\leqslant hr^k+\ell r^{k-1}}G_{X_i,X_j}(b_{r^{k-1}})\right]^+<\infty,$

(g) $\displaystyle\sum_{m=0}^{\infty}\sum_{n=r^m}^{r^{m+1}-1}\sum_{k=1}^{m+1}\frac{c_n}{a_{m,k}^2}\sum_{h=0}^{r^{m-k+1}-1}\left[\sum_{1+hr^k\leqslant i<j\leqslant hr^k+r^k}H_{X_i,X_j}(b_{r^{k-1}},b_{r^k})\right]^+<\infty,$

then

$$\sum_{n=1}^{\infty}c_n\mathbb{P}\left\{\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}(X_k-\mathbb{E}\,X_k)\right|>\varepsilon b_n\right\}<\infty \tag{3.1}$$

for all $\varepsilon>0$.

Proof. Let us fix $\varepsilon>0$ and an integer $r>1$. Hence,

$$\begin{aligned}
&\sum_{n=1}^{\infty}c_n\mathbb{P}\left\{\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}(X_k-\mathbb{E}\,X_k)\right|>\varepsilon b_n\right\}\\
&\quad=\sum_{m=0}^{\infty}\sum_{n=r^m}^{r^{m+1}-1}c_n\mathbb{P}\left\{\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}(X_k-\mathbb{E}\,X_k)\right|>\varepsilon b_n\right\}\\
&\quad\leqslant\sum_{m=0}^{\infty}\mathbb{P}\left\{\max_{1\leqslant j<r^{m+1}}\left|\sum_{k=1}^{j}(X_k-\mathbb{E}\,X_k)\right|>\varepsilon b_{r^m}\right\}\sum_{n=r^m}^{r^{m+1}-1}c_n
\end{aligned}$$

and the conclusion is a direct consequence of Corollary 1. □

Corollary 2 *Let* $r>1$ *be an integer,* $1\leqslant p<2$, *and* $\{X_n,\,n\geqslant 1\}$ *be a sequence of random variables stochastically dominated by a random variable* $X\in\mathscr{L}_p$. *If*

$$\sum_{1\leqslant i<j\leqslant\infty}\sum_{k=1}^{\infty}r^{(2/\alpha-2/p)k-2(k\vee\log_r j)/\alpha}\left[G^+_{X_i,X_j}\big(r^{(k-1)/p}\big)+H^+_{X_i,X_j}\big(r^{(k-1)/p},r^{k/p}\big)\right]<\infty \tag{3.2}$$

for some $p<\alpha<2$, *then* $\sum_{k=1}^{n}(X_k-\mathbb{E}\,X_k)/n^{1/p}\xrightarrow{\text{a.s.}}0$.

Proof. Consider $1\leqslant p<\alpha<2$, a (fixed) integer $r>1$, $c_n=1/n$, $b_n=n^{1/p}$, and $a_{n,k}=r^{n/\alpha+(1/p-1/\alpha)k}$. Hence,

$$\begin{aligned}
\limsup_{m\to\infty}\frac{b_{r^{m+1}}}{b_{r^m}}&=\limsup_{m\to\infty}\frac{r^{(m+1)/p}}{r^{m/p}}=r^{1/p},\\
\liminf_{m\to\infty}\sum_{n=r^m}^{r^{m+1}-1}c_n&\geqslant\liminf_{m\to\infty}\int_{r^m}^{r^{m+1}}\frac{\mathrm{d}u}{u}=\log r
\end{aligned}$$

and from Theorem 2.2 of [10], it suffices to show

$$\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{P}\left\{\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}(X_k-\mathbb{E}\,X_k)\right|>\varepsilon n^{1/p}\right\}<\infty \tag{3.3}$$

for all $\varepsilon > 0$. We have

$$\sum_{k=1}^{n+1} a_{n,k} = r^{n/\alpha} \cdot \frac{r^{n/p-n/\alpha+2/p-2/\alpha} - r^{1/p-1/\alpha}}{r^{1/p-1/\alpha} - 1} \leqslant \frac{r^{2/p-2/\alpha}}{r^{1/p-1/\alpha} - 1} \cdot r^{n/p}$$

so that assumption (a) of Theorem 2 holds. On the other hand, the dominated convergence theorem guarantees

$$\mathbb{E}|X|^p I_{\left\{|X|>r^{(n-1)/p}\right\}} = o(1), \quad n \to \infty.$$

Moreover, for each $k \geqslant 1$,

$$r^{k+(1-p)(k-1)/p-n/p} = o(1), \quad n \to \infty$$

and

$$\sum_{k=1}^{n+1} r^{k+(1-p)(k-1)/p-n/p} = \frac{r^{1+1/p} - r^{1-n/p}}{r^{1/p} - 1} \leqslant \frac{r^{1+1/p}}{r^{1/p} - 1}$$

which implies

$$\sum_{k=1}^{n+1} r^{k-n/p}\mathbb{E}|X|I_{\left\{|X|>r^{(k-1)/p}\right\}} \leqslant \sum_{k=1}^{n+1} r^{k+(1-p)(k-1)/p-n/p}\mathbb{E}|X|^p I_{\left\{|X|>r^{(k-1)/p}\right\}} = o(1), \quad n \to \infty$$

by Toeplitz's lemma (see [16]) proving assumption (b') of Theorem 2. According to

$$\sum_{n=r^m}^{r^{m+1}-1} c_n \leqslant \frac{r^{m+1} - r^m}{r^m} = r - 1 \tag{3.4}$$

one still get

$$\begin{aligned}
\sum_{m=0}^{\infty} \sum_{n=r^m}^{r^{m+1}-1} r^m c_n \mathbb{P}\left\{|X| > b_{r^{m+1}}\right\} &\leqslant (r-1) \sum_{m=0}^{\infty} r^m \mathbb{P}\left\{|X| > r^{(m+1)/p}\right\} \\
&= (r-1) \sum_{m=0}^{\infty} \sum_{j=m}^{\infty} r^m \mathbb{P}\left\{r^{j+1} < |X|^p \leqslant r^{j+2}\right\} \\
&= (r-1) \sum_{j=0}^{\infty} \sum_{m=0}^{j} r^m \mathbb{P}\left\{r^{j+1} < |X|^p \leqslant r^{j+2}\right\} \\
&\leqslant \sum_{j=0}^{\infty} r^{j+1} \mathbb{P}\left\{r^{j+1} < |X|^p \leqslant r^{j+2}\right\} \\
&= \sum_{j=0}^{\infty} \mathbb{E}\left(r^{j+1} I_{\{r^{j+1}<|X|^p \leqslant r^{j+2}\}}\right) \\
&\leqslant \sum_{j=0}^{\infty} \mathbb{E}|X|^p I_{\{r^{j+1}<|X|^p \leqslant r^{j+2}\}} \\
&= \mathbb{E}|X|^p I_{\{|X|^p > r\}}
\end{aligned}$$

$$\leqslant \mathbb{E}|X|^p$$

and (c) of Theorem 2 holds. By (3.4), we also obtain

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=r^m}^{r^{m+1}-1}\sum_{k=1}^{m+1}\frac{c_n r^m}{a_{m,k}^2}\mathbb{E}X^2 I_{\{|X|\leqslant b_{r^k}\}}\\
&\quad\leqslant (r-1)\sum_{m=0}^{\infty}\sum_{k=1}^{m+1} r^{(1-2/\alpha)m} r^{2(1/\alpha-1/p)k}\mathbb{E}X^2 I_{\{|X|\leqslant r^{k/p}\}}\\
&\quad= (r-1)\sum_{k=1}^{\infty}\sum_{m=k-1}^{\infty} r^{(1-2/\alpha)m} r^{2(1/\alpha-1/p)k}\mathbb{E}X^2 I_{\{|X|\leqslant r^{k/p}\}}\\
&\quad= \frac{(r-1)r^{2/\alpha-1}}{1-r^{1-2/\alpha}}\sum_{k=1}^{\infty} r^{(1-2/p)k}\mathbb{E}X^2 I_{\{|X|\leqslant r^{k/p}\}}\\
&\quad= \frac{(r-1)r^{2/\alpha-1}}{1-r^{1-2/\alpha}}\sum_{k=1}^{\infty} r^{(1-2/p)k}\mathbb{E}X^2 I_{\{|X|\leqslant 1\}}\\
&\qquad+ \frac{(r-1)r^{2/\alpha-1}}{1-r^{1-2/\alpha}}\sum_{k=1}^{\infty}\sum_{j=1}^{k} r^{(1-2/p)k}\mathbb{E}X^2 I_{\{r^{(j-1)/p}<|X|\leqslant r^{j/p}\}}\\
&\quad\leqslant \frac{(r-1)r^{2/\alpha-2/p}}{\left(1-r^{1-2/\alpha}\right)\left(1-r^{1-2/p}\right)} + \frac{(r-1)r^{2/\alpha-1}}{1-r^{1-2/\alpha}}\sum_{j=1}^{\infty}\underbrace{\sum_{k=j}^{\infty} r^{k(1-2/p)}}_{=\frac{r^{j(1-2/p)}}{1-r^{1-2/p}}}\mathbb{E}X^2 I_{\{r^{(j-1)/p}<|X|\leqslant r^{j/p}\}}\\
&\quad\leqslant \frac{(r-1)r^{2/\alpha-2/p}}{\left(1-r^{1-2/\alpha}\right)\left(1-r^{1-2/p}\right)} + \frac{(r-1)r^{2/\alpha-1}}{\left(1-r^{1-2/\alpha}\right)\left(1-r^{1-2/p}\right)}\sum_{j=1}^{\infty}\mathbb{E}|X|^p I_{\{r^{(j-1)/p}<|X|\leqslant r^{j/p}\}}\\
&\quad= \frac{(r-1)r^{2/\alpha-2/p}}{\left(1-r^{1-2/\alpha}\right)\left(1-r^{1-2/p}\right)} + \frac{(r-1)r^{2/\alpha-1}}{\left(1-r^{1-2/\alpha}\right)\left(1-r^{1-2/p}\right)}\mathbb{E}|X|^p I_{\{|X|>1\}}\\
&\quad\leqslant \frac{(r-1)r^{2/\alpha-2/p}}{\left(1-r^{1-2/\alpha}\right)\left(1-r^{1-2/p}\right)} + \frac{(r-1)r^{2/\alpha-1}}{\left(1-r^{1-2/\alpha}\right)\left(1-r^{1-2/p}\right)}\mathbb{E}|X|^p
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=r^m}^{r^{m+1}-1}\sum_{k=1}^{m+1}\frac{c_n r^m r^{2k/p}}{a_{m,k}^2}\mathbb{P}\left\{|X|>b_{r^{(k-1)/p}}\right\}\\
&\quad\leqslant (r-1)\sum_{m=0}^{\infty}\sum_{k=1}^{m+1} r^{(2/\alpha-2/p)k-2m/\alpha} r^m r^{2k/p}\mathbb{P}\left\{|X|>r^{(k-1)/p}\right\}\\
&\quad= (r-1)\sum_{k=1}^{\infty}\sum_{m=k-1}^{\infty} r^{(1-2/\alpha)m} r^{2k/\alpha}\mathbb{P}\left\{|X|>r^{(k-1)/p}\right\}\\
&\quad= \frac{(r-1)r^{2/\alpha-1}}{1-r^{1-2/\alpha}}\sum_{k=1}^{\infty} r^k\mathbb{P}\left\{|X|>r^{(k-1)/p}\right\}\\
&\quad= \frac{(r-1)r^{2/\alpha-1}}{1-r^{1-2/\alpha}}\sum_{k=1}^{\infty}\sum_{j=k-1}^{\infty} r^k\mathbb{P}\left\{r^{j/p}<|X|\leqslant r^{(j+1)/p}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(r-1)r^{2/\alpha-1}}{1-r^{1-2/\alpha}} \sum_{j=0}^{\infty} \sum_{k=1}^{j+1} r^k \mathbb{P}\big\{r^{j/p} < |X| \leqslant r^{(j+1)/p}\big\} \\
&\leqslant \frac{r^{2/\alpha+1}}{1-r^{1-2/\alpha}} \sum_{j=0}^{\infty} r^j \mathbb{P}\big\{r^{j/p} < |X| \leqslant r^{(j+1)/p}\big\} \\
&\leqslant \frac{r^{2/\alpha+1}}{1-r^{1-2/\alpha}} \sum_{j=0}^{\infty} \mathbb{E}|X|^p I_{\{r^{j/p}<|X|\leqslant r^{(j+1)/p}\}} \\
&= \frac{r^{2/\alpha+1}}{1-r^{1-2/\alpha}} \mathbb{E}|X|^p I_{\{|X|>1\}} \\
&\leqslant \frac{r^{2/\alpha+1}}{1-r^{1-2/\alpha}} \mathbb{E}|X|^p
\end{aligned}
$$

so that assumptions (d), (e) of Theorem 2 both hold. Finally, we have

$$
\begin{aligned}
&\sum_{m=0}^{\infty} \sum_{n=r^m}^{r^{m+1}-1} \sum_{k=1}^{m+1} \frac{c_n}{a_{m,k}^2} \max_{1\leqslant \ell \leqslant r-1} \sum_{h=0}^{r^{m-k+1}-1} \left[\sum_{1+hr^k \leqslant i<j \leqslant hr^k+\ell r^{k-1}} G_{X_i,X_j}(b_{r^{k-1}}) \right]^+ \\
&\quad \leqslant \sum_{m=0}^{\infty} \sum_{n=r^m}^{r^{m+1}-1} \sum_{k=1}^{m+1} \frac{r^{(2/\alpha-2/p)k-2m/\alpha}}{n} \max_{1\leqslant \ell \leqslant r-1} \sum_{h=0}^{r^{m-k+1}-1} \sum_{1+hr^k \leqslant i<j \leqslant hr^k+\ell r^{k-1}} G^+_{X_i,X_j}\big(r^{(k-1)/p}\big) \\
&\quad \leqslant (r-1) \sum_{m=0}^{\infty} \sum_{k=1}^{m+1} r^{(2/\alpha-2/p)k-2m/\alpha} \sum_{1\leqslant i<j\leqslant r^{m+1}} G^+_{X_i,X_j}\big(r^{(k-1)/p}\big) \\
&\quad = (r-1) r^{\alpha/2} \sum_{m=1}^{\infty} \sum_{k=1}^{m} r^{(2/\alpha-2/p)k-2m/\alpha} \sum_{1\leqslant i<j\leqslant r^m} G^+_{X_i,X_j}\big(r^{(k-1)/p}\big) \\
&\quad = (r-1) r^{\alpha/2} \sum_{1\leqslant i<j\leqslant \infty} \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} r^{(2/\alpha-2/p)k-2m/\alpha} I_{\{j\leqslant r^m\}} G^+_{X_i,X_j}\big(r^{(k-1)/p}\big) \\
&\quad \leqslant \frac{(r-1)r^{\alpha/2}}{1-r^{-2/\alpha}} \sum_{1\leqslant i<j\leqslant \infty} \sum_{k=1}^{\infty} r^{(2/\alpha-2/p)k-2(k\vee \log_r j)/\alpha} G^+_{X_i,X_j}\big(r^{(k-1)/p}\big) < \infty
\end{aligned}
\tag{3.5}
$$

as well as

$$
\begin{aligned}
&\sum_{m=0}^{\infty} \sum_{n=r^m}^{r^{m+1}-1} \sum_{k=1}^{m+1} \frac{c_n}{a_{m,k}^2} \sum_{h=0}^{r^{m-k+1}-1} \left[\sum_{1+hr^k \leqslant i<j \leqslant hr^k+r^k} H_{X_i,X_j}(b_{r^{k-1}}, b_{r^k}) \right]^+ \\
&\quad \leqslant \sum_{m=0}^{\infty} \sum_{n=r^m}^{r^{m+1}-1} \sum_{k=1}^{m+1} \frac{r^{(2/\alpha-2/p)k-2m/\alpha}}{n} \sum_{h=0}^{r^{m-k+1}-1} \sum_{1+hr^k \leqslant i<j \leqslant hr^k+r^k} H^+_{X_i,X_j}\big(r^{(k-1)/p}, r^{k/p}\big) \\
&\quad \leqslant (r-1) \sum_{m=0}^{\infty} \sum_{k=1}^{m+1} r^{(2/\alpha-2/p)k-2m/\alpha} \sum_{1\leqslant i<j\leqslant r^{m+1}} H^+_{X_i,X_j}\big(r^{(k-1)/p}, r^{k/p}\big) \\
&\quad = (r-1) r^{\alpha/2} \sum_{m=1}^{\infty} \sum_{k=1}^{m} r^{(2/\alpha-2/p)k-2m/\alpha} \sum_{1\leqslant i<j\leqslant r^m} H^+_{X_i,X_j}\big(r^{(k-1)/p}, r^{k/p}\big)
\end{aligned}
\tag{3.6}
$$

$$= (r-1)r^{\alpha/2} \sum_{1\leqslant i<j\leqslant\infty} \sum_{k=1}^{\infty}\sum_{m=k}^{\infty} r^{(2/\alpha-2/p)k-2m/\alpha} I_{\{j\leqslant r^m\}} H^{+}_{X_i,X_j}\big(r^{(k-1)/p}, r^{k/p}\big)$$

$$\leqslant \frac{(r-1)r^{\alpha/2}}{1-r^{-2/\alpha}} \sum_{1\leqslant i<j\leqslant\infty} \sum_{k=1}^{\infty} r^{(2/\alpha-2/p)k-2(k\vee\log_r j)/\alpha} H^{+}_{X_i,X_j}\big(r^{(k-1)/p}, r^{k/p}\big) < \infty,$$

entailing assumptions (f) and (g) of Theorem 2. The proof is complete. □

3.1 Sufficient conditions for assumptions (f) and (g)

We now present sufficient (moment) conditions which guarantee assumptions (f) and (g) of Theorem 2. Let $\{X_n,\, n\geqslant 1\}$ $\{X_n,\, n\geqslant 1\}$ be a sequence of random variables stochastically dominated by a random variable $X\in\mathscr{L}_p$, $1\leqslant p<2$. By supposing that there is a (fixed) constant $C>0$ such that

$$\left|\sum_{m\leqslant i<j\leqslant n} \mathrm{Cov}\big(g_{b_{r^{k-1}}}(X_i), g_{b_{r^{k-1}}}(X_j)\big)\right| \leqslant C\sum_{j=m}^{n} \mathbb{E}[g_{b_{r^{k-1}}}(X_j)]^2 \tag{3.7}$$

for all $1\leqslant m\leqslant n$ (recall that $G_{X_i,X_j}(b_{r^{k-1}}) := \mathrm{Cov}(g_{b_{r^{k-1}}}(X_i), g_{b_{r^{k-1}}}(X_j))$), it follows

$$\begin{aligned}
&\max_{1\leqslant\ell\leqslant r-1} \sum_{h=0}^{r^{m-k+1}-1} \left[\sum_{1+hr^k\leqslant i<j\leqslant hr^k+\ell r^{k-1}} G_{X_i,X_j}(b_{r^{k-1}})\right]^{+} \\
&\quad\leqslant \max_{1\leqslant\ell\leqslant r-1} \sum_{h=0}^{r^{m-k+1}-1} \left|\sum_{1+hr^k\leqslant i<j\leqslant hr^k+\ell r^{k-1}} G_{X_i,X_j}(b_{r^{k-1}})\right| \\
&\quad= \max_{1\leqslant\ell\leqslant r-1} \sum_{h=0}^{r^{m-k+1}-1} \left|\sum_{1+hr^k\leqslant i<j\leqslant hr^k+\ell r^{k-1}} \mathrm{Cov}\big(g_{b_{r^{k-1}}}(X_i), g_{b_{r^{k-1}}}(X_j)\big)\right| \\
&\quad\underset{(3.7)}{\leqslant} \max_{1\leqslant\ell\leqslant r-1} C \sum_{h=0}^{r^{m-k+1}-1} \sum_{j=1+hr^k}^{hr^k+\ell r^{k-1}} \mathbb{E}[g_{b_{r^{k-1}}}(X_j)]^2 \\
&\quad\leqslant C\sum_{j=1}^{r^{m+1}} \mathbb{E}[g_{b_{r^{k-1}}}(X_j)]^2 \\
&\quad= C\sum_{j=1}^{r^{m+1}} \Big[\mathbb{E}X_j^2 I_{\{|X_j|\leqslant b_{r^{k-1}}\}} + b^2_{r^{k-1}}\mathbb{P}\big\{|X_j|>b_{r^{k-1}}\big\}\Big] \\
&\quad\leqslant Cr^{m+1}\mathbb{E}X^2 I_{\{|X|\leqslant b_{r^k}\}} + Cr^{m+1} b^2_{r^{k-1}}\mathbb{P}\big\{|X|>b_{r^{k-1}}\big\}
\end{aligned}$$

where the last inequality is a consequence of Lemma 1 in [15]. On the other hand, if one admits that there exists a (fixed) constant $C>0$ such that, for all $1\leqslant m\leqslant n$, the following inequality holds

$$\left|\sum_{m\leqslant i<j\leqslant n} \mathrm{Cov}\big(h_{b_{r^{k-1}},b_{r^k}}(X_i), h_{b_{r^{k-1}},b_{r^k}}(X_j)\big)\right| \leqslant C\sum_{j=m}^{n} \mathbb{E}[h_{b_{r^{k-1}},b_{r^k}}(X_j)]^2 \tag{3.8}$$

we get

$$
\begin{aligned}
&\sum_{h=0}^{r^{m-k+1}-1}\left[\sum_{1+hr^k\leqslant i<j\leqslant hr^k+r^k} H_{X_i,X_j}\big(b_{r^{k-1}},b_{r^k}\big)\right]^+\\
&\leqslant \sum_{h=0}^{r^{m-k+1}-1}\left|\sum_{1+hr^k\leqslant i<j\leqslant hr^k+r^k} H_{X_i,X_j}\big(b_{r^{k-1}},b_{r^k}\big)\right|\\
&= \sum_{h=0}^{r^{m-k+1}-1}\left|\sum_{1+hr^k\leqslant i<j\leqslant hr^k+r^k} \operatorname{Cov}\big(h_{b_{r^{k-1}},b_{r^k}}(X_i),h_{b_{r^{k-1}},b_{r^k}}(X_j)\big)\right|\\
&\underset{(3.8)}{\leqslant} C\sum_{h=0}^{r^{m-k+1}-1}\sum_{j=1+hr^k}^{hr^k+r^k}\mathbb{E}[h_{b_{r^{k-1}},b_{r^k}}(X_j)]^2\\
&= C\sum_{j=1}^{r^{m+1}}\mathbb{E}[h_{b_{r^{k-1}},b_{r^k}}(X_j)]^2\\
&= C\sum_{j=1}^{r^{m+1}}\Big[\mathbb{E}\big(|X_j|-b_{r^{k-1}}\big)^2 I_{\{b_{r^{k-1}}<|X_j|\leqslant b_{r^k}\}}+\big(b_{r^k}-b_{r^{k-1}}\big)^2\mathbb{P}\big\{|X_j|>b_{r^k}\big\}\Big]\\
&\leqslant Cr^{m+1}\mathbb{E}X^2 I_{\{|X|\leqslant b_{r^k}\}}+Cr^{m+1}b_{r^k}^2\mathbb{P}\big\{|X|>b_{r^k}\big\}
\end{aligned}
$$

Thus, if we additionally assume (3.7) and (3.8) in Theorem 2, then its assumptions (f) and (g) can both be dropped, since they are assured by the already stated ones, (d) and (e).

Furthermore, it is worthy to note that the existence of a constant $C>0$ (which is not necessarily the same on each appearance) such that the following inequalities hold

$$
\mathbb{E}\left\{\sum_{j=m}^{n}\Big[g_{b_{r^{k-1}}}(X_j)-\mathbb{E}g_{b_{r^{k-1}}}(X_j)\Big]\right\}^2\leqslant C\sum_{j=m}^{n}\mathbb{E}\Big[g_{b_{r^{k-1}}}(X_j)\Big]^2 \tag{3.9a}
$$

$$
\mathbb{E}\left\{\sum_{j=m}^{n}\Big[h_{b_{r^{k-1}},b_{r^k}}(X_j)-\mathbb{E}h_{b_{r^{k-1}},b_{r^k}}(X_j)\Big]\right\}^2\leqslant C\sum_{j=m}^{n}\mathbb{E}\Big[h_{b_{r^{k-1}},b_{r^k}}(X_j)\Big]^2 \tag{3.9b}
$$

for all $1\leqslant m\leqslant n$, implies both (3.7) and (3.8). Indeed,

$$
\begin{aligned}
&2\left|\sum_{m\leqslant i<j\leqslant n}\operatorname{Cov}(g_{b_{r^{k-1}}}(X_i),g_{b_{r^{k-1}}}(X_j))\right|-\sum_{j=m}^{n}\mathbb{E}\Big[g_{b_{r^{k-1}}}(X_j)\Big]^2\\
&\quad\leqslant 2\left|\sum_{m\leqslant i<j\leqslant n}\operatorname{Cov}(g_{b_{r^{k-1}}}(X_i),g_{b_{r^{k-1}}}(X_j))\right|-\sum_{j=m}^{n}\mathbb{V}\Big[g_{b_{r^{k-1}}}(X_j)\Big]\\
&\quad= 2\left|\sum_{m\leqslant i<j\leqslant n}\mathbb{E}[g_{b_{r^{k-1}}}(X_i)-\mathbb{E}g_{b_{r^{k-1}}}(X_i)][g_{b_{r^{k-1}}}(X_j)-\mathbb{E}g_{b_{r^{k-1}}}(X_j)]\right|\\
&\qquad-\sum_{j=m}^{n}\mathbb{E}\Big[g_{b_{r^{k-1}}}(X_j)-\mathbb{E}g_{b_{r^{k-1}}}(X_j)\Big]^2
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \Bigg| \sum_{j=m}^{n} \mathbb{E}\left[g_{b_{r^k-1}}(X_j) - \mathbb{E}g_{b_{r^k-1}}(X_j)\right]^2 \\
&\quad + 2 \sum_{m\leqslant i<j\leqslant n} \mathbb{E}[g_{b_{r^k-1}}(X_i) - \mathbb{E}g_{b_{r^k-1}}(X_i)][g_{b_{r^k-1}}(X_j) - \mathbb{E}g_{b_{r^k-1}}(X_j)]\Bigg| \\
&= \mathbb{E}\Bigg\{ \sum_{j=m}^{n} \left[g_{b_{r^k-1}}(X_j) - \mathbb{E}g_{b_{r^k-1}}(X_j)\right] \Bigg\}^2 \\
&\underset{(3.9a)}{\leqslant} C \sum_{j=m}^{n} \mathbb{E}\left[g_{b_{r^k-1}}(X_j)\right]^2
\end{aligned}
$$

and, similarly, one can conclude that (3.9b) entails (3.8).

On the other hand, (3.9a) and (3.9b), by their turn, can be assured by the familiar moment inequality (3.10) presented below. Let φ be an arbitrary nondecreasing Borel-measurable real-valued function defined on whole real line. All sequences of random variables $\{X_n,\, n \geqslant 1\}$ satisfying $\mathbb{E}[\varphi(X_n)]^2 < \infty$ for each $n \geqslant 1$, and for which there exists a (fixed) constant $C > 0$ such that

$$
\mathbb{E}\Bigg\{ \sum_{j=m}^{n} [\varphi(X_j) - \mathbb{E}\varphi(X_j)] \Bigg\}^2 \leqslant C \sum_{j=m}^{n} \mathbb{E}\,[\varphi(X_j) - \mathbb{E}\varphi(X_j)]^2 \tag{3.10}
$$

for all $1 \leqslant m \leqslant n$, fulfill conditions (3.9a) and (3.9b), because

$$
\begin{aligned}
\mathbb{E}\Bigg\{ \sum_{j=m}^{n} \left[g_{b_{r^k-1}}(X_j) - \mathbb{E}g_{b_{r^k-1}}(X_j)\right] \Bigg\}^2 &\underset{(3.10)}{\leqslant} C \sum_{j=m}^{n} \mathbb{E}\left[g_{b_{r^k-1}}(X_j) - \mathbb{E}g_{b_{r^k-1}}(X_j)\right]^2 \\
&\leqslant C \sum_{j=m}^{n} \mathbb{E}[g_{b_{r^k-1}}(X_j)]^2,
\end{aligned}
$$

and putting $f_{L,K}(t) := g_K(t) - g_L(t)$, it follows

$$
\begin{aligned}
&\mathbb{E}\Bigg\{ \sum_{j=m}^{n} \left[h_{b_{r^k-1},b_{r^k}}(X_j) - \mathbb{E}h_{b_{r^k-1},b_{r^k}}(X_j)\right] \Bigg\}^2 \\
&\quad = \mathbb{E}\Bigg\{ \sum_{j=m}^{n} \left[f^{+}_{b_{r^k-1},b_{r^k}}(X_j) - \mathbb{E}f^{+}_{b_{r^k-1},b_{r^k}}(X_j)\right] + \sum_{j=m}^{n} \left[f^{-}_{b_{r^k-1},b_{r^k}}(X_j) - \mathbb{E}f^{-}_{b_{r^k-1},b_{r^k}}(X_j)\right] \Bigg\}^2 \\
&\quad \leqslant 2\mathbb{E}\Bigg\{ \sum_{j=m}^{n} \left[f^{+}_{b_{r^k-1},b_{r^k}}(X_j) - \mathbb{E}f^{+}_{b_{r^k-1},b_{r^k}}(X_j)\right] \Bigg\}^2 \\
&\qquad + 2\mathbb{E}\Bigg\{ \sum_{j=m}^{n} \left[\big(-f^{-}_{b_{r^k-1},b_{r^k}}(X_j)\big) - \mathbb{E}\big(-f^{-}_{b_{r^k-1},b_{r^k}}(X_j)\big)\right] \Bigg\}^2 \\
&\quad \underset{(3.10)}{\leqslant} 2C \sum_{j=m}^{n} \mathbb{E}\left[f^{+}_{b_{r^k-1},b_{r^k}}(X_j) - \mathbb{E}f^{+}_{b_{r^k-1},b_{r^k}}(X_j)\right]^2 \\
&\qquad + 2C \sum_{j=m}^{n} \mathbb{E}\left[\big(-f^{-}_{b_{r^k-1},b_{r^k}}(X_j)\big) - \mathbb{E}\big(-f^{-}_{b_{r^k-1},b_{r^k}}(X_j)\big)\right]^2
\end{aligned}
$$

$$\leqslant 2C\sum_{j=m}^{n}\mathbb{E}\left[f^{+}_{b_{r^{k-1}},b_{r^k}}(X_j)\right]^2+2C\sum_{j=m}^{n}\mathbb{E}\left[f^{-}_{b_{r^{k-1}},b_{r^k}}(X_j)\right]^2$$
$$\leqslant 4C\sum_{j=m}^{n}\mathbb{E}\left[h_{b_{r^{k-1}},b_{r^k}}(X_j)\right]^2$$

by taking $\varphi = g_{b_{r^{k-1}}}$ and $\varphi = f^{+}_{b_{r^{k-1}},b_{r^k}}$, $\varphi = -f^{-}_{b_{r^{k-1}},b_{r^k}}$ in (3.10), respectively.

Thereby, excluding (3.5) and (3.6), all steps in the proof of Corollary 2 remain valid when $\{X_n,\, n\geqslant 1\}$ is a sequence of random variables stochastically dominated by a random variable $X\in\mathscr{L}_p$, $1\leqslant p<2$ verifying (3.10), i.e. the following statement holds:

Corollary 3 *Let $1\leqslant p<2$, and $\{X_n,\, n\geqslant 1\}$ be a sequence of random variables stochastically dominated by a random variable $X\in\mathscr{L}_p$ satisfying* (3.10) *for any nondecreasing Borel-measurable real-valued function φ defined on whole real line such that $\mathbb{E}[\varphi(X_n)]^2<\infty$ for each $n\geqslant 1$. Then, $\sum_{k=1}^{n}(X_k-\mathbb{E}\,X_k)/n^{1/p}\xrightarrow{\text{a.s.}}0$.*

Remark 1 It should be noted that Corollary 3's assumptions entail conditions ($\mathcal{H}1$) and ($\mathcal{H}2$) in Theorem 1 of [22].

3.2 SLLN for quadrant dependent random variables

The notion of quadrant dependence was introduced by Lehmann in [14]. A sequence $\{X_n,\, n\geqslant 1\}$ of random variables is said to be *pairwise positively quadrant dependent* (pairwise PQD) if

$$\mathbb{P}\{X_k\leqslant x_k, X_j\leqslant x_j\}-\mathbb{P}\{X_k\leqslant x_k\}\,\mathbb{P}\{X_j\leqslant x_j\}\geqslant 0$$

for all reals x_k, x_j and all positive integers k, j such that $k\neq j$. A sequence $\{X_n,\, n\geqslant 1\}$ of random variables is said to be *pairwise negatively quadrant dependent* (pairwise NQD) if

$$\mathbb{P}\{X_k\leqslant x_k, X_j\leqslant x_j\}-\mathbb{P}\{X_k\leqslant x_k\}\,\mathbb{P}\{X_j\leqslant x_j\}\leqslant 0$$

for all reals x_k, x_j and all positive integers k, j such that $k\neq j$.

Remarkable examples of the above statistical dependence concept can be found in reliability theory, where most bivariate distributions are positively quadrant dependent and the disregarding of this assumption lead to underestimation/overestimation of the system reliability [11].

The following is a SLLN for sequences of pairwise PQD random variables under both sharpen norming constants and moment condition.

Corollary 4 *Let $1<p<2$ and $\{X_n,\, n\geqslant 1\}$ be a sequence of pairwise PQD random variables stochastically dominated by a random variable $X\in\mathscr{L}_p$. If*

$$\sum_{1\leqslant i<j\leqslant\infty}\sum_{n=j}^{\infty}n^{-1-2/\alpha}G_{X_i,X_j}\big(n^{1/p}\big)<\infty \tag{3.11}$$

for some $p<\alpha<2$, then $\sum_{k=1}^{n}(X_k-\mathbb{E}\,X_k)/n^{1/p}\xrightarrow{\text{a.s.}}0$.

Proof. Since $\Delta_{X_i,X_j}(u,v) \geqslant 0$ for every $i \neq j$ and all reals u, v, we have

$$G_{X_i,X_j}\big(r^{k/p}\big) \geqslant G_{X_i,X_j}\big(r^{(k-1)/p}\big) \geqslant 0$$

which leads to

$$G^+_{X_i,X_j}\big(r^{(k-1)/p}\big) + H^+_{X_i,X_j}\big(r^{(k-1)/p}, r^{k/p}\big) \leqslant G_{X_i,X_j}\big(r^{k/p}\big).$$

From Lemma 4 of [18], it follows

$$\begin{aligned}
&\sum_{1\leqslant i<j\leqslant\infty}\sum_{k=1}^{\infty} r^{(2/\alpha-2/p)k-2(k\vee\log_r j)/\alpha}\Big[G^+_{X_i,X_j}\big(r^{(k-1)/p}\big) + H^+_{X_i,X_j}\big(r^{(k-1)/p}, r^{k/p}\big)\Big] \\
&\quad\leqslant \sum_{1\leqslant i<j\leqslant\infty}\sum_{k=1}^{\infty} r^{(2/\alpha-2/p)k-2(k\vee\log_r j)/\alpha} G_{X_i,X_j}\big(r^{k/p}\big) \\
&\quad= \sum_{1\leqslant i<j\leqslant\infty}\sum_{k=1}^{\infty} r^{(2/\alpha-2/p)k-2(k\vee\log_r j)/\alpha} I_{\{r^k<j\}} G_{X_i,X_j}\big(r^{k/p}\big) \\
&\qquad+ \sum_{1\leqslant i<j\leqslant\infty}\sum_{k=1}^{\infty} r^{(2/\alpha-2/p)k-2(k\vee\log_r j)/\alpha} I_{\{r^k\geqslant j\}} G_{X_i,X_j}\big(r^{k/p}\big) \\
&\quad= \sum_{1\leqslant i<j\leqslant\infty} j^{-2/\alpha}\sum_{k=1}^{\infty} r^{(2/\alpha-2/p)k} I_{\{r^k<j\}} G_{X_i,X_j}\big(r^{k/p}\big) \\
&\qquad+ \sum_{1\leqslant i<j\leqslant\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{k=1}^{\infty} r^{-2k/p} I_{\{k\geqslant\log_r(|u|^p\vee|v|^p\vee j)\}}\Delta_{X_i,X_j}(u,v)\,\mathrm{d}u\mathrm{d}v \\
&\quad\leqslant \sum_{1\leqslant i<j\leqslant\infty} j^{-2/\alpha}\sum_{k=1}^{\infty} r^{(2/\alpha-2/p)k} G_{X_i,X_j}\big(j^{1/p}\big) \\
&\qquad+ \frac{1}{1-r^{-2/p}}\sum_{1\leqslant i<j\leqslant\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|u|^p\vee|v|^p\vee j)^{-2/p}\Delta_{X_i,X_j}(u,v)\,\mathrm{d}u\mathrm{d}v \\
&\quad= \frac{r^{2/\alpha-2/p}}{1-r^{2/\alpha-2/p}}\sum_{1\leqslant i<j\leqslant\infty} j^{-2/\alpha} G_{X_i,X_j}\big(j^{1/p}\big) + \frac{2}{1-r^{-2/p}}\sum_{1\leqslant i<j\leqslant\infty}\int_{j^{1/p}}^{\infty} t^{-3} G_{X_i,X_j}(t)\,\mathrm{d}t.
\end{aligned} \tag{3.12}$$

Since there are positive constants C_1, C_2 such that

$$C_1 n^{-1-2/p} G_{X_i,X_j}\big(n^{1/p}\big) \leqslant \int_{n^{1/p}}^{(n+1)^{1/p}} t^{-3} G_{X_i,X_j}(t)\,\mathrm{d}t \leqslant C_2 (n+1)^{-1-2/p} G_{X_i,X_j}\big((n+1)^{1/p}\big)$$

for each $n \geqslant 1$, we have

$$\begin{aligned}
\sum_{1\leqslant i<j\leqslant\infty}\int_{j^{1/p}}^{\infty} t^{-3} G_{X_i,X_j}(t)\,\mathrm{d}t &\leqslant C_2 \sum_{1\leqslant i<j\leqslant\infty}\sum_{n=j}^{\infty} n^{-1-2/p} G_{X_i,X_j}\big(n^{1/p}\big) \\
&\leqslant C_2 \sum_{1\leqslant i<j\leqslant\infty}\sum_{n=j}^{\infty} n^{-1-2/\alpha} G_{X_i,X_j}\big(n^{1/p}\big) < \infty.
\end{aligned} \tag{3.13}$$

Moreover,

$$\sum_{1\leqslant i<j\leqslant\infty} j^{-2/\alpha} G_{X_i,X_j}\big(j^{1/p}\big) \leqslant C(\alpha) \sum_{1\leqslant i<j\leqslant\infty} \sum_{n=j}^{\infty} n^{-1-2/\alpha} G_{X_i,X_j}\big(n^{1/p}\big) < \infty. \tag{3.14}$$

Therefore, (3.12), (3.13), (3.14) imply condition (3.2) and the thesis is established by employing Corollary 2. □

Remark 2 By comparing Corollary 4 and Theorem 1 of [18], one can see that the former allows us to consider a wider structure of dependent random variables because any sequence of associated random variables is a sequence of pairwise PQD random variables (the reciprocal does not hold). In the latter statement, the mild covariance condition

$$\sum_{1\leqslant i<j\leqslant\infty} \int_{j^{1/p}}^{\infty} v^{-3} G_{X_i,X_j}(v)\,\mathrm{d}v < \infty \tag{3.15}$$

is equivalent to $\sum_{1\leqslant i<j\leqslant\infty} \sum_{n=j}^{\infty} n^{-1-2/p} G_{X_i,X_j}(n^{1/p}) < \infty$, thus being weaker than (3.11). Hence, it stands as an open problem to know if Corollary 4 remains true under condition (3.15) instead of (3.11).

Furthermore, in light of previous subsection 3.1, among the families of random variables for which (3.10) holds, one can find well-known dependence structures, namely, sequences of pairwise NQD random variables or sequences of extended negatively dependent random variables (for definition and basic properties, see [14] and [17], respectively); with regard to the corresponding proof, we refer [26] and [28].

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Conflict of interest

Author states no conflict of interest.

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