



ALGEBRAIC THEORY OF QUASI-CRYSTALS

A GENERALIZATION OF THE HYPOPLACTIC MONOID
AND A LITTELMANN PATH MODEL

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Master in Mathematics and Applications

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Algebraic Theory of Quasi-crystals

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ABSTRACT

The plactic monoid was first introduced by Lascoux and Schützenberger based on work by Schensted and Knuth, resulting in a description of the plactic monoid by a presentation and by Young tableaux and an insertion algorithm. Then, due to work by Kashiwara on crystal bases, the plactic monoid was obtained by identifying vertices in the crystal graph associated to the general linear Lie algebra. This process proved to be fruitful, as it enriched the structure of the already acclaimed plactic monoid and allowed its generalization based on other crystal graphs.

Analogously, the hypoplactic monoid was introduced by Krob and Thibon via a presentation and via quasi-ribbon tableaux and an insertion algorithm. Recent work by Cain and Malheiro showed a construction of the hypoplactic monoid by identifying vertices in a quasi-crystal graph derived from the crystal graph associated to the general linear Lie algebra. Although this construction is based on Kashiwara's work, it cannot be extended to other crystal graphs, and it lacks connections to representation theory such as a Littelmann path model.

In this thesis we present an algebraic theory of quasi-crystals. We associate a quasi-crystal graph to each quasi-crystal and describe a one-to-one correspondence between seminormal quasi-crystals and quasi-crystal graphs. We show that a hypoplactic congruence can be defined over any seminormal quasi-crystal, from which a general notion of hypoplactic monoid emerges. We prove that the construction of the hypoplactic monoid proposed by Cain and Malheiro can be placed in context as the hypoplactic monoid associated with the general linear Lie algebra. Based on this framework, we make a study of the hypoplactic monoid associated to the symplectic Lie algebra. Finally, We present a Littelmann path model for quasi-crystals.

Keywords: quasi-crystal, hypoplactic monoid, Littelmann path model, crystal, plactic monoid, Kashiwara operator.

RESUMO

O monoide pláctico foi inicialmente introduzido por Lascoux e Schützenberger baseado em trabalhos de Schensted e Knuth, resultando numa descrição do monoide pláctico por uma apresentação e pelos quadros de Young com um algoritmo de inserção. Posteriormente, pelos trabalhos de Kashiwara em bases cristal, o monoide pláctico foi obtido através da identificação de vértices no grafo cristal associado à álgebra de Lie linear geral. Este processo mostrou-se frutuoso, dado que enriqueceu a estrutura do já reconhecido monoide pláctico e, adicionalmente, permitiu a sua generalização associando-o a outros grafos cristal.

Analogamente, o monoide hipopláctico foi introduzido por Krob e Thibon através de uma apresentação e de quadros quase-fita com um algoritmo de inserção. Em trabalhos recentes de Cain e Malheiro, o monoide hipopláctico foi obtido pela identificação de vértices num grafo quase-cristal, derivado do grafo cristal associado à álgebra de Lie linear geral. Embora com base em trabalhos de Kashiwara, esta construção não se aplica a outros grafos cristal e não tem conexões à Teoria das Representações, tais como um modelo de caminhos de Littelmann.

Nesta tese, apresenta-se uma Teoria Algébrica dos Quase-cristais. Associa-se um grafo quase-cristal a cada quase-cristal e descreve-se uma correspondência bijectiva entre quase-cristais seminormais e grafos quase-cristal. Mostra-se que se pode definir uma congruência hipoplactica sobre um qualquer quase-cristal seminormal, donde resulta um conceito geral de monoide hipopláctico. Prova-se que a construção proposta por Cain e Malheiro é equivalente à construção do monoide hipopláctico associado à álgebra de Lie linear geral. Com base nesta abordagem, estuda-se o monoide hipopláctico associado à álgebra de Lie simpléctica. Por fim, descreve-se um modelo de caminhos de Littelmann para os quase-cristais.

Palavras-chave: quase-cristal, monoide hipopláctico, modelo de caminhos de Littelmann, cristal, monoide pláctico, operador de Kashiwara.

CONTENTS

1	Introduction	1
2	Preliminaries	3
2.1	Monoids, words and presentations	3
2.2	Graphs	6
2.3	Root systems	8
2.3.1	Simple roots and weights	10
2.3.2	The Weyl group and the Weyl chambers	13
2.3.3	Cartan types $\mathfrak{A}_n$ and $\mathfrak{C}_n$	16
3	The classical plactic monoid	19
3.1	Young tableaux	19
3.2	The insertion algorithm	21
3.3	Knuth relations	23
4	Crystals	25
4.1	Crystals and homomorphisms	26
4.2	Crystal graphs	31
4.3	Tensor product	33
4.4	The signature rule for tensor product	36
5	Crystalizing the plactic monoid	39
5.1	Crystal monoids and homomorphisms	39
5.2	The free crystal monoid	44
5.3	Congruences and quotients	49
5.4	The plactic congruence	54
5.5	The plactic monoid of type $\mathfrak{A}_n$	56

6	The classical hypoplactic monoid	59
6.1	Quasi-ribbon tableaux	60
6.2	The insertion algorithm	61
6.3	A presentation for the classical hypoplactic monoid	67
6.4	Quasi-crystals for the classical hypoplactic monoid	68
7	Quasi-crystals	73
7.1	Quasi-crystals and homomorphisms	73
7.2	Quasi-crystal graphs	83
7.3	Quasi-tensor product	90
7.4	The signature rule for quasi-tensor product	100
8	Crystalizing the hypoplactic monoid	105
8.1	Quasi-crystal monoids and homomorphisms	105
8.2	The free quasi-crystal monoid	113
8.3	Congruences and quotients	118
8.4	The hypoplactic congruence	122
8.5	The hypoplactic monoid of type $\mathfrak{A}_n$	125
9	The hypoplactic monoid of type $\mathfrak{C}_n$	129
9.1	The case $n = 2$	129
9.1.1	The definition of $\text{hypo}(\mathfrak{C}_2)$	129
9.1.2	Highest-weight words	133
9.1.3	Isolated words	136
9.1.4	A presentation	137
9.1.5	Identities	144
9.2	The case $n \geq 3$	146
9.2.1	The definition of $\text{hypo}(\mathfrak{C}_n)$	146
9.2.2	Highest-weight words	150
9.2.3	Isolated words	151
9.2.4	Knuth relations on $\text{hypo}(\mathfrak{C}_n)$	153
9.2.5	From hypo_{n-1} to $\text{hypo}(\mathfrak{C}_n)$	160
10	Littelmann path models	163
10.1	A model for crystals	163
10.2	A model for quasi-crystals	166
	Bibliography	173

INTRODUCTION

The plactic monoid, introduced by Lascoux and Schützenberger [LS81], is an algebraic object of great interest, with connections to several fields such as representation theory, combinatorics [Ful97], symmetric functions, and Schubert polynomials [LS85; LS89]. It was also used to give a first rigorous proof of the Littlewood–Richardson rule [LR34]. This led Schützenberger [Sch97] to consider it one of the most fundamental monoids in algebra. There are numerous ways of obtaining the plactic monoid, we highlight three of them. It originally emerged from the study of Young tableaux and the Schensted insertion algorithm [Sch61]. It also has a presentation by the so-called Knuth relations [Knu70].

Kashiwara [Kas90; Kas91] introduced crystal bases for modules of quantized universal enveloping algebras [Jim85], and showed that the plactic monoid arises from the crystal basis associated with the quantized universal enveloping general linear Lie algebra. This result allowed a deeper study of the plactic monoid, and its generalization, because the construction still results in a monoid for crystal bases associated with other quantized universal enveloping algebras. Thus, in [KN94] the crystal graphs for Cartan types $\mathfrak{A}_n$, $\mathfrak{B}_n$, $\mathfrak{C}_n$ and $\mathfrak{D}_n$ were studied, from which a notion of Kashiwara–Nakashima tableaux emerged. Based on this, Lecouvey [Lec02; Lec03] presented comprehensive descriptions of the plactic monoids for types $\mathfrak{B}_n$, $\mathfrak{C}_n$, and $\mathfrak{D}_n$, which later appeared in a survey [Lec07]. Recently, Cain, Gray and Malheiro [CGM15a; CGM19] presented rewriting systems and biautomatic structures for these monoids.

In an independent work, Littelmann [Lit94; Lit95] introduced a path model to give a generalization of the Littlewood–Richardson rule for symmetrizable Kac–Moody algebras. Kashiwara [Kas94] showed that the crystal bases from quantized universal enveloping algebras coincide with crystals from Littelmann paths. However, Littelmann paths are of particular interest in representation theory, and their study gave rise to the notion of plactic algebra [Lit96].

The hypoplactic monoid was introduced by Krob and Thibon [KT97] from representation theoretical interpretations of quasi-symmetric functions and noncommutative symmetric functions. It emerged from a noncommutative realization of quasi-symmetric functions analogous to the realization of symmetric functions by the plactic monoid presented by Lascoux and Schützenberger [LS81]. This led to a construction of the hypoplactic monoid by quasi-ribbon tableaux and an insertion algorithm, and to a presentation by the Knuth relations and the quartic relations. A detailed study of the hypoplactic monoid was done by Novelli [Nov00], and a comparative study with other monoids was done in [CGM15b].

Cain and Malheiro [CM17] described a quasi-crystal graph, derived from the crystal graph for type $\mathfrak{A}_n$, that allows a construction of the hypoplactic monoid analogous to the construction of the plactic monoid presented by Kashiwara. However, this construction is purely combinatorial and does not have an algebraic foundation. It therefore cannot be used to construct a monoid starting with a crystal graph of other type. Cain and Malheiro also asked if there was a Littelmann path model for this construction.

In this thesis we address the problems described for the construction of the hypoplactic monoid proposed by Cain and Malheiro. In Chapter 2 we introduce basic notation, and give the basic background on root systems. In Chapter 3, we present the two original constructions of the plactic monoid. In Chapter 4, we introduce the theory of crystals. Then, in Chapter 5 we present a method of constructing the plactic monoid from crystals. Although the results in this chapter are in the literature, our approach is different, as it has a strong algebraic structure and allows us to form analogous constructions, as we see in the sequel.

In Chapter 6, we present the three constructions of the hypoplactic monoid: by a presentation, by quasi-ribbon tableaux and an insertion algorithm [KT97; Nov00], and the one obtained by Cain and Malheiro [CM17]. In Chapter 7, we introduce the algebraic theory of quasi-crystals, study their properties, give a purely combinatorial description through quasi-crystal graphs, and introduce the notion of quasi-tensor product. Then, in Chapter 8 we develop a theory of quasi-crystal monoids in parallel with the approach done in Chapter 5. We then reach a definition of a hypoplactic monoid associated to a seminormal quasi-crystal, and prove that for type $\mathfrak{A}_n$, our construction is equivalent to the construction of the hypoplactic monoid proposed by Cain and Malheiro.

Since our construction yields a monoid for any seminormal quasi-crystal, in Chapter 9 we study the hypoplactic monoid associated to the standard quasi-crystal of type $\mathfrak{C}_n$. Finally, in Chapter 10 we show that our theory of quasi-crystals admits a Littelmann path model.

PRELIMINARIES

In this chapter we give the definitions of the algebraic structures that will be used throughout this thesis. We assume the reader is familiar with the basic concepts on the theory of monoids and the theory of graphs, so our main focus in Sections 2.1 and 2.2 is to introduce notation and clarify terminology. We give a more detailed introduction to root systems in Section 2.3, because they will be used as the underlying algebraic structure for defining crystals and quasi-crystals, and since other algebraic structures can be used for this purpose, it will help if one wants to exchange root systems with some of them. Finally, we draw attention to Examples 2.16 and 2.17, where we describe the root systems for Cartan types $\mathfrak{A}_n$ and $\mathfrak{C}_n$, which will be the only non-arbitrary root systems considered in this thesis.

2.1 Monoids, words and presentations

In this section we briefly review some of the basic concepts relating to monoids. Our main goal is to introduce notation and clarify some terminology which will be used later. For a comprehensive study on semigroup theory, and in particular monoids, see for example [Hig92; How95].

A *monoid* consists of a set M together with a binary operation $\cdot : M \times M \rightarrow M$, where $(x, y) \mapsto x \cdot y$ ($x, y \in M$), with an *identity* element $1 \in M$ satisfying the associative law

$$(x \cdot y) \cdot z = x \cdot (y \cdot z),$$

and the identity law

$$1 \cdot x = x \cdot 1 = x,$$

for all $x, y, z \in M$. As usual, we denote a monoid with the same letter as its set, since it is clear from the context which object is being considered. Also, we usually

omit the binary operation $\cdot$ writing xy instead of $x \cdot y$.

A *zero monoid* is a monoid M where there is an *absorbing* element $0 \in M$ such that

$$0x = x0 = 0,$$

for all $x \in M$.

A (*monoid*) *congruence* on a monoid M is an equivalence relation $\theta \subseteq M \times M$ such that if $(x_1, y_1), (x_2, y_2) \in \theta$, then $(x_1 x_2, y_1 y_2) \in \theta$. Sometimes we write $x \theta y$ instead of $(x, y) \in \theta$. With this notation we have that an equivalence relation $\theta \subseteq M \times M$ is a congruence on M if $x_1 \theta y_1$ and $x_2 \theta y_2$ imply $x_1 x_2 \theta y_1 y_2$. This notation also allows a shorter and clear writing, as for instance $x_1 \theta x_2 \theta x_3 \theta x_4$ stands for $(x_1, x_2), (x_2, x_3), (x_3, x_4) \in \theta$. Denote by $[x]_\theta$, or simply $[x]$ when the congruence is clear from the context, the θ -equivalence class containing x , and denote by M/θ the set of all θ -equivalence classes. Then, the *quotient monoid* of M by θ consists of M/θ together with the binary operation given by $[x] \cdot [y] \mapsto [x \cdot y]$ ($x, y \in M$), where the identity is $[1]$.

Given monoids M and M' with identities 1_M and $1_{M'}$, respectively, a map $\psi : M \rightarrow M'$ is called a (*monoid*) *homomorphism* if

$$\psi(xy) = \psi(x)\psi(y),$$

for all $x, y \in M$, and

$$\psi(1_M) = 1_{M'}.$$

And if ψ is also bijective, then it is called a *monoid isomorphism*. The monoids M and M' are said to be *isomorphic* if there exists a monoid isomorphism between them.

The *opposite monoid* of a monoid M is a monoid M^{op} consisting of the set M and a binary operation $\odot$ given by

$$x \odot y = y \cdot x$$

($x, y \in M$), where the identity is the same as in the monoid M . The properties satisfied by the monoids M and M^{op} are similar, as for instance M has an absorbing element if and only if M^{op} has an absorbing element, or M is left cancellative if and only if M^{op} is right cancellative. Therefore, we say that a monoid M' is *anti-isomorphic* to a monoid M if M' is isomorphic to M^{op} . Equivalently, M and M' are anti-isomorphic if there exists a bijective map $\psi : M \rightarrow M'$ such that

$$\psi(xy) = \psi(y)\psi(x),$$

for any $x, y \in M$. This concept will be crucial to understand the subsequent chapters, because there are two main constructions of the classical plactic monoid, which originate monoids that are anti-isomorphic to each other, and although we will only work with one of the constructions, the bibliographic references may consider the other approach.

For a monoid homomorphism $\psi : M \rightarrow M'$, the *kernel* of ψ is a congruence $\ker \psi$ on M given by

$$\ker \psi = \{(x, y) \in M \times M \mid \psi(x) = \psi(y)\}.$$

This leads to the well-known result which relates congruences and homomorphisms, as for a monoid M , we have that a relation $\theta \subseteq M \times M$ is a congruence on M if and only if there exist a monoid M' and a homomorphism $\psi : M \rightarrow M'$ such that $\theta = \ker \psi$.

Let X be a set. Let X^* to be the set of all sequences of the form $x_1 x_2 \dots x_m$, with $x_1, x_2, \dots, x_m \in X$ and $m \geq 0$, where ϵ denotes the empty sequence. In this context, the set X is called the *alphabet*, its elements are called *letters*, and the elements of X^* are called *words*. For a word $w \in X^*$, the *length* of w , which corresponds to the number of letters occurring in w , is denoted by $|w|$. And for each letter $x \in X$, the number of times x occurs in w is denoted by $|w|_x$.

The set X^* of all words over X together with the concatenation of words forms a monoid called the *free monoid* over X . The term “free” is justified by the following universal property that X^* satisfies.

Theorem 2.1. *Let X be a set, and let M be a monoid. For each map $\psi : X \rightarrow M$, there exists a unique monoid homomorphism $\hat{\psi} : X^* \rightarrow M$ such that $\hat{\psi}(x) = \psi(x)$, for any $x \in X$.*

Let M be a monoid, and let X be a generating set of M . Denote by $\iota : X^* \rightarrow M$ the (unique) homomorphism satisfying $\iota(x) = x$, for any $x \in X$. Set $\theta = \ker \iota$. Recall that M is isomorphic to X^*/θ . Now, a sequence $w = x_1 x_2 \dots x_m$, with $x_1, x_2, \dots, x_m \in X$ and $m \geq 0$, can be seen either as a word in X^* or an element of M . Thus, to avoid confusion, given two sequences u and v (possibly empty) of elements of X , we write $u = v$, whenever u and v are the same word, and write $u =_M v$ or $u \theta v$, whenever u and v represent the same element of M .

Let X be a set, and let $R \subseteq X^* \times X^*$ be a relation on X^* . Denote by θ_R the smallest congruence on X^* containing R , that is,

$$\theta_R = \bigcap \{\theta \subseteq X^* \times X^* \mid R \subseteq \theta, \theta \text{ is a congruence on } X^*\}.$$

We say that X and R form a *presentation* of a monoid M , and write $M = \langle X \mid R \rangle$, if M is isomorphic to X^*/θ_R . For example, a presentation of the *bicyclic monoid* Z is

$$Z = \langle -, + \mid (+-, \epsilon) \rangle.$$

Thus, in the bicyclic monoid we have that

$$-++--++ =_Z -+^2.$$

In general, given $w \in \{-, +\}^*$, we have that $w =_Z -^a +^b$, for some $a, b \in \mathbb{Z}_{\geq 0}$. And so, we say that the elements of the bicyclic monoid are of the form $-^a +^b$, for $a, b \in \mathbb{Z}_{\geq 0}$. The bicyclic monoid will play an important role in Section 4.4.

When giving a presentation $\langle X \mid R \rangle$ for a zero monoid M , we always assume that X contains an element 0 , and the pairs $(0x, 0)$ and $(x0, 0)$ lie in R , for all $x \in X$. Thus, for the sake of simplicity, we usually omit these facts. For example, a full presentation of the zero monoid Z_0 is

$$Z_0 = \langle 0, -, + \mid (00, 0), (0-, 0), (0+, 0), (-0, 0), (+0, 0), (+-, 0) \rangle,$$

but in this context it is sufficient to state

$$Z_0 = \langle -, + \mid (+-, 0) \rangle.$$

In contrast with the bicyclic monoid, now we have that

$$-++--++ =_{Z_0} 0.$$

In general, given a word $w \in \{-, +\}^*$ if w has a decomposition of the form $w = w_1 + w_2$, for some $w_1, w_2 \in \{-, +\}^*$, then $w =_{Z_0} 0$, otherwise $w =_{Z_0} -^a +^b$, for some $a, b \in \mathbb{Z}_{\geq 0}$. Thus, an element of Z_0 other than 0 is of the form $-^a +^b$, for $a, b \in \mathbb{Z}_{\geq 0}$. The monoid Z_0 will play an important role in Section 7.4.

For a word $w = x_1 x_2 \dots x_m$, denote by $(^{\leftarrow} w) = x_m x_{m-1} \dots x_1$ the *reversed* word of w . Then, if $\langle X \mid R \rangle$ is a presentation for a monoid M , by defining $R^{\text{op}} = \{((^{\leftarrow} u), (^{\leftarrow} v)) \mid (u, v) \in R\}$, we obtain a presentation $\langle X \mid R^{\text{op}} \rangle$ for the opposite monoid M^{op} . In other words, $\langle X \mid R \rangle$ and $\langle X \mid R^{\text{op}} \rangle$ are anti-isomorphic.

2.2 Graphs

In this section we introduce a class of graphs that will be used to obtain a combinatorial description of seminormal crystals and seminormal quasi-crystals. Although no prior knowledge on graph theory is required, we assume some familiarity with the basic concepts, which is given in any introductory course on this subject. See for example [Bol98] for a general approach.

A *directed graph* Γ consists of a set X , whose elements are called *vertices*, and a set $E \subseteq X \times X$, whose elements are called *edges*. As usual, we say that $x \longrightarrow y$ is an edge of Γ instead of $(x, y) \in E$. The vertices x and y are respectively called the *start* and the *end* of an edge $x \longrightarrow y$, and if $x = y$, we say that x has a *loop*. The directed graph Γ is said to be *simple* if its vertices do not have loops, and is said to be *connected* if for any two vertices x and y , there exists a sequence of vertices $x = x_0, x_1, \dots, x_m = y$ such that $x_i \longrightarrow x_{i+1}$ or $x_{i+1} \longrightarrow x_i$ are edges of Γ , for $i = 0, 1, \dots, m-1$.

For example, the directed graph with vertex set $\{1, 2, 3, 4, 5\}$, and for each $x \leq 4$, an edge with start at x and end at $x+1$ has the following representation

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5. \quad (2.1)$$

A *walk* on a directed graph Γ is a sequence of vertices $x_0, x_1, \dots, x_m$, such that $x_0 \longrightarrow x_1 \longrightarrow \dots \longrightarrow x_m$ are edges of Γ . A *path* is a walk $x_0, x_1, \dots, x_m$ whose vertices are pairwise distinct. The *length* of a walk $x_0, x_1, \dots, x_m$ is m , the *start* is x_0 , and the *end* is x_m .

Let Γ be a directed graph with vertex set X and edge set E . A *subgraph* Γ' of Γ consists of a vertex set $X' \subseteq X$ and an edge set $E' \subseteq E \cap (X' \times X')$. A *connected component*, or simply *component*, of Γ is a subgraph Γ' satisfying the following properties:

- (i) if x and y are vertices of Γ' , then there exists a sequence $x = x_0, x_1, \dots, x_m = y$ such that $x_i \longrightarrow x_{i+1}$ or $x_{i+1} \longrightarrow x_i$ are edges of Γ' , for $i = 0, 1, \dots, m-1$;
- (ii) if x and y are vertices of Γ' and $x \longrightarrow y$ is an edge of Γ , then $x \longrightarrow y$ is an edge of Γ' ;
- (iii) if x is a vertex of Γ' and $x \longrightarrow y$ is an edge of Γ , then y is a vertex of Γ' .

Note that the notion of connected component differs in the literature, as some authors only require conditions (i) and (ii), and say that the connected component is maximal if (iii) is verified. For a vertex x of Γ , we denote by $\Gamma(x)$ the connected component of Γ containing x .

Let Γ and Γ' be directed graphs with vertex sets X and X' , respectively. A *graph homomorphism* from Γ to Γ' is a map $\psi : X \rightarrow X'$ that preserves edges, i.e., $\psi(x) \longrightarrow \psi(y)$ is an edge of Γ' , whenever $x \longrightarrow y$ is an edge of Γ .

Given a set I , an *I-labelling* of a directed graph Γ is a map $\ell : E \rightarrow I$. We say that an edge $x \longrightarrow y$ of Γ is labelled by $i \in I$, if $\ell(x, y) = i$, and denote it by $x \xrightarrow{i} y$. For example, the directed graph in (2.1) can be labelled by the

set $\{1, 2, 3, 4\}$ by defining $\ell(x, x+1) = x$, for each $x = 1, \dots, 4$, which induces the following representation

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} 4 \xrightarrow{4} 5.$$

When Γ and Γ' are I -labelled directed graphs, if we do not say otherwise, we require graph homomorphisms between Γ and Γ' to preserve labels. That is, if ψ is a graph homomorphism from Γ to Γ' , then $\psi(x) \xrightarrow{i} \psi(y)$ is an edge of Γ' , whenever $x \xrightarrow{i} y$ is an edge of Γ .

Given a set Λ , a *weight map* on a directed graph Γ is a map $\text{wt} : X \rightarrow \Lambda$. For a vertex x of Γ , $\text{wt}(x)$ is called the *weight* of x .

If we do not say otherwise, graph homomorphisms between weighted directed graphs are assumed to preserve weights. That is, given weighted directed graphs Γ and Γ' with weights in a set Λ , a graph homomorphism ψ from Γ to Γ' satisfies $\text{wt}(\psi(x)) = \text{wt}(x)$, for any vertex x of Γ .

In the subsequent chapters we consider Λ -*weighted I-labelled directed graphs*, that is, directed graphs where an I -labelling and a weight map are defined.

2.3 Root systems

In this section we introduce the background on root systems required to study crystals and quasi-crystals. Root systems are commonly found on representation theory, in particular, they usually arise on the study of Lie groups and Lie algebras, but we will detach them from this context, as our only aim is to construct an algebraic structure where crystals and quasi-crystals can be defined. For further context see for example [Bou02; Bum13; EW06; FH91].

Let V be a Euclidean space, that is, a real vector space with an inner product which is a bilinear form $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ satisfying $\langle u, v \rangle = \langle v, u \rangle$, for $u, v \in V$, and $\langle v, v \rangle > 0$ whenever $v \neq 0$.

For $\alpha \in V$ other than 0, set

$$\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} \alpha.$$

The *reflection* in the hyperplane orthogonal to α is the linear map $r_\alpha : V \rightarrow V$ given by

$$r_\alpha(v) = v - \langle v, \alpha^\vee \rangle \alpha,$$

for each $v \in V$. Note that r_α is bijective, as $r_\alpha(r_\alpha(v)) = v$, for all $v \in V$. Also, r_α preserves the inner product, as $\langle r_\alpha(u), r_\alpha(v) \rangle = \langle u, v \rangle$ for any $u, v \in V$.

Definition 2.2. A root system in V is a subset Φ of V satisfying the following conditions

- (i) Φ is nonempty, finite, and $0 \notin \Phi$;
- (ii) $r_\alpha(\beta) \in \Phi$, for all $\alpha, \beta \in \Phi$;
- (iii) $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$, for all $\alpha, \beta \in \Phi$,
- (iv) if $\alpha \in \Phi$ and $k\alpha \in \Phi$, then $k = \pm 1$.

The elements of Φ are called *roots*, and the elements $\alpha^\vee$, with $\alpha \in \Phi$, are called *coroots*.

Note that the definition of root system may differ in the literature. For instance, some authors do not require condition (iii) and say that a root system is crystallographic when it is satisfied. Similarly, some authors say that a root system is reduced when condition (iv) is satisfied.

For $\alpha \in \Phi$, since $r_\alpha \circ r_\alpha$ is the identity map on V , we have that r_α induces a permutation of order 2 on Φ , by condition (ii). Also, from conditions (ii) and (iv), we have that $-\alpha = r_\alpha(\alpha) \in \Phi$, and so, condition (iv) is equivalent to state that the only scalar multiples of α in Φ are exactly α and $-\alpha$.

Since we have $(\alpha^\vee)^\vee = \alpha$ and $r_{\alpha^\vee}(\beta^\vee) = (r_\alpha(\beta))^\vee$, it is easy to see that the set of coroots $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ is also a root system.

Definition 2.3. Let Φ be a root system in V .

- (i) Φ is said to be *semisimple* if it spans V .
- (ii) Φ is said to be *reducible* if there exists two nonempty subsets Φ_1 and Φ_2 of Φ such that $\Phi = \Phi_1 \cup \Phi_2$ and $\langle \alpha, \beta \rangle = 0$ for any $\alpha \in \Phi_1$ and $\beta \in \Phi_2$. If Φ is not reducible, it is called *irreducible* or *simple*.
- (iii) Φ is said to be *simply-laced* if all roots have the same length.

When we do not say explicitly, we consider Φ to be an arbitrary root system in V without necessarily satisfying any of these properties. Note that some authors only consider semisimple root systems, with the condition of Φ spanning V is included in the definition of root system. In this context the term simple is frequently used, but to avoid confusion we shall always use the term irreducible, because a simple root system may not be semisimple.

For $\alpha, \beta \in \Phi$, since $\langle \alpha, r_\alpha(\beta) \rangle = -\langle \alpha, \beta \rangle$, we have that $\langle \alpha, r_\alpha(\beta) \rangle = 0$ if and only if $r_\alpha(\beta) = \beta$. So, if $\Phi = \Phi_1 \cup \Phi_2$ is reducible, where Φ_1 and Φ_2 are nonempty

orthogonal subsets of Φ , then $r_\alpha(\beta) \in \Phi_i$, whenever $\alpha, \beta \in \Phi_i$, for $i = 1, 2$. This implies that Φ_1 and Φ_2 are root systems in V .

From the following result we can see that a root system has a very particular configuration, in the sense that the position and length of a root restrict the position and length of other roots.

Lemma 2.4 (Finiteness Lemma). *Let $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$. Then,*

$$\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in \{0, 1, 2, 3\}.$$

Therefore, if $\langle \alpha, \beta \rangle \neq 0$, then $\langle \alpha, \beta^\vee \rangle = \pm 1$ or $\langle \beta, \alpha^\vee \rangle = \pm 1$.

Proof. By Definition 2.2(iv), we have that $\beta \neq k\alpha$, for all $k \in \mathbb{R}$. Then, by the Cauchy–Schwarz inequality, we get that

$$\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = 4 \frac{\langle \alpha, \beta \rangle^2}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} \in [0, 4[. \quad (2.2)$$

By Definition 2.2(iii), we have that $\langle \alpha, \beta^\vee \rangle, \langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$, from which the statement follows. $\square$

Note that from (2.2), it follows that the angle θ between two roots α and β , with $\beta \neq \pm\alpha$, is such that

$$4\cos^2(\theta) \in \{0, 1, 2, 3\}.$$

Thus, θ can only be $\pi/6, \pi/4, \pi/3, \pi/2, 2\pi/3, 3\pi/4$ or $5\pi/6$. If we have $\theta \neq \pi/2$ and $\|\alpha\| \leq \|\beta\|$, then $\langle \alpha, \beta^\vee \rangle = \pm 1$, which implies that

$$\frac{\|\alpha\|}{\|\beta\|} = \frac{1}{2|\cos \theta|}.$$

2.3.1 Simple roots and weights

Throughout this section, fix a nonzero vector $v_0 \in V$ which is not orthogonal to any root, that is, $\langle v_0, \alpha \rangle \neq 0$, for all $\alpha \in \Phi$. An easy method to obtain such a vector is as follows. Given a vector $v \in V$, set $\Phi' = \{\alpha \in \Phi \mid \langle v, \alpha \rangle \neq 0\}$. If there exists $\beta \in \Phi$ such that $\langle v, \beta \rangle = 0$, since Φ' is finite, take $k \in \mathbb{R} \setminus \{0\}$ such that $\langle v, \alpha \rangle \neq -k\langle \beta, \alpha \rangle$, for any $\alpha \in \Phi'$. Set $v' = v + k\beta$. Then, $\langle v', \alpha \rangle \neq 0$, for all $\alpha \in \Phi'$, and $\langle v', \beta \rangle = k\langle \beta, \beta \rangle \neq 0$, and so, Φ' is a proper subset of $\Phi'' = \{\alpha \in \Phi \mid \langle v', \alpha \rangle \neq 0\}$. Since Φ is finite, by iterating this process we eventually obtain a vector v_0 such that $\langle v_0, \alpha \rangle \neq 0$, for all $\alpha \in \Phi$.

A root $\alpha \in \Phi$ is called a *positive root* if $\langle v_0, \alpha \rangle > 0$, otherwise it is called a *negative root*. Denote by Φ^+ the set of all positive roots, and denote by Φ^- the set

of all negative roots. So, we have that $\Phi^- = -\Phi^+$, $\Phi^+ \cap \Phi^- = \emptyset$, and $\Phi = \Phi^+ \cup \Phi^-$. In other words, we are taking a hyperplane of codimension 1 (the hyperplane orthogonal to v_0) and considering the roots on one side as positive, and the roots on the other side as negative.

It is immediate that the set of positive roots Φ^+ satisfies the following properties:

- for each $\alpha \in \Phi$, exactly one of the roots α and $-\alpha$ lie in Φ^+ ;
- if $\alpha, \beta \in \Phi^+$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Phi^+$.

On the other hand, we have that if a subset Φ' of Φ satisfies these two conditions, then there exists a vector on V for which Φ' is the set of positive roots.

Definition 2.5. A *simple root* is a positive root which cannot be obtained as the sum of two positive roots. Denote by Σ the set of all simple roots.

It is immediate from the definition that we have

$$\Sigma = \Phi^+ \setminus \{\alpha + \beta \mid \alpha, \beta \in \Phi^+\}.$$

Lemma 2.6. Let $\alpha, \beta \in \Sigma$ with $\alpha \neq \beta$. Then, $\langle \alpha, \beta \rangle \leq 0$.

Proof. Suppose there exist $\alpha, \beta \in \Sigma$ such that $\alpha \neq \beta$ and $\langle \alpha, \beta \rangle > 0$. By Lemma 2.4, we have that $\langle \alpha, \beta^\vee \rangle = 1$ or $\langle \beta, \alpha^\vee \rangle = 1$. Without loss of generality, assume that $\langle \beta, \alpha^\vee \rangle = 1$. Then, $r_\alpha(\beta) = \beta - \alpha$. If $r_\alpha(\beta) \in \Phi^+$, then $\beta = r_\alpha(\beta) + \alpha$, which is a contradiction, because β is a simple root. Otherwise, we must have that $-r_\alpha(\beta) \in \Phi^+$, and since $\alpha = \beta - r_\alpha(\beta)$, we obtain another contradiction, because α is a simple root. $\square$

Fix an index set I for the simple roots, and denote them by α_i ($i \in I$). So, we have that $\Sigma = \{\alpha_i \mid i \in I\}$, where $\alpha_i \neq \alpha_j$, whenever $i \neq j$.

Theorem 2.7. The set of simple roots Σ satisfies the following properties:

- (i) Σ is a linearly independent subset of V ; and
- (ii) for every positive root $\beta \in \Phi^+$, there exist nonnegative integers $k_i \in \mathbb{Z}_{\geq 0}$ ($i \in I$) such that $\beta = \sum_{i \in I} k_i \alpha_i$.

Proof. (i) Let $k_i \in \mathbb{R}$ ($i \in I$) be such that $\sum_{i \in I} k_i \alpha_i = 0$. Set $I_1 = \{i \in I \mid k_i \geq 0\}$ and $I_2 = \{i \in I \mid k_i < 0\}$. Define $v = \sum_{i \in I_1} k_i \alpha_i$. Since

$$\sum_{i \in I_1} k_i \alpha_i = \sum_{j \in I_2} (-k_j) \alpha_j,$$

we have that

$$0 \leq \langle v, v \rangle = \sum_{\substack{i \in I_1 \\ j \in I_2}} k_i (-k_j) \langle \alpha_i, \alpha_j \rangle.$$

For $i \in I_1$ and $j \in I_2$, since $I_1 \cap I_2 = \emptyset$, then $\langle \alpha_i, \alpha_j \rangle \leq 0$, by Lemma 2.6, and so, $k_i (-k_j) \langle \alpha_i, \alpha_j \rangle \leq 0$, as $k_i \geq 0$ and $k_j < 0$. Hence, $\langle v, v \rangle = 0$, which implies that $v = 0$, and thus,

$$\sum_{i \in I_1} k_i \langle v_0, \alpha_i \rangle = \sum_{j \in I_2} (-k_j) \langle v_0, \alpha_j \rangle = \langle v_0, v \rangle = 0.$$

As $\langle v_0, \alpha_i \rangle > 0$, for all $i \in I$, it follows that $k_i = 0$, for all $i \in I$. Therefore, Σ is a linearly independent subset of V .

(ii) Let Φ' be the set of positive roots that cannot be expressed as a linear combination of the simple roots with coefficients in $\mathbb{Z}_{\geq 0}$. Suppose Φ' is nonempty. Since Φ' is finite, take $\beta \in \Phi'$ such that $\langle v_0, \beta \rangle \leq \langle v_0, \beta' \rangle$, for any $\beta' \in \Phi'$. By the definition of Φ' , we have that β is not a simple root, and so, $\beta = \beta_1 + \beta_2$, for some $\beta_1, \beta_2 \in \Phi^+$. As $\langle v_0, \beta \rangle = \langle v_0, \beta_1 \rangle + \langle v_0, \beta_2 \rangle$, where $\langle v_0, \beta_1 \rangle, \langle v_0, \beta_2 \rangle > 0$, we have that $\langle v_0, \beta_1 \rangle, \langle v_0, \beta_2 \rangle < \langle v_0, \beta \rangle$, which implies that β_1 and β_2 do not lie in Φ' , by the choice of β . Therefore, β_1 and β_2 can be expressed as a linear combination of the simple roots with coefficients in $\mathbb{Z}_{\geq 0}$, contradicting the fact that $\beta = \beta_1 + \beta_2$ could not be obtained in that form. Hence, $\Phi' = \emptyset$. $\square$

Note that if a subset Σ' of Φ satisfies the two conditions in Theorem 2.7, then there exists a vector in V for which Σ' is the set of simple roots.

Given a negative root $\beta \in \Phi^-$, we have that $-\beta \in \Phi^+$, and by Theorem 2.7(ii), there exist $k_i \in \mathbb{Z}_{\leq 0}$ ($i \in I$) such that $\beta = \sum_{i \in I} k_i \alpha_i$. Therefore, any root can be expressed as a linear combination of the simple roots where coefficients are either all nonnegative integers or all nonpositive integers.

Definition 2.8. A *weight lattice* Λ is a $\mathbb{Z}$ -submodule of V satisfying the following conditions

- (i) Λ spans V ;
- (ii) $\Phi \subseteq \Lambda$;
- (iii) $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for any $\lambda \in \Lambda$ and $\alpha \in \Phi$.

The elements of Λ are called *weights*.

We define a partial order on a weight lattice Λ as follows. For $\lambda, \mu \in \Lambda$, we have that

$$\lambda \geq \mu \iff \lambda - \mu = \sum_{i \in I} k_i \alpha_i, \text{ for some } k_i \in \mathbb{R}_{\geq 0} \text{ } (i \in I). \quad (2.3)$$

Consider the $\mathbb{Z}$ -submodule

$$\Lambda_{\text{root}} = \sum_{i \in I} \mathbb{Z} \alpha_i$$

of V generated by the simple roots. We call Λ_{root} the *root lattice*. Note that if Φ is semisimple, then Λ_{root} is a weight lattice, and $\Lambda_{\text{root}} \subseteq \Lambda$, for any weight lattice Λ .

Definition 2.9. Let $\lambda \in \Lambda$ be a weight.

- (i) λ is said to be *dominant* if $\langle \lambda, \alpha_i \rangle \geq 0$, for all $i \in I$. The set of dominant weights is denoted by Λ^+ .
- (ii) λ is said to be *strictly dominant* if $\langle \lambda, \alpha_i \rangle > 0$, for all $i \in I$.

Since the simple roots are linearly independent, we can take nonzero vectors $\omega_i \in V$ ($i \in I$) such that for each $j \in I$,

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij},$$

where $\delta_{ij} = 1$, if $i = j$, and $\delta_{ij} = 0$, if $i \neq j$. Although these vectors may not lie in Λ , we call them *fundamental weights*. Note that if Φ is semisimple, the fundamental weights are completely determined by their defining condition, otherwise we make a choice.

If Φ is semisimple, consider the $\mathbb{Z}$ -submodule

$$\Lambda_{\text{sc}} = \sum_{i \in I} \mathbb{Z} \omega_i$$

of V generated by the fundamental weights. Then, for any weight lattice Λ , we have that

$$\Lambda_{\text{root}} \subseteq \Lambda \subseteq \Lambda_{\text{sc}}.$$

The weight lattice Λ is said to be of *adjoint type* if $\Lambda = \Lambda_{\text{root}}$, and is said to be of *simply-connected type* if $\Lambda = \Lambda_{\text{sc}}$.

2.3.2 The Weyl group and the Weyl chambers

First we show how a root system can be reconstructed from its simple roots, and as a consequence, we get that the set of simple roots completely determines the root system.

For each $i \in I$, we denote by s_i the reflection r_{α_i} in the hyperplane orthogonal to the simple root α_i . We call s_i a *simple reflection*.

Lemma 2.10. *Let $i \in I$. Then, $s_i(\beta) \in \Phi^+ \setminus \{\alpha_i\}$, for any $\beta \in \Phi^+ \setminus \{\alpha_i\}$. Therefore, s_i induces a permutation on $\Phi^+ \setminus \{\alpha_i\}$.*

Proof. Let $\beta \in \Phi^+ \setminus \{\alpha_i\}$. By Theorem 2.7(ii), there exist $k_j \in \mathbb{Z}_{\geq 0}$ ($j \in I$) such that $\beta = \sum_{j \in I} k_j \alpha_j$. Since $\beta \neq \alpha_i$, then $k_{i'} > 0$, for some $i' \neq i$. We have that

$$s_i(\beta) = \beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i = (k_i - \langle \beta, \alpha_i^\vee \rangle) \alpha_i + \sum_{j \in I \setminus \{i\}} k_j \alpha_j,$$

where the coefficient of $\alpha_{i'}$ is again $k_{i'} > 0$. Hence, $s_i(\beta) \neq \alpha_i$, and since a consequence of Theorem 2.7 is that any root can be expressed as a linear combination of the simple roots with all coefficients nonnegative or nonpositive, we get that $s_i(\beta) \in \Phi^+ \setminus \{\alpha_i\}$. $\square$

It is immediate from Definition 2.2(ii) that for any $i_1, \dots, i_m, j \in I$, we have $s_{i_1} \cdots s_{i_m}(\alpha_j) \in \Phi$. In the following result we show that any root can be obtained by applying simple reflections to a simple root.

Proposition 2.11. *Let $\beta \in \Phi$. Then, there exist $i_1, \dots, i_m, j \in I$ such that*

$$\beta = s_{i_1} \cdots s_{i_m}(\alpha_j).$$

Proof. Let Φ' be the set of positive roots that cannot be obtained by applying simple reflections to a simple root. Suppose that Φ' is nonempty. As Φ' is finite, take $\beta \in \Phi'$ such that $\langle v_0, \beta \rangle \leq \langle v_0, \beta' \rangle$, for all $\beta' \in \Phi'$. By Theorem 2.7(ii), we have that $\beta = \sum_{i \in I} k_i \alpha_i$, for some $k_i \in \mathbb{Z}_{\geq 0}$ ($i \in I$). If we had $\langle \beta, \alpha_i \rangle \leq 0$, for any $i \in I$, then we would get

$$\langle \beta, \beta \rangle = \sum_{i \in I} k_i \langle \beta, \alpha_i \rangle \leq 0$$

leading to a contradiction, because $\beta \neq 0$. Thus, $\langle \beta, \alpha_i \rangle > 0$, for some $i \in I$. By Lemma 2.10, $s_i(\beta)$ is a positive root, and since

$$\langle v_0, s_i(\beta) \rangle = \langle v_0, \beta \rangle - \langle \beta, \alpha_i^\vee \rangle \langle v_0, \alpha_i \rangle < \langle v_0, \beta \rangle,$$

we have that $s_i(\beta) \in \Phi^+ \setminus \Phi'$. Hence, there exist $i_1, \dots, i_m, j \in I$ such that $s_i(\beta) = s_{i_1} \cdots s_{i_m}(\alpha_j)$. This implies that $\beta = s_i s_{i_1} \cdots s_{i_m}(\alpha_j)$, a contradiction which comes from assuming that Φ' is nonempty.

Finally, if γ is a negative root, then $-\gamma \in \Phi^+$, and so, there exist $i_1, \dots, i_m, j \in I$ such that $-\gamma = s_{i_1} \cdots s_{i_m}(\alpha_j)$. Since each simple reflection is a linear map and $s_j(\alpha_j) = -\alpha_j$, we get that $\gamma = s_{i_1} \cdots s_{i_m} s_j(\alpha_j)$. $\square$

From the previous result we have that by successively applying reflections to the simple roots, we obtain any root. In other words, we apply an element of the following group.

Definition 2.12. The Weyl group W of Φ is the subgroup of linear isomorphisms on V generated by the reflections r_α ($\alpha \in \Phi$).

We now show that the simple reflections s_i ($i \in I$) form a set of generators for the Weyl group.

Lemma 2.13. Let $\alpha \in \Phi$ and $i \in I$. Then, $s_i r_\alpha s_i = r_{s_i(\alpha)}$.

Proof. Let $v \in V$. We have that

$$s_i r_\alpha s_i(v) = s_i(s_i(v) - \langle s_i(v), \alpha^\vee \rangle \alpha) = v - \langle s_i(v), \alpha^\vee \rangle s_i(\alpha),$$

as s_i is a linear map and $s_i s_i(v) = v$. Since s_i preserves the inner product, we get that

$$\langle s_i(v), \alpha^\vee \rangle = \langle s_i(v), s_i s_i(\alpha^\vee) \rangle = \langle v, s_i(\alpha^\vee) \rangle.$$

Finally, as

$$s_i(\alpha^\vee) = \frac{2}{\langle \alpha, \alpha \rangle} s_i(\alpha) = \frac{2}{\langle s_i(v), s_i(v) \rangle} s_i(v) = (s_i(\alpha))^\vee,$$

we obtain

$$s_i r_\alpha s_i(v) = v - \langle v, (s_i(\alpha))^\vee \rangle s_i(\alpha) = r_{s_i(\alpha)}(v).$$

And therefore, $s_i r_\alpha s_i = r_{s_i(\alpha)}$. □

Theorem 2.14. The Weyl group W is generated by the simple reflections s_i ($i \in I$).

Proof. Since the Weyl group is generated by the reflections r_α ($\alpha \in \Phi$), it suffices to show that any reflection can be obtained by composing simple reflections. Let $\alpha \in \Phi$. By Proposition 2.11, there exist $i_1, \dots, i_m, j \in I$ such that $\alpha = s_{i_1} \cdots s_{i_m}(\alpha_j)$, and so,

$$r_{s_{i_1} \cdots s_{i_m}(\alpha_j)} = s_{i_1} r_{s_{i_2} \cdots s_{i_m}(\alpha_j)} s_{i_1} = \cdots = s_{i_1} \cdots s_{i_m} s_j s_{i_m} \cdots s_{i_1},$$

by Lemma 2.13. □

Let $g \in W$ be an element of the Weyl group such that $g(\alpha_i) = \alpha_i$, for any $i \in I$. Given a vector $v \in V$, as the simple roots are linearly independent, there exist $k_i \in \mathbb{R}$ ($i \in I$) and $v' \in V$ such that $v = \sum_{i \in I} k_i \alpha_i + v'$ and $\langle v', \alpha_i \rangle = 0$, for any $i \in I$. Thus, $s_i(v') = v'$, for any $i \in I$, and since the simple reflections generate W , we have that $g(v') = v'$, which implies that $g(v) = v$. Hence, g is the identity map on V . It is immediate from this fact that the map from W to the symmetric group S_Φ of Φ , which maps each $g \in W$ to its restriction $g|_\Phi$ to Φ , is an injective group homomorphism. This implies that the Weyl group W is finite. In particular, for any $i, j \in I$, there exists $m \geq 2$ such that $(s_i s_j)^m$ is the identity map on V . Also, $s_i s_i$

is the identity map on V , for any $i \in I$. And therefore, the Weyl group W is a finite Coxeter group.

Consider the set

$$\{v \in V \mid \langle v, \alpha \rangle \neq 0 \text{ for all } \alpha \in \Phi\},$$

which is the complement in V of the set of all hyperplanes orthogonal to some root. Clearly, this is a disconnected subset of V . We call each connected component a *Weyl chamber*. For example, the set

$$\{v \in V \mid \langle v, \alpha_i \rangle > 0 \text{ for all } i \in I\}$$

is a Weyl chamber, and although its closure is not formally a Weyl chamber, the following definition is commonly found in the literature.

Definition 2.15. The set $\{v \in V \mid \langle v, \alpha_i \rangle \geq 0 \text{ for all } i \in I\}$ is called the *dominant Weyl chamber*.

One direct consequence of this definition is that a weight is dominant if and only if it lies in the dominant Weyl chamber.

2.3.3 Cartan types $\mathfrak{A}_n$ and $\mathfrak{C}_n$

Let $n \geq 2$. Now, we introduce the root systems associated to Cartan types $\mathfrak{A}_n$ and $\mathfrak{C}_n$, which will be the only non-arbitrary root systems considered in the subsequent chapters.

There are two distinct approaches to the Cartan type $\mathfrak{A}_n$, one from the general linear Lie algebra $\mathfrak{gl}_n$, and other from the special linear Lie algebra $\mathfrak{sl}_n$. We will only consider the $\mathfrak{gl}_n$ approach. Note that in representation theory the Cartan type $\mathfrak{A}_n$ is usually associated with the general linear Lie algebra $\mathfrak{gl}_{n+1}$, but associating it with $\mathfrak{gl}_n$ will simplify our approach to the plactic monoid and the hypoplactic monoid on the subsequent chapters.

Example 2.16. The root system for the Cartan type $\mathfrak{A}_n$ based on $\mathfrak{gl}_n$ is the following. Let $V = \mathbb{R}^n$ with the usual inner product. Consider $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ to be the vectors forming the usual basis for $\mathbb{R}^n$, where $\mathbf{e}_i$ consists of the n -tuple with 1 in the i -th position, and 0 elsewhere ($i = 1, 2, \dots, n$). The root system is

$$\Phi = \{\mathbf{e}_i - \mathbf{e}_j \mid i \neq j\},$$

and the set of positive roots is

$$\Phi^+ = \{\mathbf{e}_i - \mathbf{e}_j \mid i < j\}.$$

The index set for the simple roots is $I = \{1, \dots, n-1\}$, and the simple roots are

$$\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1},$$

for $i = 1, \dots, n-1$. The weight lattice is $\Lambda = \mathbb{Z}^n$, and the fundamental weights are

$$\omega_i = \mathbf{e}_1 + \dots + \mathbf{e}_i,$$

for $i = 1, \dots, n-1$. A weight $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \Lambda$ is dominant if $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. The dominant Weyl chamber is

$$\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \dots \geq x_n\}.$$

This root system is called the *root system of type $\mathfrak{A}_n$* .

Note that the root system of type $\mathfrak{A}_n$ is irreducible and simply-laced, but it is not semisimple.

The Cartan type $\mathfrak{C}_n$ is associated with the symplectic Lie algebra $\mathfrak{sp}_{2n}$, and gives rise to the following root system.

Example 2.17. The root system for the Cartan type $\mathfrak{C}_n$ based on $\mathfrak{sp}_{2n}$ is the following. Let $V = \mathbb{R}^n$ with the usual inner product. Consider $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ as described in Example 2.16. The root system is

$$\Phi = \{\pm \mathbf{e}_i \pm \mathbf{e}_j \mid i < j\} \cup \{\pm 2\mathbf{e}_i \mid i = 1, 2, \dots, n\},$$

and the set of positive roots is

$$\Phi^+ = \{\mathbf{e}_i \pm \mathbf{e}_j \mid i < j\} \cup \{2\mathbf{e}_i \mid i = 1, 2, \dots, n\}.$$

The index set for the simple roots is $I = \{1, 2, \dots, n\}$, and the simple roots are

$$\begin{aligned} \alpha_i &= \mathbf{e}_i - \mathbf{e}_{i+1}, & \text{for } i = 1, \dots, n-1, \\ \alpha_n &= 2\mathbf{e}_n. \end{aligned}$$

The weight lattice is $\Lambda = \mathbb{Z}^n$, and the fundamental weights are

$$\begin{aligned} \omega_i &= \mathbf{e}_1 + \dots + \mathbf{e}_i, & \text{for } i = 1, \dots, n-1, \\ \omega_n &= \frac{1}{2}(\mathbf{e}_1 + \dots + \mathbf{e}_n) \end{aligned}$$

A weight $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \Lambda$ is dominant if $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. The dominant Weyl chamber is

$$\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \dots \geq x_n \geq 0\}.$$

This root system is called the *root system of type $\mathfrak{C}_n$* .

Note that the root system of type $\mathfrak{C}_n$ is semisimple and irreducible, but it is not simply-laced.

THE CLASSICAL PLACTIC MONOID

In this chapter, we introduce the classical notion of plactic monoid, which arises from Young tableaux and Schensted insertion algorithm [Sch61]. A presentation for the plactic monoid is given by the Knuth relations [Knu70]. An in-depth study of the plactic monoid was done by Lascoux and Schützenberger in [LS81].

In Chapter 5, the classical plactic monoid will be placed in context as the plactic monoid associated to the Cartan type $\mathfrak{A}_n$. Thus, we will be using notation that will make more sense then, and may be confusing for now. For instance, for $n \geq 2$, we set $A_n = \{1, 2, \dots, n\}$. Also, the plactic monoid obtained from crystals is anti-isomorphic to the original one [LS81], and therefore, some of the concepts introduced in this chapter are adaptations of the original ones.

3.1 Young tableaux

In this section we introduce the notion of semi-standard Young tableaux over a totally ordered set X , and define a method of reading that maps a semi-standard Young tableau to a word in X^* . Young tableaux appear in different contexts, but the only purpose of this section is to introduce some basic concepts needed to construct the plactic monoid. For further background, see for example [Ful97].

A *partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ is a sequence of nonnegative integers such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m > 0$. A *Young diagram* of shape λ is a collection of left-justified boxes such that the i -th row has λ_i boxes, $i = 1, 2, \dots, m$. For example, we have that

$$\begin{array}{c} \square \\ \square \end{array} \quad \begin{array}{ccc} \square & \square & \square \\ \square & \square & \\ \square & & \end{array} \quad \begin{array}{cc} \square & \square \\ \square & \square \\ \square & \end{array} \quad (3.1)$$

are Young diagrams of shape $(1, 1)$, $(3, 2, 1)$ and $(2, 2, 1)$, respectively.

Given a totally ordered set X and a partition λ , a *semi-standard Young tableau* of shape λ over X corresponds to a Young diagram of shape λ where boxes are filled with elements of X such that each column is strictly increasing from top to bottom, and each row is weakly increasing from left to right. A *column* over X is a semi-standard Young tableau over X whose shape is of the form $(1, 1, \dots, 1)$. For example, the Young diagrams in (3.1) filled as follows

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 2 & 4 & \\ \hline 3 & & \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \hline \end{array} \qquad (3.2)$$

are semi-standard Young tableaux over A_5 . In particular, the first is a column.

A *standard Young tableau* of shape $\lambda = (\lambda_1, \dots, \lambda_m)$ is a semi-standard Young tableau of shape λ where the numbers $1, 2, \dots, \lambda_1 + \dots + \lambda_m$ occur exactly once. The third tableau in (3.2) is a standard Young tableau.

Note that a partition can be of the form $\lambda = ()$. In this case a semi-standard Young tableau of shape λ is called *empty* as there are no boxes to fill.

We also denote a semi-standard Young tableau $T = C_1 \cdots C_m$, where $C_1, \dots, C_m$ are the columns of T .

Given a semi-standard Young tableau T over a totally ordered set X , the *reading* of T is a word $w(T)$ in X^* defined as follows. If T is empty, then $w(T)$ is the empty word ϵ . If T is a column, then $w(T)$ is the concatenation from top to bottom of the elements in the boxes of T . If $T = C_1 C_2 \cdots C_m$, where C_i is a column ($i = 1, 2, \dots, m$), then $w(T) = w(C_m)w(C_{m-1}) \cdots w(C_1)$. For example, the readings of the semi-standard Young tableaux in (3.2) are 12, 224123 and 35124, respectively.

Note that there are various ways of reading a Young tableau, the one defined above is commonly called *Japanese reading* [Kas95] or *far-eastern reading* [HK02], and it is commonly found in the theory of crystals due to its compatibility with the tensor product of crystals. This choice will have impact on the order by which letters are taken from a word in Algorithm 3.2 (from left to right), and ultimately, on the plactic monoid, because its construction relies on this algorithm. In [HK02, §7.3], a notion of admissible readings is introduced, which identifies different methods of reading Young tableaux that result in the same plactic monoid as the far-eastern reading. However, because of the compatibility of the far-eastern reading with the tensor product of crystals, the resulting plactic monoid is anti-isomorphic to the original plactic monoid introduced in [LS81] with the Knuth relations as originally defined [Knu70]. There are other methods of reading Young tableaux that result in the original plactic monoid, namely, the *western reading*

used in [BS17], but at the cost of considering an alternative definition for the tensor product of crystals.

3.2 The insertion algorithm

In this section, we present the Schensted insertion algorithm for adding a new element to a semi-standard Young tableau, and an algorithm to compute the P-symbol and then Q-symbol introduced in [Sch61].

The following algorithm computes Schensted column insertion algorithm, also known as Schensted bumping algorithm [Lec07].

Algorithm 3.1. *Input:* A semi-standard Young tableau T and a letter $x \in X$.

Output: A semi-standard Young tableau $x \rightarrow T$.

Procedure:

- (1) If T is empty, then $x \rightarrow T := \boxed{x}$.
- (2) If $T = C_1 \cdots C_m$, where $C_1, \dots, C_m$ are columns, and C_1 consists of the elements $x_1 < x_2 < \cdots < x_k$, then

$$(i) \text{ If } x_m < x, \text{ then set } C'_1 := \begin{array}{|c|} \hline x_1 \\ \hline \vdots \\ \hline x_m \\ \hline x \\ \hline \end{array} \text{ and } x \rightarrow C'_1 C_2 \cdots C_m;$$

- (ii) Otherwise, set $i \in \{1, \dots, k\}$ minimal such that $x \leq x_i$, and set $C'_1 :=$

$$\begin{array}{|c|} \hline x_1 \\ \hline \vdots \\ \hline x_{i-1} \\ \hline x \\ \hline x_{i+1} \\ \hline \vdots \\ \hline x_k \\ \hline \end{array}$$

and $x \rightarrow T := C'_1(x \rightarrow C_2 \cdots C_m)$.

Note that the recursion described in step (2) may end inserting an element in the empty tableau as described in step (1). For example, we have that

$$3 \rightarrow \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

and

$$2 \rightarrow \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 3 & \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array}.$$

Now, we present an algorithm to compute the P-symbol and the Q-symbol associated with the insertion algorithm.

Algorithm 3.2. *Input:* A word $w \in X^*$.

Output: A semi-standard Young tableau $P(w)$ and a standard Young tableau $Q(w)$.

Procedure:

- (1) Set P_0 and Q_0 to be empty Young tableaux.
- (2) If $w = \epsilon$, then $P(\epsilon) := P_0$ and $Q(w) := Q_0$.
- (3) Else, let $x_1, \dots, x_m \in X$ be such that $w = x_1 \dots x_m$. For each $i = 1, \dots, m$, do
 - (i) Set $P_i := x_i \rightarrow P_{i-1}$;
 - (ii) Set Q_i by adding a box $\boxed{i}$ filled with i to Q_{i-1} such that Q_i has the same shape as P_i .
- (4) Set $P(w) := P_m$ and $Q(w) := Q_m$.

For example, if $w = 24123$, the semi-standard Young tableaux $P_1, \dots, P_5$ are respectively

$$\begin{array}{|c|} \hline 2 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline 3 & \\ \hline \end{array},$$

and the standard Young tableaux $Q_1, \dots, Q_5$ are respectively

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline \end{array}.$$

Therefore,

$$P(24123) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline 3 & \\ \hline \end{array} \quad \text{and} \quad Q(24123) = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline \end{array}.$$

Due to work by Robinson [Rob38], Schensted [Sch61], and Knuth [Knu70], we have the following result.

Theorem 3.3 (Robinson–Schensted–Knuth correspondence). *Let X be a totally ordered set. The map that sends each word $w \in X^*$ to $(P(w), Q(w))$ is an one-to-one correspondence between the set of all words over X and the set of ordered pairs (P, Q) where P is a semi-standard Young tableau and Q is a standard Young tableau of the same shape over X .*

If we consider the ordered set $A_n = \{1 < 2 < \dots < n\}$, and restrict our study to the correspondence between words $w \in A_n^*$ and semi-standard Young tableaux $P(w)$, we obtain a first construction of the plactic monoid as in [Knu70; LS81].

Theorem 3.4. *Let $\sim_{\text{plac}}$ be a relation on A_n^* given by*

$$u \sim_{\text{plac}} v \iff P(u) = P(v).$$

Then, $\sim_{\text{plac}}$ is a monoid congruence on A_n^ .*

We call $\sim_{\text{plac}}$ the *plactic congruence*. And now, we have reached a construction of the plactic monoid as it originally emerged.

Definition 3.5. The *plactic monoid* plac_n of rank n corresponds to the quotient monoid $A_n^*/\sim_{\text{plac}}$ of the free monoid A_n^* over A_n by the plactic congruence $\sim_{\text{plac}}$.

3.3 Knuth relations

In this section we give a presentation for the plactic monoid plac_n based on the Knuth relations [Knu70].

Definition 3.6. Let $n \geq 1$. Consider the relations $R_1^{\text{plac}(\mathfrak{A}_n)}$ and $R_2^{\text{plac}(\mathfrak{A}_n)}$ on A_n^* given by

$$\begin{aligned} R_1^{\text{plac}(\mathfrak{A}_n)} &= \{(yzx, yxz), (xzy, zxy) \mid x < y < z\}, \\ R_2^{\text{plac}(\mathfrak{A}_n)} &= \{(xyx, xxy), (xzy, yxz) \mid x < y\}. \end{aligned}$$

The relation $R^{\text{plac}(\mathfrak{A}_n)} = R_1^{\text{plac}(\mathfrak{A}_n)} \cup R_2^{\text{plac}(\mathfrak{A}_n)}$ is formed by what are called the *Knuth relations*.

As we mentioned in the introduction of this chapter, the classical plactic monoid will later be placed in context as the plactic monoid associated to the standard crystal of Cartan type $\mathfrak{A}_n$. This justifies the superscript $\text{plac}(\mathfrak{A}_n)$ on the relations above. Also, due to this fact, we chose the far-eastern reading of Young tableaux which results in a plactic monoid anti-isomorphic to the original definition, as remarked at the end of Section 3.1. And therefore, the words in the Knuth

relations are reversed when compared to the original definition of the relations [Knu70, Formula (6.7)].

Note that for any pair (u, v) in the Knuth relations $R^{\text{plac}(\mathfrak{A}_n)}$, we have that $P(u) = P(v)$. Let $x \leq y \leq z$. If $y < z$, then

$$P(yzx) = P(yxz) = \begin{array}{|c|c|} \hline x & y \\ \hline z & \\ \hline \end{array}.$$

If $x < y$, then

$$P(xzy) = P(zxy) = \begin{array}{|c|c|} \hline x & z \\ \hline y & \\ \hline \end{array}.$$

In fact, we have the following result which shows that the Knuth relations and Young tableaux together with the insertion algorithm induce the same monoid.

Theorem 3.7. *Let $n \geq 1$. Let θ to be the monoid congruence on A_n^* generated by $R^{\text{plac}(\mathfrak{A}_n)}$. Then, $\theta = \sim_{\text{plac}}$.*

Therefore, we have that $\langle A_n \mid R^{\text{plac}(\mathfrak{A}_n)} \rangle$ is a presentation for the plactic monoid plac_n of rank n .

CRYSTALS

In this chapter we introduce the background on the theory of crystals, which will be the framework to construct a general notion of plactic monoid on the following chapter. Crystals, also known as Kashiwara crystals, arise from detaching the fundamental algebraic properties of crystal bases, introduced by Kashiwara [Kas90; Kas91] from an algebraic limit of modules for quantized universal enveloping algebras at $q = 0$. Since our purpose is to describe the plactic monoid from crystals, we will not give more context on the origin of crystals, although we remark that many properties and applications of crystals result from their connection with crystal bases. For a comprehensive study of crystals and crystal bases see [HK02], and for a deep study of crystals see [BS17].

In Section 4.1 we introduce the definition of crystals, present some of their basic properties, and define the notion of crystal homomorphism. In Section 4.2 we describe a combinatorial approach to crystals through the notion of crystal graphs, which in some contexts is used as an alternative definition of crystals. In Section 4.3 we define the tensor product of crystals, and present some basic results that will allow the construction of a special monoid over a crystal in Section 5.2. Finally, as the notion of tensor product of crystals will play an important role in constructing a plactic monoid from crystals, in Section 4.4 we present the signature rule for the tensor product of crystals, which is a combinatorial method to compute the crystal structure of the tensor product.

The results presented in this chapter can be found in any introductory course on the theory of crystals, and so, we omit their proofs. See for example [HK02] or [BS17]. Also, some of the results in Sections 4.1 and 4.2 are proven in a more general framework in Chapter 7.

4.1 Crystals and homomorphisms

In this section we introduce the notion of crystal associated to a root system. In Example 4.4 we describe the standard crystals associated to the Cartan types $\mathfrak{A}_n$ and $\mathfrak{C}_n$. Finally, we introduce the notion of crystal homomorphism.

Although we use root systems (Section 2.3), crystals can be defined based on other algebraic structures such as an algebraic data [Kas95] or a Cartan datum [HK02]. Since we only make use of properties of root systems that are also satisfied by these algebraic structures, all subsequent definitions and results can be reinterpreted based on them.

Consider $\mathbb{Z} \cup \{-\infty\}$ to be the usual set of integers where we add a minimal element $-\infty$, that is, $-\infty < m$, for all $m \in \mathbb{Z}$. Also, set $m + (-\infty) = (-\infty) + m = -\infty$, for all $m \in \mathbb{Z}$.

Definition 4.1. Let Φ be a root system with weight lattice Λ and index set I for the simple roots $(\alpha_i)_{i \in I}$. A *crystal* $\mathcal{B}$ of type Φ consists of a set B together with maps $\text{wt} : B \rightarrow \Lambda$, $\tilde{e}_i, \tilde{f}_i : B \rightarrow B \sqcup \{\perp\}$ and $\tilde{\varepsilon}_i, \tilde{\varphi}_i : B \rightarrow \mathbb{Z} \cup \{-\infty\}$, for each $i \in I$, satisfying the following conditions:

$$(i) \quad \tilde{\varphi}_i(x) = \tilde{\varepsilon}_i(x) + \langle \text{wt}(x), \alpha_i^\vee \rangle;$$

(ii) if $\tilde{e}_i(x) \in B$, then

$$\text{wt}(\tilde{e}_i(x)) = \text{wt}(x) + \alpha_i,$$

$$\tilde{\varepsilon}_i(\tilde{e}_i(x)) = \tilde{\varepsilon}_i(x) - 1,$$

$$\tilde{\varphi}_i(\tilde{e}_i(x)) = \tilde{\varphi}_i(x) + 1;$$

(iii) if $\tilde{f}_i(x) \in B$, then

$$\text{wt}(\tilde{f}_i(x)) = \text{wt}(x) - \alpha_i,$$

$$\tilde{\varepsilon}_i(\tilde{f}_i(x)) = \tilde{\varepsilon}_i(x) + 1,$$

$$\tilde{\varphi}_i(\tilde{f}_i(x)) = \tilde{\varphi}_i(x) - 1;$$

(iv) $\tilde{e}_i(x) = y$ if and only if $x = \tilde{f}_i(y)$;

(v) if $\tilde{\varepsilon}_i(x) = -\infty$ then $\tilde{e}_i(x) = \tilde{f}_i(x) = \perp$;

for $x, y \in B$ and $i \in I$. The set B is called the *underlying set* of $\mathcal{B}$, and the maps wt , $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{\varepsilon}_i$ and $\tilde{\varphi}_i$ ($i \in I$) form the *crystal structure* of $\mathcal{B}$. Also, the map wt is called the *weight map*, where $\text{wt}(x)$ is said to be the *weight* of $x \in B$, and the maps $\tilde{e}_i$ and $\tilde{f}_i$ ($i \in I$) are called the *raising* and *lowering Kashiwara operators*, respectively.

In this definition, $\perp$ is an auxiliary symbol. In the literature, 0 is usually used instead of $\perp$, but since some well-known crystals have 0 as an element, we have adopted this notation to avoid ambiguity. By $\tilde{e}_i(x) = \perp$ (or $\tilde{f}_i(x) = \perp$) we mean that $\tilde{e}_i$ (resp., $\tilde{f}_i$) is *undefined* on x . On the other hand, we say that $\tilde{e}_i$ (or $\tilde{f}_i$) is *defined* on x whenever $\tilde{e}_i(x) \in B$ (resp., $\tilde{f}_i(x) \in B$). So, alternatively one can consider the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ ($i \in I$) to be partial maps from B to B .

From condition (i), we get that $\tilde{e}_i(x) = -\infty$ if and only if $\tilde{\varphi}_i(x) = -\infty$. Thus, condition (v) is equivalent to stating that if $\tilde{\varphi}_i(x) = -\infty$, then $\tilde{e}_i(x) = \tilde{f}_i(x) = \perp$.

Note that in the definition of a crystal we could have only stated one of conditions (ii) and (iii), as each one together with (iv) implies the other. Moreover, by conditions (i) and (iv), we have that a crystal is determined by a set B and the weight map wt together with either $\tilde{e}_i$ or $\tilde{f}_i$, and either $\tilde{e}_i$ or $\tilde{\varphi}_i$, for each $i \in I$. However, for a purpose of clarity, we usually give explicit definitions for each map when defining a crystal.

Recall the partial order defined on a weight lattice Λ described in (2.3). Let $x \in B$ and $i \in I$. If $\tilde{e}_i(x)$ is defined, then

$$\text{wt}(\tilde{e}_i(x)) - \text{wt}(x) = \text{wt}(x) + \alpha_i - \text{wt}(x) = \alpha_i,$$

and so, $\text{wt}(\tilde{e}_i(x)) > \text{wt}(x)$. Hence, $\tilde{e}_i$ increases the weight of an element, whenever it is defined. Similarly, if $\tilde{f}_i$ is defined, then

$$\text{wt}(x) - \text{wt}(\tilde{f}_i(x)) = \text{wt}(x) - (\text{wt}(x) - \alpha_i) = \alpha_i,$$

and so, $\text{wt}(x) > \text{wt}(\tilde{f}_i(x))$. Hence, $\tilde{f}_i$ decreases the weight of an element, whenever it is defined. These facts justify the characterization of the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ ($i \in I$) as *raising* and *lowering*.

Definition 4.2. Let $x \in B$ be an element of a crystal $\mathcal{B}$.

- (i) x is said to be of *highest weight* if $\tilde{e}_i(x) = \perp$, for all $i \in I$.
- (ii) x is said to be of *lowest weight* if $\tilde{f}_i(x) = \perp$, for all $i \in I$.

Notice that in a crystal $\mathcal{B}$ there may exist elements $x, y \in B$ such that x is of highest weight, y is not of highest weight, and $\text{wt}(y) > \text{wt}(x)$ (the crystal in Example 4.15(ii) has elements in these conditions). But if we extend this definition to the weights as follows, we get some more natural results.

Definition 4.3. Let $\mathcal{B}$ be a crystal, and let $\lambda \in \Lambda$ be a weight.

- (i) λ is called a *highest weight* if there exists a highest-weight element $x \in B$ such that $\lambda = \text{wt}(x)$.

- (ii) λ is called a *lowest weight* if there exists a lowest-weight element $x \in B$ such that $\lambda = \text{wt}(x)$.

In a crystal $\mathcal{B}$, we have that if $\tilde{e}_i$ is defined on an element $x \in B$, then $\text{wt}(\tilde{e}_i(x)) > \text{wt}(x)$. Thus, if a weight λ in $\text{wt}(B) = \{\text{wt}(x) \mid x \in B\}$ is maximal among weights in $\text{wt}(B)$, then it is a highest weight, and any element $x \in B$ such that $\text{wt}(x) = \lambda$ is of highest weight. Similarly, we have that $\text{wt}(x) > \text{wt}(\tilde{f}_i(x))$, whenever $\tilde{f}_i$ is defined on $x \in B$. Then, if a weight $\mu \in \text{wt}(B)$ is minimal among weights in $\text{wt}(B)$, then μ is a lowest weight, and any element $x \in B$ such that $\text{wt}(x) = \mu$ is of lowest weight.

In the following example we introduce two well-known crystals which will be used throughout this thesis.

Example 4.4. (i) Let $n \geq 2$. Consider the root system of Cartan type $\mathfrak{A}_n$ described in Example 2.16. The *standard crystal of type $\mathfrak{A}_n$* is a crystal $\mathcal{A}_n$ consisting of a set $A_n = \{1, 2, \dots, n\}$ and maps given by

$$\text{wt}(x) = \mathbf{e}_x,$$

and for $i = 1, \dots, n-1$,

$$\begin{aligned} \tilde{e}_i(x) &= \begin{cases} x-1 & \text{if } x = i+1 \\ \perp & \text{otherwise,} \end{cases} & \tilde{f}_i(x) &= \begin{cases} x+1 & \text{if } x = i \\ \perp & \text{otherwise,} \end{cases} \\ \tilde{\varepsilon}_i(x) &= \delta_{x,i+1}, & \tilde{\varphi}_i(x) &= \delta_{x,i}, \end{aligned}$$

$x \in A_n$, where $\delta_{k,l} = 1$ if $k = l$, and $\delta_{k,l} = 0$ whenever $k \neq l$.

(ii) Let $n \geq 2$. Consider the root system of Cartan type $\mathfrak{C}_n$ described in Example 2.17. The *standard crystal of type $\mathfrak{C}_n$* is a crystal $\mathcal{C}_n$ consisting of a set $C_n = \{1, 2, \dots, n, \bar{n}, \overline{n-1}, \dots, \bar{1}\}$ and maps given as follows. For $x \in \{1, 2, \dots, n\}$,

$$\text{wt}(x) = \mathbf{e}_x \quad \text{and} \quad \text{wt}(\bar{x}) = -\mathbf{e}_x.$$

For $x \in C_n$ and $i = 1, \dots, n$, $\tilde{e}_i$ and $\tilde{f}_i$ are defined on x only in the following cases

$$\tilde{e}_i(x) = \begin{cases} i & \text{if } x = i+1 \\ \overline{i+1} & \text{if } x = \bar{i} \neq \bar{n} \\ n & \text{if } x = \bar{n} \text{ and } i = n, \end{cases}$$

and

$$\tilde{f}_i(x) = \begin{cases} i+1 & \text{if } x = i \neq n \\ \bar{i} & \text{if } x = \overline{i+1} \\ \bar{n} & \text{if } x = n \text{ and } i = n. \end{cases}$$

Also,

$$\tilde{\varepsilon}_i(x) = \delta_{x,i+1} + \delta_{x,\bar{i}} \quad \text{and} \quad \tilde{\varphi}_i(x) = \delta_{x,i} + \delta_{x,\overline{i+1}}.$$

The standard crystals of types $\mathfrak{A}_n$ and $\mathfrak{C}_n$ have a unique highest-weight element and a unique highest weight, which are 1 and $\mathbf{e}_1$, respectively. In $\mathfrak{A}_n$, we have that n and $\mathbf{e}_n$ are the unique lowest-weight element and lowest weight, respectively. And in $\mathfrak{C}_n$, we have that $\bar{1}$ and $-\mathbf{e}_1$ are the unique lowest-weight element and lowest weight, respectively.

Since the Kashiwara operators of a crystal $\mathcal{B}$ can be seen as partial maps from B to B , we can compose them in a natural way. For the sake of simplicity, we usually omit the composition operator $\circ$. For instance, $\tilde{e}_i \tilde{f}_j$ stands for $\tilde{e}_i \circ \tilde{f}_j$, where $\tilde{e}_i \tilde{f}_j(x) = \tilde{e}_i(\tilde{f}_j(x))$, for $x \in B$ and $i, j \in I$. Also, set $\tilde{e}_i^0$ and $\tilde{f}_i^0$ to be the identity map on B , and recursively, define $\tilde{e}_i^{k+1} = \tilde{e}_i \tilde{e}_i^k$ and $\tilde{f}_i^{k+1} = \tilde{f}_i \tilde{f}_i^k$, for $k \geq 0$.

Definition 4.5. A crystal $\mathcal{B}$ is said to be *seminormal* if

$$\tilde{\varepsilon}_i(x) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^k(x) \in B\}$$

and

$$\tilde{\varphi}_i(x) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^k(x) \in B\},$$

for all $x \in B$ and $i \in I$.

The standard crystals of types $\mathfrak{A}_n$ and $\mathfrak{C}_n$ described in Example 4.4 are seminormal.

If a crystal $\mathcal{B}$ is seminormal, then in particular, $\tilde{\varepsilon}_i(x)$ and $\tilde{\varphi}_i(x)$ lie in $\mathbb{Z}_{\geq 0}$, for any $x \in B$ and $i \in I$. From Definition 4.1(i), we obtain that

$$\langle \text{wt}(x), \alpha_i^\vee \rangle = \tilde{\varphi}_i(x) - \tilde{\varepsilon}_i(x), \quad (4.1)$$

for all $x \in B$ and $i \in I$. Thus, if Φ is semisimple, the weight $\text{wt}(x)$ of an element $x \in B$ is completely determined by the values of $\tilde{\varepsilon}_i$ and $\tilde{\varphi}_i$ ($i \in I$) on x . Hence, for seminormal crystals whose type is semisimple the maps $\tilde{\varepsilon}_i$ and $\tilde{\varphi}_i$ ($i \in I$) completely determine the weight map wt .

From (4.1) and the fact that $\tilde{\varepsilon}_i(x), \tilde{\varphi}_i(x) \in \mathbb{Z}_{\geq 0}$, we have that if $\langle \text{wt}(x), \alpha_i^\vee \rangle < 0$, then $\tilde{\varepsilon}_i(x) > 0$, which implies that $\tilde{e}_i$ is defined on x . Also, if $\langle \text{wt}(x), \alpha^\vee \rangle > 0$, then $\tilde{\varphi}_i(x) > 0$, which implies that $\tilde{f}_i$ is defined on x . This leads to the following result.

Proposition 4.6. *Let $x \in B$ be an element of a seminormal crystal $\mathcal{B}$. If x is of highest weight, then $\text{wt}(x)$ is a dominant weight. If x is of lowest weight, then $-\text{wt}(x)$ is a dominant weight.*

A consequence of this result is that the highest weights of a seminormal crystal lie in the dominant Weyl chamber.

Finally, we introduce the notion of homomorphism between crystals.

Definition 4.7. Let $\mathcal{B}$ and $\mathcal{B}'$ be crystals of the same type. A *crystal homomorphism* ψ from $\mathcal{B}$ to $\mathcal{B}'$, denoted by $\psi : \mathcal{B} \rightarrow \mathcal{B}'$, is a map $\psi : B \sqcup \{\perp\} \rightarrow B' \sqcup \{\perp\}$ that satisfies the following conditions:

(i) $\psi(\perp) = \perp$;

(ii) if $\psi(x) \in B'$, then

$$\text{wt}(\psi(x)) = \text{wt}(x),$$

$$\tilde{e}_i(\psi(x)) = \tilde{e}_i(x),$$

$$\tilde{\varphi}_i(\psi(x)) = \tilde{\varphi}_i(x);$$

(iii) if $\tilde{e}_i(x) \in B$ and $\psi(x), \psi(\tilde{e}_i(x)) \in B'$, then

$$\psi(\tilde{e}_i(x)) = \tilde{e}_i(\psi(x));$$

(iv) if $\tilde{f}_i(x) \in B$ and $\psi(x), \psi(\tilde{f}_i(x)) \in B'$, then

$$\psi(\tilde{f}_i(x)) = \tilde{f}_i(\psi(x));$$

for $x \in B$ and $i \in I$.

A *crystal isomorphism* ψ between $\mathcal{B}$ and $\mathcal{B}'$ is a bijection $\psi : B \sqcup \{\perp\} \rightarrow B' \sqcup \{\perp\}$ such that $\psi : \mathcal{B} \rightarrow \mathcal{B}'$ and $\psi^{-1} : \mathcal{B}' \rightarrow \mathcal{B}$ are crystal homomorphisms.

We say that $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic if there exists a crystal isomorphism between $\mathcal{B}$ and $\mathcal{B}'$.

As we remarked after Definition 4.1, by $\psi(x) = \perp$ we mean that ψ is undefined on x . So, alternatively a crystal homomorphism $\psi : \mathcal{B} \rightarrow \mathcal{B}'$ can be defined as a partial map from B to B' satisfying conditions (ii) to (iv) above.

Also, by condition (i), when defining a crystal homomorphism $\psi : \mathcal{B} \rightarrow \mathcal{B}'$, we only give its value on each $x \in B$. By the same reason, we also consider a map $\psi : B \rightarrow B'$ satisfying conditions (ii) to (iv) above to be a crystal homomorphism from $\mathcal{B}$ to $\mathcal{B}'$.

We say that a crystal homomorphism $\psi : \mathcal{B} \rightarrow \mathcal{B}'$ is injective, surjective or bijective if the map $\psi : B \sqcup \{\perp\} \rightarrow B' \sqcup \{\perp\}$ is injective, surjective or bijective, respectively.

In general, a bijective crystal homomorphism may not be a crystal isomorphism (in Example 7.15 we shall see this in a more general framework). However, for seminormal crystals we have the following result.

Theorem 4.8. *Let $\mathcal{B}$ and $\mathcal{B}'$ be seminormal crystals of the same type, and let $\psi : \mathcal{B} \rightarrow \mathcal{B}'$ be a bijective crystal homomorphism. Then, ψ is a crystal isomorphism.*

4.2 Crystal graphs

In this section we present a combinatorial approach to crystals by the construction of a graph. From this framework we are able to identify some properties of crystals with combinatorial properties of the resulting graph. Moreover, for a seminormal crystal, the resulting graph completely determines its crystal structure, which allows a purely combinatorial description of seminormal crystals. This also allows some studies of seminormal crystals to be mainly based on their graphs, see for example [CGM19; KN94].

Definition 4.9. The *crystal graph* $\Gamma_{\mathcal{B}}$ of a crystal $\mathcal{B}$ is an I -labelled directed simple graph with vertex set B and an edge $x \xrightarrow{i} y$ from $x \in B$ to $y \in B$ labelled by $i \in I$ whenever $\tilde{f}_i(x) = y$. For $x \in B$, let $\Gamma_{\mathcal{B}}(x)$ denote the connected component of $\Gamma_{\mathcal{B}}$ containing the vertex x .

By Definition 4.1(iv), we have an edge $x \xrightarrow{i} y$ in a crystal graph $\Gamma_{\mathcal{B}}$ if and only if $x = \tilde{e}_i(y)$. Thus, it is immediate that an element $x \in B$ is of highest weight if and only if x is not the end of an edge in $\Gamma_{\mathcal{B}}$. Also, x is of lowest weight if and only if x is not the start of an edge in $\Gamma_{\mathcal{B}}$.

By associating graphs with crystals, we obtain a combinatorial framework to study crystals, as exemplified above with highest-weight and lowest-weight elements, and to consider structures as follows.

Definition 4.10. Let $\mathcal{B}$ be a crystal. A *connected component* of $\mathcal{B}$ is a subset B' of B that satisfies the following conditions:

- (i) for each $x, y \in B'$ there exists $g_1, \dots, g_m \in \{\tilde{e}_i, \tilde{f}_i \mid i \in I\}$ such that $g_1 \cdots g_m(x) = y$;
- (ii) $\tilde{e}_i(x), \tilde{f}_i(x) \in B' \sqcup \{\perp\}$ for all $x \in B'$ and $i \in I$.

We also call a *connected component* the crystal $\mathcal{B}'$ of the same type as $\mathcal{B}$ consisting of B' together with the maps wt , $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{\varepsilon}_i$ and $\tilde{\varphi}_i$ ($i \in I$) of $\mathcal{B}$ restricted to B' .

For each $x \in B$, the connected component of $\mathcal{B}$ containing x is denoted by $B(x)$, and the associated crystal is denoted by $\mathcal{B}(x)$.

This terminology is justified by the fact that a set B' is a connected component of a crystal $\mathcal{B}$ if and only if B' is the vertex set of a connected component of

the crystal graph $\Gamma_{\mathcal{B}}$. Also, for $x \in B$, the crystal graph $\Gamma_{\mathcal{B}(x)}$ coincides with the connected component $\Gamma_{\mathcal{B}}(x)$.

The crystals described in Example 4.4 have the following crystal graphs.

Example 4.11. (i) The crystal graph $\Gamma_{\mathcal{A}_n}$ of the standard crystal of type $\mathfrak{A}_n$ is the following.

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} \cdots \xrightarrow{n-1} n$$

(ii) The crystal graph $\Gamma_{\mathcal{C}_n}$ of the standard crystal of type $\mathfrak{C}_n$ is the following.

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-1} n \xrightarrow{n} \bar{n} \xrightarrow{n-1} \overline{n-1} \xrightarrow{n-2} \cdots \xrightarrow{1} \bar{1}$$

Let $\psi : \mathcal{B} \rightarrow \mathcal{B}'$ be a crystal homomorphism between crystals $\mathcal{B}$ and $\mathcal{B}'$ of the same type satisfying $\psi(B) \subseteq B'$. By Definition 4.7(iv), we have that if $x \xrightarrow{i} y$ is an edge of $\Gamma_{\mathcal{B}}$, then $\psi(x) \xrightarrow{i} \psi(y)$ is an edge of $\Gamma_{\mathcal{B}'}$. Thus, ψ is also a graph homomorphism from $\Gamma_{\mathcal{B}}$ to $\Gamma_{\mathcal{B}'}$. Moreover, if ψ is a crystal isomorphism between $\mathcal{B}$ and $\mathcal{B}'$, then a set $B_0 \subseteq B$ is a connected component of $\mathcal{B}$ if and only if $\psi(B_0)$ is a connected component of $\mathcal{B}'$. In particular, for each $x \in B$, the restriction of ψ to $B(x)$ is a crystal isomorphism between $\mathcal{B}(x)$ and $\mathcal{B}'(x)$. These results will be studied in a more general framework in Section 7.2.

Let $\mathcal{B}$ be a seminormal crystal. From Definition 4.5, for $x \in B$ and $i \in I$, we have that $\tilde{\varepsilon}_i(x)$ corresponds to the maximum number of times $\tilde{e}_i$ can be applied to x and be defined. Thus, the path

$$\tilde{e}_i^{\tilde{\varepsilon}_i(x)}(x) \xrightarrow{i} \tilde{e}_i^{\tilde{\varepsilon}_i(x)-1}(x) \xrightarrow{i} \cdots \xrightarrow{i} x$$

lies in $\Gamma_{\mathcal{B}}$, where $\tilde{e}_i^{\tilde{\varepsilon}_i(x)}(x)$ is not the end of an i -labelled edge. And so, $\tilde{\varepsilon}_i(x)$ corresponds to the length of a maximal path ending in x with edges labelled by i . Similarly, the path

$$x \xrightarrow{i} \tilde{f}_i(x) \xrightarrow{i} \cdots \xrightarrow{i} \tilde{f}_i^{\tilde{\varphi}_i(x)}(x)$$

lies in $\Gamma_{\mathcal{B}}$, where $\tilde{f}_i^{\tilde{\varphi}_i(x)}(x)$ is not the start of an i -labelled edge, because $\mathcal{B}$ is seminormal. And thus, $\tilde{\varphi}_i(x)$ corresponds to the length of a maximal path starting in x with edges labelled by i . Therefore, the crystal graph $\Gamma_{\mathcal{B}}$ together with the weight map wt completely determines $\mathcal{B}$, and if the type of $\mathcal{B}$ is a semisimple root system, $\Gamma_{\mathcal{B}}$ suffices to determine $\mathcal{B}$, because the weight map wt can be obtained from the maps $\tilde{\varepsilon}_i$ and $\tilde{\varphi}_i$ ($i \in I$), as we have seen in (4.1). From this fact, seminormal crystals are alternatively described as (weighted) I -labelled graphs satisfying some properties, which will be presented in a more general framework in Section 7.2.

From the connection between crystals and graphs, we also characterize some elements of a crystal based on their properties in the crystal graph.

Definition 4.12. Let $\mathcal{B}$ be a crystal. An element $x \in B$ is said to be *isolated* if $B(x) = \{x\}$.

Note that an element $x \in B$ of a crystal $\mathcal{B}$ is isolated if and only if x is of highest and lowest weight. And if $\mathcal{B}$ is seminormal, an element $x \in B$ is isolated if and only if $\tilde{\varepsilon}_i(x) = \tilde{\varphi}_i(x) = 0$, for all $i \in I$.

4.3 Tensor product

In this section we introduce the notion of tensor product between two crystals of the same type. As crystals derive from crystal bases of quantum groups, the notion of tensor product for quantum groups gives rise to the following construction.

Definition 4.13. Consider a root system Φ with weight lattice Λ and index set I for the simple roots $(\alpha_i)_{i \in I}$. Let $\mathcal{B}$ and $\mathcal{B}'$ be crystals of type Φ . The *tensor product* of $\mathcal{B}$ and $\mathcal{B}'$ is a crystal $\mathcal{B} \otimes \mathcal{B}'$ of type Φ consisting of a set $B \otimes B'$, which corresponds to the Cartesian product $B \times B'$ whose ordered pairs are denoted by $x \otimes x'$ ($x \in B$ and $x' \in B'$), and crystal structure given as follows. For $x \in B$ and $x' \in B'$, set

$$\text{wt}(x \otimes x') = \text{wt}(x) + \text{wt}(x'),$$

and for $i \in I$, set

$$\begin{aligned} \tilde{e}_i(x \otimes x') &= \begin{cases} \tilde{e}_i(x) \otimes x' & \text{if } \tilde{\varphi}_i(x) \geq \tilde{\varepsilon}_i(x') \\ x \otimes \tilde{e}_i(x') & \text{if } \tilde{\varphi}_i(x) < \tilde{\varepsilon}_i(x'), \end{cases} \\ \tilde{f}_i(x \otimes x') &= \begin{cases} \tilde{f}_i(x) \otimes x' & \text{if } \tilde{\varphi}_i(x) > \tilde{\varepsilon}_i(x') \\ x \otimes \tilde{f}_i(x') & \text{if } \tilde{\varphi}_i(x) \leq \tilde{\varepsilon}_i(x'), \end{cases} \\ \tilde{\varepsilon}_i(x \otimes x') &= \max\{\tilde{\varepsilon}_i(x), \tilde{\varepsilon}_i(x') - \langle \text{wt}(x), \alpha_i^\vee \rangle\}, \\ \tilde{\varphi}_i(x \otimes x') &= \max\{\tilde{\varphi}_i(x) + \langle \text{wt}(x'), \alpha_i^\vee \rangle, \tilde{\varphi}_i(x')\}, \end{aligned}$$

where $x \otimes \perp = \perp \otimes x' = \perp$;

Note that this is the original notion of tensor product introduced by Kashiwara [Kas90; Kas91; KN94]. It is due to this notion of tensor product that we considered the far-eastern reading of Young tableaux in Chapter 3, and made the

necessary adaptations to introduce a plactic monoid anti-isomorphic to the original one. Another definition of tensor product of crystals is introduced in [BS17, Section 2.3], which is there described as opposite to the one introduced by Kashiwara. Although this second notion agrees with the western reading, resulting in the construction of the original plactic monoid through crystals, it lacks a connection with the theory of crystal bases of quantum groups, which motivates the development of the theory of crystals.

Let $\mathcal{B}$ be a seminormal crystal. Note that Definition 4.1(i) implies that

$$\langle \text{wt}(x), \alpha_i^\vee \rangle = \tilde{\varphi}_i(x) - \tilde{\varepsilon}_i(x),$$

for any $x \in B$ and $i \in I$. Thus, if $\mathcal{B}'$ is also a seminormal crystal, we have that

$$\tilde{\varepsilon}_i(x \otimes x') = \max\{\tilde{\varepsilon}_i(x), \tilde{\varepsilon}_i(x) + \tilde{\varepsilon}_i(x') - \tilde{\varphi}_i(x)\}$$

and

$$\tilde{\varphi}_i(x \otimes x') = \max\{\tilde{\varphi}_i(x) + \tilde{\varphi}_i(x') - \tilde{\varepsilon}_i(x'), \tilde{\varphi}_i(x')\},$$

for any $x \in B$, $x' \in B'$ and $i \in I$. From these equalities we can show that the tensor product preserves the seminormal property in the following sense.

Theorem 4.14. *Let $\mathcal{B}$ and $\mathcal{B}'$ be seminormal crystals of the same type. Then, $\mathcal{B} \otimes \mathcal{B}'$ is seminormal.*

As we saw in Section 4.2, a seminormal crystal is completely determined by its crystal graph (as a weighted graph). Therefore, in the following example we give the crystal graphs of tensor products of the standard crystals of types $\mathcal{A}_n$ and $\mathcal{C}_n$, described in Example 4.4.

Example 4.15. (i) The crystal graph of $\mathcal{A}_3 \otimes \mathcal{A}_3$ is

$$\begin{array}{ccccc} 1 \otimes 1 & \xrightarrow{1} & 2 \otimes 1 & \xrightarrow{2} & 3 \otimes 1 \\ & & \downarrow 1 & & \downarrow 1 \\ 1 \otimes 2 & & 2 \otimes 2 & \xrightarrow{2} & 3 \otimes 2 \\ \downarrow 2 & & & & \downarrow 2 \\ 1 \otimes 3 & \xrightarrow{1} & 2 \otimes 3 & & 3 \otimes 3 \end{array}$$

where $\text{wt}(x \otimes y) = \mathbf{e}_x + \mathbf{e}_y$, for each $x, y \in \{1, 2, 3\}$.

(ii) The crystal graph of $\mathcal{C}_2 \otimes \mathcal{C}_2$ is

$$\begin{array}{ccccccc}
 1 \otimes 1 & \xrightarrow{1} & 2 \otimes 1 & \xrightarrow{2} & \bar{2} \otimes 1 & \xrightarrow{1} & \bar{1} \otimes 1 \\
 & & \downarrow 1 & & & & \downarrow 1 \\
 1 \otimes 2 & & 2 \otimes 2 & \xrightarrow{2} & \bar{2} \otimes 2 & & \bar{1} \otimes 2 \\
 \downarrow 2 & & & & \downarrow 2 & & \downarrow 2 \\
 1 \otimes \bar{2} & \xrightarrow{1} & 2 \otimes \bar{2} & & \bar{2} \otimes \bar{2} & \xrightarrow{1} & \bar{1} \otimes \bar{2} \\
 & & \downarrow 1 & & & & \downarrow 1 \\
 1 \otimes \bar{1} & & 2 \otimes \bar{1} & \xrightarrow{2} & \bar{2} \otimes \bar{1} & & \bar{1} \otimes \bar{1}
 \end{array}$$

where $\text{wt}(x \otimes y) = \mathbf{e}_x + \mathbf{e}_y$, $\text{wt}(x \otimes \bar{y}) = \text{wt}(\bar{y} \otimes x) = \mathbf{e}_x - \mathbf{e}_y$, and $\text{wt}(\bar{x} \otimes \bar{y}) = -\mathbf{e}_x - \mathbf{e}_y$, for each $x, y \in \{1, 2\}$.

The tensor product of crystals is an associative binary operation as shown by the following result.

Theorem 4.16. *Let $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_3$ be crystals of the same type. The map $(B_1 \otimes B_2) \otimes B_3 \rightarrow B_1 \otimes (B_2 \otimes B_3)$, given by $(x_1 \otimes x_2) \otimes x_3 \mapsto x_1 \otimes (x_2 \otimes x_3)$, is a crystal isomorphism between $(\mathcal{B}_1 \otimes \mathcal{B}_2) \otimes \mathcal{B}_3$ and $\mathcal{B}_1 \otimes (\mathcal{B}_2 \otimes \mathcal{B}_3)$.*

Due to the previous result, we may omit parenthesis for the tensor product of crystals and simply write $\mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_3$ with elements $x_1 \otimes x_2 \otimes x_3$, for $x_1 \in B_1$, $x_2 \in B_2$ and $x_3 \in B_3$. For instance, we have the following result where we omit parenthesis.

Proposition 4.17. *Let $\mathcal{B}_1, \dots, \mathcal{B}_m$ be seminormal crystals of the same type, and let $x_1 \in B_1, \dots, x_m \in B_m$. Then,*

$$\text{wt}(x_1 \otimes \dots \otimes x_m) = \text{wt}(x_1) + \dots + \text{wt}(x_m).$$

For $i \in I$,

$$\tilde{\varepsilon}_i(x_1 \otimes \dots \otimes x_m) = \max_{1 \leq p \leq m} \left(\sum_{k=1}^p \tilde{\varepsilon}_i(x_k) - \sum_{l=1}^{p-1} \tilde{\varphi}_i(x_l) \right),$$

and if p is the smallest integer for which the maximum is attained, then

$$\tilde{e}_i(x_1 \otimes \dots \otimes x_m) = x_1 \otimes \dots \otimes x_{p-1} \otimes \tilde{e}_i(x_p) \otimes x_{p+1} \otimes \dots \otimes x_m.$$

Also,

$$\tilde{\varphi}_i(x_1 \otimes \dots \otimes x_m) = \max_{1 \leq q \leq m} \left(\sum_{k=q}^m \tilde{\varphi}_i(x_k) - \sum_{l=q+1}^m \tilde{\varepsilon}_i(x_l) \right),$$

and if q is the largest integer for which the maximum is attained, then

$$\tilde{f}_i(x_1 \otimes \cdots \otimes x_m) = x_1 \otimes \cdots \otimes x_{q-1} \otimes \tilde{f}_i(x_q) \otimes x_{q+1} \otimes \cdots \otimes x_m.$$

In the following result, we show that crystal homomorphisms can also be extended to tensor products of their domains and images.

Proposition 4.18. *Let $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}'_1$ and $\mathcal{B}'_2$ be crystals of the same type, and let $\psi_1 : \mathcal{B}_1 \rightarrow \mathcal{B}'_1$ and $\psi_2 : \mathcal{B}_2 \rightarrow \mathcal{B}'_2$ be crystal homomorphisms. Then, the map $\psi_1 \otimes \psi_2 : \mathcal{B}_1 \otimes \mathcal{B}_2 \rightarrow \mathcal{B}'_1 \otimes \mathcal{B}'_2$, defined by $x_1 \otimes x_2 \mapsto \psi_1(x_1) \otimes \psi_2(x_2)$ for each $x_1 \in \mathcal{B}_1$ and $x_2 \in \mathcal{B}_2$, is a crystal homomorphism from $\mathcal{B}_1 \otimes \mathcal{B}_2$ to $\mathcal{B}'_1 \otimes \mathcal{B}'_2$. Moreover, if ψ_1 and ψ_2 are crystal isomorphisms, then $\psi_1 \otimes \psi_2$ is a crystal isomorphism.*

Finally, we remark that crystals, crystal homomorphisms and tensor product of crystals form a tensor category [Kas95].

4.4 The signature rule for tensor product

In this section we describe a combinatorial method that makes easier the computation of the crystal structure of the tensor product of seminormal crystals. This method is essentially a combinatorial interpretation of Proposition 4.17.

Consider the bicyclic monoid Z with the following presentation

$$Z = \langle -, + \mid (+-, \epsilon) \rangle,$$

So, an element of Z has the form $-^a +^b$ with $a, b \in \mathbb{Z}_{\geq 0}$.

Let $\mathcal{B}$ be a seminormal crystal. For each $i \in I$ define a map $\text{sgn}_i^\otimes : \mathcal{B} \rightarrow Z$ by

$$\text{sgn}_i^\otimes(x) = -^{\tilde{\epsilon}_i(x)} +^{\tilde{\varphi}_i(x)},$$

for each $x \in \mathcal{B}$. The map $\text{sgn}_i^\otimes$ is called the i -signature map for the tensor product $\otimes$, and $\text{sgn}_i^\otimes(x)$ is called the i -signature of $x \in \mathcal{B}$. The signature map interacts with the tensor product in the following way.

Proposition 4.19. *Let $\mathcal{B}$ and $\mathcal{B}'$ be seminormal crystals of the same type. Then,*

$$\text{sgn}_i^\otimes(x \otimes x') = \text{sgn}_i^\otimes(x) \text{sgn}_i^\otimes(x'),$$

for any $x \in \mathcal{B}, x' \in \mathcal{B}'$ and $i \in I$.

Given seminormal crystals $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_m$ of the same type, and elements $x_1 \in \mathcal{B}_1, x_2 \in \mathcal{B}_2, \dots, x_m \in \mathcal{B}_m$, by the previous result, we can easily compute the i -signature of $x_1 \otimes x_2 \otimes \cdots \otimes x_m$ as

$$\text{sgn}_i^\otimes(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = \text{sgn}_i^\otimes(x_1) \text{sgn}_i^\otimes(x_2) \cdots \text{sgn}_i^\otimes(x_m)$$

($i \in I$), and since $+- = \epsilon$ in Z , we get

$$\text{sgn}_i^\otimes(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = -^a +^b,$$

for some $a, b \in \mathbb{Z}_{\geq 0}$. Then, $\tilde{\epsilon}_i(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = a$ and $\tilde{\varphi}_i(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = b$. By Proposition 4.17, if $a \geq 1$, then

$$\tilde{\epsilon}_i(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = x_1 \otimes \cdots \otimes x_{p-1} \otimes \tilde{\epsilon}_i(x_p) \otimes x_{p+1} \otimes \cdots \otimes x_m,$$

where x_p originates the right-most $-$ in $\text{sgn}_i^\otimes(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = -^{a-1} +^b$. Also, if $b \geq 1$, then

$$\tilde{f}_i(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = x_1 \otimes \cdots \otimes x_{q-1} \otimes \tilde{f}_i(x_q) \otimes x_{q+1} \otimes \cdots \otimes x_m,$$

where x_q originates the left-most $+$ in $\text{sgn}_i^\otimes(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = -^a +^{b-1}$. This process is called the *signature rule for the tensor product*, and can be translated in the following algorithm.

Algorithm 4.20. *Input:* Elements $x_1 \in B_1, \dots, x_m \in B_m$ of seminormal crystals $\mathcal{B}_1, \dots, \mathcal{B}_m$ of the same type, and an index $i \in I$.

Output: The values of $\tilde{\epsilon}_i$, $\tilde{f}_i$, $\tilde{\epsilon}_i$ and $\tilde{\varphi}_i$ on $x_1 \otimes \cdots \otimes x_m$.

Procedure:

- (1) Set a word $w := -\tilde{\epsilon}_i(x_1) + \tilde{\varphi}_i(x_1) - \tilde{\epsilon}_i(x_2) + \tilde{\varphi}_i(x_2) \dots - \tilde{\epsilon}_i(x_{m-1}) + \tilde{\varphi}_i(x_{m-1}) - \tilde{\epsilon}_i(x_m) + \tilde{\varphi}_i(x_m)$ over the alphabet $\{-, +\}$, and keep record of which x_k originates each $-$ and $+$ on w .
- (2) While there exist $u, v \in \{-, +\}^*$ such that $w = u + -v$, set $w := uv$.
- (3) Set $a, b \in \mathbb{Z}_{\geq 0}$ such that $w = -^a +^b$.
- (4) Set $\tilde{\epsilon}_i(x_1 \otimes \cdots \otimes x_m) := a$, and $\tilde{\varphi}_i(x_1 \otimes \cdots \otimes x_m) := b$.
- (5) If $a = 0$, set $\tilde{\epsilon}_i(x_1 \otimes \cdots \otimes x_m) := \perp$.
- (6) If $a \geq 1$, retrieve p from the record such that x_p originated the right-most $-$ on w , and set $\tilde{\epsilon}_i(x_1 \otimes \cdots \otimes x_m) := x_1 \otimes \cdots \otimes x_{p-1} \otimes \tilde{\epsilon}_i(x_p) \otimes x_{p+1} \otimes \cdots \otimes x_m$.
- (7) If $b = 0$, set $\tilde{f}_i(x_1 \otimes \cdots \otimes x_m) := \perp$.
- (8) If $b \geq 1$, retrieve q from the record such that x_q originated the left-most $+$ on w , and set $\tilde{f}_i(x_1 \otimes \cdots \otimes x_m) := x_1 \otimes \cdots \otimes x_{q-1} \otimes \tilde{f}_i(x_q) \otimes x_{q+1} \otimes \cdots \otimes x_m$.
- (9) Return $\tilde{\epsilon}_i(x_1 \otimes \cdots \otimes x_m)$, $\tilde{f}_i(x_1 \otimes \cdots \otimes x_m)$, $\tilde{\epsilon}_i(x_1 \otimes \cdots \otimes x_m)$, and $\tilde{\varphi}_i(x_1 \otimes \cdots \otimes x_m)$.

In the following example, we apply the signature rule to some elements.

Example 4.21. Consider the crystal $\mathcal{A}_4 \otimes \mathcal{A}_4 \otimes \mathcal{A}_4 \otimes \mathcal{A}_4 \otimes \mathcal{A}_4$, where $\mathcal{A}_4$ is the standard crystal of type $\mathfrak{A}_4$. We compute $\tilde{e}_2, \tilde{f}_2, \tilde{\varepsilon}_2$ and $\tilde{\varphi}_2$ on $3 \otimes 1 \otimes 2 \otimes 2 \otimes 3$. We start with the word $-_1+_3+_4-_5$, where the subscript is an auxiliary notation to keep record of the position of the element which originates each $-$ and $+$. Next, we obtain the word $-_1+_3$, as $+_4-_5$ is deleted. Hence,

$$\begin{aligned} \tilde{e}_2(3 \otimes 1 \otimes 2 \otimes 2 \otimes 3) &= 2 \otimes 1 \otimes 2 \otimes 2 \otimes 3, & \tilde{\varepsilon}_2(3 \otimes 1 \otimes 2 \otimes 2 \otimes 3) &= 1, \\ \tilde{f}_2(3 \otimes 1 \otimes 2 \otimes 2 \otimes 3) &= 3 \otimes 1 \otimes 3 \otimes 2 \otimes 3, & \tilde{\varphi}_2(3 \otimes 1 \otimes 2 \otimes 2 \otimes 3) &= 1. \end{aligned}$$

Now we compute $\tilde{e}_1, \tilde{f}_1, \tilde{\varepsilon}_1$ and $\tilde{\varphi}_1$ on $1 \otimes 1 \otimes 2 \otimes 2 \otimes 1$. Using the same notation as above, we start with $+_1+_2-_3-_4+_5$, then $+_1-_4+_5$, and finally, just $+_5$. Hence,

$$\begin{aligned} \tilde{e}_1(1 \otimes 1 \otimes 2 \otimes 2 \otimes 1) &= \perp, & \tilde{\varepsilon}_1(1 \otimes 1 \otimes 2 \otimes 2 \otimes 1) &= 0, \\ \tilde{f}_1(1 \otimes 1 \otimes 2 \otimes 2 \otimes 1) &= 1 \otimes 1 \otimes 2 \otimes 2 \otimes 2, & \tilde{\varphi}_1(1 \otimes 1 \otimes 2 \otimes 2 \otimes 1) &= 1. \end{aligned}$$

CRYSTALIZING THE PLACTIC MONOID

In this chapter we present the method of obtaining a monoid from a seminormal crystal. The resulting monoid is called a plactic monoid, and this process was introduced by Kashiwara [Kas90; Kas91], where the classical plactic monoid was obtained from the study of crystal bases for the quantized universal enveloping general linear algebra. This led to the study of crystal bases for other celebrated Lie algebras [KN94], and the study of plactic monoids for the Cartan types $\mathcal{B}_n$, $\mathcal{C}_n$ and $\mathcal{D}_n$ [Lec02; Lec03].

In Sections 5.1 to 5.3, we present an algebraic framework relating crystals and monoids. Although the results in these sections are already present in the literature, our approach put them in a strong algebraic framework that allows an analogue study for the hypoplactic monoid in Chapter 8. In Section 5.4, we show that the plactic congruence can be obtained from this algebraic framework. Finally, in Section 5.5, we describe the construction of the classical plactic monoid from the plactic congruence over the free crystal monoid $\mathcal{A}_n$ of Cartan type $\mathcal{A}_n$.

5.1 Crystal monoids and homomorphisms

In this section we introduce the notion of crystal monoid, and present some results showing how the crystal and monoid structures interact. In particular, we describe an analogue of the signature rule, resulting in an algorithm to compute the crystal structure maps on a product based only on their values on each parcel. Finally, we introduce the notion of homomorphism between crystal monoids, and obtain an algebraic framework that will be studied on the subsequent sections.

Definition 5.1. Let Φ be a root system with weight lattice Λ and index set I for the simple roots $(\alpha_i)_{i \in I}$. A $\otimes$ -crystal monoid $\mathcal{M}$ of type Φ consists of a set M together with maps $\text{wt} : M \rightarrow \Lambda$, $\tilde{e}_i, \tilde{f}_i : M \rightarrow M \sqcup \{\perp\}$, $\tilde{\varepsilon}_i, \tilde{\varphi}_i : M \rightarrow \mathbb{Z} \cup \{-\infty\}$ ($i \in I$) and a binary operation $\cdot : M \times M \rightarrow M$ satisfying the following conditions:

- (i) M together with wt , $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{\varepsilon}_i$ and $\tilde{\varphi}_i$ ($i \in I$) is a seminormal crystal;
- (ii) M together with $\cdot$ is a monoid;
- (iii) the map $M \otimes M \rightarrow M$, given by $x \otimes y \mapsto x \cdot y$ for $x, y \in M$, is a crystal homomorphism from $\mathcal{M} \otimes \mathcal{M}$ to $\mathcal{M}$.

Since we only consider the definition of tensor product of crystals given in Definition 4.13, we will omit $\otimes$ and just say that $\mathcal{M}$ is a crystal monoid. We included this fact in the definition above to emphasize that we have different notions of crystal monoid depending on which tensor product is being considered.

Let $\mathcal{M}$ be a crystal monoid. Define $\psi : M \otimes M \rightarrow M$ by $\psi(x \otimes y) = xy$, for $x, y \in M$. From Definition 5.1(iii), we have that ψ is a crystal homomorphism from $\mathcal{M} \otimes \mathcal{M}$ to $\mathcal{M}$ (Definition 4.7). Then, for $x, y \in M$, we get

$$\text{wt}(xy) = \text{wt}(x \otimes y) = \text{wt}(x) + \text{wt}(y), \quad (5.1)$$

and since $\mathcal{M}$ is a seminormal crystal,

$$\begin{aligned} \tilde{\varepsilon}_i(xy) &= \tilde{\varepsilon}_i(x \otimes y) = \max\{\tilde{\varepsilon}_i(x), \tilde{\varepsilon}_i(y) - \langle \text{wt}(x), \alpha_i^\vee \rangle\} \\ &= \max\{\tilde{\varepsilon}_i(x), \tilde{\varepsilon}_i(x) + \tilde{\varepsilon}_i(y) - \tilde{\varphi}_i(x)\} \end{aligned} \quad (5.2)$$

and

$$\begin{aligned} \tilde{\varphi}_i(xy) &= \tilde{\varphi}_i(x \otimes y) = \max\{\tilde{\varphi}_i(x) + \langle \text{wt}(y), \alpha_i^\vee \rangle, \tilde{\varphi}_i(y)\} \\ &= \max\{\tilde{\varphi}_i(x) + \tilde{\varphi}_i(y) - \tilde{\varepsilon}_i(y), \tilde{\varphi}_i(y)\}, \end{aligned} \quad (5.3)$$

for $i \in I$. Since $\mathcal{M} \otimes \mathcal{M}$ is also seminormal (Theorem 4.14) and $\psi(M \otimes M) \subseteq M$, we have that $\tilde{e}_i(x \otimes y)$ is defined if and only if $\psi(\tilde{e}_i(x \otimes y))$ is defined, which implies that

$$\tilde{e}_i(xy) = \psi(\tilde{e}_i(x \otimes y)) = \begin{cases} \tilde{e}_i(x) \cdot y & \text{if } \tilde{\varphi}_i(x) \geq \tilde{\varepsilon}_i(y) \\ x \cdot \tilde{e}_i(y) & \text{if } \tilde{\varphi}_i(x) < \tilde{\varepsilon}_i(y), \end{cases} \quad (5.4)$$

where $x \cdot \perp = \perp \cdot y = \perp$. Similarly,

$$\tilde{f}_i(xy) = \psi(\tilde{f}_i(x \otimes y)) = \begin{cases} \tilde{f}_i(x) \cdot y & \text{if } \tilde{\varphi}_i(x) > \tilde{\varepsilon}_i(y) \\ x \cdot \tilde{f}_i(y) & \text{if } \tilde{\varphi}_i(x) \leq \tilde{\varepsilon}_i(y). \end{cases} \quad (5.5)$$

In fact, Definition 5.1(iii) is equivalent to require the equalities on (5.1) to (5.5) to be satisfied. These equalities can be generalized directly from Proposition 4.17.

Proposition 5.2. *Let $\mathcal{M}$ be a crystal monoid, and let $x_1, \dots, x_m \in M$. Then,*

$$\text{wt}(x_1 \cdots x_m) = \text{wt}(x_1) + \cdots + \text{wt}(x_m).$$

For $i \in I$,

$$\tilde{\varepsilon}_i(x_1 \cdots x_m) = \max_{1 \leq p \leq m} \left(\sum_{k=1}^p \tilde{\varepsilon}_i(x_k) - \sum_{l=1}^{p-1} \tilde{\varphi}_i(x_l) \right),$$

and if p is the smallest integer for which the maximum is attained, then

$$\tilde{\varepsilon}_i(x_1 \cdots x_m) = x_1 \cdots x_{p-1} \cdot \tilde{\varepsilon}_i(x_p) \cdot x_{p+1} \cdots x_m.$$

Also,

$$\tilde{\varphi}_i(x_1 \cdots x_m) = \max_{1 \leq q \leq m} \left(\sum_{k=q}^m \tilde{\varphi}_i(x_k) - \sum_{l=q+1}^m \tilde{\varepsilon}_i(x_l) \right),$$

and if q is the largest integer for which the maximum is attained, then

$$\tilde{f}_i(x_1 \cdots x_m) = x_1 \cdots x_{q-1} \cdot \tilde{f}_i(x_q) \cdot x_{q+1} \cdots x_m.$$

Proof. We proceed by induction on m . If $m = 1$, then the result is trivial. If $m = 2$, then we obtain the equalities on (5.1) to (5.5). Thus, assume as induction hypothesis (IH) that the result holds for any $x_1, \dots, x_m \in M$. Let $y_1, \dots, y_m, y_{m+1} \in M$. Since $\cdot$ is associative, we have that $y_1 \cdots y_m y_{m+1} = (y_1 \cdots y_m) y_{m+1}$, where

$$\text{wt}((y_1 \cdots y_m) y_{m+1}) = \text{wt}(y_1 \cdots y_m) + \text{wt}(y_{m+1}),$$

and as $\text{wt}(y_1 \cdots y_m) = \text{wt}(y_1) + \cdots + \text{wt}(y_m)$ by (IH), we obtain that

$$\text{wt}(y_1 \cdots y_m y_{m+1}) = \text{wt}(y_1) + \cdots + \text{wt}(y_m) + \text{wt}(y_{m+1}).$$

Let $i \in I$. We have that

$$\tilde{\varepsilon}_i((y_1 \cdots y_m) y_{m+1}) = \max\{\tilde{\varepsilon}_i(y_1 \cdots y_m), \tilde{\varepsilon}_i(y_{m+1} - \langle \text{wt}(y_1 \cdots y_m), \alpha_i^\vee \rangle)\},$$

where

$$\tilde{\varepsilon}_i(y_1 \cdots y_m) = \max_{1 \leq p \leq m} \left(\sum_{k=1}^p \tilde{\varepsilon}_i(y_k) - \sum_{l=1}^{p-1} \tilde{\varphi}_i(y_l) \right)$$

and

$$\langle \text{wt}(y_1 \cdots y_m), \alpha_i^\vee \rangle = \sum_{k=1}^m \langle \text{wt}(y_k), \alpha_i^\vee \rangle = \sum_{k=1}^m \tilde{\varphi}_i(y_k) - \tilde{\varepsilon}_i(y_k)$$

by (IH) and because $\mathcal{M}$ is seminormal. Hence,

$$\tilde{\varepsilon}_i(y_1 \cdots y_m y_{m+1}) = \max_{1 \leq p \leq m+1} \left(\sum_{k=1}^p \tilde{\varepsilon}_i(y_k) - \sum_{l=1}^{p-1} \tilde{\varphi}_i(y_l) \right),$$

and since $\tilde{\varphi}_i(y_1 \cdots y_m y_{m+1}) = \tilde{\varepsilon}_i(y_1 \cdots y_m y_{m+1}) + \langle \text{wt}(y_1 \cdots y_m y_{m+1}), \alpha_i^\vee \rangle$,

$$\tilde{\varphi}_i(y_1 \cdots y_m y_{m+1}) = \max_{1 \leq q \leq m+1} \left(\sum_{k=q}^{m+1} \tilde{\varphi}_i(y_k) - \sum_{l=q+1}^{m+1} \tilde{\varepsilon}_i(y_l) \right).$$

Let p be the smallest integer in $\{1, \dots, m+1\}$ such that

$$\tilde{\varepsilon}_i(y_1 \cdots y_m y_{m+1}) = \sum_{k=1}^p \tilde{\varepsilon}_i(y_k) - \sum_{l=1}^{p-1} \tilde{\varphi}_i(y_l).$$

If $\tilde{\varphi}_i(y_1 \cdots y_m) \geq \tilde{\varepsilon}_i(y_{m+1})$, then $\tilde{\varepsilon}_i(y_1 \cdots y_m y_{m+1}) = \tilde{\varepsilon}_i(y_1 \cdots y_m)$, and so, $p \leq m$ and

$$\begin{aligned} \tilde{\varepsilon}_i((y_1 \cdots y_m) y_{m+1}) &= \tilde{\varepsilon}_i(y_1 \cdots y_m) y_{m+1} \\ &= y_1 \cdots y_{p-1} \cdot \tilde{\varepsilon}_i(y_p) \cdot y_{p+1} \cdots y_m y_{m+1}, \end{aligned}$$

by (IH). Otherwise, we have that $\tilde{\varphi}_i(y_1 \cdots y_m) < \tilde{\varepsilon}_i(y_{m+1})$, which implies that

$$\tilde{\varepsilon}_i(y_1 \cdots y_m) + \tilde{\varepsilon}_i(y_{m+1}) - \tilde{\varphi}_i(y_1 \cdots y_m) > \tilde{\varepsilon}_i(y_1 \cdots y_m),$$

and so, $p = m+1$ and

$$\tilde{\varepsilon}_i((y_1 \cdots y_m) y_{m+1}) = y_1 \cdots y_m \cdot \tilde{\varepsilon}_i(y_{m+1}).$$

The fact that $\tilde{f}_i(y_1 \cdots y_m y_{m+1}) = y_1 \cdots y_{q-1} \cdot \tilde{f}_i(y_q) \cdot y_{q+1} \cdots y_{m+1}$, where q is the largest integer in $\{1, \dots, m+1\}$ such that

$$\tilde{\varphi}_i(y_1 \cdots y_m y_{m+1}) = \sum_{k=q}^{m+1} \tilde{\varphi}_i(y_k) - \sum_{l=q+1}^{m+1} \tilde{\varepsilon}_i(y_l),$$

follows analogously. □

In the previous result we saw how the monoid operation interacts with the crystal structure maps. Now we show that the monoid identity also satisfies some properties.

Proposition 5.3. *Let $\mathcal{M}$ be a quasi-crystal monoid where the monoid identity is denoted by 1. Then, $\text{wt}(1) = 0$, and for each $i \in I$, $\tilde{\varepsilon}_i(1) = \tilde{\varphi}_i(1) = 0$. Therefore, $\tilde{e}_i(1) = \tilde{f}_i(1) = \perp$, for all $i \in I$.*

Proof. We have that

$$\text{wt}(1) = \text{wt}(1 \cdot 1) = \text{wt}(1) + \text{wt}(1),$$

which implies that $\text{wt}(1) = 0$.

Let $i \in I$. For any $x \in M$, we have that

$$\tilde{\varepsilon}_i(x) = \tilde{\varepsilon}_i(1x) = \max\{\tilde{\varepsilon}_i(1), \tilde{\varepsilon}_i(x) + \tilde{\varepsilon}_i(1) - \tilde{\varphi}_i(1)\} \geq \tilde{\varepsilon}_i(1).$$

Thus, given $x \in M$, we obtain that $\tilde{\varepsilon}_i(1) \leq \tilde{\varepsilon}_i(\tilde{e}_i^{\tilde{\varepsilon}_i(x)}(x)) = 0$, and so, $\tilde{\varepsilon}_i(1) = 0$, because $\mathcal{M}$ is seminormal. We also get that

$$\tilde{\varphi}_i(1) = \tilde{\varepsilon}_i(1) + \langle \text{wt}(1), \alpha_i^\vee \rangle = 0.$$

As $\tilde{\varepsilon}_i(1) = \tilde{\varphi}_i(1) = 0$ and $\mathcal{M}$ is seminormal, it is immediate that $\tilde{e}_i$ and $\tilde{f}_i$ are undefined on 1. $\square$

Due to the interaction between the binary operation $\cdot$ of a crystal monoid $\mathcal{M}$ and the crystal structure of the tensor product $\mathcal{M} \otimes \mathcal{M}$, we can extend the signature rule described in Section 4.4 to crystal monoids. Let $x, y \in M$ and $i \in I$. Since $\tilde{\varepsilon}_i(xy) = \tilde{\varepsilon}_i(x \otimes y)$ and $\tilde{\varphi}_i(xy) = \tilde{\varphi}_i(x \otimes y)$, we have that the i -signature of xy and $x \otimes y$ coincide. Hence,

$$\text{sgn}_i^\otimes(xy) = \text{sgn}_i^\otimes(x) \text{sgn}_i^\otimes(y),$$

by Proposition 4.19. Also, $\text{sgn}_i^\otimes(1) = \epsilon$, by Proposition 5.3. Therefore, we obtain the following result, which is an improvement of Proposition 4.19.

Proposition 5.4. *Let $\mathcal{M}$ be a crystal monoid. Then, the signature maps $\text{sgn}_i^\otimes$ ($i \in I$) are monoid homomorphisms from M to the bicyclic monoid Z .*

The previous result together with Proposition 5.2 leads to an analogue of Algorithm 4.20 to compute the values of the maps $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{\varepsilon}_i$ and $\tilde{\varphi}_i$ ($i \in I$) on an element of the form $x_1 \cdots x_m$, where $x_1, \dots, x_m \in M$.

Algorithm 5.5. *Input:* Elements $x_1, \dots, x_m \in M$ of a crystal monoid $\mathcal{M}$, and an index $i \in I$.

Output: The values of $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{\varepsilon}_i$ and $\tilde{\varphi}_i$ on $x_1 \cdots x_m$.

Procedure:

- (1) Set a word $w := -\tilde{\varepsilon}_i(x_1) + \tilde{\varphi}_i(x_1) - \tilde{\varepsilon}_i(x_2) + \tilde{\varphi}_i(x_2) \dots - \tilde{\varepsilon}_i(x_{m-1}) + \tilde{\varphi}_i(x_{m-1}) - \tilde{\varepsilon}_i(x_m) + \tilde{\varphi}_i(x_m)$ over the alphabet $\{-, +\}$, and keep record of which x_k originates each $-$ and $+$ on w .
- (2) While there exist $u, v \in \{-, +\}^*$ such that $w = u + -v$, set $w := uv$.
- (3) Set $a, b \in \mathbb{Z}_{\geq 0}$ such that $w = -^a +^b$.
- (4) Set $\tilde{\varepsilon}_i(x_1 \cdots x_m) := a$, and $\tilde{\varphi}_i(x_1 \cdots x_m) := b$.
- (5) If $a = 0$, set $\tilde{e}_i(x_1 \cdots x_m) := \perp$.

- (6) If $a \geq 1$, retrieve p from the record such that x_p originated the right-most $-$ on w , and set $\tilde{e}_i(x_1 \cdots x_m) := x_1 \cdots x_{p-1} \cdot \tilde{e}_i(x_p) \cdot x_{p+1} \cdots x_m$.
- (7) If $b = 0$, set $\tilde{f}_i(x_1 \cdots x_m) := \perp$.
- (8) If $b \geq 1$, retrieve q from the record such that x_q originated the left-most $+$ on w , and set $\tilde{f}_i(x_1 \cdots x_m) := x_1 \cdots x_{q-1} \cdot \tilde{f}_i(x_q) \cdot x_{q+1} \cdots x_m$.
- (9) Return $\tilde{e}_i(x_1 \cdots x_m)$, $\tilde{f}_i(x_1 \cdots x_m)$, $\tilde{e}_i(x_1 \cdots x_m)$, and $\tilde{\varphi}_i(x_1 \cdots x_m)$.

Finally, we introduce the notion of homomorphism between crystal monoids.

Definition 5.6. Let $\mathcal{M}$ and $\mathcal{M}'$ be quasi-crystal monoids of the same type. A *crystal monoid homomorphism* ψ from $\mathcal{M}$ to $\mathcal{M}'$, denoted by $\psi : \mathcal{M} \rightarrow \mathcal{M}'$, is a map $\psi : M \rightarrow M'$ that satisfies the following conditions:

- (i) ψ is a crystal homomorphism;
- (ii) ψ is a monoid homomorphism.

If ψ is also bijective, it is called a *crystal monoid isomorphism*.

Note that in the previous definition we only consider maps from M to M' , but as we remarked after Definition 4.7, when we state that ψ is a crystal homomorphism in condition (i) we mean that the map $\psi' : M \sqcup \{\perp\} \rightarrow M' \sqcup \{\perp\}$, given by $\psi'(\perp) = \perp$ and $\psi'(x) = \psi(x)$, for each $x \in M$, is a crystal homomorphism.

Also, if $\psi : \mathcal{M} \rightarrow \mathcal{M}'$ is a crystal monoid isomorphism, then ψ is a crystal isomorphism, by Theorem 4.8, and a monoid isomorphism. The converse is immediate, because a monoid isomorphism implies the map to be bijective. Hence, a map $\psi : M \rightarrow M'$ is a crystal monoid isomorphism if and only if ψ is a crystal isomorphism and a monoid isomorphism. This implies that if $\psi : \mathcal{M} \rightarrow \mathcal{M}'$ is a crystal monoid isomorphism, then ψ^{-1} is a crystal monoid isomorphism between $\mathcal{M}'$ and $\mathcal{M}$.

5.2 The free crystal monoid

In this section we first present a construction of a free crystal monoid. Then, we study some properties satisfied by the free crystal monoids. In particular, we show that free crystal monoids satisfy a universal property commonly associated to the term free in algebra. Finally, by this universal property, we present an alternative method of defining a free crystal monoid up to isomorphism.

Let $\mathcal{B}$ be a seminormal crystal. For $k \geq 1$, set

$$\mathcal{B}^{\otimes k} = \underbrace{\mathcal{B} \otimes \cdots \otimes \mathcal{B}}_{k \text{ times}}.$$

As the tensor product of crystals is associative (Theorem 4.16), we have that for $k, l \geq 1$ the map $\mathcal{B}^{\otimes k} \otimes \mathcal{B}^{\otimes l} \rightarrow \mathcal{B}^{\otimes(k+l)}$, given by

$$(x_1 \otimes \cdots \otimes x_k) \otimes (y_1 \otimes \cdots \otimes y_l) \mapsto x_1 \otimes \cdots \otimes x_k \otimes y_1 \otimes \cdots \otimes y_l$$

for $x_1, \dots, x_k, y_1, \dots, y_l \in B$, is a crystal isomorphism.

Let ϵ be an element that does not lie in B . Set $\mathcal{B}^{\otimes 0}$ to be the seminormal crystal of the same type as $\mathcal{B}$ formed by the set $B^{\otimes 0} = \{\epsilon\}$ and maps given by $\text{wt}(\epsilon) = 0$, $\tilde{e}_i(\epsilon) = \tilde{f}_i(\epsilon) = \perp$ and $\tilde{e}_i(\epsilon) = \tilde{\varphi}_i(\epsilon) = 0$ ($i \in I$). Note that $\mathcal{B}^{\otimes 0} \otimes \mathcal{B}^{\otimes 0}$ is crystal isomorphic to $\mathcal{B}^{\otimes 0}$, as both crystals consist of a single element where the crystal structure maps coincide. For any $x \in B$ and $i \in I$, we have that $\text{wt}(x \otimes \epsilon) = \text{wt}(x)$, $\tilde{e}_i(x \otimes \epsilon) = \tilde{e}_i(x)$ and $\tilde{\varphi}_i(x \otimes \epsilon) = \tilde{\varphi}_i(x)$. Also, by the signature rule, since $\text{sgn}_i^{\otimes}(\epsilon) = \epsilon$, it is immediate that $\tilde{e}_i$ (or $\tilde{f}_i$) is defined on $x \otimes \epsilon$ if and only if $\tilde{e}_i$ (resp., $\tilde{f}_i$) is defined on x . And if so, $\tilde{e}_i(x \otimes \epsilon) = \tilde{e}_i(x) \otimes \epsilon$ (resp., $\tilde{f}_i(x \otimes \epsilon) = \tilde{f}_i(x) \otimes \epsilon$). Therefore, the map $B \otimes \mathcal{B}^{\otimes 0} \rightarrow B$, given by $y \otimes \epsilon \mapsto y$ for each $y \in B$, is a crystal isomorphism. Similarly, the map $\mathcal{B}^{\otimes 0} \otimes B \rightarrow B$, given by $\epsilon \otimes y \mapsto y$ for each $y \in B$, is a crystal isomorphism. Since the tensor product of crystals is associative, we get that $\mathcal{B}^{\otimes k} \otimes \mathcal{B}^{\otimes 0}$ and $\mathcal{B}^{\otimes 0} \otimes \mathcal{B}^{\otimes k}$ are isomorphic to $\mathcal{B}^{\otimes k}$, for any $k \geq 0$.

The sets $B^{\otimes k}$ and $B^{\otimes l}$ are disjoint, whenever $k \neq l$. Thus, we can extend the maps wt , $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{e}_i$ and $\tilde{\varphi}_i$ ($i \in I$) defined on each $B^{\otimes k}$ to the set

$$M = \bigcup_{k \geq 0} B^{\otimes k},$$

obtaining a seminormal crystal $\mathcal{M}$. By the crystal isomorphisms $\mathcal{B}^{\otimes k} \otimes \mathcal{B}^{\otimes l} \rightarrow \mathcal{B}^{\otimes(k+l)}$ ($k, l \geq 0$) defined above, we get that $\mathcal{M}$ becomes a crystal monoid with the binary operation $\cdot : M \times M \rightarrow M$ given by

$$(x_1 \otimes \cdots \otimes x_k) \cdot (y_1 \otimes \cdots \otimes y_l) = x_1 \otimes \cdots \otimes x_k \otimes y_1 \otimes \cdots \otimes y_l$$

and

$$(x_1 \otimes \cdots \otimes x_k) \cdot \epsilon = \epsilon \cdot (x_1 \otimes \cdots \otimes x_k) = x_1 \otimes \cdots \otimes x_k,$$

for $x_1, \dots, x_k, y_1, \dots, y_l \in B$. If we identify an element of the form $x_1 \otimes \cdots \otimes x_k$ with the word $x_1 \dots x_k$ over the alphabet B , we get a monoid isomorphism between M and the free monoid B^* over B , and through this identification we can also define a crystal structure on B^* . Therefore, we constructed a crystal monoid that leads to the following definition.

Definition 5.7. Let $\mathcal{B}$ be a seminormal crystal. The *free crystal monoid* $\mathcal{B}^*$ over $\mathcal{B}$ is a crystal monoid of the same type as $\mathcal{B}$ consisting of the set B^* of all words over B , the usual concatenation of words, and maps defined as follows. For $i \in I$, set

$$\text{wt}(\epsilon) = 0, \quad \tilde{e}_i(\epsilon) = \tilde{f}_i(\epsilon) = \perp, \quad \tilde{\epsilon}_i(\epsilon) = \tilde{\varphi}_i(\epsilon) = 0,$$

where ϵ denotes the empty word, and for $u, v \in B^*$, set

$$\begin{aligned} \text{wt}(uv) &= \text{wt}(u) + \text{wt}(v), \\ \tilde{e}_i(uv) &= \begin{cases} (\tilde{e}_i(u))v & \text{if } \tilde{\varphi}_i(u) \geq \tilde{\epsilon}_i(v) \\ u(\tilde{e}_i(v)) & \text{if } \tilde{\varphi}_i(u) < \tilde{\epsilon}_i(v), \end{cases} \\ \tilde{f}_i(uv) &= \begin{cases} (\tilde{f}_i(u))v & \text{if } \tilde{\varphi}_i(u) > \tilde{\epsilon}_i(v) \\ u(\tilde{f}_i(v)) & \text{if } \tilde{\varphi}_i(u) \leq \tilde{\epsilon}_i(v), \end{cases} \\ \tilde{\epsilon}_i(uv) &= \max\{\tilde{\epsilon}_i(u), \tilde{\epsilon}_i(v) - \langle \text{wt}(u), \alpha_i^\vee \rangle\}, \\ \tilde{\varphi}_i(uv) &= \max\{\tilde{\varphi}_i(u) + \langle \text{wt}(v), \alpha_i^\vee \rangle, \tilde{\varphi}_i(v)\}, \end{aligned}$$

where $u\perp = \perp v = \perp$.

As we constructed the free crystal monoid $\mathcal{B}^*$ based on the tensor product of crystals (Definition 4.13), we described its crystal structure based on the definition of the crystal structure for a tensor product. Notice that the values of the crystal structure maps on a word of the form uv depend only on their values on u and v . Thus, we can obtain the values of the crystal structure maps on a word $w = x_1 \dots x_m$, where $x_1, \dots, x_m \in B$, based on their values on each letter $x_1, \dots, x_m$, which are given by the crystal structure of $\mathcal{B}$. This also follows from Proposition 5.2 and Algorithm 5.5, which describe how to get the values of the crystal structure maps on w based on their values on its letters.

Proposition 5.8. Let $\mathcal{B}$ be a seminormal crystal. For any $w \in B^*$ and $i \in I$, if $\tilde{e}_i(w) \in B^*$ then $|\tilde{e}_i(w)| = |w|$, and if $\tilde{f}_i(w) \in B^*$ then $|\tilde{f}_i(w)| = |w|$. Therefore, $|u| = |v|$, whenever u and v lie in the same connected component of $\mathcal{B}^*$.

Proof. Let $w \in B^*$ and $i \in I$ such that $\tilde{f}_i$ is defined on w , then by Definition 5.7, we have that $w \neq \epsilon$, and so, $w = x_1 \dots x_m$, for some $x_1, \dots, x_m \in B$ and $m \geq 1$. By Proposition 5.2, there exists $q \in \{1, \dots, m\}$ such that

$$\tilde{f}_i(w) = x_1 \dots x_{q-1}(\tilde{f}_i(x_q))x_{q+1} \dots x_m.$$

Since $x_q \in B$, then $\tilde{f}_i(x_q) \in B$, which implies that $|\tilde{f}_i(w)| = |w|$. Hence, for any $w \in B^*$ and $i \in I$, $|\tilde{f}_i(w)| = |w|$, whenever $\tilde{f}_i(w) \in B^*$. Similarly, for any $w \in B^*$ and $i \in I$, if $\tilde{e}_i$ is defined on w , then $|\tilde{e}_i(w)| = |w|$.

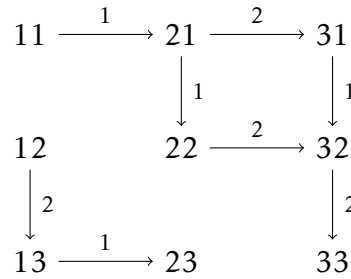
Finally, let $u, v \in B^*$ be such that u and v lie in the same connected component of $\mathcal{B}^*$. Then, there exist $g_1, \dots, g_k \in \{\tilde{e}_i, \tilde{f}_i \mid i \in I\}$ such that $g_1 \cdots g_k(u) = v$. By applying recursively what we have proven above, we obtain that $|g_1 \cdots g_k(u)| = |u|$. $\square$

Example 5.9. (i) Consider the standard crystal $\mathcal{A}_n$ of type $\mathfrak{A}_n$ as described in Example 4.4(i). The weight of a word $w \in A_n^*$ is given by

$$\text{wt}(w) = |w|_1 \mathbf{e}_1 + |w|_2 \mathbf{e}_2 + \cdots + |w|_n \mathbf{e}_n.$$

For $i \in \{1, \dots, n-1\}$, we can compute the values of the maps $\tilde{e}_i, \tilde{f}_i, \tilde{\varepsilon}_i$ and $\tilde{\varphi}_i$ on w using the signature rule (Algorithm 5.5). As we checked in Example 4.21, $\text{sgn}_2^\otimes(31223) = -1+3+4-5 = -1+3$, and so, $\tilde{e}_2(31223) = 21223$, $\tilde{f}_2(31223) = 31323$, $\tilde{\varepsilon}_2(31223) = 1$, and $\tilde{\varphi}_2(31223) = 1$. The word 123 is of highest weight, and the word $12 \dots n$ is isolated.

From Example 4.15(i), We have that the words of length 2 form the following subgraph of the crystal graph of $\mathcal{A}_3^*$.



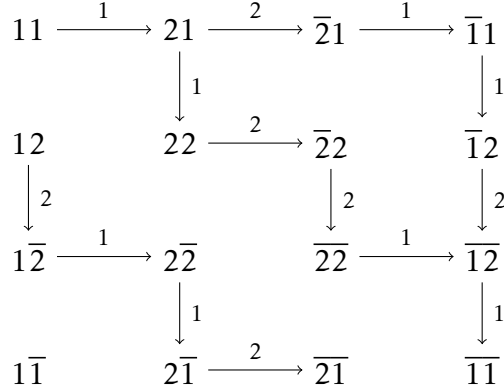
(ii) Consider the standard crystal $\mathcal{C}_n$ of type $\mathfrak{C}_n$ as described in Example 4.4(ii). The weight of a word $w \in C_n^*$ is given by

$$\text{wt}(w) = (|w|_1 - |w|_{\bar{1}}) \mathbf{e}_1 + (|w|_2 - |w|_{\bar{2}}) \mathbf{e}_2 + \cdots + (|w|_n - |w|_{\bar{n}}) \mathbf{e}_n.$$

By applying the signature rule (Algorithm 5.5), we can compute the values of the maps $\tilde{e}_2, \tilde{f}_2, \tilde{\varepsilon}_2$ and $\tilde{\varphi}_2$ on $\bar{2}\bar{2}\bar{3}32$. As $\text{sgn}_2^\otimes(\bar{2}\bar{2}\bar{3}32) = -1+2+3-4+5 = -1+2+5$, we get that $\tilde{e}_2(\bar{2}\bar{2}\bar{3}32) = \bar{3}\bar{2}\bar{3}32$, $\tilde{f}_2(\bar{2}\bar{2}\bar{3}32) = \bar{2}\bar{3}\bar{3}32$, $\tilde{\varepsilon}_2(\bar{2}\bar{2}\bar{3}32) = 1$, and $\tilde{\varphi}_2(\bar{2}\bar{2}\bar{3}32) = 2$. The word $12\bar{2}$ is of highest weight, and the word $12\bar{2}\bar{1}$ is isolated.

From Example 4.15(ii), We have that the words of length 2 form the following

subgraph of the crystal graph of $\mathcal{C}_2^*$.



We characterized the crystal monoid $\mathcal{B}^*$ over a seminormal crystal $\mathcal{B}$ as *free*, because of the following property.

Theorem 5.10. *Let $\mathcal{B}$ be a seminormal crystal and $\mathcal{M}$ be a crystal monoid of the same type. Then, for each crystal homomorphism $\psi : \mathcal{B} \rightarrow \mathcal{M}$ satisfying $\psi(B) \subseteq M$, there exists a unique crystal monoid homomorphism $\hat{\psi} : \mathcal{B}^* \rightarrow \mathcal{M}$ such that $\hat{\psi}(x) = \psi(x)$, for all $x \in B$.*

Proof. Let $\psi : \mathcal{B} \rightarrow \mathcal{M}$ be a crystal homomorphism such that $\psi(B) \subseteq M$. So, we can consider ψ as a map from B to M . It is well-known that there exists a unique monoid homomorphism $\hat{\psi} : B^* \rightarrow M$ such that $\hat{\psi}(x) = \psi(x)$, for all $x \in B$. We also have that $\hat{\psi}$ is given by

$$\hat{\psi}(\epsilon) = 1 \quad \text{and} \quad \hat{\psi}(x_1 \dots x_m) = \psi(x_1) \cdots \psi(x_m),$$

for $x_1, \dots, x_m \in B$. It remains to show that $\hat{\psi}$ is a crystal homomorphism.

Let $i \in I$. By Proposition 5.3 and Definition 5.7, we have that $\text{wt}(\epsilon) = 0 = \text{wt}(1)$, $\tilde{e}_i(\epsilon) = \tilde{f}_i(\epsilon) = \perp = \tilde{e}_i(1) = \tilde{f}_i(1)$, and $\tilde{\epsilon}_i(\epsilon) = \tilde{\varphi}_i(\epsilon) = 0 = \tilde{\epsilon}_i(1) = \tilde{\varphi}_i(1)$. Let $x_1, \dots, x_m \in B$, $m \geq 1$, and set $w = x_1 \dots x_m$. We have that $\hat{\psi}(w) = \psi(x_1) \cdots \psi(x_m)$, where $\text{wt}(\psi(x_k)) = \text{wt}(x_k)$, $\tilde{e}_i(\psi(x_k)) = \tilde{e}_i(x_k)$ and $\tilde{\varphi}_i(\psi(x_k)) = \tilde{\varphi}_i(x_k)$, for $k = 1, \dots, m$, because ψ is a crystal homomorphism. By Proposition 5.2, we obtain that

$$\text{wt}(\hat{\psi}(w)) = \text{wt}(\psi(x_1)) + \cdots + \text{wt}(\psi(x_m)) = \text{wt}(x_1) + \cdots + \text{wt}(x_m) = \text{wt}(w),$$

and

$$\begin{aligned}
 \tilde{\epsilon}_i(\hat{\psi}(w)) &= \max_{1 \leq p \leq m} \left(\sum_{k=1}^p \tilde{\epsilon}_i(\psi(x_k)) - \sum_{l=1}^{p-1} \tilde{\varphi}_i(\psi(x_l)) \right) \\
 &= \max_{1 \leq p \leq m} \left(\sum_{k=1}^p \tilde{\epsilon}_i(x_k) - \sum_{l=1}^{p-1} \tilde{\varphi}_i(x_l) \right) = \tilde{\epsilon}_i(w),
 \end{aligned}$$

and since both maxima range over the same numbers in the same order, we have a common p which corresponds to the smallest integer where each maximum is attained, implying that

$$\begin{aligned}\tilde{e}_i(\hat{\psi}(w)) &= \psi(x_1) \cdots \psi(x_{p-1}) \cdot \tilde{e}_i(\psi(x_p)) \cdot \psi(x_{p+1}) \cdots \psi(x_m) \\ &= \psi(x_1) \cdots \psi(x_{p-1}) \cdot \psi(\tilde{e}_i(x_p)) \cdot \psi(x_{p+1}) \cdots \psi(x_m) = \hat{\psi}(\tilde{e}_i(w)),\end{aligned}$$

because $\mathcal{B}$ and $\mathcal{M}$ are seminormal, ψ is a crystal homomorphism and $\psi(B) \subseteq M$. Analogously, we get that $\tilde{f}_i(\hat{\psi}(w)) = \tilde{f}_i(w)$ and $\tilde{f}_i(\hat{\psi}(w)) = \hat{\psi}(\tilde{f}_i(w))$. $\square$

The property described in the previous result can be used to give a definition of free crystal monoid up to isomorphism.

Corollary 5.11. *Let $\mathcal{B}$ be a seminormal crystal and $\mathcal{M}$ be a crystal monoid of the same type, where $B \subseteq M$. If for each crystal monoid $\mathcal{M}'$ and each crystal homomorphism $\psi : \mathcal{B} \rightarrow \mathcal{M}'$ satisfying $\psi(B) \subseteq M'$, there exists a unique crystal monoid homomorphism $\hat{\psi} : \mathcal{M} \rightarrow \mathcal{M}'$ such that $\hat{\psi}(x) = \psi(x)$, for all $x \in B$, then there exists a crystal monoid isomorphism between $\mathcal{M}$ and $\mathcal{B}^*$.*

Proof. Defined $\psi : B \rightarrow B^*$ by $\psi(x) = x$, for each $x \in B$. As the crystal structure maps of $\mathcal{B}^*$ agree with the crystal structure maps of $\mathcal{B}$ on words of length 1, we have that ψ is a crystal homomorphism from $\mathcal{B}$ to $\mathcal{B}^*$. Thus, there exists a unique crystal monoid homomorphism $\psi_1 : \mathcal{M} \rightarrow \mathcal{B}^*$ such that $\psi_1(x) = x$, for all $x \in B$. In particular, by restricting ψ_1 to B , we get that the crystal structure maps of $\mathcal{M}$ restricted to B agree with the crystal structure maps of $\mathcal{B}$, and so, the map $\psi' : B \rightarrow M$, given by $\psi'(x) = x$ for each $x \in B$, is a crystal homomorphism. By Theorem 5.10, there exists a unique crystal monoid homomorphism $\psi_2 : \mathcal{B}^* \rightarrow \mathcal{M}$ such that $\psi_2(x) = x$, for all $x \in B$. Then, $\psi_1 \circ \psi_2$ is a crystal monoid homomorphism from $\mathcal{B}^*$ to $\mathcal{B}^*$ such that $\psi_1 \psi_2(x) = x$, for any $x \in B$, and by Theorem 5.10, we obtain that $\psi_1 \circ \psi_2$ must be the identity map on B^* . Also, $\psi_2 \circ \psi_1$ is a crystal monoid homomorphism from $\mathcal{M}$ to $\mathcal{M}$ such that $\psi_2 \psi_1(x) = x$, for any $x \in B$, and by uniqueness, we get that $\psi_2 \circ \psi_1$ must be the identity map on M . Hence, ψ_1 is a crystal monoid isomorphism. $\square$

5.3 Congruences and quotients

In this section we introduce the notion of a congruence on a crystal monoid. This leads to the definition of a quotient crystal monoid. We then show how congruences relate with homomorphisms, and deduce a homomorphism theorem for crystal monoids.

Definition 5.12. Let $\mathcal{M}$ be a crystal monoid. A *crystal monoid congruence* on $\mathcal{M}$ is an equivalence relation $\theta \subseteq M \times M$ satisfying the following conditions:

- (i) if $(x, y) \in \theta$, then $\text{wt}(x) = \text{wt}(y)$, $\tilde{\varepsilon}_i(x) = \tilde{\varepsilon}_i(y)$ and $\tilde{\varphi}_i(x) = \tilde{\varphi}_i(y)$, for all $i \in I$;
- (ii) if $(x, y) \in \theta$ and $\tilde{e}_i(x) \in M$, then $(\tilde{e}_i(x), \tilde{e}_i(y)) \in \theta$;
- (iii) if $(x, y) \in \theta$ and $\tilde{f}_i(x) \in M$, then $(\tilde{f}_i(x), \tilde{f}_i(y)) \in \theta$;
- (iv) if $(x_1, y_1), (x_2, y_2) \in \theta$, then $(x_1 x_2, y_1 y_2) \in \theta$.

Let $\mathcal{M}$ be a crystal monoid. It is immediate from the definition that the equality relation $\Delta = \{(x, x) \mid x \in M\}$ is a crystal monoid congruence on $\mathcal{M}$. We have that $\Delta \subseteq \theta$, for any crystal monoid congruence θ on $\mathcal{M}$. Also, given a nonempty family Θ of crystal monoid congruences on $\mathcal{M}$, it is straightforward to show that $\bigcap \Theta$ is a crystal monoid congruence on $\mathcal{M}$. From this and the following result, we are able to show that the crystal monoid congruences on a crystal monoid form a lattice.

Lemma 5.13. Let $\mathcal{M}$ be a crystal monoid, and let $R \subseteq M \times M$ be a relation on M such that for any $(x, y) \in R$ and $i \in I$, the following conditions are satisfied:

- (i) $\text{wt}(x) = \text{wt}(y)$, $\tilde{\varepsilon}_i(x) = \tilde{\varepsilon}_i(y)$, and $\tilde{\varphi}_i(x) = \tilde{\varphi}_i(y)$;
- (ii) if $\tilde{e}_i(x) \in M$, then $(\tilde{e}_i(x), \tilde{e}_i(y)) \in R$; and
- (iii) if $\tilde{f}_i(x) \in M$, then $(\tilde{f}_i(x), \tilde{f}_i(y)) \in R$.

Then, the monoid congruence θ_R generated by R is a crystal monoid congruence on $\mathcal{M}$.

Proof. We check that in every step of constructing θ_R from R , properties (i) to (iii) are preserved. Set

$$R_1 = \{(ux, uy) \mid (x, y) \in R, u \in M\}.$$

For $(x, y) \in R$, $u \in M$ and $i \in I$, since $\text{wt}(x) = \text{wt}(y)$, $\tilde{\varepsilon}_i(x) = \tilde{\varepsilon}_i(y)$ and $\tilde{\varphi}_i(x) = \tilde{\varphi}_i(y)$, we get that $\text{wt}(ux) = \text{wt}(uy)$, $\tilde{\varepsilon}_i(ux) = \tilde{\varepsilon}_i(uy)$ and $\tilde{\varphi}_i(ux) = \tilde{\varphi}_i(uy)$, by (5.1) to (5.3). As $\mathcal{M}$ is seminormal, we have that $\tilde{e}_i(ux) \in M$ if and only if $\tilde{e}_i(uy) \in M$. And if so, by (5.4), we obtain if $\tilde{\varphi}_i(u) \geq \tilde{\varepsilon}_i(x)$ that $\tilde{e}_i(ux) = \tilde{e}_i(u) \cdot x$ and $\tilde{e}_i(uy) = \tilde{e}_i(u) \cdot y$ implying $(\tilde{e}_i(ux), \tilde{e}_i(uy)) \in R_1$, or otherwise, we obtain that $\tilde{e}_i(ux) = u \cdot \tilde{e}_i(x)$ and $\tilde{e}_i(uy) = u \cdot \tilde{e}_i(y)$ implying $(\tilde{e}_i(ux), \tilde{e}_i(uy)) \in R_1$, because $(\tilde{e}_i(x), \tilde{e}_i(y)) \in R$. Similarly, by (5.5), if $\tilde{f}_i(ux) \in M$, we have either $\tilde{f}_i(ux) = \tilde{f}_i(u) \cdot x$ and $\tilde{f}_i(uy) = \tilde{f}_i(u) \cdot y$, or $\tilde{f}_i(ux) = u \cdot \tilde{f}_i(x)$ and $\tilde{f}_i(uy) = u \cdot \tilde{f}_i(y)$, which results in $(\tilde{f}_i(ux), \tilde{f}_i(uy)) \in R_1$ in any case. Hence, R_1 satisfies conditions (i) to (iii).

Set

$$R_2 = \{(xv, yv) \mid (x, y) \in R_1, v \in M\}.$$

For $(x, y) \in R_1$, $v \in M$ and $i \in I$, we have that $\text{wt}(x) = \text{wt}(y)$, $\tilde{\varepsilon}_i(x) = \tilde{\varepsilon}_i(y)$ and $\tilde{\varphi}_i(x) = \tilde{\varphi}_i(y)$. As shown for R_1 , from (5.1) to (5.5) we can analogously deduce that R_2 satisfies conditions (i) to (iii).

It is immediate that the reflexive closure

$$R_3 = \{(x, y) \in M \times M \mid (x, y) \in R_2 \text{ or } x = y\}$$

of R_2 satisfies conditions (i) to (iii).

Set

$$R_4 = \{(x, y) \in M \times M \mid (x, y) \in R_3 \text{ or } (y, x) \in R_3\},$$

which corresponds to the symmetric closure of R_3 . Since R_3 satisfies condition (i), so it does R_4 . For $(x, y) \in R_3$, if $\tilde{\varepsilon}_i(y) \in M$, then $\tilde{\varepsilon}_i(x) \in M$, because $\tilde{\varepsilon}_i(x) = \tilde{\varepsilon}_i(y)$ and $\mathcal{M}$ is seminormal, and so, $(\tilde{\varepsilon}_i(x), \tilde{\varepsilon}_i(y)) \in R_3$. Similarly, if $\tilde{f}_i(y) \in M$, then $(\tilde{f}_i(x), \tilde{f}_i(y)) \in R_3$. Hence, R_4 satisfies conditions (ii) and (iii).

Finally, θ_R corresponds to the transitive closure of R_4 . For $(x, y) \in \theta_R$, there exist $x_0, x_1, \dots, x_m \in M$ such that $x = x_0$, $y = x_m$ and $(x_k, x_{k+1}) \in R_4$, for $k = 0, \dots, m-1$. For $i \in I$, since R_4 satisfies condition (i), we get that

$$\text{wt}(x) = \text{wt}(x_0) = \text{wt}(x_1) = \dots = \text{wt}(x_m) = \text{wt}(y),$$

and similarly, $\tilde{\varepsilon}_i(x) = \tilde{\varepsilon}_i(y)$ and $\tilde{\varphi}_i(x) = \tilde{\varphi}_i(y)$. If $\tilde{\varepsilon}_i(x) \in M$, then for $k = 0, \dots, m-1$, since $\tilde{\varepsilon}_i(x_k) = \tilde{\varepsilon}_i(x)$ and $\mathcal{M}$ is seminormal, we have that $\tilde{\varepsilon}_i(x_k) \in M$, and as R_4 satisfies condition (ii), $(\tilde{\varepsilon}_i(x_k), \tilde{\varepsilon}_i(x_{k+1})) \in R_4$. Hence, $(\tilde{\varepsilon}_i(x), \tilde{\varepsilon}_i(y)) = (\tilde{\varepsilon}_i(x_0), \tilde{\varepsilon}_i(x_m)) \in \theta_R$. Analogously, as R_4 satisfies condition (iii), if $\tilde{f}_i(x) \in M$, then $(\tilde{f}_i(x), \tilde{f}_i(y)) \in \theta_R$. Therefore, θ_R satisfies conditions (i) to (iii). Moreover, by construction θ_R is a monoid congruence on M , and so, θ_R is a crystal monoid congruence on $\mathcal{M}$. $\square$

Theorem 5.14. *Let $\mathcal{M}$ be a crystal monoid. Then, the crystal monoid congruences on $\mathcal{M}$ form a lattice with respect to the partial order $\subseteq$ of inclusion.*

Proof. Let Θ be the set of all crystal monoid congruences on $\mathcal{M}$. Since the equality relation Δ lies in Θ , we have that Θ is nonempty. Clearly, $\subseteq$ is a partial order on Θ . Let $\theta, \sigma \in \Theta$. Since $\theta \cap \sigma$ is a crystal monoid congruence and the largest set contained in θ and σ , we have that $\theta \cap \sigma$ is the infimum of θ and σ in Θ . Finally, the relation $R = \theta \cup \sigma$ satisfies conditions (i) to (iii) of Lemma 5.13, and so, the monoid congruence θ_R generated by R is a crystal monoid congruence on $\mathcal{M}$. Furthermore, θ_R is the smallest equivalence on M satisfying Definition 5.12(iv) and containing R . Hence, θ_R is the supremum of θ and σ in Θ . $\square$

Let θ be a crystal monoid congruence on a crystal monoid $\mathcal{M}$. From the conditions (i) to (iii) of Definition 5.12, it follows that the crystal structure maps wt ,

$\tilde{e}_i$, $\tilde{f}_i$, $\tilde{\varepsilon}_i$ and $\tilde{\varphi}_i$ ($i \in I$) of $\mathcal{M}$ can be defined on the set of all θ -equivalence classes M/θ in a natural way, because they do not depend on the equivalence class representative. If $x, y \in M$ are such that $[x] = [y]$, then $x \theta y$, and by Definition 5.12(i), $\text{wt}(x) = \text{wt}(y)$, $\tilde{\varepsilon}_i(x) = \tilde{\varepsilon}_i(y)$ and $\tilde{\varphi}_i(x) = \tilde{\varphi}_i(y)$. As $\mathcal{M}$ is seminormal and $\tilde{\varepsilon}_i(x) = \tilde{\varepsilon}_i(y)$, we have that $\tilde{e}_i$ is defined on x if and only if $\tilde{e}_i$ is defined on y . And if so, by Definition 5.12(ii), we have that $[\tilde{e}_i(x)] = [\tilde{e}_i(y)]$. Similarly, by Definition 5.12(iii), $[\tilde{f}_i(x)] = [\tilde{f}_i(y)]$, whenever $\tilde{f}_i$ is defined on x or y . Finally, if $x_1, x_2, y_1, y_2 \in M$ are such that $[x_1] = [y_1]$ and $[x_2] = [y_2]$, then $[x_1 x_2] = [y_1 y_2]$, by Definition 5.12(iv). Therefore, we obtain the following construction.

Definition 5.15. Let θ be a crystal monoid congruence on a crystal monoid $\mathcal{M}$. The *quotient crystal monoid* of $\mathcal{M}$ by θ is a crystal monoid $\mathcal{M}/\theta$ of the same type as $\mathcal{M}$ consisting of the set M/θ and maps given by

$$\begin{aligned} \text{wt}([x]) &= \text{wt}(x) & \tilde{e}_i([x]) &= [\tilde{e}_i(x)] & \tilde{f}_i([x]) &= [\tilde{f}_i(x)] \\ \tilde{\varepsilon}_i([x]) &= \tilde{\varepsilon}_i(x) & \tilde{\varphi}_i([x]) &= \tilde{\varphi}_i(x) & [x] \cdot [y] &= [x \cdot y], \end{aligned}$$

where $[\perp] = \perp$, for $x, y \in M$ and $i \in I$.

The following result follows directly from the previous definition.

Lemma 5.16. Let θ be a congruence on a crystal monoid $\mathcal{M}$. Then, the map $\pi : M \rightarrow M/\theta$, given by $\pi(x) = [x]$ for each $x \in M$, is a surjective crystal monoid homomorphism from $\mathcal{M}$ to $\mathcal{M}/\theta$.

This leads to the following relation between congruences and homomorphisms of crystal monoids.

Theorem 5.17. Let $\mathcal{M}$ be a crystal monoid, and let $\theta \subseteq M \times M$. Then, θ is a crystal monoid congruence on $\mathcal{M}$ if and only if there exist a crystal monoid $\mathcal{M}'$ and a crystal monoid homomorphism $\psi : \mathcal{M} \rightarrow \mathcal{M}'$ such that $\theta = \ker \psi$.

Proof. If θ is a crystal monoid congruence on $\mathcal{M}$, set $\pi : \mathcal{M} \rightarrow \mathcal{M}/\theta$ to be the crystal monoid homomorphism defined in Lemma 5.16. Then, for $x, y \in M$, we have that $\pi(x) = \pi(y)$ if and only if $x \theta y$, and so, $\theta = \ker \pi$.

Conversely, let $\psi : \mathcal{M} \rightarrow \mathcal{M}'$ be a crystal monoid homomorphism. Clearly $\ker \psi$ is an equivalence on M . Let $(x, y) \in \ker \psi$ and $i \in I$. As $\psi(x) = \psi(y)$, we have that

$$\text{wt}(x) = \text{wt}(\psi(x)) = \text{wt}(\psi(y)) = \text{wt}(y),$$

and similarly, $\tilde{\varepsilon}_i(x) = \tilde{\varepsilon}_i(y)$ and $\tilde{\varphi}_i(x) = \tilde{\varphi}_i(y)$. If $\tilde{e}_i(x) \in M$, then $\tilde{e}_i(y) \in M$, because $\mathcal{M}$ is seminormal and $\tilde{\varepsilon}_i(y) = \tilde{\varepsilon}_i(x)$. Also, as $\psi(M) \subseteq M'$, we have that

$\psi(\tilde{e}_i(x)), \psi(\tilde{e}_i(y)) \in M'$, and as ψ is a crystal monoid homomorphism,

$$\psi(\tilde{e}_i(x)) = \tilde{e}_i(\psi(x)) = \tilde{e}_i(\psi(y)) = \psi(\tilde{e}_i(y)),$$

which implies that $(\tilde{e}_i(x), \tilde{e}_i(y)) \in \ker \psi$. Analogously, $(\tilde{f}_i(x), \tilde{f}_i(y)) \in \ker \psi$, whenever $\tilde{f}_i(x) \in M$. Finally, for $(x_1, y_1), (x_2, y_2) \in \ker \psi$, we have that

$$\psi(x_1 x_2) = \psi(x_1) \cdot \psi(x_2) = \psi(y_1) \cdot \psi(y_2) = \psi(y_1 y_2),$$

and thus, $(x_1 x_2, y_1 y_2) \in \ker \psi$. Hence, $\ker \psi$ is a crystal monoid congruence on $\mathcal{M}$. $\square$

Now, we introduce the homomorphism theorems for crystal monoids.

Theorem 5.18. *Let $\mathcal{M}$ and $\mathcal{M}'$ be crystal monoids of the same type, and let $\psi : \mathcal{M} \rightarrow \mathcal{M}'$ be a crystal monoid homomorphism. Then, for each crystal monoid congruence θ on $\mathcal{M}$ satisfying $\theta \subseteq \ker \psi$, there exists a unique crystal monoid homomorphism $\hat{\psi} : \mathcal{M}/\theta \rightarrow \mathcal{M}'$ such that $\hat{\psi}([x]_\theta) = \psi(x)$, for any $x \in M$.*

Furthermore, if ψ is surjective, then there exists a unique crystal monoid isomorphism $\hat{\psi} : \mathcal{M}/\ker \psi \rightarrow \mathcal{M}'$ such that $\hat{\psi}([x]) = \psi(x)$, for all $x \in M$.

Proof. Let θ be a crystal monoid congruence on $\mathcal{M}$ such that $\theta \subseteq \ker \psi$. For $x, y \in M$, if $[x]_\theta = [y]_\theta$, then $x \theta y$, which implies that $\psi(x) = \psi(y)$, because $\theta \subseteq \ker \psi$. Thus, we can define a map $\hat{\psi} : M/\theta \rightarrow M'$ by $\hat{\psi}([x]_\theta) = \psi(x)$, for each $x \in M$. Since ψ is a crystal monoid homomorphism, it is immediate from Definition 5.15 that $\hat{\psi}$ is a crystal monoid homomorphism from $\mathcal{M}/\theta$ to $\mathcal{M}'$.

Assume that $\theta = \ker \psi$ and that ψ is surjective. Then, given $x' \in M'$, there exists $x \in M$ such that $\psi(x) = x'$, and so, $\hat{\psi}([x]) = x'$. Also, if $y, z \in M$ are such that $\hat{\psi}([y]) = \hat{\psi}([z])$, then $\psi(y) = \psi(z)$, or equivalently, $(y, z) \in \ker \psi$, which implies that $[y] = [z]$. Hence, $\hat{\psi}$ is bijective, and therefore, a crystal monoid isomorphism. $\square$

Theorem 5.19. *Let θ and σ be crystal monoid congruences on a crystal monoid $\mathcal{M}$ such that $\theta \subseteq \sigma$. Define a relation σ/θ on M/θ by*

$$\sigma/\theta = \{([x]_\theta, [y]_\theta) \mid (x, y) \in \sigma\}.$$

Then, σ/θ is a crystal monoid congruence on $\mathcal{M}/\theta$. Moreover, the map $(M/\theta)/(\sigma/\theta) \rightarrow M/\sigma$, given by $[x]_\theta]_{\sigma/\theta} \mapsto [x]_\sigma$ for each $x \in M$, is a crystal monoid isomorphism between $(\mathcal{M}/\theta)/(\sigma/\theta)$ and $\mathcal{M}/\sigma$.

Proof. Since $\theta \subseteq \sigma$, then for $x, y \in M$, if $[x]_\theta = [y]_\theta$, then $[x]_\sigma = [y]_\sigma$. Thus, we can define a map $\psi : M/\theta \rightarrow M/\sigma$ by $\psi([x]_\theta) = [x]_\sigma$, for each $x \in M$. By Definition 5.15,

it is immediate that ψ is a surjective crystal monoid homomorphism from $\mathcal{M}/\theta$ to $\mathcal{M}/\sigma$. For $x, y \in M$, we have that $\psi([x]_\theta) = \psi([y]_\theta)$ if and only if $(x, y) \in \sigma$. Hence, $\ker \psi = \theta/\sigma$. By Theorem 5.18, we obtain that the map $(M/\theta)/(\sigma/\theta) \rightarrow M/\sigma$, given by $[[x]_\theta]_{\sigma/\theta} \mapsto [x]_\sigma$ for each $x \in M$, is a crystal monoid isomorphism between $(\mathcal{M}/\theta)/(\sigma/\theta)$ and $\mathcal{M}/\sigma$. $\square$

5.4 The plactic congruence

In this section we define the plactic monoid associated to a seminormal crystal. This definition is based on the framework introduced on the previous sections, as the plactic monoid will be obtained as a quotient crystal monoid of the free crystal monoid by the following congruence.

Definition 5.20. Let $\mathcal{B}$ be a seminormal crystal. The *plactic congruence* on $\mathcal{B}^*$ is a relation $\approx$ on B^* given as follows. For $u, v \in B^*$, $u \approx v$ if and only if there exists a crystal isomorphism $\psi : \mathcal{B}^*(u) \rightarrow \mathcal{B}^*(v)$ such that $\psi(u) = v$.

To prove that $\approx$ is a crystal monoid congruence on $\mathcal{B}^*$ we first show the following result.

Lemma 5.21. Let $\mathcal{B}$ be a seminormal crystal, and let $u, v \in B^*$. Then, the map from $(B^*(u) \otimes B^*(v))(u \otimes v)$ to $B^*(uv)$, given by $x \otimes y \mapsto xy$, is a crystal isomorphism between $(\mathcal{B}^*(u) \otimes \mathcal{B}^*(v))(u \otimes v)$ and $\mathcal{B}^*(uv)$.

Proof. Since $\mathcal{B}^*$ is a crystal monoid, the map $B^* \otimes B^* \rightarrow B^*$, defined by $x \otimes y \mapsto xy$ for each $x, y \in B^*$, is a crystal homomorphism (Definition 5.1(iii)). Then, by (5.4) and (5.5), we have a surjective crystal homomorphism $\psi : (\mathcal{B}^* \otimes \mathcal{B}^*)(u \otimes v) \rightarrow \mathcal{B}^*(uv)$ given by $\psi(x \otimes y) = xy$ for each $x \otimes y \in (B^* \otimes B^*)(u \otimes v)$.

By Definition 4.10, if $x \otimes y \in (B^* \otimes B^*)(u \otimes v)$, then there exist $g_1, \dots, g_m \in \{\tilde{e}_i, \tilde{f}_i \mid i \in I\}$ such that $x \otimes y = g_1 \cdots g_m(u \otimes v)$, and by Definition 4.13, $x = g'_1 \cdots g'_k(u)$ and $y = g''_1 \cdots g''_l(v)$ for some $g'_1, \dots, g'_k, g''_1, \dots, g''_l \in \{g_1, \dots, g_m\}$. Hence, $x \in B^*(u)$ and $y \in B^*(v)$. And therefore, $(B^* \otimes B^*)(u \otimes v) \subseteq B^*(u) \otimes B^*(v)$. This implies that $(B^* \otimes B^*)(u \otimes v) = (B^*(u) \otimes B^*(v))(u \otimes v)$, and so, $(\mathcal{B}^* \otimes \mathcal{B}^*)(u \otimes v)$ and $(\mathcal{B}^*(u) \otimes \mathcal{B}^*(v))(u \otimes v)$ correspond to the same crystal. In particular, ψ is a surjective crystal homomorphism from $(\mathcal{B}^*(u) \otimes \mathcal{B}^*(v))(u \otimes v)$ to $\mathcal{B}^*(uv)$.

Finally, if $x_1 \otimes y_1, x_2 \otimes y_2 \in (B^*(u) \otimes B^*(v))(u \otimes v)$ are such that $x_1 y_1 = x_2 y_2$, then $|x_1| = |x_2|$ and $|y_1| = |y_2|$, by Proposition 5.8, which implies that $x_1 = x_2$ and $y_1 = y_2$. Hence, $x_1 \otimes y_1 = x_2 \otimes y_2$, and therefore, ψ is injective. By Theorem 4.8, ψ is a crystal isomorphism between $(\mathcal{B}^*(u) \otimes \mathcal{B}^*(v))(u \otimes v)$ and $\mathcal{B}^*(uv)$. $\square$

Theorem 5.22. *Let $\mathcal{B}$ be a seminormal crystal. Then, the plactic congruence $\approx$ on $\mathcal{B}^*$ is a crystal monoid congruence on $\mathcal{B}^*$.*

Proof. Let $u, v, u', v' \in B^*$ with $u \approx v$ and $u' \approx v'$, and so, let $\psi : \mathcal{B}^*(u) \rightarrow \mathcal{B}^*(v)$ and $\psi' : \mathcal{B}^*(u') \rightarrow \mathcal{B}^*(v')$ be crystal isomorphisms such that $\psi(u) = v$ and $\psi'(u') = v'$. We have that $u \approx u$ as the identity map on $B^*(u)$ is a crystal isomorphism on $\mathcal{B}^*(u)$. We also have that $v \approx u$ since ψ^{-1} is a crystal isomorphism between $\mathcal{B}^*(v)$ and $\mathcal{B}^*(u)$ satisfying $\psi^{-1}(v) = u$. And finally, if $v = u'$, we have that $u \approx v'$ because $\psi' \circ \psi$ is a crystal isomorphism between $\mathcal{B}^*(u)$ and $\mathcal{B}^*(v')$ that satisfies $\psi'(\psi(u)) = v'$. Hence, $\approx$ is an equivalence on B^* .

Let $i \in I$. Since ψ is a crystal isomorphism, we get that

$$\text{wt}(u) = \text{wt}(\psi(u)) = \text{wt}(v),$$

and similarly, $\tilde{\varepsilon}_i(u) = \tilde{\varepsilon}_i(v)$ and $\tilde{\varphi}_i(u) = \tilde{\varphi}_i(v)$. If $\tilde{e}_i(u) \in B^*$, then

$$\psi(\tilde{e}_i(u)) = \tilde{e}_i(\psi(u)) = \tilde{e}_i(v),$$

and since $B^*(\tilde{e}_i(u)) = B^*(u)$ and $B^*(\tilde{e}_i(v)) = B^*(v)$, we obtain $\tilde{e}_i(u) \approx \tilde{e}_i(v)$. Analogously, if $\tilde{f}_i(u) \in B^*$, then $\tilde{f}_i(u) \approx \tilde{f}_i(v)$.

By Lemma 5.21, we can take crystal isomorphisms $\psi_1 : \mathcal{B}^*(uv) \rightarrow (\mathcal{B}^*(u) \otimes \mathcal{B}^*(v))(u \otimes v)$ and $\psi_2 : (\mathcal{B}^*(u') \otimes \mathcal{B}^*(v'))(u' \otimes v') \rightarrow \mathcal{B}^*(u'v')$ such that $\psi_1(uv) = u \otimes v$ and $\psi_2(u' \otimes v') = u'v'$. By Proposition 4.18, the crystal isomorphism $\psi \otimes \psi'$ between $\mathcal{B}^*(u) \otimes \mathcal{B}^*(v)$ and $\mathcal{B}^*(u') \otimes \mathcal{B}^*(v')$ satisfies $(\psi \otimes \psi')(u \otimes v) = u' \otimes v'$. Thus, the restriction ψ_3 of $\psi \otimes \psi'$ to $(B^*(u) \otimes B^*(v))(u \otimes v)$ is a crystal isomorphism between $(\mathcal{B}^*(u) \otimes \mathcal{B}^*(v))(u \otimes v)$ and $(\mathcal{B}^*(u') \otimes \mathcal{B}^*(v'))(u' \otimes v')$. Then, $\psi_2 \circ \psi_3 \circ \psi_1$ is a crystal isomorphism between $\mathcal{B}^*(uv)$ and $\mathcal{B}^*(u'v')$ that satisfies

$$\psi_2 \psi_3 \psi_1(uv) = \psi_2 \psi_3(u \otimes v) = \psi_2(u' \otimes v') = u'v',$$

and so, $uv \approx u'v'$. Therefore, $\approx$ is a crystal monoid congruence on $\mathcal{B}^*$. $\square$

We have set the framework to present the following definition.

Definition 5.23. Let $\mathcal{B}$ be a seminormal crystal, and let $\approx$ be the plactic congruence on $\mathcal{B}^*$. The quotient crystal monoid $\mathcal{B}^*/\approx$ is called the *plactic crystal monoid*, or simply the *plactic monoid*, associated to $\mathcal{B}$, and is denoted by $\text{plac}(\mathcal{B})$.

Note that although $\text{plac}(\mathcal{B})$ is a crystal monoid, we are interested in studying its properties as a monoid, and thus, we just refer it as the plactic monoid associated to $\mathcal{B}$. However, we will be constantly considering its crystal structure, as it plays a fundamental role in the construction of $\text{plac}(\mathcal{B})$.

This terminology will make more sense in the following section, where we see how the classical plactic monoid can be accommodated as the plactic monoid associated to some crystal.

5.5 The plactic monoid of type $\mathfrak{A}_n$

In this section we show that the classical plactic monoid corresponds to the plactic monoid associated to the standard crystal $\mathcal{A}_n$ of type $\mathfrak{A}_n$. This follows from the work in [Kas90; Kas91; KN94], and a detailed construction is done in [HK02, Chapter 7]. Thus, we will omit the proofs, and just present the connection between isomorphic components of $\mathcal{A}_n^*$ and semi-standard Young tableaux over A_n , from which follows that plac_n and $\text{plac}(\mathcal{A}_n)$ correspond to the same monoid.

Let $n \geq 2$. Consider the free crystal monoid $\mathcal{A}_n^*$ over the standard crystal $\mathcal{A}_n$ of type $\mathfrak{A}_n$. Denote by Y_{A_n} the set of all semi-standard Young tableaux over A_n (see Section 3.1). The crystal structure of $\mathcal{A}_n^*$ induces a crystal structure on Y_{A_n} through the reading of semi-standard Young tableaux, resulting in a crystal $\mathcal{Y}_{\mathcal{A}_n}$ of type $\mathfrak{A}_n$. For a semi-standard Young tableau T over A_n and for $i \in \{1, \dots, n-1\}$, set

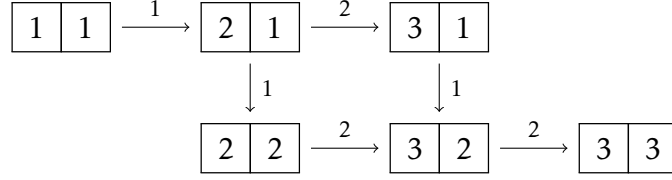
$$\text{wt}(T) = \text{wt}(w(T)), \quad \tilde{e}_i(T) = \tilde{e}_i(w(T)), \quad \tilde{\varphi}_i(T) = \tilde{\varphi}_i(w(T)).$$

If $\tilde{e}_i(w(T)) = \perp$, then set $\tilde{e}_i(T) = \perp$. Otherwise, by Proposition 5.2, $\tilde{e}_i(w(T))$ is obtained from $w(T)$ by changing an $i+1$ by an i , set $\tilde{e}_i(T)$ to be obtained from T by replacing the box $\boxed{i+1}$ which originated the changed $i+1$ by $\boxed{i}$. Equivalently, $\tilde{e}_i(T)$ is the unique semi-standard Young tableau over A_n with the same shape as T that satisfies $w(\tilde{e}_i(T)) = \tilde{e}_i(w(T))$. Similarly, if $\tilde{f}_i(w(T)) = \perp$, set $\tilde{f}_i(T) = \perp$. Otherwise, $\tilde{f}_i(w(T))$ is obtained from $w(T)$ by changing an i by an $i+1$, and set $\tilde{f}_i(T)$ by replacing the box $\boxed{i}$ which originates the i changed by $\boxed{i+1}$. Again, we have that $\tilde{f}_i(T)$ is the unique semi-standard Young tableau over A_n with the same shape as T such that $w(\tilde{f}_i(T)) = \tilde{f}_i(w(T))$.

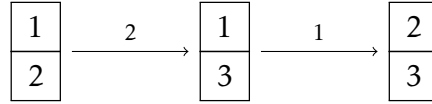
Note that the Kashiwara operators do not change the shape of the tableau. Also, given a partition λ , and a semi-standard Young tableau T of shape λ over A_n , we can apply the rising Kashiwara operators $\tilde{e}_1, \dots, \tilde{e}_{n-1}$ successively on T , and obtain a semi-standard Young tableau of shape λ whose i -th row is filled with i , for $i = 1, \dots, n$. This implies the following result.

Proposition 5.24. *Let T and T' be semi-standard Young tableaux over A_n . Then, T and T' lie in the same connected component of $\mathcal{Y}_{\mathcal{A}_n}$ if and only if T and T' have the same shape.*

Example 5.25. In $\mathcal{Y}_{\mathcal{A}_3}$, The semi-standard Young tableaux of shape (2) over A_3 form the following connected component.



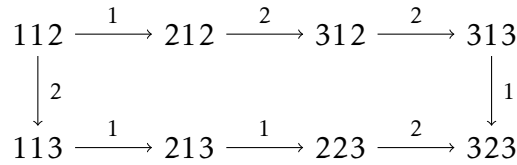
The semi-standard Young tableaux of shape (1,1) over A_3 form the following connected component.



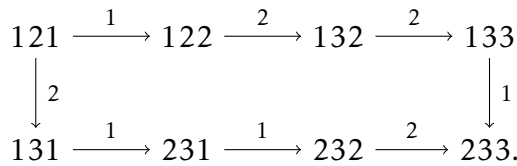
Notice that the connected components of the previous example are crystal isomorphic to the components formed by words of length 2 in $\mathcal{A}_3^*$ described in Example 5.9(i). In fact, the crystal $\mathcal{Y}_{\mathcal{A}_n}$ was defined so that the reading w is an injective crystal homomorphism from $\mathcal{Y}_{\mathcal{A}_n}$ to $\mathcal{A}_n^*$. And therefore, the restriction of the reading w to a connected component of $\mathcal{Y}_{\mathcal{A}_n}$ induces a crystal isomorphism.

The map P from A_n^* to Y_{A_n} induced by the insertion algorithm in Algorithm 3.2 is also a crystal homomorphism from $\mathcal{A}_n^*$ to $\mathcal{Y}_{\mathcal{A}_n}$.

Example 5.26. Consider the connected component $\mathcal{A}_3^*(112)$ containing 112 of $\mathcal{A}_3^*$

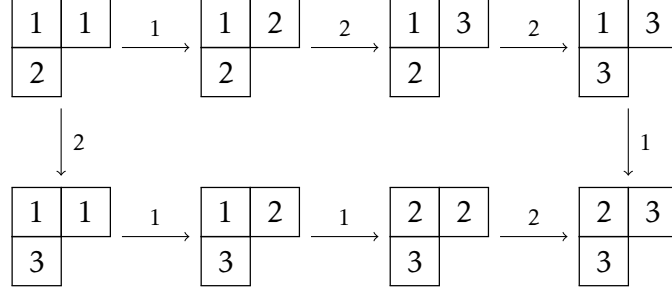


and consider the connected component $\mathcal{A}_3^*(121)$ containing 121 of $\mathcal{A}_3^*$



Clearly, $\mathcal{A}_3^*(112)$ and $\mathcal{A}_3^*(121)$ are crystal isomorphic. If we apply P to these connected components, we get the same connected component of $\mathcal{Y}_{\mathcal{A}_3}$, which is the

following.



The previous example is a consequence of the following result.

Theorem 5.27. *Let $\approx$ be the plactic congruence on $\mathcal{A}_n^*$. For any $u, v \in A_n^*$, we have that*

$$u \approx v \iff P(u) = P(v).$$

And therefore, $\approx = \sim_{\text{plac}}$.

This result justifies why we gave the same name to $\approx$ and $\sim_{\text{plac}}$. We also have that the plactic monoid $\text{plac}(\mathcal{A}_n)$ associated to the standard crystal $\mathcal{A}_n$ of type $\mathfrak{A}_n$ corresponds to the classical plactic monoid plac_n of rank n .

This process of obtaining the classical plactic monoid from crystals is referred as *crystalizing the plactic monoid*. Since Definition 5.23 only requires a seminormal crystal, the notion of plactic monoid is generalized. From the work of Kashiwara and Nakashima [KN94], notions of admissible semi-standard Young tableaux emerged, later called Kashiwara–Nakashima tableaux, which were then used by Lecouvey [Lec02; Lec03; Lec07] to give a description of the plactic monoids associated to the standard crystals of Cartan types $\mathcal{B}_n$, $\mathcal{C}_n$ and $\mathcal{D}_n$ analogous to the construction in this section for the plactic monoid associated to $\mathcal{A}_n$.

THE CLASSICAL HYPOPLACTIC MONOID

The classical hypoplactic monoid was first introduced by Krob and Thibon [KT97] from representation theoretical interpretations of quasi-symmetric functions and noncommutative symmetric functions. It emerged from a noncommutative realization of quasi-symmetric functions analogous to the realization of symmetric functions by the plactic monoid presented by Lascoux and Schützenberger [LS81]. In-depth study of the hypoplactic monoid was done by Novelli [Nov00]. Cain and Malheiro [CM17] presented an alternative description of the classical hypoplactic monoid analogous to the construction of the classical plactic monoid by crystals.

In this chapter, we introduce three constructions of the classical hypoplactic monoid. In Sections 6.1 and 6.2, we give a definition of the classical hypoplactic monoid based on quasi-ribbon tableaux and an insertion algorithm. In Section 6.3, we introduce a presentation for the classical hypoplactic monoid by the Knuth relations and the quartic relations. Finally, in Section 6.4, we describe the approach in [CM17], where the classical hypoplactic monoid is obtained from the crystal graph of the free crystal monoid $\mathcal{A}_n^*$, and show in Remark 6.17 why it cannot be directly used for other crystal monoids.

In Chapter 8, the classical hypoplactic monoid will be placed in context as the hypoplactic monoid associated to the Cartan type $\mathfrak{A}_n$. This justifies the term classical, because we will have various types of hypoplactic monoids. Due to this fact, we will be using notation that will make more sense then, and may be confusing for now. Also, as occurs in the construction of the classical plactic monoid by crystals, the hypoplactic monoid obtained from quasi-crystals is anti-isomorphic to the original one [KT97], and therefore, some of the concepts introduced in this chapter are adaptations of the original ones.

6.1 Quasi-ribbon tableaux

In this section we introduce the notion of quasi-ribbon tableaux over a totally ordered set X , and define a method of reading that maps a quasi-ribbon tableau to a word in X^* .

A *composition* $\mu = (\mu_1, \mu_2, \dots, \mu_m)$, where $m \geq 0$, is a sequence of positive integers $\mu_1, \mu_2, \dots, \mu_m > 0$. A *ribbon diagram* of shape μ is a collection of boxes such that the i -th row has μ_i boxes, for $i = 1, 2, \dots, m$, and the right-most box of the j -th row stands over the left-most box of the $(j + 1)$ -th row, for $j = 1, \dots, m - 1$. For example, we have that

(6.1)

are ribbon diagrams of shape $(1, 1)$, $(2, 1, 1)$ and $(2, 3, 1)$, respectively.

Given a totally ordered set X and a composition μ . A *quasi-ribbon tableau* of shape μ over X corresponds to a ribbon diagram of shape μ where boxes are filled with elements of X such that each column is strictly increasing from top to bottom, and rows are weakly increasing from left to right. For example, the ribbon diagrams in (6.1) filled as follows

(6.2)

are quasi-ribbon tableaux over A_6 .

Notice that columns of quasi-ribbon tableaux are strictly increasing from top to bottom, which differs from ribbon tableaux where columns are strictly decreasing from top to bottom. On the other hand, the order in rows and columns of quasi-ribbon tableaux coincides with the order in rows and columns of semi-standard Young tableaux.

A *recording quasi-ribbon* of shape μ corresponds to a ribbon diagram of shape μ where boxes are filled with elements of $\{1, 2, \dots, \mu_1 + \dots + \mu_m\}$ such that each element occurs exactly once, each column is increasing from top to bottom, and each row is decreasing from left to right. For example, the ribbon diagrams in (6.1) filled as follows

(6.3)

are recording quasi-ribbons.

In general, a recording quasi-ribbon is not a quasi-ribbon tableau, as rows in a recording ribbon are decreasing from left to right, while rows in a quasi-ribbon tableau are weakly increasing. Thus, the only recording quasi-ribbons that are quasi-ribbon tableaux are columns. Also, recording quasi-ribbons differ from recording ribbons in the order taken for rows and columns, because in a recording ribbon rows are increasing from left to right, and columns are decreasing from top to bottom.

Note that a composition can be of the form $\mu = ()$. In this case a quasi-ribbon tableau or a recording quasi-ribbon of shape μ is called *empty* as there are no boxes to fill.

Given a quasi-ribbon tableau T over a totally ordered set X , the *reading* of T is a word $w(T)$ in X^* defined as for semi-standard Young tableaux (Section 3.1). Thus, $w(T)$ is obtained by concatenating the elements in T starting in the right-most column and ending in the left-most, where the elements in each column are concatenated from top to bottom. For example, the readings of the quasi-ribbon tableaux in (6.2) are 12, 3461 and 353121, respectively.

We choose this method of reading a quasi-ribbon tableau, because it agrees with the far-eastern reading of semi-standard Young tableaux used for constructing the plactic monoid in Chapter 3. Since we make a generalization of the theory of crystals by quasi-crystals in the subsequent chapters, we also need a reading compatible with the original definition of tensor product of crystals. However, this notion of reading a quasi-ribbon tableau results in a reversed word when compared with the word obtained by the method of reading commonly found in the literature [CM17; KT97]. This choice has impacts on Algorithms 6.1 and 6.2, and ultimately, in the resulting hypoplactic monoid which is anti-isomorphic to the original one [KT97; Nov00].

6.2 The insertion algorithm

In this section, we present an analogue of the Schensted insertion algorithm for inserting a new element into a quasi-ribbon tableau, and analogues of the P-symbol and the Q-symbol. This leads to a bijection similar to the Robinson–Schensted–Knuth correspondence for quasi-ribbon tableaux with respect to the far-eastern reading, and a construction of the hypoplactic monoid analogous to the construction of the plactic monoid in Section 3.2.

The following algorithm computes a quasi-ribbon tableau by adding a new

element to a quasi-ribbon tableau. It is an adaptation of the algorithms given in [KT97, §7.2], [Nov00, Algorithm 4.4] and [CM17, Algorithm 4.2], which makes them compatible with the far-eastern reading and the notation introduced in Algorithm 3.1.

Algorithm 6.1. *Input:* A quasi-ribbon tableau T over a totally ordered set X , and a letter $x \in X$.

Output: A quasi-ribbon tableau $x \rightarrow T$.

Procedure:

- (1) If T is empty, then $x \rightarrow T := \boxed{x}$.
- (2) If x is smaller than or equal to the top-left-most element in T , then set $x \rightarrow T$ to be the quasi-ribbon tableau obtained by attaching $\boxed{x}$ to the left of the first row of T .

$$x \rightarrow T := \boxed{x} \quad \boxed{[\quad T \quad]}$$

- (3) If x is greater than the bottom-right-most element in T , then set $x \rightarrow T$ to be the quasi-ribbon tableau obtained by attaching $\boxed{x}$ to the bottom of the bottom-right-most box of T .

$$x \rightarrow T := \begin{array}{|c|c|c|} \hline [& T &] \\ \hline & & x \\ \hline \end{array}$$

- (4) Otherwise, set y and z be adjacent elements in T such that $y < x \leq z$, or equivalently, y is the bottom-right-most element in T smaller than x , and z is the top-left-most element in T greater than or equal to x . Break T between y and z . Then, set $x \rightarrow T$ to be the quasi-ribbon tableau obtained by attaching $\boxed{x}$ to the bottom of $\boxed{y}$ and to the left of $\boxed{z}$.

$$x \rightarrow T := \begin{array}{|c|c|c|} \hline [& \cdots &] \\ \hline & & y \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|} \hline x & z & [& \cdots &] \\ \hline \end{array}$$

In the previous algorithm note that x is always attach on the left of an element greater than or equal to it and/or below an element smaller than it. Any other adjacent elements in $x \rightarrow T$ are already adjacent elements in T . Therefore, $x \rightarrow T$ is indeed a quasi-ribbon tableau.

For example, we have that

$$1 \rightarrow \begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline & 4 \\ \hline \end{array}$$

$$\begin{array}{c}
 5 \rightarrow \begin{array}{|c|c|c|c|} \hline 1 & 1 & & \\ \hline & 2 & 3 & 3 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & & \\ \hline & 2 & 3 & 3 \\ \hline & & & 5 \\ \hline \end{array} \\
 3 \rightarrow \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 1 & 2 & \\ \hline & 3 & 4 \\ \hline \end{array}
 \end{array}$$

and

$$2 \rightarrow \begin{array}{|c|c|c|} \hline 1 & 1 & \\ \hline & 3 & 5 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & & \\ \hline & 2 & 3 & 5 \\ \hline \end{array}.$$

Now, we present an algorithm to compute analogues of the P-symbol and the Q-symbol as in Algorithm 3.2. This algorithm is an adaptation of the algorithms in [KT97, §7.2] and [CM17, Algorithm 4.3].

Algorithm 6.2. *Input:* A word $w \in X^*$.

Output: A quasi-ribbon tableau $QR(w)$ and a recording quasi-ribbon $RR(w)$.

Procedure:

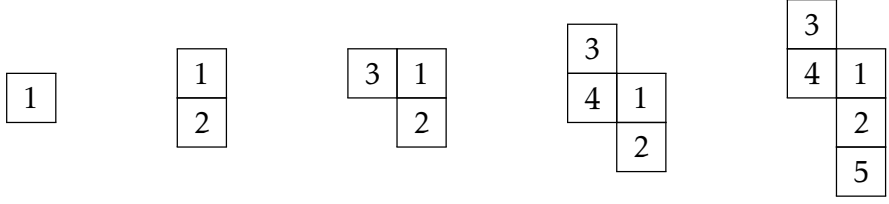
- (1) Set P_0 to be the empty quasi-ribbon tableau and set Q_0 to be the empty recording quasi-ribbon.
- (2) If $w = \epsilon$, then $QR(\epsilon) := P_0$ and $RR(w) := Q_0$.
- (3) Else, let $x_1, \dots, x_m \in X$ such that $w = x_1 \dots x_m$. For each $i = 1, \dots, m$, do
 - (i) Set $P_i := x_i \rightarrow P_{i-1}$;
 - (ii) Set Q_i by Adding a box $\boxed{i}$ filled with i to Q_{i-1} such that Q_i has the same shape as T_i and $\boxed{i}$ occurs in the same position as $\boxed{x_i}$.
- (4) Set $QR(w) := P_m$ and $RR(w) := Q_m$.

In the previous algorithm it is clear that $QR(w)$ is a quasi-ribbon tableau, as it uses Algorithm 6.1 in each step. Also, in step i we have that $\boxed{i}$ is always attached to the left of some row and to the bottom of some column, and recursively, we get that rows in Q_i are decreasing from left to right, and columns are increasing from top to bottom. Therefore, $RR(w)$ is a recording quasi-ribbon.

If $w = 24125$, the quasi-ribbon tableaux $P_1, \dots, P_5$ are respectively

$$\begin{array}{|c|} \hline 2 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 2 \\ \hline & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 2 \\ \hline & 4 \\ \hline & 5 \\ \hline \end{array},$$

and the recording ribbons $Q_1, \dots, q_5$ are respectively



Hence,

$$\text{QR}(24125) = \begin{array}{|c|c|} \hline 1 & \\ \hline 2 & 2 \\ \hline & 4 \\ \hline & 5 \\ \hline \end{array} \quad \text{and} \quad \text{RR}(24125) = \begin{array}{|c|c|} \hline 3 & \\ \hline 4 & 1 \\ \hline & 2 \\ \hline & 5 \\ \hline \end{array}. \quad (6.4)$$

Let $w = x_1 \dots x_m$ with $x_1, \dots, x_m \in X$. In each step of Algorithm 6.2 we have that each recording quasi-ribbon Q_i is obtained by attaching $\boxed{i}$ to Q_{i-1} exactly as P_i was constructed by attaching $\boxed{x_i}$ to P_{i-1} . Thus, for each $i \in \{1, \dots, m\}$, we have that x_i occurs in $\text{QR}(w)$ in the same position as i occurs in $\text{RR}(w)$. And therefore, if $w' \in X^*$ is such that $\text{QR}(w') = \text{QR}(w)$ and $\text{RR}(w') = \text{RR}(w)$, then $w' = w$.

Conversely, given a quasi-ribbon tableau P and a recording ribbon Q with the same shape, consisting of a total of m boxes, we can obtain a word $x_1 \dots x_m$ where x_i occurs in P exactly in the same position as i occurs in Q , $i = 1, \dots, m$. Then, we have described a bijection analogous to the one in [KT97, §7.2], which is similar to the Robinson–Schensted–Knuth correspondence.

Theorem 6.3. *Let X be a totally ordered set. The map that sends each word $w \in X^*$ to $(\text{QR}(w), \text{RR}(w))$ is an one-to-one correspondence between the set of all words over X and the set of ordered pairs (P, Q) where P is a quasi-ribbon tableau and Q is a recording quasi-ribbon of the same shape over X .*

We now compare Algorithms 6.1 and 6.2 to the methods introduced in [KT97, §7.2], based on the following algorithms from [CM17].

Algorithm 6.4 ([CM17, Algorithm 4.2]). *Input:* A quasi-ribbon tableau T over a totally ordered set X , and a letter $x \in X$.

Output: A quasi-ribbon tableau $T \leftarrow x$.

Procedure:

- (1) If T is empty, then $T \leftarrow x := \boxed{x}$.

- (2) If x is smaller than all elements in T , then set $T \leftarrow x$ to be the quasi-ribbon tableau obtained by attaching $\boxed{x}$ to the top of the top-left-most box in T .

$$T \leftarrow x := \begin{array}{|c|c|c|} \hline x & & \\ \hline \boxed{} & T & \boxed{} \\ \hline \end{array}$$

- (3) If x is greater than or equal to all elements in T , then set $T \leftarrow x$ to be the quasi-ribbon tableau obtained by attaching $\boxed{x}$ to the right of the bottom row of T .

$$T \leftarrow x := \begin{array}{|c|c|c|c|} \hline \boxed{} & T & \boxed{} & x \\ \hline \end{array}$$

- (4) Otherwise, set y and z be adjacent elements in T such that $y \leq x < z$, or equivalently, y is the bottom-right-most element in T smaller than or equal to x , and z is the top-left-most element in T greater than x . Break T between y and z . Then, set $T \leftarrow x$ to be the quasi-ribbon tableau obtained by attaching $\boxed{x}$ to the right of $\boxed{y}$ and to the top of $\boxed{z}$.

$$T \leftarrow x := \begin{array}{|c|c|c|c|c|} \hline \boxed{} & \cdots & \boxed{} & y & x \\ \hline & & & z & \boxed{} & \cdots & \boxed{} \\ \hline \end{array}$$

Note that the following algorithm only contains the part of the original one relevant for the purpose of constructing the hypoplactic monoid.

Algorithm 6.5 ([CM17, Algorithm 4.3]). *Input:* A word $w \in X^*$.

Output: A quasi-ribbon tableau $\text{QR}'(w)$.

Procedure:

- (1) Set P_0 to be the empty quasi-ribbon tableau.
- (2) If $w = \epsilon$, then $\text{QR}'(\epsilon) := P_0$.
- (3) Else, let $x_1, \dots, x_m \in X$ such that $w = x_1 \dots x_m$. For each $i = 1, \dots, m$, set $P_i := P_{i-1} \leftarrow x_i$.
- (4) Set $\text{QR}'(w) := P_m$.

Proposition 6.6. *Let $x_1, \dots, x_m \in X$. Then,*

$$\text{QR}(x_1 x_2 \dots x_m) = \text{QR}'(x_m x_{m-1} \dots x_1).$$

Proof. Set $w = x_1 x_2 \dots x_m$ and $w' = x_m x_{m-1} \dots x_1$. Note that $\text{QR}(w)$ and $\text{QR}'(w')$ have m boxes each, filled with the elements $x_1, \dots, x_m$. Since rows are weakly increasing, and columns are strictly increasing, we have that given two consecutive

rows in $QR(w)$ or $QR'(w')$ beginning with y and z , where $y < z$, then the row beginning with y has to be filled increasingly with the elements x_i such that $y \leq x_i < z$. Thus, it suffices to show that $QR(w)$ and $QR'(w')$ have the same number of rows and each row begins with the same element.

The first row of $QR(w)$ and the first row of $QR'(w')$ begin with $\min\{x_1, \dots, x_m\}$, because $QR(w)$ and $QR'(w')$ are quasi-ribbon tableaux. If $QR(w)$ has a row other than the first beginning with z , set y to be the element in the box on top of $\boxed{z}$. By Algorithm 6.2, there is no element x_i such that $y < x_i < z$, and there exist $j < k$ such that $x_j = y$, $y \notin \{x_1, \dots, x_{j-1}\}$, $x_k = z$, and $z \notin \{x_{k+1}, \dots, x_m\}$. By Algorithm 6.4, $QR'(x_m \dots x_{j+1}) \leftarrow x_j$ is a quasi-ribbon tableau with x_k in the beginning of some row with $\boxed{x_j}$ on top of $\boxed{x_k}$. Since there is no element x_i such that $x_j < x_i < x_k$, by Algorithm 6.5, the quasi-ribbon tableau $QR'(w')$ has some row beginning with $z = x_k$.

Conversely, if $QR'(w')$ has a row other than the first beginning with z' , set y' to be the element in the box on top of $\boxed{z'}$. By Algorithm 6.5, there is no element x_i such that $y' < x_i < z'$, and there exist $k' > j'$ such that $x_{k'} = z'$, $z' \notin \{x_{k'+1}, \dots, x_m\}$, $x_{j'} = y'$, and $y' \notin \{x_1, \dots, x_{j'-1}\}$. By Algorithm 6.1, $x_k \rightarrow QR(x_1 \dots x_{k'-1})$ is a quasi-ribbon tableau with $x_{k'}$ in the beginning of some row with $\boxed{x_{j'}}$ on top of $\boxed{x_{k'}}$. Since there is no element x_i such that $x_{j'} < x_i < x_{k'}$, by Algorithm 6.2, the quasi-ribbon tableau $QR(w)$ has some row beginning with $z = x_{k'}$. $\square$

If we consider the ordered set $A_n = \{1 < 2 < \dots < n\}$, from [Nov00] we have a monoid congruence on A_n^* where two words $u, v \in A_n^*$ are congruent whenever $QR'(u) = QR'(v)$. Since the relation obtained by reversing words in a monoid congruence is still a monoid congruence, from Proposition 6.6 we get the following result.

Theorem 6.7. *Let $\sim_{\text{hypo}}$ be a relation on A_n^* given by*

$$u \sim_{\text{hypo}} v \iff QR(u) = QR(v).$$

Then, $\sim_{\text{hypo}}$ is a monoid congruence on A_n^ .*

We call $\sim_{\text{hypo}}$ the *hypoplactic congruence*. And now, we have reached a construction of the hypoplactic monoid through quasi-ribbon tableaux and an insertion algorithm.

Definition 6.8. The *hypoplactic monoid* hypo_n of rank n corresponds to the quotient monoid $A_n^*/\sim_{\text{hypo}}$ of the free monoid A_n^* over A_n by the hypoplactic congruence $\sim_{\text{hypo}}$.

6.3 A presentation for the classical hypoplactic monoid

In this section we give a presentation for the hypoplactic monoid hypo_n based on the relations introduced in [Nov00, §4.1].

Definition 6.9. Let $n \geq 1$. Consider the relations $R_1^{\text{hypo}(\mathfrak{A}_n)}$ and $R_2^{\text{hypo}(\mathfrak{A}_n)}$ on A_n^* given by

$$\begin{aligned} R_1^{\text{hypo}(\mathfrak{A}_n)} &= \{(x_2 x_4 x_1 x_3, x_4 x_2 x_3 x_1) \mid x_1 \leq x_2 < x_3 \leq x_4\}, \\ R_2^{\text{hypo}(\mathfrak{A}_n)} &= \{(x_3 x_1 x_4 x_2, x_1 x_3 x_2 x_4) \mid x_1 < x_2 \leq x_3 < x_4\}. \end{aligned}$$

The relation $R_1^{\text{hypo}(\mathfrak{A}_n)} \cup R_2^{\text{hypo}(\mathfrak{A}_n)}$ consists of what are called the *quartic relations*.

The *hypoplactic relations* $R^{\text{hypo}(\mathfrak{A}_n)} = R_1^{\text{plac}(\mathfrak{A}_n)} \cup R_2^{\text{plac}(\mathfrak{A}_n)} \cup R_1^{\text{hypo}(\mathfrak{A}_n)} \cup R_2^{\text{hypo}(\mathfrak{A}_n)}$ are formed by the Knuth relations (Definition 3.6) and the quartic relations.

As we mentioned in the introduction of this chapter, the classical hypoplactic monoid will later be placed in context as the hypoplactic monoid associated to the standard quasi-crystal of Cartan type $\mathfrak{A}_n$. This justifies the superscript $\text{hypo}(\mathfrak{A}_n)$ on the quartic relations above. Also, due to this fact, we chose a reading of quasi-ribbon tableaux which results in a hypoplactic monoid anti-isomorphic to the original definition, as shown in Proposition 6.6. And therefore, the words on the hypoplactic relations are reversed when compared to the definition of the relations in [Nov00, §4.1].

Note that for any pair (u, v) in the hypoplactic relations $R^{\text{hypo}(\mathfrak{A}_n)}$, we have that $\text{QR}(u) = \text{QR}(v)$. For $x \leq y < z$, we get that

$$\text{QR}(yzx) = \text{QR}(yxz) = \begin{array}{|c|c|} \hline x & y \\ \hline & z \\ \hline \end{array}.$$

For $x < y \leq z$, we also get that

$$\text{QR}(xzy) = \text{QR}(zxy) = \begin{array}{|c|} \hline x \\ \hline y \quad z \\ \hline \end{array}.$$

For $x_1 \leq x_2 < x_3 \leq x_4$, we obtain that

$$\text{QR}(x_2 x_4 x_1 x_3) = \text{QR}(x_4 x_2 x_3 x_1) = \begin{array}{|c|c|} \hline x_1 & x_2 \\ \hline & x_3 \quad x_4 \\ \hline \end{array}.$$

And for $x_1 < x_2 \leq x_3 < x_4$, we also obtain that

$$\text{QR}(x_3 x_1 x_4 x_2) = \text{QR}(x_1 x_3 x_2 x_4) = \begin{array}{|c|} \hline x_1 \\ \hline x_2 \quad x_3 \\ \hline & x_4 \\ \hline \end{array}.$$

This leads to an analogue of [KT97, Proposition 7.1] or [Nov00, Proposition 4.6 and Theorem 4.7] as we are considering a hypoplactic monoid anti-isomorphic to the classical one.

Theorem 6.10. *Let $n \geq 1$. Let θ to be the monoid congruence on A_n^* generated by $R^{\text{hypo}(\mathfrak{A}_n)}$. Then, $\theta = \sim_{\text{hypo}}$.*

Therefore, we have that $\langle A_n \mid R^{\text{hypo}(\mathfrak{A}_n)} \rangle$ is a presentation for the hypoplactic monoid hypo_n of rank n .

6.4 Quasi-crystals for the classical hypoplactic monoid

In this section we present a construction of the classical hypoplactic monoid from the crystal graph of the free crystal monoid $\mathcal{A}_n^*$ over the standard crystal $\mathcal{A}_n$ of type $\mathfrak{A}_n$. This method of obtaining the classical hypoplactic monoid was first introduced in [CM17], where a detailed construction can be found. Since our definition of classical hypoplactic monoid is anti-isomorphic to the one in [CM17], we do some adaptations, which are justified by Proposition 6.6. Thus, we will omit proofs, and focus on some examples.

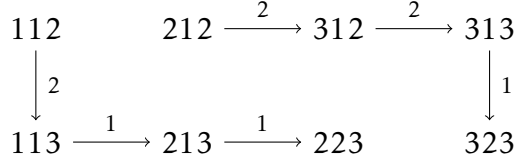
Let $n \geq 2$. Consider Γ to be the crystal graph of the free crystal monoid $\mathcal{A}_n^*$ over the standard crystal $\mathcal{A}_n$ of type $\mathfrak{A}_n$. Recall that Γ has vertex set A_n^* and an edge $u \xrightarrow{i} v$ labelled by $i \in \{1, \dots, n-1\}$, whenever $\tilde{f}_i(u) = v$. The following property allows us to distinguish vertices of Γ .

Definition 6.11. Let $i \in \{1, \dots, n-1\}$. A word $w \in A_n^*$ is said to have an *i-inversion* if there exist words $w_1, w_2, w_3 \in A_n^*$ such that $w = w_1 i w_2 (i+1) w_3$. A word $w \in A_n^*$ is said to be *i-inversion-free* if w does not have an *i-inversion*.

We define a directed graph Γ' with edges labelled by $\{1, \dots, n-1\}$ as follows. The vertex set of Γ' is the set A_n^* of all words over A_n . For $u, v \in A_n^*$ and $i \in \{1, \dots, n-1\}$, $u \xrightarrow{i} v$ is an edge of Γ' , if u and v are *i-inversion-free* and $u \xrightarrow{i} v$ is an edge of Γ . Equivalently, Γ' is obtained from Γ by removing for each $i \in \{1, \dots, n-1\}$, all *i*-labelled edges with start or end in a word with an *i-inversion*.

Example 6.12. If $n = 3$, then the connected component of Γ containing 112, described in Example 5.26, originate the following two connected components of

Γ' .



Note that as 112 has an 1-inversion and 223 has a 2-inversion, the edges $11 \xrightarrow{1} 212$ and $223 \xrightarrow{2} 323$ were removed from Γ to obtain the connected components of Γ' above.

For each $i \in \{1, \dots, n-1\}$, we define partial maps $\dot{e}_i, \dot{f}_i : A_n^* \rightarrow A_n^*$ as follows. For each edge $u \xrightarrow{i} v$ of Γ' labelled by i , set $\dot{e}_i(v) = u$ and $\dot{f}_i(u) = v$. Equivalently, for $w \in A_n^*$, if w is i -inversion-free, then $\dot{e}_i(w) = \tilde{e}_i(w)$, whenever $\tilde{e}_i$ is defined on w , and $\dot{f}_i(w) = \tilde{f}_i(w)$, whenever $\tilde{f}_i$ is defined on w .

By the signature rule (Algorithm 5.5), we obtain the following alternative definition of $\dot{e}_i$ and $\dot{f}_i$ ($i \in \{1, \dots, n-1\}$). For a word $w \in A_n^*$, if w is i -inversion-free and $|w|_{i+1} \geq 1$, then $\dot{e}_i(w)$ is obtained from w by replacing the right-most $i+1$ by i . If w is i -inversion-free and $|w|_i \geq 1$, then $\dot{f}_i(w)$ is obtained from w by replacing the left-most i by $i+1$.

In [CM17], Γ' is called a quasi-crystal graph. The partial maps $\dot{e}_1, \dots, \dot{e}_{n-1}$ and $\dot{f}_1, \dots, \dot{f}_{n-1}$ are called quasi-Kashiwara operators. A quasi-crystal is then formed by the set A_n^* of all words over A_n , the quasi-Kashiwara operators, the weight map $\text{wt} : A_n^* \rightarrow \mathbb{Z}^n$ given by

$$\text{wt}(w) = |w|_1 \mathbf{e}_1 + |w|_2 \mathbf{e}_2 + \dots + |w|_n \mathbf{e}_n,$$

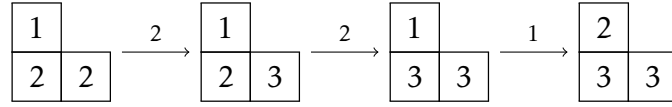
($w \in A_n^*$), and maps $\dot{e}_i, \dot{\varphi}_i : A_n^* \rightarrow \mathbb{Z}$ defined by

$$\begin{aligned}
 \dot{e}_i(w) &= \max\{k \in \mathbb{Z}_{\geq 0} \mid \dot{e}_i^k(w) \text{ is defined}\}, \\
 \dot{\varphi}_i(w) &= \max\{k \in \mathbb{Z}_{\geq 0} \mid \dot{f}_i^k(w) \text{ is defined}\}
 \end{aligned}$$

($w \in A_n^*$), $i = 1, \dots, n-1$. We will not adopt this terminology introduced in [CM17], as in the following chapters it will be used for designate other concepts, which generalize this approach.

Denote by Q_R the set of all quasi-ribbon tableaux over A_n . Through the method of reading quasi-ribbon tableaux introduced in Section 6.1, we can define the weight of a quasi-ribbon tableau T over A_n as the weight of $w(T)$, that is, $\text{wt}(T) = \text{wt}(w(T))$. Also, Γ' induces a directed graph Γ_{QR} with edges labelled by $\{1, \dots, n-1\}$ and vertex set Q_R , where $T \xrightarrow{i} T'$ is an i -labelled edge of Γ_{QR} if $w(T) \xrightarrow{i} w(T')$ is an edge of Γ' .

Example 6.13. If $n = 3$, from Example 6.12, we obtain that the quasi-ribbon tableaux of shape $(1, 2)$ over A_3 form the following connected component of Γ_{QR} .



In fact, the previous example is a consequence of the following result.

Proposition 6.14. *Let T and T' be quasi-ribbon tableaux over A_n . Then, T and T' lie in the same connected component of Γ_{QR} if and only if T and T' have the same shape.*

By the method that Γ_{QR} was constructed from Γ' , it follows that the reading w of quasi-ribbon tableaux is an injective graph homomorphism from Γ_{QR} to Γ' that preserves weights. On the other hand, the map QR from A_n^* to Q_R induced by the insertion algorithm in Algorithm 6.2 is a surjective graph homomorphism from Γ' to Γ_{QR} which preserves weights.

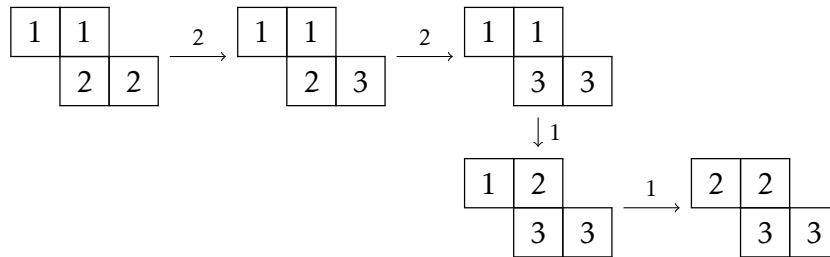
Example 6.15. Suppose $n = 3$. Consider the connected component $\Gamma'(2121)$ containing 2121 of Γ' .

$$2121 \xrightarrow{2} 3121 \xrightarrow{2} 3131 \xrightarrow{1} 3231 \xrightarrow{1} 3232$$

and consider the connected component $\Gamma'(1212)$ containing 1212 of Γ'

$$1212 \xrightarrow{2} 1312 \xrightarrow{2} 1313 \xrightarrow{1} 2313 \xrightarrow{1} 2323.$$

Clearly, we have a graph isomorphism between $\Gamma'(2121)$ and $\Gamma'(1212)$ that preserves weights. If we apply QR to these connected components, we get the same connected component of Γ_{QR} , which is the following.



The previous example reflects the fact that for a word $w \in A_n^*$, QR induces a weight-preserving graph isomorphism between $\Gamma'(w)$ and $\Gamma_{QR}(QR(w))$. This leads to the following result.

Theorem 6.16. *Let $u, v \in A_n^*$. Then, $u \sim_{\text{hypo}} v$ if and only if there exists a weight-preserving graph isomorphism between $\Gamma'(u)$ and $\Gamma'(v)$ mapping u to v .*

From the previous result, we have that the elements of the classical hypoplactic monoid hypo_n of rank n can be obtained by identifying elements in the same position of isomorphic connected components of Γ' . In [CM17], this process of obtaining hypo_n is referred as *crystalizing the hypoplactic monoid*. Although this construction is analogous to the one described in Section 5.5 of obtaining the classical plactic monoid through identifying elements by crystal isomorphisms between connected components of $\mathcal{A}_n^*$, it cannot be directly applied for other crystals other than $\mathcal{A}_n^*$.

Remark 6.17. Consider the free crystal monoid $\mathcal{C}_3^*$ over the standard crystal $\mathcal{C}_3$ of type $\mathcal{C}_3$. Let $\Gamma'_{\mathcal{C}_3^*}$ be a subgraph of $\Gamma_{\mathcal{C}_3^*}$ with vertex set C_3^* and satisfying the following properties:

- (i) the path $1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} \bar{3} \xrightarrow{2} \bar{2} \xrightarrow{1} \bar{1}$ lies in $\Gamma'_{\mathcal{C}_3^*}$;
- (ii) if a word $w \in C_3^*$ has a decomposition of the form $w = w_1 1 w_2 2 w_3$, for some $w_1, w_2, w_3 \in C_3^*$, then w is neither the start nor the end of an edge labelled by 1 in $\Gamma'_{\mathcal{C}_3^*}$.

Define a relation $\sim$ on $\mathcal{C}_3^*$ as follows. For $u, v \in C_3^*$, we have that $u \sim v$ if and only if there exists a weight-preserving graph isomorphism between $\Gamma'_{\mathcal{C}_3^*}(u)$ and $\Gamma'_{\mathcal{C}_3^*}(v)$ mapping u to v .

We have that ϵ and $1\bar{1}$ are isolated vertices in $\Gamma_{\mathcal{C}_3^*}$, and so, they also are isolated vertices in $\Gamma'_{\mathcal{C}_3^*}$. Since $\text{wt}(1\bar{1}) = \text{wt}(\epsilon) = 0$, then $1\bar{1} \sim \epsilon$. We also have that $2 \sim 2$, because the identity map induces a weight-preserving graph isomorphism between $\Gamma'_{\mathcal{C}_3^*}(2)$ and $\Gamma'_{\mathcal{C}_3^*}(2)$. By condition (i), 2 is the end of an 1-labelled edge in $\Gamma'_{\mathcal{C}_3^*}$, and by condition (ii), $1\bar{1}2$ cannot be the end of an 1-labelled edge in $\Gamma'_{\mathcal{C}_3^*}$. Hence, $1\bar{1}2 \not\sim 2$, and therefore, $\sim$ is not a monoid congruence on C_3^* .

In the following chapters we present an algebraic framework that accommodates the construction of the classical hypoplactic monoid from the standard crystal $\mathcal{A}_n$ of type $\mathcal{A}_n$ (see Section 8.5) and allows an analogous construction for any other seminormal crystal.

QUASI-CRYSTALS

In Remark 6.17, we showed that the method of obtaining the classical hypoplactic monoid from the crystal graph of type $\mathfrak{A}_n$, introduced by Cain and Malheiro in [CM17], does not result in a monoid when applied to crystal graphs of other types. In this chapter we present a new algebraic theory of quasi-crystals. These objects are obtained by allowing a slight relaxing on the defining conditions of crystals, which is sufficient to place in context the construction proposed by Cain and Malheiro and allow its generalization, as we shall see in Chapter 8.

In Section 7.1, we begin by introducing the definition of quasi-crystals and quasi-Kashiwara operators. Then, we make a study of quasi-crystals, where we see that many properties satisfied by crystals also hold for quasi-crystals. In Section 7.2, we introduce the notion of quasi-crystal graphs, which allow a combinatorial description of quasi-crystals. Analogous to crystal graphs, we show that quasi-crystal graphs can be seen as an alternative definition of quasi-crystals. In Section 7.3, we define the quasi-tensor product of quasi-crystals, and study its properties. Quasi-crystals together with the quasi-tensor product form the algebraic framework that we will use on the subsequent chapters. Therefore, in Section 7.4 we introduce a combinatorial method to compute the quasi-crystal structure of a quasi-tensor product of quasi-crystals, analogous to the signature rule for the tensor product of crystals.

7.1 Quasi-crystals and homomorphisms

In this section we introduce the notion of a quasi-crystal associated to a root system. Then, we study some basic properties satisfied by quasi-crystals, where some correspond to generalizations of properties of crystals studied in Chapter 4. Finally, we introduce the notion of quasi-crystal homomorphisms and study their properties.

Although we rely on root systems (Section 2.3) to define quasi-crystals, we only make use of properties that are also satisfied by other algebraic structures commonly used to define crystals. Thus, all subsequent definitions and results can be reinterpreted using an algebraic datum as in [Kas95] or a Cartan datum as in [HK02].

Consider $\mathbb{Z} \cup \{-\infty, +\infty\}$ to be the usual set of integers where we add a minimal element $-\infty$ and a maximal element $+\infty$, that is, $-\infty < m < +\infty$ for all $m \in \mathbb{Z}$. Also, set $m + (-\infty) = (-\infty) + m = -\infty$ and $m + (+\infty) = (+\infty) + m = +\infty$, for all $m \in \mathbb{Z}$.

Definition 7.1. Let Φ be a root system with weight lattice Λ and index set I for the simple roots $(\alpha_i)_{i \in I}$. A *quasi-crystal* $\mathbb{Q}$ of type Φ consists of a set Q together with maps $\text{wt} : Q \rightarrow \Lambda$, $\check{e}_i, \check{f}_i : Q \rightarrow Q \sqcup \{\perp\}$ and $\check{\varepsilon}_i, \check{\varphi}_i : Q \rightarrow \mathbb{Z} \cup \{-\infty, +\infty\}$, for each $i \in I$, satisfying the following conditions:

$$(i) \quad \check{\varphi}_i(x) = \check{\varepsilon}_i(x) + \langle \text{wt}(x), \alpha_i^\vee \rangle;$$

(ii) if $\check{e}_i(x) \in Q$, then

$$\begin{aligned} \text{wt}(\check{e}_i(x)) &= \text{wt}(x) + \alpha_i, \\ \check{\varepsilon}_i(\check{e}_i(x)) &= \check{\varepsilon}_i(x) - 1, \\ \check{\varphi}_i(\check{e}_i(x)) &= \check{\varphi}_i(x) + 1; \end{aligned}$$

(iii) if $\check{f}_i(x) \in Q$, then

$$\begin{aligned} \text{wt}(\check{f}_i(x)) &= \text{wt}(x) - \alpha_i, \\ \check{\varepsilon}_i(\check{f}_i(x)) &= \check{\varepsilon}_i(x) + 1, \\ \check{\varphi}_i(\check{f}_i(x)) &= \check{\varphi}_i(x) - 1; \end{aligned}$$

(iv) $\check{e}_i(x) = y$ if and only if $x = \check{f}_i(y)$;

(v) if $\check{\varepsilon}_i(x) = -\infty$ then $\check{e}_i(x) = \check{f}_i(x) = \perp$;

(vi) if $\check{\varepsilon}_i(x) = +\infty$ then $\check{e}_i(x) = \check{f}_i(x) = \perp$;

for $x, y \in Q$ and $i \in I$. The set Q is called the *underlying set* of $\mathbb{Q}$, and the maps wt , $\check{e}_i$, $\check{f}_i$, $\check{\varepsilon}_i$ and $\check{\varphi}_i$ ($i \in I$) form the *quasi-crystal structure* of $\mathbb{Q}$. Also, the map wt is called the *weight map*, where $\text{wt}(x)$ is said to be the *weight* of $x \in Q$, and the maps $\check{e}_i$ and $\check{f}_i$ ($i \in I$) are called the *raising* and *lowering quasi-Kashiwara operators*, respectively.

In comparison with the definition of a crystal (Definition 4.1), we have that $\check{\varepsilon}_i$ and $\check{\varphi}_i$ ($i \in I$) can take the value $+\infty$, which is not allowed for ε_i and φ_i . This leads to the addition of condition (vi), as conditions (i) to (v) coincide in both definitions. Thus, we could take the following as the definition of a crystal.

Remark 7.2. A crystal is a quasi-crystal $\mathbb{Q}$ where $\check{\varepsilon}_i(x) \neq +\infty$ and $\check{\varphi}_i(x) \neq +\infty$, for all $x \in Q$ and $i \in I$.

As discussed for Kashiwara operators after Definition 4.1, alternatively one can consider the quasi-Kashiwara operators $\check{e}_i$ and $\check{f}_i$ ($i \in I$) of a quasi-crystal $\mathbb{Q}$ to be partial maps from Q to Q . Similarly, we say that $\check{e}_i$ (or $\check{f}_i$) is *defined* on $x \in Q$ if $\check{e}_i(x) \in Q$ (resp., $\check{f}_i(x) \in Q$), otherwise we say that $\check{e}_i$ (resp. $\check{f}_i$) is *undefined* on x .

From condition (i), we get that $\check{\varepsilon}_i(x) = \pm\infty$ if and only if $\check{\varphi}_i(x) = \pm\infty$. And if so, $\check{\varepsilon}_i(x) = \check{\varphi}_i(x)$. Thus, conditions (v) and (vi) could have been stated replacing $\check{\varepsilon}_i$ by $\check{\varphi}_i$, as the universe of objects would remain the same.

Moreover, similar to the definition of a crystal, in the definition of a quasi-crystal we could have only stated one of conditions (ii) and (iii) as justified by the following result.

Proposition 7.3. *Let Φ be a root system with weight lattice Λ and index set I for the simple roots $(\alpha_i)_{i \in I}$. Consider a set Q and maps $\text{wt} : Q \rightarrow \Lambda$, $\check{e}_i, \check{f}_i : Q \rightarrow Q \sqcup \{\perp\}$ and $\check{\varepsilon}_i, \check{\varphi}_i : Q \rightarrow \mathbb{Z} \cup \{-\infty, +\infty\}$, for each $i \in I$, satisfying Definition 7.1(iv). Then, Definition 7.1(ii) holds if and only if Definition 7.1(iii) holds.*

Proof. Assume Definition 7.1(ii). Let $x \in Q$ and $i \in I$ such that $\check{f}_i(x) \in Q$. By Definition 7.1(iv), we have that $\check{e}_i(\check{f}_i(x)) = x$, and so,

$$\begin{aligned} \text{wt}(x) &= \text{wt}(\check{e}_i(\check{f}_i(x))) = \text{wt}(\check{f}_i(x)) - \alpha_i, \\ \check{\varepsilon}_i(x) &= \check{\varepsilon}_i(\check{e}_i(\check{f}_i(x))) = \check{\varepsilon}_i(\check{f}_i(x)) - 1, \\ \check{\varphi}_i(x) &= \check{\varphi}_i(\check{e}_i(\check{f}_i(x))) = \check{\varphi}_i(\check{f}_i(x)) + 1, \end{aligned}$$

by Definition 7.1(ii). Hence, Definition 7.1(iii) holds.

The converse implication is analogous. □

In the same way, by Definition 7.1(i) and (iv) we have that a quasi-crystal is determined by a set Q and the weight map wt together with either $\check{e}_i$ or $\check{f}_i$, and either $\check{\varepsilon}_i$ or $\check{\varphi}_i$, for each $i \in I$. However, for a purpose of clarity, we usually give explicit definitions for each map when defining a quasi-crystal.

Example 7.4. (i) By Remark 7.2, the crystals described in Example 4.4 are also examples of quasi-crystals. Thus, we also call $\mathcal{A}_n$ the *standard quasi-crystal of type $\mathfrak{A}_n$* and $\mathcal{C}_n$ the *standard quasi-crystal of type $\mathfrak{C}_n$* .

(ii) Consider the root system of type $\mathfrak{A}_3$ described in Example 2.16. Set $A_3 = \{1, 2, 3\}$. We have a quasi-crystal $\mathcal{A}_3^2$ of type $\mathfrak{A}_3$ whose underlying set is the Cartesian product $A_3 \times A_3$ and whose quasi-crystal structure is given as follows.

x	$\text{wt}(x)$	$\check{e}_1(x)$	$\check{e}_2(x)$	$\check{f}_1(x)$	$\check{f}_2(x)$	$\check{e}_1(x)$	$\check{e}_2(x)$	$\check{\varphi}_1(x)$	$\check{\varphi}_2(x)$
(1, 1)	$2\mathbf{e}_1$	$\perp$	$\perp$	(2, 1)	$\perp$	0	0	2	0
(1, 2)	$\mathbf{e}_1 + \mathbf{e}_2$	$\perp$	$\perp$	$\perp$	(1, 3)	$+\infty$	0	$+\infty$	1
(1, 3)	$\mathbf{e}_1 + \mathbf{e}_3$	$\perp$	(1, 2)	(2, 3)	$\perp$	0	1	1	0
(2, 1)	$\mathbf{e}_1 + \mathbf{e}_2$	(1, 1)	$\perp$	(2, 2)	(3, 1)	1	0	1	1
(2, 2)	$2\mathbf{e}_2$	(2, 1)	$\perp$	$\perp$	(3, 2)	2	0	0	2
(2, 3)	$\mathbf{e}_2 + \mathbf{e}_3$	(1, 3)	$\perp$	$\perp$	$\perp$	1	$+\infty$	0	$+\infty$
(3, 1)	$\mathbf{e}_1 + \mathbf{e}_3$	$\perp$	(2, 1)	(3, 2)	$\perp$	0	1	1	0
(3, 2)	$\mathbf{e}_2 + \mathbf{e}_3$	(3, 1)	(2, 2)	$\perp$	(3, 3)	1	1	0	1
(3, 3)	$2\mathbf{e}_3$	$\perp$	(3, 2)	$\perp$	$\perp$	0	2	0	0

(iii) Consider the root system of type $\mathfrak{A}_2$ described in Example 2.16. We have a quasi-crystal $\mathcal{Q}$ of type $\mathfrak{A}_2$ consisting of a set $Q = \{a, b\}$ and maps defined as follows.

x	$\text{wt}(x)$	$\check{e}(x)$	$\check{f}(x)$	$\check{e}(x)$	$\check{\varphi}(x)$
a	$\mathbf{e}_1$	$\perp$	$\perp$	0	1
b	$\mathbf{e}_2$	$\perp$	$\perp$	1	0

Since the root system of type $\mathfrak{A}_2$ has exactly one simple root, we omit the subscript index in the maps, for instance $\check{e}$ instead of $\check{e}_1$.

Recall the partial order defined on a weight lattice Λ described in (2.3). The following result justifies the terminology of *raising* and *lowering* used to characterize the quasi-Kashiwara operators $\check{e}_i$ and $\check{f}_i$ ($i \in I$).

Proposition 7.5. *Let $\mathcal{Q}$ be a quasi-crystal, and let $x \in Q$ and $i \in I$. If $\check{e}_i(x) \in Q$, then $\text{wt}(\check{e}_i(x)) > \text{wt}(x)$. If $\check{f}_i(x) \in Q$, then $\text{wt}(x) > \text{wt}(\check{f}_i(x))$.*

Proof. If $\check{e}_i(x) \in Q$, then

$$\text{wt}(\check{e}_i(x)) - \text{wt}(x) = \text{wt}(x) + \alpha_i - \text{wt}(x) = \alpha_i,$$

by Definition 7.1(ii), and so, $\text{wt}(\check{e}_i(x)) \geq \text{wt}(x)$. Since α_i is a root, we have that $\alpha_i \neq 0$, which implies that $\text{wt}(\check{e}_i(x)) \neq \text{wt}(x)$. Hence, $\text{wt}(\check{e}_i(x)) > \text{wt}(x)$.

If $\check{f}_i(x) \in Q$, then $x = \check{e}_i(\check{f}_i(x))$, by Definition 7.1(iv). As proved above, we have that $\text{wt}(x) = \text{wt}(\check{e}_i(\check{f}_i(x))) > \text{wt}(\check{f}_i(x))$. $\square$

From the previous result, we have that, like the Kashiwara operators in crystals, the raising quasi-Kashiwara operators $\tilde{e}_i$ ($i \in I$) increase the weight of elements, whenever defined, and the lowering quasi-Kashiwara operators $\tilde{f}_i$ ($i \in I$) decrease the weight of elements, whenever defined. Thus, the notions of highest- and lowest-weight elements (Definition 4.2) can be generalized in a natural way.

Definition 7.6. Let $x \in Q$ be an element of a quasi-crystal $\mathbb{Q}$.

- (i) x is said to be of *highest weight* if $\tilde{e}_i(x) = \perp$, for all $i \in I$.
- (ii) x is said to be of *lowest weight* if $\tilde{f}_i(x) = \perp$, for all $i \in I$.

Notice that in a quasi-crystal $\mathbb{Q}$ there may exist elements $x, y \in Q$ such that x is of highest weight, y is not of highest weight, and $\text{wt}(y) > \text{wt}(x)$. For instance, consider the crystal $\mathbb{C}_2 \otimes \mathbb{C}_2$ described in Example 4.15(ii), where the elements $x = 1 \otimes \bar{1}$ and $y = 1 \otimes \bar{2}$ satisfy these conditions. Moreover, if a quasi-crystal $\mathbb{Q}$ has an element $x \in Q$ such that $\tilde{e}_i(x) \in \{-\infty, +\infty\}$, for all $i \in I$, then we can change the weight $\text{wt}(x)$ of x to any weight in Λ , and the resulting structure is still a quasi-crystal. However, if we extend this definition to the weights as follows, we get some more natural results.

Definition 7.7. Let $\mathbb{Q}$ be a quasi-crystal, and let $\lambda \in \Lambda$ be a weight.

- (i) λ is called a *highest weight* if there exists a highest-weight element $x \in Q$ such that $\lambda = \text{wt}(x)$.
- (ii) λ is called a *lowest weight* if there exists a lowest-weight element $x \in Q$ such that $\lambda = \text{wt}(x)$.

Proposition 7.8. Let $\mathbb{Q}$ be a quasi-crystal, and let λ be a weight in $\text{wt}(Q) = \{\text{wt}(x) \mid x \in Q\}$.

- (i) If λ is maximal among weights in $\text{wt}(Q)$, then λ is a highest weight, and any element $x \in Q$ such that $\text{wt}(x) = \lambda$ is of highest weight.
- (ii) If λ is minimal among weights in $\text{wt}(Q)$, then λ is a lowest weight, and any element $x \in Q$ such that $\text{wt}(x) = \lambda$ is of lowest weight.

Proof. (i) Let $x \in Q$ be such that $\text{wt}(x) = \lambda$. If x is not of highest weight, then $\tilde{e}_i(x) \in Q$, for some $i \in I$, and by Proposition 7.5, $\text{wt}(\tilde{e}_i(x)) > \text{wt}(x)$. Therefore, λ is not maximal among weights in $\text{wt}(Q)$.

(ii) Let $x \in Q$ be such that $\text{wt}(x) = \lambda$. If x is not of lowest weight, then $\tilde{f}_i(x) \in Q$, for some $i \in I$, and by Proposition 7.5, $\text{wt}(x) > \text{wt}(\tilde{f}_i(x))$. Hence, λ is not minimal among weights in $\text{wt}(Q)$. $\square$

Since the quasi-Kashiwara operators of a quasi-crystal $\mathbb{Q}$ can be regarded as partial maps from Q to Q , we can compose them in a natural way. As usual, for $i \in I$, set $\check{e}_i^0$ and $\check{f}_i^0$ to be the identity map on Q , and recursively, defined $\check{e}_i^{k+1} = \check{e}_i \check{e}_i^k$ and $\check{f}_i^{k+1} = \check{f}_i \check{f}_i^k$, for $k \geq 0$.

Definition 7.9. A quasi-crystal $\mathbb{Q}$ is said to be *seminormal* if for any $x \in Q$ and $i \in I$,

$$\check{\varepsilon}_i(x) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \check{e}_i^k(x) \in Q\}$$

and

$$\check{\varphi}_i(x) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \check{f}_i^k(x) \in Q\},$$

whenever $\check{\varepsilon}_i(x) \neq +\infty$.

The quasi-crystals described in Example 7.4(i) and (ii) are seminormal. The quasi-crystal described in Example 7.4(iii) is not seminormal.

As we intend the seminormal property for quasi-crystals to generalize the same property for crystals, we first show that Definition 7.9 for crystals coincides with Definition 4.5 in the following sense.

Proposition 7.10. *Let $\mathcal{B}$ be a crystal. Then, $\mathcal{B}$ is a seminormal crystal (Definition 4.5) if and only if $\mathcal{B}$ is a seminormal quasi-crystal (Definition 7.9).*

Proof. Since $\mathcal{B}$ is a crystal, then as a quasi-crystal we have that $\check{\varepsilon}_i(x) \neq +\infty \neq \check{\varphi}_i(x)$, for all $x \in B$ and $i \in I$ (see Remark 7.2). Hence, $\mathcal{B}$ satisfies Definition 4.5 if and only if it satisfies Definition 7.9. $\square$

By the previous result, we have that the seminormal property for quasi-crystals is consistent with the same property for crystals. On the other hand, one could argue if we could have just stated a condition as in Definition 4.5 without allowing exceptions. However, in Definition 7.9, if we did not allow $\check{\varepsilon}_i$ and $\check{\varphi}_i$ ($i \in I$) to take the value $+\infty$, then a seminormal quasi-crystal would be always a seminormal crystal (Remark 7.2), resulting in the class of seminormal quasi-crystals and the class of seminormal crystals to coincide, and thus, we would not have a generalization of the seminormal property as intended. Allowing this exception is needed, but it has deep implications, as some common results for seminormal crystals are not satisfied by seminormal quasi-crystals. For example, we can no longer guarantee the weight of a highest-weight element to be dominant (Proposition 4.6), instead we have the following result.

Proposition 7.11. *Let $x \in Q$ be an element of a seminormal quasi-crystal $\mathbb{Q}$.*

(i) If x is of highest weight and $\langle \text{wt}(x), \alpha_i^\vee \rangle < 0$, for some $i \in I$, then $\check{e}_i(x) = \check{\varphi}_i(x) = +\infty$.

(ii) If x is of lowest weight and $\langle \text{wt}(x), \alpha_i^\vee \rangle > 0$, for some $i \in I$, then $\check{e}_i(x) = \check{\varphi}_i(x) = +\infty$.

Proof. (i) Suppose that $\check{e}_i(x) \neq +\infty$ (or equivalently, $\check{\varphi}_i(x) \neq +\infty$), for some $i \in I$. As $\mathbb{Q}$ is seminormal, we have that $\check{e}_i(x), \check{\varphi}_i(x) \in \mathbb{Z}_{\geq 0}$ and by Definition 7.1(i),

$$\langle \text{wt}(x), \alpha_i^\vee \rangle = \check{\varphi}_i(x) - \check{e}_i(x).$$

Thus, if $\langle \text{wt}(x), \alpha_i^\vee \rangle < 0$, then $\check{e}_i(x) > 0$, which implies that $\check{e}_i(x) \in Q$, because $\mathbb{Q}$ is seminormal. Hence, x is not of highest weight.

(ii) As above, if $\check{e}_i(x) \neq +\infty$ and $\langle \text{wt}(x), \alpha_i^\vee \rangle > 0$, then $\check{\varphi}_i(x) \in \mathbb{Z}_{>0}$, which implies that $\check{f}_i(x) \in Q$. And therefore, x is not of lowest weight. $\square$

Now, we introduce the definition of homomorphism between quasi-crystals, which is analogous to the one for crystals (see Definition 4.7).

Definition 7.12. Let $\mathbb{Q}$ and $\mathbb{Q}'$ be quasi-crystals of the same type. A *quasi-crystal homomorphism* ψ from $\mathbb{Q}$ to $\mathbb{Q}'$, denoted by $\psi : \mathbb{Q} \rightarrow \mathbb{Q}'$, is a map $\psi : Q \sqcup \{\perp\} \rightarrow Q' \sqcup \{\perp\}$ that satisfies the following conditions:

(i) $\psi(\perp) = \perp$;

(ii) if $\psi(x) \in Q'$, then

$$\text{wt}(\psi(x)) = \text{wt}(x),$$

$$\check{e}_i(\psi(x)) = \check{e}_i(x),$$

$$\check{\varphi}_i(\psi(x)) = \check{\varphi}_i(x);$$

(iii) if $\check{e}_i(x) \in Q$ and $\psi(x), \psi(\check{e}_i(x)) \in Q'$, then

$$\psi(\check{e}_i(x)) = \check{e}_i(\psi(x));$$

(iv) if $\check{f}_i(x) \in Q$ and $\psi(x), \psi(\check{f}_i(x)) \in Q'$, then

$$\psi(\check{f}_i(x)) = \check{f}_i(\psi(x));$$

for $x \in Q$ and $i \in I$.

A *quasi-crystal isomorphism* ψ between $\mathbb{Q}$ and $\mathbb{Q}'$ is a bijection $\psi : Q \sqcup \{\perp\} \rightarrow Q' \sqcup \{\perp\}$ such that $\psi : \mathbb{Q} \rightarrow \mathbb{Q}'$ and $\psi^{-1} : \mathbb{Q}' \rightarrow \mathbb{Q}$ are quasi-crystal homomorphisms.

We say that $\mathbb{Q}$ and $\mathbb{Q}'$ are isomorphic if there exists a quasi-crystal isomorphism between $\mathbb{Q}$ and $\mathbb{Q}'$.

Note that the conditions in the definition of a quasi-crystal homomorphism coincide with the conditions in the definition of a crystal homomorphism (Definition 4.7). Thus, from Remark 7.2, we observe the following.

Remark 7.13. A quasi-crystal homomorphism between two crystals is a crystal homomorphism.

As $\perp$ is an auxiliary symbol which stands for undefinition, alternatively a quasi-crystal homomorphism $\psi : \mathbb{Q} \rightarrow \mathbb{Q}'$ can be defined as a partial map ψ from Q to Q' satisfying conditions (ii) to (iv) of Definition 7.12.

By Definition 7.12(i), when defining a quasi-crystal homomorphism $\psi : \mathbb{Q} \rightarrow \mathbb{Q}'$, we only give its value on each $x \in Q$. Also, by the same reason, we consider a map $\psi : Q \rightarrow Q'$ satisfying conditions (ii) to (iv) of Definition 7.12 to be a quasi-crystal homomorphism from $\mathbb{Q}$ to $\mathbb{Q}'$. In other words, a map $\psi : Q \rightarrow Q'$ is a quasi-crystal homomorphism from $\mathbb{Q}$ to $\mathbb{Q}'$ whenever the map $\psi' : Q \sqcup \{\perp\} \rightarrow Q' \sqcup \{\perp\}$, given by $\psi'(\perp) = \perp$ and $\psi'(x) = \psi(x)$, for each $x \in Q$, is a quasi-crystal homomorphism from $\mathbb{Q}$ to $\mathbb{Q}'$.

At this point we defined quasi-crystals and homomorphisms between them. It is immediate from Definition 7.12 that given a quasi-crystal $\mathbb{Q}$, the identity map on Q is a quasi-crystal homomorphism from $\mathbb{Q}$ to $\mathbb{Q}$. Thus, to obtain a category whose objects are quasi-crystals and morphisms are quasi-crystal homomorphisms, it remains to show that the composition of quasi-crystal homomorphisms is a quasi-crystal homomorphism.

Proposition 7.14. *Let $\mathbb{Q}_1, \mathbb{Q}_2$ and $\mathbb{Q}_3$ be quasi-crystals of the same type, and let $\psi_1 : \mathbb{Q}_1 \rightarrow \mathbb{Q}_2$ and $\psi_2 : \mathbb{Q}_2 \rightarrow \mathbb{Q}_3$ be quasi-crystal homomorphisms. Then, $\psi_2 \circ \psi_1$ is a quasi-crystal homomorphism from $\mathbb{Q}_1$ to $\mathbb{Q}_3$.*

Proof. Since ψ_1 and ψ_2 are quasi-crystal homomorphisms, we have that $\psi_1(\perp) = \psi_2(\perp) = \perp$, and so,

$$(\psi_2 \circ \psi_1)(\perp) = \psi_2(\psi_1(\perp)) = \psi_2(\perp) = \perp.$$

Let $x \in Q_1$ and $i \in I$. If $(\psi_2 \circ \psi_1)(x) \in Q_3$, then $\psi_1(x) \in Q_2$, and so,

$$\text{wt}((\psi_2 \circ \psi_1)(x)) = \text{wt}(\psi_2(\psi_1(x))) = \text{wt}(\psi_1(x)) = \text{wt}(x),$$

because ψ_1 and ψ_2 satisfy Definition 7.12(ii). Similarly, if $(\psi_2 \circ \psi_1)(x) \in Q_3$, then $\check{e}_i((\psi_2 \circ \psi_1)(x)) = \check{e}_i(x)$ and $\check{f}_i((\psi_2 \circ \psi_1)(x)) = \check{f}_i(x)$. If $\check{e}_i(x) \in Q_1$ and $(\psi_2 \circ \psi_1)(x), (\psi_2 \circ \psi_1)(\check{e}_i(x)) \in Q_3$, then $\psi_1(x), \psi_1(\check{e}_i(x)) \in Q_2$, and since ψ_1 satisfies Definition 7.12(iii), we have that $\psi_1(\check{e}_i(x)) = \check{e}_i(\psi_1(x))$, and since ψ_2 also satisfies Definition 7.12(iii), we get that

$$\psi_2(\psi_1(\check{e}_i(x))) = \psi_2(\check{e}_i(\psi_1(x))) = \check{e}_i(\psi_2(\psi_1(x))).$$

Hence, $(\psi_2 \circ \psi_1)(\ddot{e}_i(x)) = \ddot{e}_i((\psi_2 \circ \psi_1)(x))$. Analogously, if $\ddot{f}_i(x) \in Q_1$ and $(\psi_2 \circ \psi_1)(x), (\psi_2 \circ \psi_1)(\ddot{f}_i(x)) \in Q_3$, then

$$\psi_2(\psi_1(\ddot{f}_i(x))) = \psi_2(\ddot{f}_i(\psi_1(x))) = \ddot{f}_i(\psi_2(\psi_1(x))),$$

because ψ_1 and ψ_2 satisfy Definition 7.12(iv). And so, $(\psi_2 \circ \psi_1)(\ddot{f}_i(x)) = \ddot{f}_i((\psi_2 \circ \psi_1)(x))$. Therefore, $\psi_2 \circ \psi_1$ is a quasi-crystal homomorphism from $\mathbb{Q}_1$ to $\mathbb{Q}_3$. $\square$

We say that a quasi-crystal homomorphism $\psi : \mathbb{Q} \rightarrow \mathbb{Q}'$ is injective, surjective or bijective if the map $\psi : Q \sqcup \{\perp\} \rightarrow Q' \sqcup \{\perp\}$ is injective, surjective or bijective, respectively.

Example 7.15. Let $\mathcal{A}_2$ be the standard quasi-crystal of type $\mathfrak{A}_2$, and let $\mathbb{Q}$ be the quasi-crystal of type $\mathfrak{A}_2$ defined in Example 7.4(iii). Define a map $\psi : Q \rightarrow A_2$ by $\psi(a) = 1$ and $\psi(b) = 2$. Then, ψ is a quasi-crystal homomorphism from $\mathbb{Q}$ to $\mathcal{A}_2$.

In this last example, although ψ is a quasi-crystal homomorphism which is bijective, it is not a quasi-crystal isomorphism as ψ^{-1} does not verify Definition 7.12(iii) and (iv). Moreover, by Remarks 7.2 and 7.13, we have that ψ is a bijective crystal homomorphism that is not a crystal isomorphism. So, in the following results we present an alternative characterization of quasi-crystal isomorphisms analogous to Theorem 4.8.

Lemma 7.16. *Let $\mathbb{Q}$ and $\mathbb{Q}'$ be quasi-crystals of the same type, and let $\psi : \mathbb{Q} \rightarrow \mathbb{Q}'$ be a bijective quasi-crystal homomorphism. The following conditions are equivalent*

- (i) $\psi(\ddot{e}_i(x)) = \ddot{e}_i(\psi(x))$ for all $x \in Q$ and $i \in I$;
- (ii) $\psi(\ddot{f}_i(x)) = \ddot{f}_i(\psi(x))$ for all $x \in Q$ and $i \in I$.

Proof. Suppose that ψ satisfies (i). Let $x \in Q$ and $i \in I$. Since ψ is bijective and $\psi(\perp) = \perp$ by Definition 7.12(i), then $\psi(y) \in Q'$ for all $y \in Q$. Thus, if $\ddot{f}_i(x) \in Q$, then $\psi(\ddot{f}_i(x)) \in Q'$ which implies $\psi(\ddot{f}_i(x)) = \ddot{f}_i(\psi(x))$ by Definition 7.12(iv). If $\ddot{f}_i(\psi(x)) \in Q'$, or equivalently, $\psi^{-1}(\ddot{f}_i(\psi(x))) \in Q$, we have that

$$\psi(\ddot{e}_i(\psi^{-1}(\ddot{f}_i(\psi(x))))) = \ddot{e}_i(\psi(\psi^{-1}(\ddot{f}_i(\psi(x))))) = \ddot{e}_i(\ddot{f}_i(\psi(x))) = \psi(x)$$

as ψ satisfies (i), and so, $x = \ddot{e}_i(\psi^{-1}(\ddot{f}_i(\psi(x))))$ which implies

$$\psi(\ddot{f}_i(x)) = \psi(\ddot{f}_i(\ddot{e}_i(\psi^{-1}(\ddot{f}_i(\psi(x))))) = \psi(\psi^{-1}(\ddot{f}_i(\psi(x)))) = \ddot{f}_i(\psi(x)).$$

Hence, ψ satisfies (ii).

The fact that (ii) implies (i) follows analogously. $\square$

Theorem 7.17. *Let $\mathbb{Q}$ and $\mathbb{Q}'$ be quasi-crystals of the same type, and let $\psi : \mathbb{Q} \rightarrow \mathbb{Q}'$ be a quasi-crystal homomorphism. Then, ψ is a quasi-crystal isomorphism if and only if ψ is bijective and satisfies (i) or (ii) of Lemma 7.16.*

Proof. Suppose that ψ is a quasi-crystal isomorphism. By Definition 7.12, ψ is bijective. Let $x \in Q$ and $i \in I$. If $\ddot{e}_i(x) \in Q$, we also have that $\psi(x), \psi(\ddot{e}_i(x)) \in Q'$ as ψ is bijective and $\psi(\perp) = \perp$, and so, $\psi(\ddot{e}_i(x)) = \ddot{e}_i(\psi(x))$ by Definition 7.12(iii). Similarly, since ψ^{-1} is also a quasi-crystal isomorphism, we have that if $\ddot{e}_i(\psi(x)) \in Q'$, then

$$\psi^{-1}(\ddot{e}_i(\psi(x))) = \ddot{e}_i(\psi^{-1}(\psi(x))) = \ddot{e}_i(x)$$

which implies $\ddot{e}_i(\psi(x)) = \psi(\ddot{e}_i(x))$. Hence, ψ satisfies Lemma 7.16(i).

Conversely, by Lemma 7.16, we can assume that ψ is bijective and satisfies conditions (i) and (ii) of that lemma. Clearly, $\psi^{-1}(\perp) = \perp$. Let $x' \in Q'$ and $i \in I$. Since ψ is a quasi-crystal homomorphism, by Definition 7.12(ii) we have that

$$\text{wt}(\psi^{-1}(x')) = \text{wt}(\psi(\psi^{-1}(x'))) = \text{wt}(x').$$

Similarly, we get $\ddot{e}_i(\psi^{-1}(x')) = \ddot{e}_i(x')$ and $\ddot{\varphi}_i(\psi^{-1}(x')) = \ddot{\varphi}_i(x')$. Since ψ satisfies Lemma 7.16(i), then

$$\ddot{e}_i(\psi^{-1}(x')) = \psi^{-1}(\psi(\ddot{e}_i(\psi^{-1}(x')))) = \psi^{-1}(\ddot{e}_i(\psi(\psi^{-1}(x')))) = \psi^{-1}(\ddot{e}_i(x')).$$

And since ψ satisfies Lemma 7.16(ii), then

$$\ddot{f}_i(\psi^{-1}(x')) = \psi^{-1}(\psi(\ddot{f}_i(\psi^{-1}(x')))) = \psi^{-1}(\ddot{f}_i(\psi(\psi^{-1}(x')))) = \psi^{-1}(\ddot{f}_i(x')).$$

Hence, ψ^{-1} is a quasi-crystal homomorphism, and therefore, ψ is a quasi-crystal isomorphism. $\square$

Corollary 7.18. *Let $\mathbb{Q}$ and $\mathbb{Q}'$ be seminormal quasi-crystals of the same type, and let $\psi : \mathbb{Q} \rightarrow \mathbb{Q}'$ be a bijective quasi-crystal homomorphism. Then, ψ is a quasi-crystal isomorphism.*

Proof. Let $x \in Q$ and $i \in I$. Since $\mathbb{Q}$ and $\mathbb{Q}'$ are seminormal and $\ddot{e}_i(\psi(x)) = \ddot{e}_i(x)$, we have that $\ddot{e}_i(x) \in Q$ if and only if $\ddot{e}_i(\psi(x)) \in Q'$. So, if $\ddot{e}_i(x) \in Q$, then $\psi(\ddot{e}_i(x)) \in Q$, as ψ is bijective, and $\psi(\ddot{e}_i(x)) = \ddot{e}_i(\psi(x))$ by Definition 7.12(iii). Otherwise, we have that $\ddot{e}_i(x) = \perp = \ddot{e}_i(\psi(x))$, which implies that $\psi(\ddot{e}_i(x)) = \perp = \ddot{e}_i(\psi(x))$, by Definition 7.12(i). Hence, ψ satisfies Lemma 7.16(i), and by Theorem 7.17, ψ is a quasi-crystal isomorphism. $\square$

7.2 Quasi-crystal graphs

In this section we show a combinatorial approach to quasi-crystals, which is a generalization of the notion of crystal graph introduced in Section 4.2. From this approach we are able to characterize some substructures of a quasi-crystal, generalizing similar structures described for crystals based on crystal graphs. Finally, as a crystal graph of a seminormal crystal completely determines its crystal structure, we show a similar connection for seminormal quasi-crystals.

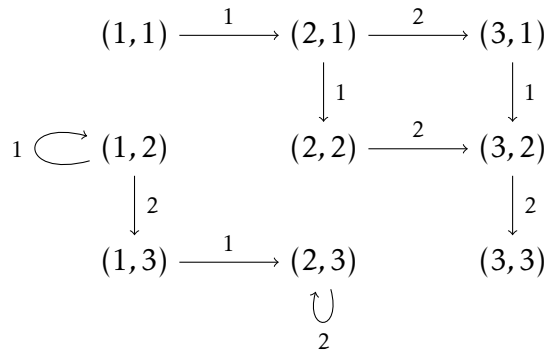
Definition 7.19. The *quasi-crystal graph* $\Gamma_{\mathbb{Q}}$ of a quasi-crystal $\mathbb{Q}$ is a Λ -weighted I -labelled directed graph with vertex set Q and an edge $x \xrightarrow{i} y$ from $x \in Q$ to $y \in Q$ labelled by $i \in I$ whenever $\check{f}_i(x) = y$, and a loop on $x \in Q$ labelled by $i \in I$ whenever $\check{e}_i(x) = +\infty$. For $x \in Q$, let $\Gamma_{\mathbb{Q}}(x)$ denote the connected component of $\Gamma_{\mathbb{Q}}$ containing the vertex x .

In comparison with crystal graphs (see Definition 4.9), we accommodate the weight map wt directly in the definition of quasi-crystal graphs. Also, we have that a quasi-crystal graph may not be simple. Moreover, a quasi-crystal graph is simple if the maps $\check{e}_i$ ($i \in I$) do not take the value $+\infty$, and from Remark 7.2, we observe the following.

Remark 7.20. For a quasi-crystal $\mathbb{Q}$, the quasi-crystal graph $\Gamma_{\mathbb{Q}}$ is simple if and only if $\mathbb{Q}$ is a crystal. Also, the quasi-crystal graph of a crystal coincides with its crystal graph.

As crystal graphs are also quasi-crystal graphs, the quasi-crystal graphs for the standard quasi-crystals of types $\mathfrak{A}_n$ and $\mathfrak{C}_n$ can be found in Example 4.11.

Example 7.21. The quasi-crystal graph of the quasi-crystal $\mathcal{A}_3^2$ of type $\mathfrak{A}_3$ described in Example 7.21(ii) is the following.



Let $x \xrightarrow{i} y$ be an edge of a quasi-crystal graph $\Gamma_{\mathbb{Q}}$ of a quasi-crystal $\mathbb{Q}$. If $x \neq y$, then $y = \check{f}_i(x)$ by Definition 7.19, or equivalently, $x = \check{e}_i(y)$ due to Definition 7.1(iv). Otherwise, we have that $x = y$, that is, x has an i -labelled loop, and

so, $\check{e}_i(x) = +\infty$ by Definition 7.19, which implies that $\check{e}_i$ and $\check{f}_i$ are undefined on x , by Definition 7.1(vi). In either case, we have that if $x \xrightarrow{i} y'$ is an edge of $\Gamma_{\mathbb{Q}}$, then $y = y'$, and similarly, if $x' \xrightarrow{i} y$ is an edge of $\Gamma_{\mathbb{Q}}$, then $x = x'$. Hence, for $i \in I$, a vertex of $\Gamma_{\mathbb{Q}}$ is the start of at most an edge, and is the end of at most an edge labelled by i .

As crystal graphs form a combinatorial framework to study crystals (see Section 4.2), quasi-crystal graphs allow a similar approach to quasi-crystals. For instance, Proposition 7.5 is equivalent to stating that for a quasi-crystal $\mathbb{Q}$, if $x \xrightarrow{i} y$ is an edge of $\Gamma_{\mathbb{Q}}$ with $x \neq y$, then $\text{wt}(x) > \text{wt}(y)$. Also, Definition 7.6 is equivalent to state that an element $x \in Q$ is of highest (resp., lowest) weight if the only edges of $\Gamma_{\mathbb{Q}}$ ending (resp., starting) at x are loops. Moreover, from Remark 7.20, the combinatorial framework formed by quasi-crystal graphs is a coherent generalization of the framework formed by crystal graphs. And therefore, the notions of connected components (Definition 4.10) can be generalized as follows.

Definition 7.22. Let $\mathbb{Q}$ be a quasi-crystal. A *connected component* of $\mathbb{Q}$ is a subset Q' of Q that satisfies the following conditions:

- (i) for each $x, y \in Q'$ there exist $g_1, \dots, g_m \in \{\check{e}_i, \check{f}_i \mid i \in I\}$ such that $g_1 \cdots g_m(x) = y$;
- (ii) $\check{e}_i(x), \check{f}_i(x) \in Q' \sqcup \{\perp\}$ for all $x \in Q'$ and $i \in I$.

We also call a *connected component* the quasi-crystal $\mathbb{Q}'$ consisting of Q' together with the maps wt , $\check{e}_i$, $\check{f}_i$, $\check{\epsilon}_i$ and $\check{\varphi}_i$ ($i \in I$) of $\mathbb{Q}$ restricted to Q' .

For each $x \in Q$, the connected component of $\mathbb{Q}$ containing x is denoted by $Q(x)$, and the associated quasi-crystal is denoted by $\mathbb{Q}(x)$.

As a justification for this terminology, we check that connected components of a quasi-crystal $\mathbb{Q}$ and the vertex sets of connected components of the quasi-crystal graph $\Gamma_{\mathbb{Q}}$ identify the same subsets of Q .

Proposition 7.23. Let $\mathbb{Q}$ be a quasi-crystal, and let $Q' \subseteq Q$. Then, Q' is a connected component of $\mathbb{Q}$ if and only if Q' is the vertex set of a connected component of $\Gamma_{\mathbb{Q}}$.

Proof. Assume that Q' is a connected component of $\mathbb{Q}$. Let $x, y \in Q'$. By Definition 7.22(i), there exist $g_1, \dots, g_m \in \{\check{e}_i, \check{f}_i \mid i \in I\}$ such that $g_1 \cdots g_m(x) = y$. Set $x_0 = x$, and $x_{k+1} = g_{m-k}(x_k)$, for $k = 0, 1, \dots, m-1$. Note that $x_{k+1} = g_{m-k} \cdots g_1(x) \in Q'$, by Definition 7.22(ii). In particular, $x_m = g_1 \cdots g_m(x) = y$. If $g_{m-k} = \check{f}_i$, for some $i \in I$, then $x_k \xrightarrow{i} x_{k+1}$ is an edge of $\Gamma_{\mathbb{Q}}$. Otherwise, $g_{m-k} = \check{e}_i$, for some $i \in I$, and so,

$x_{k+1} \xrightarrow{i} x_k$ is an edge of $\Gamma_{\mathbb{Q}}$. For $z \in Q$, if $x \xrightarrow{i} z$ or $z \xrightarrow{i} x$ are edges of $\Gamma_{\mathbb{Q}}$, then $z = \check{f}_i(x)$ or $z = \check{e}_i(x)$, which implies that $z \in Q'$, by Definition 7.22(ii). Therefore, the subgraph of $\Gamma_{\mathbb{Q}}$ induced by Q' is a connected component.

Conversely, assume that Q' is a vertex set of a connected component of $\Gamma_{\mathbb{Q}}$. Let $x, y \in Q'$. Then, there exist $x_0, \dots, x_m \in Q'$ such that $x = x_0$, $y = x_m$, and for $k = 0, \dots, m$, $x_k \xrightarrow{i} x_{k+1}$ or $x_{k+1} \xrightarrow{i} x_k$ are edges of $\Gamma_{\mathbb{Q}}$, for some $i \in I$. If $x_k \xrightarrow{i} x_{k+1}$ is an edge of $\Gamma_{\mathbb{Q}}$, for some $i \in I$, set $g_{m-k} = \check{f}_i$. Otherwise, $x_{k+1} \xrightarrow{i} x_k$ is an edge of $\Gamma_{\mathbb{Q}}$, for some $i \in I$, and so, set $g_{m-k} = \check{e}_i$. In any case we have that $x_{k+1} = g_{m-k}(x_k)$, which implies that $g_1 \cdots g_m(x) = y$. For $i \in I$, if $\check{f}_i(x) \in Q$, then $x \xrightarrow{i} \check{f}_i(x)$ is an edge of $\Gamma_{\mathbb{Q}}$, and so, $\check{f}_i(x) \in Q'$, because Q' is the vertex set of a connected component. Similarly, if $\check{e}_i(x) \in Q$, then $\check{e}_i(x) \xrightarrow{i} x$ is an edge of $\Gamma_{\mathbb{Q}}$, and so, $\check{e}_i(x) \in Q'$. Therefore, Q' is a connected component of $\mathbb{Q}$. $\square$

We also have the following relation between quasi-crystal homomorphisms and graph homomorphisms.

Lemma 7.24. *Let $\mathbb{Q}$ and $\mathbb{Q}'$ be quasi-crystals of the same type, and let $\psi : \mathbb{Q} \rightarrow \mathbb{Q}'$ be a quasi-crystal homomorphism such that $\psi(Q) \subseteq Q'$. Then, ψ is a graph homomorphism from $\Gamma_{\mathbb{Q}}$ to $\Gamma_{\mathbb{Q}'}$.*

Proof. Let $x, y \in Q$ and $i \in I$. If $x \xrightarrow{i} x$ is an edge of $\Gamma_{\mathbb{Q}}$, then $\check{e}_i(x) = +\infty$, and by Definition 7.12(ii), $\check{e}_i(\psi(x)) = +\infty$, which implies that $\psi(x) \xrightarrow{i} \psi(x)$ is an edge of $\Gamma_{\mathbb{Q}'}$. If $x \xrightarrow{i} y$ is an edge of $\Gamma_{\mathbb{Q}}$ with $x \neq y$, then $\check{f}_i(x) = y$, and since $\psi(x), \psi(y) \in \psi(Q) \subseteq Q'$, we obtain that $\check{f}_i(\psi(x)) = \psi(y)$. By Definition 7.12(iv), which implies that $\psi(x) \xrightarrow{i} \psi(y)$ is an edge of $\Gamma_{\mathbb{Q}'}$. Finally, by Definition 7.12(ii), we have that $\text{wt}(\psi(x)) = \text{wt}(x)$. And therefore, ψ is a graph homomorphism. $\square$

Note that a quasi-crystal isomorphism ψ between quasi-crystals $\mathbb{Q}$ and $\mathbb{Q}'$ satisfies the property $\psi(Q) = Q'$. Thus, it is immediate that ψ and ψ^{-1} are graph homomorphisms, which implies that ψ is a graph isomorphism between $\Gamma_{\mathbb{Q}}$ and $\Gamma_{\mathbb{Q}'}$. This leads to the following result.

Proposition 7.25. *Let $\mathbb{Q}$ and $\mathbb{Q}'$ be quasi-crystals of the same type, and let $\psi : \mathbb{Q} \rightarrow \mathbb{Q}'$ be a quasi-crystal isomorphism. Then, Q_0 is a connected component of $\mathbb{Q}$ if and only if $\psi(Q_0)$ is a connected component of $\mathbb{Q}'$. Furthermore, for each $x \in Q$, the restriction of ψ to $Q(x)$ is a quasi-crystal isomorphism between $\mathbb{Q}(x)$ and $\mathbb{Q}'(\psi(x))$.*

Proof. Suppose Q_0 is a connected component of $\mathbb{Q}$. Let $x', y' \in \psi(Q_0)$. Set $x = \psi^{-1}(x')$ and $y = \psi^{-1}(y')$. As $x, y \in Q_0$, there exist $g_1, \dots, g_m \in \{\check{e}_i, \check{f}_i \mid i \in I\}$ such that

$g_1 \cdots g_m(x) = y$. By Lemma 7.16 and Theorem 7.17, we have that

$$g_1 \cdots g_m(x') = g_1 \cdots g_m(\psi(x)) = \psi(g_1 \cdots g_m(x)) = \psi(y) = y'.$$

And so, $\psi(Q_0)$ satisfies Definition 7.22(i). Also, for $i \in I$, we have that $\check{e}_i(x), \check{f}_i(x) \in Q_0 \sqcup \{\perp\}$, which implies that $\psi(\check{e}_i(x)), \psi(\check{f}_i(x)) \in \psi(Q_0) \sqcup \{\perp\}$. By Lemma 7.16 and Theorem 7.17, $\psi(\check{e}_i(x)) = \check{e}_i(\psi(x)) = \check{e}_i(x')$ and $\psi(\check{f}_i(x)) = \check{f}_i(\psi(x)) = \check{f}_i(x')$, and thus, $\check{e}_i(x'), \check{f}_i(x') \in \psi(Q_0) \sqcup \{\perp\}$. Therefore, $\psi(Q_0)$ is a connected component of $\mathbb{Q}'$.

Since ψ^{-1} is a quasi-crystal isomorphism between $\mathbb{Q}'$ and $\mathbb{Q}$, by the previous implication, we get that if $\psi(Q_0)$ is a connected component of $\mathbb{Q}'$, then $\psi^{-1}(\psi(Q_0)) = Q_0$ is a connected component of $\mathbb{Q}$.

Finally, let $z \in Q$. Since $Q(z)$ is a connected component of $\mathbb{Q}$, then $\psi(Q(z))$ is also a connected component of $\mathbb{Q}'$. As $\psi(z) \in \psi(Q(z))$, we get that $\psi(Q(z)) = Q'(\psi(z))$. The restriction of ψ to $Q(z)$ is a bijective quasi-crystal homomorphism from $\mathbb{Q}(z)$ to $\mathbb{Q}'(\psi(z))$, because ψ is a quasi-crystal homomorphism from $\mathbb{Q}$ to $\mathbb{Q}'$ and $\psi(Q(z)) = Q'(\psi(z))$. Similarly, the restriction of ψ^{-1} to $Q'(\psi(z))$ is a bijective quasi-crystal homomorphism from $\mathbb{Q}'(\psi(z))$ to $\mathbb{Q}(z)$. And therefore, the restriction of ψ to $Q(z)$ is a quasi-crystal isomorphism between $\mathbb{Q}(z)$ and $\mathbb{Q}'(\psi(z))$. $\square$

For a quasi-crystal $\mathbb{Q}$, it is immediate that the quasi-Kashiwara operators $\check{e}_i$ and $\check{f}_i$ ($i \in I$) are completely determined by the quasi-crystal graph $\Gamma_{\mathbb{Q}}$, because given $x, y \in Q$ with $x \neq y$, we have that $x \xrightarrow{i} y$ is an edge of $\Gamma_{\mathbb{Q}}$ if and only if $\check{f}_i(x) = y$ and $\check{e}_i(y) = x$. Now, we show that if $\mathbb{Q}$ is seminormal, then the maps $\check{e}_i$ and $\check{f}_i$ ($i \in I$) are also completely determined by $\Gamma_{\mathbb{Q}}$.

Lemma 7.26. *Let $\mathbb{Q}$ be a quasi-crystal, and let $i \in I$. Given an i -labelled walk*

$$x_0 \xrightarrow{i} x_1 \xrightarrow{i} \cdots \xrightarrow{i} x_m$$

on $\Gamma_{\mathbb{Q}}$, then either

(i) $x_0 = x_1 = \cdots = x_m$; or

(ii) $x_k = \check{f}_i^k(x_0)$, for $k = 0, 1, \dots, m$, and thus, $x_0, x_1, \dots, x_m$ form the unique i -labelled path on $\Gamma_{\mathbb{Q}}$ starting at x_0 and ending at x_m .

Proof. If $x_0 = x_1$, then x_0 has an i -labelled loop, and so, $\check{e}_i(x_0) = +\infty$. Thus, $\check{f}_i(x_1) = \check{f}_i(x_0) = \perp$, which implies that $x_1 = x_2$. And recursively, we obtain that $x_0 = x_1 = \cdots = x_m$.

Otherwise, we have that $x_0 \neq x_1$, which implies that $\check{f}_i(x_0) = x_1$ (or equivalently, $\check{e}_i(x_1) = x_0$), because $x_0 \xrightarrow{i} x_1$ is an edge of $\Gamma_{\mathbb{Q}}$. Since $\check{e}_i$ is defined on x_1 , then

$\ddot{e}_i(x_1) \neq +\infty$, and so, x_1 does not have an i -labelled loop. Hence, $x_1 \neq x_2$, and so, $\ddot{f}_i(x_1) = x_2$, because $x_1 \xrightarrow{i} x_2$ is an edge of $\Gamma_{\mathbb{Q}}$. Recursively, we get that $\ddot{f}_i(x_k) = x_{k+1}$, for $k = 0, 1, \dots, m-1$. By Proposition 7.5, we have that $\text{wt}(x_0) > \text{wt}(x_1) > \dots > \text{wt}(x_m)$, and thus, $x_0, x_1, \dots, x_m$ are pairwise distinct. Hence, $x_0, x_1, \dots, x_m$ form an i -labelled path on $\Gamma_{\mathbb{Q}}$. If

$$x'_0 \xrightarrow{i} x'_1 \xrightarrow{i} \dots \xrightarrow{i} x'_p$$

is an i -labelled path on $\Gamma_{\mathbb{Q}}$ such that $x'_0 = x_0$ and $x'_m = x_m$, we have that

$$x'_k = \ddot{f}_i^k(x'_0) = \ddot{f}_i^k(x_0) = x_k,$$

for $k = 0, 1, \dots, m$, and as $x'_m = x_m = x'_p$, we also have that $p = m$. Therefore, $x_0, x_1, \dots, x_m$ form the unique i -labelled path on $\Gamma_{\mathbb{Q}}$ starting at x_0 and ending on x_m . $\square$

Proposition 7.27. *Let $\mathbb{Q}$ be a seminormal quasi-crystal. For $x \in Q$ and $i \in I$, we have that*

- (i) $\ddot{\varphi}_i(x)$ is the supremum among nonnegative integers $m \in \mathbb{Z}_{\geq 0}$ such that there exists an i -labelled walk on $\Gamma_{\mathbb{Q}}$ starting at x with length m ;
- (ii) $\ddot{e}_i(x)$ is the supremum among nonnegative integers $m \in \mathbb{Z}_{\geq 0}$ such that there exists an i -labelled walk on $\Gamma_{\mathbb{Q}}$ ending at x with length m .

Proof. (i) Let $z \in \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ be the supremum among nonnegative integers $m \in \mathbb{Z}_{\geq 0}$ such that there exists an i -labelled walk on $\Gamma_{\mathbb{Q}}$ starting on x with length m . If $\ddot{\varphi}_i(x) = +\infty$, then x has an i -labelled loop on $\Gamma_{\mathbb{Q}}$. And so, for any $m \in \mathbb{Z}_{\geq 0}$, the sequence $x_0, \dots, x_m$, where $x_0 = \dots = x_m = x$, is an i -labelled walk starting on x with length m . Hence, $z = +\infty = \ddot{\varphi}_i(x)$.

Otherwise, we have that

$$\ddot{\varphi}_i(x) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \ddot{f}_i^k(x) \in Q\},$$

because $\mathbb{Q}$ is seminormal. Since

$$x \xrightarrow{i} \ddot{f}_i(x) \xrightarrow{i} \dots \xrightarrow{i} \ddot{f}_i^{\ddot{\varphi}_i(x)}(x)$$

is an i -labelled path on $\Gamma_{\mathbb{Q}}$ ending on x , we have that $\ddot{\varphi}_i(x) \leq z$. Since x has no i -labelled loops, if

$$x_0 \xrightarrow{i} x_1 \xrightarrow{i} \dots \xrightarrow{i} x_m$$

is a walk on $\Gamma_{\mathbb{Q}}$ such that $x_0 = x$, then $x_k = \check{f}_i^k(x)$, for $k = 0, 1, \dots, m$, by Lemma 7.26. And since $\check{f}_i^{\check{\varphi}_i(x)+1}(x) = x$, we get that $m \leq \check{\varphi}_i(x)$. Hence, $z \leq \check{\varphi}_i(x)$, and therefore, $\check{\varphi}_i(x) = z$.

(ii) Analogously to (i), we have that if $\check{\varepsilon}_i(x) = +\infty$, then the lengths of i -labelled walks on $\Gamma_{\mathbb{Q}}$ ending on x are unbounded. And otherwise,

$$\check{\varepsilon}_i^{\check{\varepsilon}_i(x)}(x) \xrightarrow{i} \check{\varepsilon}_i^{\check{\varepsilon}_i(x)-1}(x) \xrightarrow{i} \dots \xrightarrow{i} x$$

is the longest i -labelled walk on $\Gamma_{\mathbb{Q}}$ ending on x . □

By the construction of a quasi-crystal graph, we showed how the maps $\check{\varepsilon}_i$, $\check{f}_i$, $\check{\varepsilon}_i$ and $\check{\varphi}_i$ ($i \in I$) of a seminormal quasi-crystal can be described by an I -labelled directed graph. And so, a seminormal quasi-crystal can be completely described by a Λ -weighted I -labelled directed graph. From this correspondence between seminormal quasi-crystals and weighted I -labelled directed graphs, we can identify a subclass of graphs which leads to a purely combinatorial description of seminormal quasi-crystals.

Remark 7.28. By translating Definitions 7.1 and 7.9 for weighted labelled directed graphs we obtain a subclass of graphs, whose elements are call *seminormal quasi-crystal graphs*. Consider a root system Φ with weight lattice Λ and index set I for the simple roots $(\alpha_i)_{i \in I}$. A Λ -weighted I -labelled directed graph Γ is a *seminormal quasi-crystal graph* if for any vertices x and y , and any $i \in I$, the following conditions are satisfied:

- (i) x is the start of at most an edge labelled by i , and is the end of at most an edge labelled by i ;
- (ii) any i -labelled path of Γ is finite;
- (iii) if $x \xrightarrow{i} y$ is an edge of Γ with $x \neq y$, then $\text{wt}(y) = \text{wt}(x) - \alpha_i$;
- (iv) $\check{\varphi}_i(x) = \check{\varepsilon}_i(x) + \langle \text{wt}(x), \alpha_i^\vee \rangle$, where $\check{\varphi}_i(x)$ is the supremum among nonnegative integers $k \in \mathbb{Z}_{\geq 0}$ such that there exists an i -labelled walk on Γ starting on x with length k , and $\check{\varepsilon}_i(x)$ is the supremum among nonnegative integers $l \in \mathbb{Z}_{\geq 0}$ such that there exists an i -labelled walk on Γ ending on x with length l .

Note that if Γ is a seminormal quasi-crystal with vertex set Q , we can define partial maps $\check{\varepsilon}_i$ and $\check{f}_i$ ($i \in I$) on Q , by setting $\check{\varepsilon}_i(y) = x$ and $\check{f}_i(x) = y$, whenever $x \xrightarrow{i} y$ is an edge of Γ with $x \neq y$. Then, we get a seminormal quasi-crystal $\mathbb{Q}$, and Γ coincides with $\Gamma_{\mathbb{Q}}$.

Due to this relation between seminormal quasi-crystals and weighted labelled directed graphs we obtain the following result.

Theorem 7.29. *Let $\mathbb{Q}$ and $\mathbb{Q}'$ be seminormal quasi-crystals of the same type, and let $\psi : Q \rightarrow Q'$. Then, ψ is a quasi-crystal isomorphism between $\mathbb{Q}$ and $\mathbb{Q}'$ if and only if ψ is a graph isomorphism between the weighted labelled directed graphs $\Gamma_{\mathbb{Q}}$ and $\Gamma_{\mathbb{Q}'}$.*

Proof. If ψ is a quasi-crystal isomorphism between $\mathbb{Q}$ and $\mathbb{Q}'$, then by Lemma 7.24, ψ is a graph homomorphism from $\Gamma_{\mathbb{Q}}$ to $\Gamma_{\mathbb{Q}'}$, and ψ^{-1} is a graph homomorphism from $\Gamma_{\mathbb{Q}'}$ to $\Gamma_{\mathbb{Q}}$. Hence, ψ is a graph isomorphism between $\Gamma_{\mathbb{Q}}$ and $\Gamma_{\mathbb{Q}'}$.

Conversely, suppose that ψ is a graph isomorphism between $\Gamma_{\mathbb{Q}}$ and $\Gamma_{\mathbb{Q}'}$. Let $x \in Q$ and $i \in I$. As we remarked ψ is a graph isomorphism between weighted graphs, then we require ψ to be weight-preserving, and so, $\text{wt}(\psi(x)) = \text{wt}(x)$. If $\check{f}_i(x)$ is defined, then $x \xrightarrow{i} \check{f}_i(x)$ is an edge of $\Gamma_{\mathbb{Q}}$, which implies that $\psi(x) \xrightarrow{i} \psi(\check{f}_i(x))$ is an edge of $\Gamma_{\mathbb{Q}'}$. Since $x \neq \check{f}_i(x)$ and ψ is bijective, then $\psi(x) \neq \psi(\check{f}_i(x))$, and so, $\check{f}_i(\psi(x)) = \psi(\check{f}_i(x))$. Analogously, since ψ^{-1} is also a graph isomorphism, we have that if $\check{f}_i(\psi(x))$ is defined, then $\psi(x) \xrightarrow{i} \check{f}_i(\psi(x))$ is an edge of $\Gamma_{\mathbb{Q}'}$, which implies that $\check{f}_i(x) = \psi^{-1}(\check{f}_i(\psi(x)))$, or equivalently, $\psi(\check{f}_i(x)) = \check{f}_i(\psi(x))$. Hence, $\psi(\check{f}_i(y)) = \check{f}_i(\psi(y))$, for all $y \in Q$. Since ψ is a graph isomorphism, we have that $x_0, x_1, \dots, x_m \in Q$ form an i -labelled walk on $\Gamma_{\mathbb{Q}}$ if and only if $\psi(x_0), \psi(x_1), \dots, \psi(x_m)$ form an i -labelled walk on $\Gamma_{\mathbb{Q}'}$. Thus, by Proposition 7.27, $\varepsilon_i(\psi(x)) = \varepsilon_i(x)$ and $\check{\varphi}_i(\psi(x)) = \check{\varphi}_i(x)$. Therefore, by Corollary 7.18, ψ is a quasi-crystal isomorphism between $\mathbb{Q}$ and $\mathbb{Q}'$. $\square$

In the case of seminormal quasi-crystals, we can give a stronger version of Proposition 7.25.

Proposition 7.30. *Let $\mathbb{Q}$ and $\mathbb{Q}'$ be seminormal quasi-crystals of the same type, and let $\psi : \mathbb{Q} \rightarrow \mathbb{Q}'$ be a quasi-crystal homomorphism. For each $x \in Q$, if $\psi(Q(x)) \subseteq Q'$, then the restriction of ψ to $Q(x)$ is a surjective quasi-crystal homomorphism from $\mathbb{Q}(x)$ to $\mathbb{Q}'(\psi(x))$.*

Proof. For any $y \in Q(x)$ and $i \in I$, as $\psi(Q(x)) \subseteq Q'$, we have that $\check{f}_i(y)$ is defined if and only if $\psi(\check{f}_i(y)) \in Q'$. And if so, $\psi(\check{f}_i(y)) = \check{f}_i(\psi(y))$, because ψ is a quasi-crystal homomorphism. Similarly, if $\check{e}_i(y)$ is defined, or equivalently, $\psi(\check{e}_i(y))$ is defined, then $\psi(\check{e}_i(y)) = \check{e}_i(\psi(y))$. In particular, if $g_1 \cdots g_m(x)$ is defined, where $g_1, \dots, g_m \in \{\check{e}_j, \check{f}_j \mid j \in I\}$, then $\psi(g_1 \cdots g_m(x)) = g_1 \cdots g_m(\psi(x))$ is defined, and so, $\psi(g_1 \cdots g_m(x)) \in Q'(\psi(x))$. This implies that $\psi(Q(x)) \subseteq Q'(\psi(x))$, and thus, the restriction of ψ to $Q(x)$ induces a well-defined map from $Q(x)$ to $Q'(\psi(x))$. Also,

as ψ is a quasi-crystal homomorphism, we get that the restriction of ψ to $Q(x)$ is a quasi-crystal homomorphism from $\mathbb{Q}(x)$ to $\mathbb{Q}'(\psi(x))$.

For any $y \in Q(x)$ and $i \in I$, as $\mathbb{Q}$ and $\mathbb{Q}'$ are seminormal and ψ is a quasi-crystal homomorphism, we have that if $\check{f}_i(\psi(y))$ is defined, then $\check{\varphi}_i(y) = \check{\varphi}_i(\psi(y)) \in \mathbb{Z}_{\geq 1}$, which implies that $\check{f}_i(y)$ is defined, and as we proved above, $\psi(\check{f}_i(y)) = \check{f}_i(\psi(y))$. Analogously, if $\check{e}_i(\psi(y))$ is defined, then $\check{e}_i(y)$ is also defined and $\psi(\check{e}_i(y)) = \check{e}_i(\psi(y))$. Hence, given $y' \in Q'(\psi(x))$, there exist $g_1, \dots, g_m \in \{\check{e}_j, \check{f}_j \mid j \in I\}$ such that $g_1 \cdots g_m(\psi(x)) = y'$, then $g_1 \cdots g_m(x)$ is defined, and $\psi(g_1 \cdots g_m(x)) = g_1 \cdots g_m(\psi(x)) = y'$. Therefore, ψ is surjective. $\square$

Due to the connection between quasi-crystals and graphs, an element of a quasi-crystal $\mathbb{Q}$ is also a vertex of the graph $\Gamma_{\mathbb{Q}}$, which allows us to define properties like the following.

Definition 7.31. Let $\mathbb{Q}$ be a quasi-crystal. An element $x \in Q$ is said to be *isolated* if $Q(x) = \{x\}$.

Note that an isolated element of a quasi-crystal $\mathbb{Q}$ is an isolated vertex of the quasi-crystal graph $\Gamma_{\mathbb{Q}}$. Thus, an element $x \in Q$ is isolated if and only if $\check{e}_i(x) = \check{f}_i(x) = \perp$, for all $i \in I$, or equivalently, x is isolated if and only if x is of highest and lowest weight. Furthermore, if $\mathbb{Q}$ is seminormal, then x is isolated if and only if for each $i \in I$, either $\check{e}_i(x) = \check{\varphi}_i(x) = 0$ or $\check{e}_i(x) = \check{\varphi}_i(x) = +\infty$.

7.3 Quasi-tensor product

A definition of tensor product for quasi-crystals can be given in a similar way as it was originally done for crystals (see Section 4.3). Such a definition would lead to a generalization to quasi-crystals of the construction of a plactic monoid from a crystal as in Chapter 5. Since we are interested in a general construction of the hypoplactic monoid from a quasi-crystal, in this section we introduce a slightly different definition.

Theorem 7.32. Consider a root system Φ with weight lattice Λ and index set I for the simple roots $(\alpha_i)_{i \in I}$. Let $\mathbb{Q}$ and $\mathbb{Q}'$ be seminormal quasi-crystals of type Φ . Set $Q \tilde{\otimes} Q'$ to be the Cartesian product $Q \times Q'$ whose ordered pairs are denoted by $x \tilde{\otimes} x'$ with $x \in Q$ and $x' \in Q'$. Define a map $\text{wt} : Q \tilde{\otimes} Q' \rightarrow \Lambda$ by

$$\text{wt}(x \tilde{\otimes} x') = \text{wt}(x) + \text{wt}(x'),$$

for $x \in Q$ and $x' \in Q'$. And for each $i \in I$, define maps $\check{e}_i, \check{f}_i : Q \tilde{\otimes} Q' \rightarrow (Q \tilde{\otimes} Q') \sqcup \{\perp\}$ and $\check{\varepsilon}_i, \check{\varphi}_i : Q \tilde{\otimes} Q' \rightarrow \mathbb{Z} \cup \{-\infty, +\infty\}$ as follows:

(i) if $\check{\varphi}_i(x), \check{\varepsilon}_i(x') > 0$, set

$$\check{e}_i(x \check{\otimes} x') = \check{f}_i(x \check{\otimes} x') = \perp \quad \check{\varepsilon}_i(x \check{\otimes} x') = \check{\varphi}_i(x \check{\otimes} x') = +\infty;$$

(ii) otherwise, set

$$\begin{aligned} \check{e}_i(x \check{\otimes} x') &= \begin{cases} \check{e}_i(x) \check{\otimes} x' & \text{if } \check{\varphi}_i(x) \geq \check{\varepsilon}_i(x') \\ x \check{\otimes} \check{e}_i(x') & \text{if } \check{\varphi}_i(x) < \check{\varepsilon}_i(x'), \end{cases} \\ \check{f}_i(x \check{\otimes} x') &= \begin{cases} \check{f}_i(x) \check{\otimes} x' & \text{if } \check{\varphi}_i(x) > \check{\varepsilon}_i(x') \\ x \check{\otimes} \check{f}_i(x') & \text{if } \check{\varphi}_i(x) \leq \check{\varepsilon}_i(x'), \end{cases} \\ \check{\varepsilon}_i(x \check{\otimes} x') &= \max\{\check{\varepsilon}_i(x), \check{\varepsilon}_i(x') - \langle \text{wt}(x), \alpha_i^\vee \rangle\}, \\ \check{\varphi}_i(x \check{\otimes} x') &= \max\{\check{\varphi}_i(x) + \langle \text{wt}(x'), \alpha_i^\vee \rangle, \check{\varphi}_i(x')\}, \end{aligned}$$

where $x \check{\otimes} \perp = \perp \check{\otimes} x' = \perp$;

for $x \in Q$ and $x' \in Q'$. Then, $Q \check{\otimes} Q'$ together with the maps wt , $\check{e}_i$, $\check{f}_i$, $\check{\varepsilon}_i$ and $\check{\varphi}_i$ ($i \in I$) forms a seminormal quasi-crystal of type Φ .

Proof. Let $x \in Q$, $x' \in Q'$ and $i \in I$. We have that $\check{\varepsilon}_i(x), \check{\varphi}_i(x), \check{\varepsilon}_i(x'), \check{\varphi}_i(x') \geq 0$ as $\mathbb{Q}$ and $\mathbb{Q}'$ are seminormal (Definition 7.9). If $\check{\varphi}_i(x), \check{\varepsilon}_i(x') > 0$, it is immediate that $x \check{\otimes} x'$ satisfies all conditions of Definition 7.1, namely, conditions (i) and (vi) which are the ones that apply to this case. So, assume that $\check{\varphi}_i(x) = 0$ or $\check{\varepsilon}_i(x') = 0$.

We have that

$$\begin{aligned} \check{\varphi}_i(x \check{\otimes} x') &= \max\{\check{\varphi}_i(x) + \langle \text{wt}(x'), \alpha_i^\vee \rangle, \check{\varphi}_i(x')\} \\ &= \max\{\check{\varphi}_i(x), \check{\varphi}_i(x') - \langle \text{wt}(x'), \alpha_i^\vee \rangle\} + \langle \text{wt}(x'), \alpha_i^\vee \rangle \\ &= \max\{\check{\varepsilon}_i(x) + \langle \text{wt}(x), \alpha_i^\vee \rangle, \check{\varepsilon}_i(x')\} + \langle \text{wt}(x'), \alpha_i^\vee \rangle \\ &= \max\{\check{\varepsilon}_i(x), \check{\varepsilon}_i(x') - \langle \text{wt}(x), \alpha_i^\vee \rangle\} + \langle \text{wt}(x), \alpha_i^\vee \rangle + \langle \text{wt}(x'), \alpha_i^\vee \rangle \\ &= \check{\varepsilon}_i(x \check{\otimes} x') + \langle \text{wt}(x \check{\otimes} x'), \alpha_i^\vee \rangle. \end{aligned}$$

Hence, condition (i) of Definition 7.1 is satisfied.

If $\check{\varphi}_i(x) = +\infty$ and $\check{\varepsilon}_i(x') = 0$, then $\check{\varepsilon}_i(x) = +\infty$ and $\check{e}_i(x) = \check{f}_i(x) = \perp$ as $\mathbb{Q}$ is a quasi-crystal. And so,

$$\check{\varepsilon}_i(x \check{\otimes} x') = \check{\varphi}_i(x \check{\otimes} x') = +\infty$$

and

$$\check{e}_i(x \check{\otimes} x') = \check{e}_i(x) \check{\otimes} x' = \perp \check{\otimes} x' = \perp \quad \text{and} \quad \check{f}_i(x \check{\otimes} x') = \check{f}_i(x) \check{\otimes} x' = \perp \check{\otimes} x' = \perp.$$

Also, if $\check{\varphi}_i(x) = 0$ and $\check{\varepsilon}_i(x') = +\infty$, it follows analogously that $\check{e}_i(x \check{\otimes} x') = \check{f}_i(x \check{\otimes} x') = \perp$ and $\check{\varepsilon}_i(x \check{\otimes} x') = \check{\varphi}_i(x \check{\otimes} x') = +\infty$. So, besides $\check{\varphi}_i(x) = 0$ or $\check{\varepsilon}_i(x') = 0$, we may

further assume $\check{\varphi}_i(x), \check{\varepsilon}_i(x') \in \mathbb{Z}_{\geq 0}$ which also implies $\check{\varepsilon}_i(x), \check{\varphi}_i(x') \in \mathbb{Z}_{\geq 0}$ as $\mathbb{Q}$ and $\mathbb{Q}'$ are seminormal quasi-crystals.

Since we have $\langle \text{wt}(x), \alpha_i^\vee \rangle = \check{\varphi}_i(x) - \check{\varepsilon}_i(x)$ and $\langle \text{wt}(x'), \alpha_i^\vee \rangle = \check{\varphi}_i(x') - \check{\varepsilon}_i(x')$, then we obtain

$$\check{\varepsilon}_i(x \check{\otimes} x') = \max\{\check{\varepsilon}_i(x), \check{\varepsilon}_i(x') - \check{\varphi}_i(x) + \check{\varepsilon}_i(x)\}$$

and

$$\check{\varphi}_i(x \check{\otimes} x') = \max\{\check{\varphi}_i(x) + \check{\varphi}_i(x') - \check{\varepsilon}_i(x'), \check{\varphi}_i(x')\}.$$

As we also have that $\check{\varepsilon}_i(x), \check{\varepsilon}_i(x'), \check{\varphi}_i(x), \check{\varphi}_i(x') \in \mathbb{Z}_{\geq 0}$ where $\check{\varphi}_i(x) = 0$ or $\check{\varepsilon}_i(x) = 0$, we deduce that

$$\check{\varepsilon}_i(x \check{\otimes} x') = \check{\varepsilon}_i(x) + \check{\varepsilon}_i(x') \quad \text{and} \quad \check{\varphi}_i(x \check{\otimes} x') = \check{\varphi}_i(x) + \check{\varphi}_i(x'). \quad (7.1)$$

Now, we consider the following cases.

- Case 1: $\check{\varphi}_i(x) = \check{\varepsilon}_i(x') = 0$. We have that

$$\check{\varepsilon}_i(x \check{\otimes} x') = \check{\varepsilon}_i(x) \check{\otimes} x', \quad \text{and} \quad \check{f}_i(x \check{\otimes} x') = x \check{\otimes} \check{f}_i(x').$$

Thus, $\check{\varepsilon}_i(x \check{\otimes} x')$ is defined if and only if $\check{\varepsilon}_i(x) \in Q$. If so, then we have that

$$\begin{aligned} \text{wt}(\check{\varepsilon}_i(x \check{\otimes} x')) &= \text{wt}(\check{\varepsilon}_i(x)) + \text{wt}(x') = \text{wt}(x) + \alpha_i + \text{wt}(x') \\ &= \text{wt}(x \check{\otimes} x') + \alpha_i, \end{aligned}$$

and since $\check{\varepsilon}_i(x) \check{\otimes} x'$ satisfies the conditions leading to (7.1), we deduce that

$$\begin{aligned} \check{\varepsilon}_i(\check{\varepsilon}_i(x \check{\otimes} x')) &= \check{\varepsilon}_i(\check{\varepsilon}_i(x)) + \check{\varepsilon}_i(x') = \check{\varepsilon}_i(x) - 1 + \check{\varepsilon}_i(x') \\ &= \check{\varepsilon}_i(x \check{\otimes} x') - 1 \end{aligned}$$

and

$$\begin{aligned} \check{\varphi}_i(\check{\varepsilon}_i(x \check{\otimes} x')) &= \check{\varphi}_i(\check{\varepsilon}_i(x)) + \check{\varphi}_i(x') = \check{\varphi}_i(x) + 1 + \check{\varphi}_i(x') \\ &= \check{\varphi}_i(x \check{\otimes} x') + 1. \end{aligned}$$

Also, as $\check{\varphi}_i(\check{\varepsilon}_i(x)) = \check{\varphi}_i(x) + 1 = 1$ and $\check{\varepsilon}_i(x') = 0$, we have

$$\check{f}_i(\check{\varepsilon}_i(x \check{\otimes} x')) = \check{f}_i(\check{\varepsilon}_i(x) \check{\otimes} x') = \check{f}_i(\check{\varepsilon}_i(x)) \check{\otimes} x' = x \check{\otimes} x'.$$

On the other hand, $\check{f}_i(x \check{\otimes} x')$ is defined if and only if $\check{f}_i(x') \in Q'$. If so, we have $\check{\varphi}_i(x) = 0$ and $\check{\varepsilon}_i(\check{f}_i(x')) = \check{\varepsilon}_i(x') + 1 = 1$, and thus,

$$\check{\varepsilon}_i(\check{f}_i(x \check{\otimes} x')) = \check{\varepsilon}_i(x \check{\otimes} \check{f}_i(x')) = x \check{\otimes} \check{\varepsilon}_i(\check{f}_i(x')) = x \check{\otimes} x'.$$

Hence, Definition 7.1(iii) also holds in this case by Proposition 7.3.

- Case 2: $\check{\varphi}_i(x) > 0$ and $\check{\varepsilon}_i(x') = 0$. We have that

$$\check{e}_i(x \otimes x') = \check{e}_i(x) \otimes x', \quad \text{and} \quad \check{f}_i(x \otimes x') = \check{f}_i(x) \otimes x'.$$

The results for $\check{e}_i(x \otimes x')$ follow as in case 1. On the other hand, since $\mathbb{Q}$ is seminormal and $\check{\varphi}_i(x) > 0$, we have that $\check{f}_i(x) \in Q$, and so, $\check{f}_i(x \otimes x')$ is defined. As $\check{\varphi}_i(\check{f}_i(x)) = \check{\varphi}_i(x) - 1 \geq 0$ and $\check{\varepsilon}_i(x') = 0$, we get

$$\check{e}_i(\check{f}_i(x \otimes x')) = \check{e}_i(\check{f}_i(x) \otimes x') = \check{e}_i(\check{f}_i(x)) \otimes x' = x \otimes x'.$$

Therefore, all conditions of Definition 7.1 hold for this case.

- Case 3: $\check{\varphi}_i(x) = 0$ and $\check{\varepsilon}_i(x') > 0$. We have that

$$\check{e}_i(x \otimes x') = x \otimes \check{e}_i(x'), \quad \text{and} \quad \check{f}_i(x \otimes x') = x \otimes \check{f}_i(x').$$

Since $\mathbb{Q}'$ is seminormal and $\check{\varepsilon}_i(x') > 0$, then $\check{e}_i(x') \in Q'$, and so, $\check{e}_i(x \otimes x')$ is defined. We have that

$$\begin{aligned} \text{wt}(\check{e}_i(x \otimes x')) &= \text{wt}(x) + \text{wt}(\check{e}_i(x')) = \text{wt}(x) + \text{wt}(x') + \alpha_i \\ &= \text{wt}(x \otimes x') + \alpha_i, \end{aligned}$$

and since $\check{e}_i(x) \otimes x'$ satisfies the conditions leading to (7.1), we deduce that

$$\begin{aligned} \check{\varepsilon}_i(\check{e}_i(x \otimes x')) &= \check{\varepsilon}_i(x) + \check{\varepsilon}_i(\check{e}_i(x')) = \check{\varepsilon}_i(x) + \check{\varepsilon}_i(x') - 1 \\ &= \check{\varepsilon}_i(x \otimes x') - 1 \end{aligned}$$

and

$$\begin{aligned} \check{\varphi}_i(\check{e}_i(x \otimes x')) &= \check{\varphi}_i(x) + \check{\varphi}_i(\check{e}_i(x')) = \check{\varphi}_i(x) + \check{\varphi}_i(x') + 1 \\ &= \check{\varphi}_i(x \otimes x') + 1. \end{aligned}$$

Also, as $\check{\varphi}_i(x) = 0$ and $\check{\varepsilon}_i(\check{e}_i(x')) = \check{\varepsilon}_i(x') - 1 \geq 0$, we have

$$\check{f}_i(\check{e}_i(x \otimes x')) = \check{f}_i(x \otimes \check{e}_i(x')) = x \otimes \check{f}_i(\check{e}_i(x')) = x \otimes x'.$$

The fact that $\check{e}_i(\check{f}_i(x \otimes x')) = x \otimes x'$ whenever $\check{f}_i(x \otimes x')$ is defined follows as in case 1. And again, all conditions of Definition 7.1 hold.

Therefore, $Q \otimes Q'$ together with wt , $\check{e}_i$, $\check{f}_i$, $\check{\varepsilon}_i$ and $\check{\varphi}_i$ ($i \in I$) forms a quasi-crystal.

Finally, we show that this quasi-crystal is seminormal (Definition 7.9). Assume that $\check{\varepsilon}_i(x \otimes x') \neq +\infty$. Thus, we have that $\check{\varepsilon}_i(x), \check{\varphi}_i(x), \check{\varepsilon}_i(x'), \check{\varphi}_i(x') \in \mathbb{Z}_{\geq 0}$ where $\check{\varphi}_i(x) = 0$ or $\check{\varepsilon}_i(x') = 0$, and so, we lie on one of the three cases above. Since

$\mathbb{Q}$ and $\mathbb{Q}'$ are seminormal, then $\check{e}_i^{\check{\epsilon}_i(x)}(x) \in Q$, $\check{e}_i^{\check{\epsilon}_i(x)+1}(x) = \perp$, $\check{e}_i^{\check{\epsilon}_i(x')}(x') \in Q'$ and $\check{e}_i(\check{e}_i^{\check{\epsilon}_i(x')}(x')) = 0 \leq \check{\varphi}_i(\check{e}_i^{\check{\epsilon}_i(x)}(x))$. By (7.1) and cases 1 and 3 above, we obtain

$$\begin{aligned} \check{e}_i^{\check{\epsilon}_i(x \check{\otimes} x')}(x \check{\otimes} x') &= \check{e}_i^{\check{\varphi}_i(x) + \check{\epsilon}_i(x')}(x \check{\otimes} x') = \check{e}_i^{\check{\epsilon}_i(x)}(x \check{\otimes} \check{e}_i^{\check{\epsilon}_i(x')}(x')) \\ &= \check{e}_i^{\check{\epsilon}_i(x)}(x) \check{\otimes} \check{e}_i^{\check{\epsilon}_i(x')}(x') \end{aligned}$$

and

$$\begin{aligned} \check{e}_i^{\check{\epsilon}_i(x \check{\otimes} x') + 1}(x \check{\otimes} x') &= \check{e}_i(\check{e}_i^{\check{\epsilon}_i(x)}(x) \check{\otimes} \check{e}_i^{\check{\epsilon}_i(x')}(x')) = \check{e}_i^{\check{\epsilon}_i(x) + 1}(x) \check{\otimes} \check{e}_i^{\check{\epsilon}_i(x')}(x') \\ &= \perp \check{\otimes} \check{e}_i^{\check{\epsilon}_i(x')}(x') = \perp. \end{aligned}$$

Similarly, we have that $\check{f}_i^{\check{\varphi}_i(x \check{\otimes} x')}(x \check{\otimes} x') = \check{f}_i^{\check{\varphi}_i(x)}(x) \check{\otimes} \check{f}_i^{\check{\varphi}_i(x')}(x')$ is defined, and $\check{f}_i^{\check{\varphi}_i(x \check{\otimes} x') + 1}(x \check{\otimes} x') = \check{f}_i^{\check{\varphi}_i(x)}(x) \check{\otimes} \check{f}_i^{\check{\varphi}_i(x') + 1}(x') = \perp$. Hence, $Q \check{\otimes} Q'$ together with wt , $\check{e}_i$, $\check{f}_i$, $\check{\epsilon}_i$ and $\check{\varphi}_i$ ($i \in I$) is a seminormal quasi-crystal. $\square$

Note that the quasi-crystal structure on $Q \check{\otimes} Q'$ given in Theorem 7.32 is similar to the definition of the crystal structure for the tensor product of crystals (Definition 4.13), with the exception of condition (i) when $\check{\varphi}_i(x), \check{\epsilon}_i(x') > 0$. If $\check{\varphi}_i(x) = \check{\epsilon}_i(x') = +\infty$, then $\check{e}_i(x) = \check{f}_i(x') = \perp$, and so, if we apply the maps as defined in Theorem 7.32(ii) to $x \check{\otimes} x'$, we would obtain exactly the same result as in Theorem 7.32(i). Since $\mathbb{Q}$ and $\mathbb{Q}'$ are seminormal, we have that $0 < \check{\varphi}_i(x) < +\infty$ and $0 < \check{\epsilon}_i(x') < +\infty$ if and only if $\check{f}_i(x) \in Q$ and $\check{e}_i(x') \in Q'$. Thus, condition (i) of Theorem 7.32 is distinguishing elements of the form $x \check{\otimes} x'$ where $\check{f}_i(x)$ and $\check{e}_i(x')$ are defined, and is specifying values for the maps of the quasi-crystal structure different from what they would be if the definitions in Theorem 7.32(ii) would apply to them. This is a quasi-crystal interpretation of the notion of an i -inversion in a word, introduced in Definition 6.11, which justifies the following terminology.

Definition 7.33. Let $\mathbb{Q}$ and $\mathbb{Q}'$ be seminormal quasi-crystals of the same type. The *inverse-free quasi-tensor product* of $\mathbb{Q}$ and $\mathbb{Q}'$, or simply the *quasi-tensor product* of $\mathbb{Q}$ and $\mathbb{Q}'$, is the seminormal quasi-crystal defined in Theorem 7.32 and is denoted by $\mathbb{Q} \check{\otimes} \mathbb{Q}'$.

We chose to give definitions of the maps of the quasi-crystal structure of a quasi-tensor product in Theorem 7.32 to emphasize their resemblance with the maps of the crystal structure of a tensor product of crystals (Definition 4.13), although in the proof we deduced alternative definitions. Thus, the following result is an immediate consequence of the arguments that led to (7.1) and the cases that followed it.

Proposition 7.34. *Let $\mathbb{Q}$ and $\mathbb{Q}'$ be seminormal quasi-crystals of the same type. For $x \in Q$, $x' \in Q'$ and $i \in I$ with either $\check{\varphi}_i(x) = 0$ or $\check{\varepsilon}_i(x') = 0$, we have that*

$$\begin{aligned} \check{\varepsilon}_i(x \check{\otimes} x') &= \begin{cases} \check{\varepsilon}_i(x) \check{\otimes} x' & \text{if } \check{\varepsilon}_i(x') = 0 \\ x \check{\otimes} \check{\varepsilon}_i(x') & \text{if } \check{\varepsilon}_i(x') > 0, \end{cases} \\ \check{f}_i(x \check{\otimes} x') &= \begin{cases} \check{f}_i(x) \check{\otimes} x' & \text{if } \check{\varphi}_i(x) > 0 \\ x \check{\otimes} \check{f}_i(x') & \text{if } \check{\varphi}_i(x) = 0, \end{cases} \\ \check{\varepsilon}_i(x \check{\otimes} x') &= \check{\varepsilon}_i(x) + \check{\varepsilon}_i(x'), \\ \check{\varphi}_i(x \check{\otimes} x') &= \check{\varphi}_i(x) + \check{\varphi}_i(x'). \end{aligned}$$

Example 7.35. The quasi-crystal $\mathcal{A}_3^2$ in Example 7.4(ii) is isomorphic to $\mathcal{A}_3 \check{\otimes} \mathcal{A}_3$ as the map $A_3 \times A_3 \rightarrow A_3 \check{\otimes} A_3$, given by $(x, y) \mapsto x \check{\otimes} y$ for each $x, y \in A_3$, is a quasi-crystal isomorphism. Thus, from Example 7.21, the quasi-crystal graph of $\mathcal{A}_3 \check{\otimes} \mathcal{A}_3$ is

$$\begin{array}{ccccc} 1 \check{\otimes} 1 & \xrightarrow{1} & 2 \check{\otimes} 1 & \xrightarrow{2} & 3 \check{\otimes} 1 \\ & & \downarrow 1 & & \downarrow 1 \\ 1 \check{\otimes} 2 & \xrightarrow{2} & 2 \check{\otimes} 2 & \xrightarrow{2} & 3 \check{\otimes} 2 \\ \downarrow 2 & & & & \downarrow 2 \\ 1 \check{\otimes} 3 & \xrightarrow{1} & 2 \check{\otimes} 3 & & 3 \check{\otimes} 3 \\ & & \uparrow 2 & & \end{array}$$

where $\text{wt}(x \check{\otimes} y) = \mathbf{e}_x + \mathbf{e}_y$, for $x, y \in A_3$.

The difference between $\mathcal{A}_3 \check{\otimes} \mathcal{A}_3$ and the usual crystal tensor product $\mathcal{A}_3 \otimes \mathcal{A}_3$, described in Example 4.15(i), lies in elements $1 \check{\otimes} 2$ and $2 \check{\otimes} 3$ which are mapped to $+\infty$ by $\check{\varepsilon}_1$ and $\check{\varepsilon}_2$, respectively.

From the example above, we can see that the quasi-tensor product of seminormal crystals may not be a crystal.

We only defined quasi-tensor product between quasi-crystals that are seminormal. Now, we observe that this condition is essential to model inversions in words by quasi-crystals.

Remark 7.36. Consider the standard quasi-crystal $\mathcal{A}_2$ of type $\mathfrak{A}_2$, which is seminormal. Let $\mathbb{Q}$ be a quasi-crystal of type $\mathfrak{A}_2$ consisting of a set $Q = \{-1, -2\}$ and maps given as follows:

x	$\text{wt}(x)$	$\check{\varepsilon}(x)$	$\check{f}(x)$	$\check{\varepsilon}(x)$	$\check{\varphi}(x)$
-1	$-\mathbf{e}_1$	-2	$\perp$	0	-1
-2	$-\mathbf{e}_2$	$\perp$	-1	-1	0

Clearly, $\mathbb{Q}$ is not seminormal. Nonetheless, set $A_2 \tilde{\otimes} Q$ be the set of ordered pairs $x \tilde{\otimes} y$, with $x \in A_2$ and $y \in Q$. Define maps $\tilde{e}, \tilde{f} : A_2 \tilde{\otimes} Q \rightarrow (A_2 \tilde{\otimes} Q) \sqcup \{\perp\}$ as follows. For each $x \in A_2$ and $y \in Q$, if $\tilde{f}(x) \in A_2$ and $\tilde{e}(y) \in Q$, set $\tilde{e}(x \tilde{\otimes} y) = \tilde{f}(x \tilde{\otimes} y) = \perp$, otherwise, set

$$\tilde{e}(x \tilde{\otimes} y) = \begin{cases} \tilde{e}(x) \tilde{\otimes} y & \text{if } \tilde{\varphi}(x) \geq \tilde{e}(y) \\ x \tilde{\otimes} \tilde{e}(y) & \text{if } \tilde{\varphi}(x) < \tilde{e}(y), \end{cases}$$

and

$$\tilde{f}(x \tilde{\otimes} y) = \begin{cases} \tilde{f}(x) \tilde{\otimes} y & \text{if } \tilde{\varphi}(x) > \tilde{e}(y) \\ x \tilde{\otimes} \tilde{f}(y) & \text{if } \tilde{\varphi}(x) \leq \tilde{e}(y). \end{cases}$$

So, the maps $\tilde{e}$ and $\tilde{f}$ defined on $A_2 \tilde{\otimes} Q$ follow the tensor product rule (Definition 4.13), except when applied to an element of the form $x \tilde{\otimes} y$ with $\tilde{f}(x)$ and $\tilde{e}(y)$ defined. Explicitly, $\tilde{e}$ and $\tilde{f}$ are described by the following table.

$x \tilde{\otimes} y$	$\tilde{e}(x \tilde{\otimes} y)$	$\tilde{f}(x \tilde{\otimes} y)$
$1 \tilde{\otimes} (-1)$	$\perp$	$\perp$
$2 \tilde{\otimes} (-1)$	$1 \tilde{\otimes} (-1)$	$\tilde{f}(x \tilde{\otimes} y)$
$1 \tilde{\otimes} (-2)$	$\perp$	$2 \tilde{\otimes} (-2)$
$2 \tilde{\otimes} (-2)$	$1 \tilde{\otimes} (-2)$	$\perp$

Since $\tilde{e}(2 \tilde{\otimes} (-1)) = 1 \tilde{\otimes} (-1)$ and $\tilde{f}(1 \tilde{\otimes} (-1)) = \perp$, there is no quasi-crystal with set $A_2 \tilde{\otimes} Q$ and quasi-Kashiwara operators $\tilde{e}$ and $\tilde{f}$, because Definition 7.1(iv) is not satisfied. This illustrates that requiring quasi-crystals to be seminormal is essential to give an interpretation of an inversion on a word by quasi-crystals. And therefore, (inversion-free) quasi-tensor product cannot be extended to quasi-crystals that are not seminormal.

In the following result we show that the quasi-tensor product $\tilde{\otimes}$ of quasi-crystals is an associative operation.

Theorem 7.37. *Let $\mathbb{Q}_1, \mathbb{Q}_2$ and $\mathbb{Q}_3$ be seminormal quasi-crystals of the same type. The map $(Q_1 \tilde{\otimes} Q_2) \tilde{\otimes} Q_3 \rightarrow Q_1 \tilde{\otimes} (Q_2 \tilde{\otimes} Q_3)$, given by $(x_1 \tilde{\otimes} x_2) \tilde{\otimes} x_3 \mapsto x_1 \tilde{\otimes} (x_2 \tilde{\otimes} x_3)$, is a quasi-crystal isomorphism.*

Proof. Define $\psi : (Q_1 \tilde{\otimes} Q_2) \tilde{\otimes} Q_3 \rightarrow Q_1 \tilde{\otimes} (Q_2 \tilde{\otimes} Q_3)$ by $\psi((x_1 \tilde{\otimes} x_2) \tilde{\otimes} x_3) = x_1 \tilde{\otimes} (x_2 \tilde{\otimes} x_3)$ for each $x_1 \in Q_1, x_2 \in Q_2$ and $x_3 \in Q_3$. Clearly ψ is bijective. Since $(\mathbb{Q}_1 \tilde{\otimes} \mathbb{Q}_2) \tilde{\otimes} \mathbb{Q}_3$ and $\mathbb{Q}_1 \tilde{\otimes} (\mathbb{Q}_2 \tilde{\otimes} \mathbb{Q}_3)$ are seminormal by Theorem 7.32, to prove that ψ is a quasi-crystal isomorphism, it suffices to show that ψ is a quasi-crystal homomorphism by Corollary 7.18.

Let $x_1 \in Q_1$, $x_2 \in Q_2$ and $x_3 \in Q_3$. We have that

$$\text{wt}((x_1 \ddot{\otimes} x_2) \ddot{\otimes} x_3) = \text{wt}(x_1) + \text{wt}(x_2) + \text{wt}(x_3) = \text{wt}(x_1 \ddot{\otimes} (x_2 \ddot{\otimes} x_3)).$$

Let $i \in I$. For $k = 1, 2, 3$, we have that $\ddot{\epsilon}_i(x_k), \ddot{\varphi}_i(x_k) \geq 0$ as $\mathbb{Q}_k$ is seminormal. If $\ddot{\epsilon}_i(x_k) = +\infty$ for some $k \in \{1, 2, 3\}$, then

$$\ddot{\epsilon}_i((x_1 \ddot{\otimes} x_2) \ddot{\otimes} x_3) = \ddot{\epsilon}_i(x_1 \ddot{\otimes} (x_2 \ddot{\otimes} x_3)) = \ddot{\epsilon}_i(x_k) = +\infty,$$

And therefore, $\ddot{\varphi}_i((x_1 \ddot{\otimes} x_2) \ddot{\otimes} x_3) = \ddot{\varphi}_i(x_1 \ddot{\otimes} (x_2 \ddot{\otimes} x_3)) = +\infty$, $\ddot{e}_i((x_1 \ddot{\otimes} x_2) \ddot{\otimes} x_3) = \ddot{e}_i(x_1 \ddot{\otimes} (x_2 \ddot{\otimes} x_3)) = \perp$ and $\ddot{f}_i((x_1 \ddot{\otimes} x_2) \ddot{\otimes} x_3) = \ddot{f}_i(x_1 \ddot{\otimes} (x_2 \ddot{\otimes} x_3)) = \perp$. So assume that $\ddot{\epsilon}_i(x_k) \in \mathbb{Z}_{\geq 0}$ for all $k = 1, 2, 3$.

In the following cases we show that if $\ddot{\varphi}_i(x_k), \ddot{\epsilon}_i(x_l) > 0$ for some $1 \leq k < l \leq 3$, then $\ddot{\epsilon}_i((x_1 \ddot{\otimes} x_2) \ddot{\otimes} x_3) = \ddot{\epsilon}_i(x_1 \ddot{\otimes} (x_2 \ddot{\otimes} x_3)) = +\infty$, and thus, it follows as above.

- Case 1: $\ddot{\varphi}_i(x_1), \ddot{\epsilon}_i(x_2) > 0$. Then $\ddot{\epsilon}_i(x_1 \ddot{\otimes} x_2) = +\infty$, which implies $\ddot{\epsilon}_i((x_1 \ddot{\otimes} x_2) \ddot{\otimes} x_3) = +\infty$. Since $\ddot{\epsilon}_i(x_2 \ddot{\otimes} x_3) \geq \ddot{\epsilon}_i(x_2) > 0$, we have that $\ddot{\epsilon}_i(x_1 \ddot{\otimes} (x_2 \ddot{\otimes} x_3)) = +\infty$.
- Case 2: $\ddot{\varphi}_i(x_1), \ddot{\epsilon}_i(x_3) > 0$. Since $\ddot{\varphi}_i(x_1 \ddot{\otimes} x_2) \geq \ddot{\varphi}_i(x_1) > 0$, then $\ddot{\epsilon}_i((x_1 \ddot{\otimes} x_2) \ddot{\otimes} x_3) = +\infty$. Also, $\ddot{\epsilon}_i(x_1 \ddot{\otimes} (x_2 \ddot{\otimes} x_3)) = +\infty$ because $\ddot{\epsilon}_i(x_2 \ddot{\otimes} x_3) \geq \ddot{\epsilon}_i(x_3) > 0$.
- Case 3: $\ddot{\varphi}_i(x_2), \ddot{\epsilon}_i(x_3) > 0$. Since $\ddot{\varphi}_i(x_1 \ddot{\otimes} x_2) \geq \ddot{\varphi}_i(x_1) > 0$, then $\ddot{\epsilon}_i((x_1 \ddot{\otimes} x_2) \ddot{\otimes} x_3) = +\infty$. And since $\ddot{\epsilon}_i(x_2 \ddot{\otimes} x_3) = +\infty$, then $\ddot{\epsilon}_i(x_1 \ddot{\otimes} (x_2 \ddot{\otimes} x_3)) = +\infty$.

Finally, we suppose $\ddot{\epsilon}_i(x_k), \ddot{\varphi}_i(x_k) \in \mathbb{Z}_{\geq 0}$, for $k = 1, 2, 3$, such that $\ddot{\epsilon}_i(x_k), \ddot{\varphi}_i(x_l) > 0$ implies $k \leq l$. By Proposition 7.34, we obtain

$$\ddot{\epsilon}_i((x_1 \ddot{\otimes} x_2) \ddot{\otimes} x_3) = \ddot{\epsilon}_i(x_1) + \ddot{\epsilon}_i(x_2) + \ddot{\epsilon}_i(x_3) = \ddot{\epsilon}_i(x_1 \ddot{\otimes} (x_2 \ddot{\otimes} x_3))$$

and

$$\ddot{\varphi}_i((x_1 \ddot{\otimes} x_2) \ddot{\otimes} x_3) = \ddot{\varphi}_i(x_1) + \ddot{\varphi}_i(x_2) + \ddot{\varphi}_i(x_3) = \ddot{\varphi}_i(x_1 \ddot{\otimes} (x_2 \ddot{\otimes} x_3)).$$

We also have that

$$\begin{aligned} \ddot{e}_i((x_1 \ddot{\otimes} x_2) \ddot{\otimes} x_3) &= \begin{cases} (\ddot{e}_i(x_1) \ddot{\otimes} x_2) \ddot{\otimes} x_3 & \text{if } \ddot{\epsilon}_i(x_2) = \ddot{\epsilon}_i(x_3) = 0 \\ (x_1 \ddot{\otimes} \ddot{e}_i(x_2)) \ddot{\otimes} x_3 & \text{if } \ddot{\epsilon}_i(x_2) > 0, \ddot{\epsilon}_i(x_3) = 0 \\ (x_1 \ddot{\otimes} x_2) \ddot{\otimes} \ddot{e}_i(x_3) & \text{if } \ddot{\epsilon}_i(x_3) > 0, \end{cases} \\ \ddot{e}_i(x_1 \ddot{\otimes} (x_2 \ddot{\otimes} x_3)) &= \begin{cases} \ddot{e}_i(x_1) \ddot{\otimes} (x_2 \ddot{\otimes} x_3) & \text{if } \ddot{\epsilon}_i(x_2) = \ddot{\epsilon}_i(x_3) = 0 \\ x_1 \ddot{\otimes} (\ddot{e}_i(x_2) \ddot{\otimes} x_3) & \text{if } \ddot{\epsilon}_i(x_2) > 0, \ddot{\epsilon}_i(x_3) = 0 \\ x_1 \ddot{\otimes} (x_2 \ddot{\otimes} \ddot{e}_i(x_3)) & \text{if } \ddot{\epsilon}_i(x_3) > 0, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \ddot{f}_i((x_1 \ddot{\otimes} x_2) \ddot{\otimes} x_3) &= \begin{cases} (\ddot{f}_i(x_1) \ddot{\otimes} x_2) \ddot{\otimes} x_3 & \text{if } \ddot{\varphi}_i(x_1) > 0 \\ (x_1 \ddot{\otimes} \ddot{f}_i(x_2)) \ddot{\otimes} x_3 & \text{if } \ddot{\varphi}_i(x_1) = 0, \ddot{\varphi}_i(x_2) > 0 \\ (x_1 \ddot{\otimes} x_2) \ddot{\otimes} \ddot{f}_i(x_3) & \text{if } \ddot{\varphi}_i(x_1) = \ddot{\varphi}_i(x_2) > 0, \end{cases} \\ \ddot{f}_i(x_1 \ddot{\otimes} (x_2 \ddot{\otimes} x_3)) &= \begin{cases} \ddot{f}_i(x_1) \ddot{\otimes} (x_2 \ddot{\otimes} x_3) & \text{if } \ddot{\varphi}_i(x_1) > 0 \\ x_1 \ddot{\otimes} (\ddot{f}_i(x_2) \ddot{\otimes} x_3) & \text{if } \ddot{\varphi}_i(x_1) = 0, \ddot{\varphi}_i(x_2) > 0 \\ x_1 \ddot{\otimes} (x_2 \ddot{\otimes} \ddot{f}_i(x_3)) & \text{if } \ddot{\varphi}_i(x_1) = \ddot{\varphi}_i(x_2) = 0. \end{cases} \end{aligned}$$

Hence, $\psi(\ddot{e}_i((x_1 \ddot{\otimes} x_2) \ddot{\otimes} x_3)) = \ddot{e}_i(\psi(x_1 \ddot{\otimes} (x_2 \ddot{\otimes} x_3)))$ and $\psi(\ddot{f}_i((x_1 \ddot{\otimes} x_2) \ddot{\otimes} x_3)) = \ddot{f}_i(\psi(x_1 \ddot{\otimes} (x_2 \ddot{\otimes} x_3)))$. And therefore, ψ is a quasi-crystal isomorphism. $\square$

Due to the previous result, we may omit parenthesis for the quasi-tensor product of seminormal quasi-crystals and simply write $\mathbb{Q}_1 \ddot{\otimes} \mathbb{Q}_2 \ddot{\otimes} \mathbb{Q}_3$ with elements $x_1 \ddot{\otimes} x_2 \ddot{\otimes} x_3$, for $x_1 \in Q_1$, $x_2 \in Q_2$ and $x_3 \in Q_3$.

From the proofs of Theorems 7.32 and 7.37, we deduce the following result which generalizes Proposition 7.34.

Corollary 7.38. *Let $\mathbb{Q}_1, \dots, \mathbb{Q}_m$ be seminormal quasi-crystals of the same type, and let $x_1 \in Q_1, \dots, x_m \in Q_m$. Then,*

$$\text{wt}(x_1 \ddot{\otimes} \dots \ddot{\otimes} x_m) = \text{wt}(x_1) + \dots + \text{wt}(x_m).$$

Also, for $i \in I$, by setting

$$\begin{aligned} p &= \max\{1 \leq k \leq m \mid \ddot{\varepsilon}_i(x_k) > 0\} \text{ and} \\ q &= \min\{1 \leq l \leq m \mid \ddot{\varphi}_i(x_l) > 0\}, \end{aligned}$$

we get

(i) if $p > q$ or $\ddot{\varepsilon}_i(x_k) = +\infty$ for some $1 \leq k \leq m$, then

$$\ddot{e}_i(x_1 \ddot{\otimes} \dots \ddot{\otimes} x_m) = \ddot{f}_i(x_1 \ddot{\otimes} \dots \ddot{\otimes} x_m) = \perp$$

and

$$\ddot{\varepsilon}_i(x_1 \ddot{\otimes} \dots \ddot{\otimes} x_m) = \ddot{\varphi}_i(x_1 \ddot{\otimes} \dots \ddot{\otimes} x_m) = +\infty.$$

(ii) otherwise,

$$\begin{aligned} \ddot{e}_i(x_1 \ddot{\otimes} \dots \ddot{\otimes} x_m) &= x_1 \ddot{\otimes} \dots \ddot{\otimes} x_{p-1} \ddot{\otimes} \ddot{e}_i(x_p) \ddot{\otimes} x_{p+1} \ddot{\otimes} \dots \ddot{\otimes} x_m \\ \ddot{f}_i(x_1 \ddot{\otimes} \dots \ddot{\otimes} x_m) &= x_1 \ddot{\otimes} \dots \ddot{\otimes} x_{q-1} \ddot{\otimes} \ddot{f}_i(x_q) \ddot{\otimes} x_{q+1} \ddot{\otimes} \dots \ddot{\otimes} x_m \end{aligned}$$

and

$$\begin{aligned}\check{\varepsilon}_i(x_1 \check{\otimes} \cdots \check{\otimes} x_m) &= \check{\varepsilon}_i(x_1) + \cdots + \check{\varepsilon}_i(x_m) \\ \check{\varphi}_i(x_1 \check{\otimes} \cdots \check{\otimes} x_m) &= \check{\varphi}_i(x_1) + \cdots + \check{\varphi}_i(x_m)\end{aligned}$$

Note that if $\mathbb{Q}_1, \dots, \mathbb{Q}_m$ are seminormal crystals, then the rules for the quasi-tensor product coincide with the rules for the tensor product described in Proposition 4.17, except when lying in case (i). In this context, for $x_1 \in \mathbb{Q}_1, \dots, x_m \in \mathbb{Q}_m$ and $i \in I$, we say that $x_1 \check{\otimes} \cdots \check{\otimes} x_m$ has an i -inversion if there exist $k < l$ such that $\check{f}_i(x_k)$ and $\check{e}_i(x_l)$ are defined. Thus, the maps $\check{e}_i, \check{f}_i, \check{\varepsilon}_i$ and $\check{\varphi}_i$ of $\mathbb{Q}_1 \check{\otimes} \cdots \check{\otimes} \mathbb{Q}_m$ and the maps $\tilde{e}_i, \tilde{f}_i, \tilde{\varepsilon}_i$ and $\tilde{\varphi}_i$ of $\mathbb{Q}_1 \otimes \cdots \otimes \mathbb{Q}_m$ are defined similarly in elements without i -inversions. Also, note that the weight maps coincide in the sense that $\text{wt}(x_1 \check{\otimes} \cdots \check{\otimes} x_m) = \text{wt}(x_1 \otimes \cdots \otimes x_m)$, for any $x_1 \in \mathbb{Q}_1, \dots, x_m \in \mathbb{Q}_m$.

In the following result we show that quasi-crystal homomorphisms between seminormal quasi-crystals give rise to homomorphisms between quasi-tensor products of their domains and images.

Theorem 7.39. *Let $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}'_1$ and $\mathbb{Q}'_2$ be seminormal quasi-crystals of the same type, and let $\psi_1 : \mathbb{Q}_1 \rightarrow \mathbb{Q}'_1$ and $\psi_2 : \mathbb{Q}_2 \rightarrow \mathbb{Q}'_2$ be quasi-crystal homomorphisms. Then, the map $\psi_1 \check{\otimes} \psi_2 : \mathbb{Q}_1 \check{\otimes} \mathbb{Q}_2 \rightarrow \mathbb{Q}'_1 \check{\otimes} \mathbb{Q}'_2$, defined by $x_1 \check{\otimes} x_2 \mapsto \psi_1(x_1) \check{\otimes} \psi_2(x_2)$ for each $x_1 \in \mathbb{Q}_1$ and $x_2 \in \mathbb{Q}_2$, is a quasi-crystal homomorphism from $\mathbb{Q}_1 \check{\otimes} \mathbb{Q}_2$ to $\mathbb{Q}'_1 \check{\otimes} \mathbb{Q}'_2$. Moreover, if ψ_1 and ψ_2 are quasi-crystal isomorphisms, then $\psi_1 \check{\otimes} \psi_2$ is a quasi-crystal isomorphism.*

Proof. Let $x_1 \in \mathbb{Q}_1$ and $x_2 \in \mathbb{Q}_2$. We have that

$$\text{wt}(\psi_1(x_1) \check{\otimes} \psi_2(x_2)) = \text{wt}(\psi_1(x_1)) + \text{wt}(\psi_2(x_2)) = \text{wt}(x_1) + \text{wt}(x_2) = \text{wt}(x_1 \check{\otimes} x_2).$$

Let $i \in I$ and $k \in \{1, 2\}$. Since $\check{\varepsilon}_i(\psi_k(x_k)) = \check{\varepsilon}_i(x_k)$ and $\check{\varphi}_i(\psi_k(x_k)) = \check{\varphi}_i(x_k)$, we have that if $\check{\varphi}_i(x_1), \check{\varepsilon}_i(x_2) > 0$, then

$$\check{e}_i(\psi_1(x_1) \check{\otimes} \psi_2(x_2)) = \check{f}_i(\psi_1(x_1) \check{\otimes} \psi_2(x_2)) = \check{e}_i(x_1 \check{\otimes} x_2) = \check{f}_i(x_1 \check{\otimes} x_2) = \perp$$

and

$$\check{\varepsilon}_i(\psi_1(x_1) \check{\otimes} \psi_2(x_2)) = \check{\varphi}_i(\psi_1(x_1) \check{\otimes} \psi_2(x_2)) = \check{\varepsilon}_i(x_1 \check{\otimes} x_2) = \check{\varphi}_i(x_1 \check{\otimes} x_2) = +\infty.$$

Otherwise, Since $\mathbb{Q}_k$ and $\mathbb{Q}'_k$ are seminormal, we have that $\check{e}_i(x_k) \in \mathbb{Q}_k$ if and only if $\check{e}_i(\psi_k(x_k)) \in \mathbb{Q}'_k$, and thus, $\psi_k(\check{e}_i(x_k)) = \check{e}_i(\psi_k(x_k))$. Hence,

$$(\psi_1 \check{\otimes} \psi_2)(\check{e}_i(x_1 \check{\otimes} x_2)) = \begin{cases} \psi_1(\check{e}_i(x_1)) \check{\otimes} \psi_2(x_2) & \text{if } \check{\varphi}_i(x_1) \geq \check{\varepsilon}_i(x_2) \\ \psi_1(x_1) \check{\otimes} \psi_2(\check{e}_i(x_2)) & \text{if } \check{\varphi}_i(x_1) < \check{\varepsilon}_i(x_2) \end{cases}$$

$$\begin{aligned}
 &= \begin{cases} \check{e}_i(\psi_1(x_1)) \check{\otimes} \psi_2(x_2) & \text{if } \check{\varphi}_i(\psi_1(x_1)) \geq \check{e}_i(\psi_2(x_2)) \\ \psi_1(x_1) \check{\otimes} \check{e}_i(\psi_2(x_2)) & \text{if } \check{\varphi}_i(\psi_1(x_1)) < \check{e}_i(\psi_2(x_2)) \end{cases} \\
 &= \check{e}_i(\psi_1(x_1) \check{\otimes} \psi_2(x_2))
 \end{aligned}$$

and

$$\begin{aligned}
 \check{e}_i(\psi_1(x_1) \check{\otimes} \psi_2(x_2)) &= \max\{\check{e}_i(\psi_1(x_1)), \check{e}_i(\psi_2(x_2)) - \langle \text{wt}(\psi_1(x_1)), \alpha_i^\vee \rangle\} \\
 &= \max\{\check{e}_i(x_1), \check{e}_i(x_2) - \langle \text{wt}(x_1), \alpha_i^\vee \rangle\} \\
 &= \check{e}_i(x_1 \check{\otimes} x_2).
 \end{aligned}$$

Similarly, $(\psi_1 \check{\otimes} \psi_2)(\check{f}_i(x_1 \check{\otimes} x_2)) = \check{f}_i(\psi_1(x_1) \check{\otimes} \psi_2(x_2))$ and $\check{\varphi}_i(\psi_1(x_1) \check{\otimes} \psi_2(x_2)) = \check{\varphi}_i(x_1 \check{\otimes} x_2)$. And therefore, $\psi_1 \check{\otimes} \psi_2$ is a quasi-crystal homomorphism.

If ψ_1 and ψ_2 are quasi-crystal isomorphisms, then $\psi_1 \check{\otimes} \psi_2$ is a bijective quasi-crystal homomorphism between seminormal quasi-crystals. By Corollary 7.18, $\psi_1 \check{\otimes} \psi_2$ is a quasi-crystal isomorphism. $\square$

7.4 The signature rule for quasi-tensor product

In this section we describe a combinatorial method that makes easier the computation of the quasicrystal structure of the quasi-tensor product of seminormal quasi-crystals. This method is essentially a combinatorial interpretation of Corollary 7.38, and has a procedure similar to the signature rule for the tensor product of seminormal crystals described in Section 4.4. Due to this analogy, the terminology introduced in this section for the signature rule for quasi-tensor product coincides with the terminology introduced for the signature rule for the tensor product, since it is clear from the context which one is being considered.

Consider a zero monoid Z_0 with the following presentation

$$Z_0 = \langle -, + \mid +- = 0 \rangle.$$

So, an element of Z_0 , other than 0, has the form $-^a +^b$ with $a, b \in \mathbb{Z}_{\geq 0}$.

Let $\mathbb{Q}$ be a seminormal quasi-crystal. For each $i \in I$ define a map $\text{sgn}_i^{\check{\otimes}} : Q \rightarrow Z_0$ by

$$\text{sgn}_i^{\check{\otimes}}(x) = \begin{cases} 0 & \text{if } \check{e}_i(x) = +\infty \\ -\check{e}_i(x) + \check{\varphi}_i(x) & \text{otherwise,} \end{cases}$$

for each $x \in Q$. The map $\text{sgn}_i^{\check{\otimes}}$ is called the *i-signature map* for the quasi-tensor product $\check{\otimes}$, and $\text{sgn}_i^{\check{\otimes}}(x)$ is called the *i-signature* of $x \in Q$. The signature map interacts with the quasi-tensor product in the following way.

Proposition 7.40. *Let $\mathbb{Q}$ and $\mathbb{Q}'$ be seminormal quasi-crystals of the same type. Then,*

$$\text{sgn}_i^{\otimes}(x \otimes x') = \text{sgn}_i^{\otimes}(x) \text{sgn}_i^{\otimes}(x'),$$

for all $x \in Q$, $x' \in Q'$ and $i \in I$.

Proof. Let $x \in Q$, $x' \in Q'$ and $i \in I$. By Corollary 7.38, if $\varepsilon_i(x) = +\infty$ (or equivalently, $\varphi_i(x) = +\infty$), then $\varepsilon_i(x \otimes x') = +\infty$, and so,

$$\text{sgn}_i^{\otimes}(x) \text{sgn}_i^{\otimes}(x') = 0 \text{sgn}_i^{\otimes}(x') = 0 = \text{sgn}_i^{\otimes}(x \otimes x').$$

Similarly, if $\varepsilon_i(x') = +\infty$, then $\text{sgn}_i^{\otimes}(x \otimes x') = 0 = \text{sgn}_i^{\otimes}(x) \text{sgn}_i^{\otimes}(x')$. Thus, assume that $\varphi_i(x), \varepsilon_i(x') \in \mathbb{Z}_{\geq 0}$. If $\varphi_i(x), \varepsilon_i(x') > 0$, then $\varepsilon_i(x \otimes x') = +\infty$, and

$$\begin{aligned} \text{sgn}_i^{\otimes}(x) \text{sgn}_i^{\otimes}(x') &= -\varepsilon_i(x) + \varphi_i(x) - \varepsilon_i(x') + \varphi_i(x') \\ &= -\varepsilon_i(x) + \varphi_i(x) - 1 \cdot 0 - \varepsilon_i(x') - 1 + \varphi_i(x') = 0 = \text{sgn}_i^{\otimes}(x \otimes x'). \end{aligned}$$

Finally, assume that either $\varphi_i(x) = 0$ or $\varepsilon_i(x') = 0$. If $\varphi_i(x) = 0$, then

$$\begin{aligned} \text{sgn}_i^{\otimes}(x) \text{sgn}_i^{\otimes}(x') &= -\varepsilon_i(x) - \varepsilon_i(x') + \varphi_i(x') = -\varepsilon_i(x) + \varepsilon_i(x') + \varphi_i(x') \\ &= -\varepsilon_i(x \otimes x') + \varphi_i(x \otimes x') = \text{sgn}_i^{\otimes}(x \otimes x'), \end{aligned}$$

by Proposition 7.34. If $\varepsilon_i(x') = 0$, then

$$\begin{aligned} \text{sgn}_i^{\otimes}(x) \text{sgn}_i^{\otimes}(x') &= -\varepsilon_i(x) + \varphi_i(x) + \varphi_i(x') = -\varepsilon_i(x) + \varphi_i(x) + \varphi_i(x') \\ &= -\varepsilon_i(x \otimes x') + \varphi_i(x \otimes x') = \text{sgn}_i^{\otimes}(x \otimes x'), \end{aligned}$$

by Proposition 7.34. □

From the previous result, given seminormal quasi-crystals $\mathbb{Q}_1, \mathbb{Q}_2, \dots, \mathbb{Q}_m$ of the same type and elements $x_1 \in Q_1, x_2 \in Q_2, \dots, x_m \in Q_m$, we can easily compute the i -signature of $x_1 \otimes x_2 \otimes \dots \otimes x_m$ as

$$\text{sgn}_i^{\otimes}(x_1 \otimes x_2 \otimes \dots \otimes x_m) = \text{sgn}_i^{\otimes}(x_1) \text{sgn}_i^{\otimes}(x_2) \dots \text{sgn}_i^{\otimes}(x_m)$$

($i \in I$). If $\text{sgn}_i^{\otimes}(x_1 \otimes x_2 \otimes \dots \otimes x_m) = 0$, then $\varepsilon_i(x_1 \otimes x_2 \otimes \dots \otimes x_m) = \varphi_i(x_1 \otimes x_2 \otimes \dots \otimes x_m) = +\infty$ which implies $\varepsilon_i(x_1 \otimes x_2 \otimes \dots \otimes x_m) = \check{f}_i(x_1 \otimes x_2 \otimes \dots \otimes x_m) = \perp$. Otherwise,

$$\text{sgn}_i^{\otimes}(x_1 \otimes x_2 \otimes \dots \otimes x_m) = -a + b$$

for some $a, b \in \mathbb{Z}_{\geq 0}$. Then, $\varepsilon_i(x_1 \otimes x_2 \otimes \dots \otimes x_m) = a$ and $\varphi_i(x_1 \otimes x_2 \otimes \dots \otimes x_m) = b$. From Corollary 7.38, if $a \geq 1$, then

$$\varepsilon_i(x_1 \otimes x_2 \otimes \dots \otimes x_m) = x_1 \otimes \dots \otimes x_{p-1} \otimes \varepsilon_i(x_p) \otimes x_{p+1} \otimes \dots \otimes x_m,$$

where x_p originates the right-most $-$ in $\text{sgn}_i^{\otimes}(x_1 \otimes x_2 \otimes \cdots \otimes x_m)$. Also, if $b \geq 1$, then

$$\check{f}_i(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = x_1 \otimes \cdots \otimes x_{q-1} \otimes \check{f}_i(x_q) \otimes x_{q+1} \otimes \cdots \otimes x_m,$$

where x_q originates the left-most $+$ in $\text{sgn}_i^{\otimes}(x_1 \otimes x_2 \otimes \cdots \otimes x_m)$. This process is called the *signature rule for the quasi-tensor product*.

Example 7.41. Consider the quasi-crystal $\mathcal{A}_4 \otimes \mathcal{A}_4 \otimes \mathcal{A}_4 \otimes \mathcal{A}_4 \otimes \mathcal{A}_4$, where $\mathcal{A}_4$ is the standard quasi-crystal of type $\mathcal{A}_4$. We compute $\check{e}_2, \check{f}_2, \check{\varepsilon}_2$ and $\check{\varphi}_2$ on $3 \otimes 1 \otimes 2 \otimes 2 \otimes 3$ using the signature rule. To keep track to which element originate each $-$ and $+$ we write a subscript with the position of the element, this is just an auxiliar notation and the binary operation of Z should be applied ignoring the subscripts. So we have that

$$\text{sgn}_2^{\otimes}(3 \otimes 1 \otimes 2 \otimes 2 \otimes 3) = -_1 +_3 +_4 -_5 = -_1 +_3 0 = 0,$$

and therefore,

$$\begin{aligned} \check{e}_2(3 \otimes 1 \otimes 2 \otimes 2 \otimes 3) &= \perp, & \check{\varepsilon}_2(3 \otimes 1 \otimes 2 \otimes 2 \otimes 3) &= +\infty, \\ \check{f}_2(3 \otimes 1 \otimes 2 \otimes 2 \otimes 3) &= \perp, & \check{\varphi}_2(3 \otimes 1 \otimes 2 \otimes 2 \otimes 3) &= +\infty. \end{aligned}$$

Now we compute $\check{e}_1, \check{f}_1, \check{\varepsilon}_1$ and $\check{\varphi}_1$ on $2 \otimes 3 \otimes 2 \otimes 3 \otimes 1$. Using the same notation as above, we obtain that

$$\text{sgn}_1^{\otimes}(2 \otimes 3 \otimes 2 \otimes 3 \otimes 1) = -_1 -_3 +_5,$$

and therefore,

$$\begin{aligned} \check{e}_1(2 \otimes 3 \otimes 2 \otimes 3 \otimes 1) &= 2 \otimes 3 \otimes 1 \otimes 3 \otimes 1, & \check{\varepsilon}_1(2 \otimes 3 \otimes 2 \otimes 3 \otimes 1) &= 2, \\ \check{f}_1(2 \otimes 3 \otimes 2 \otimes 3 \otimes 1) &= 2 \otimes 3 \otimes 2 \otimes 3 \otimes 2, & \check{\varphi}_1(2 \otimes 3 \otimes 2 \otimes 3 \otimes 1) &= 1. \end{aligned}$$

The signature rule for the quasi-tensor product can also be translated into the following algorithm.

Algorithm 7.42. *Input:* Elements $x_1 \in Q_1, x_2 \in Q_2, \dots, x_m \in Q_m$ of seminormal quasi-crystals $\mathbb{Q}_1, \mathbb{Q}_2, \dots, \mathbb{Q}_m$ of the same type, and an index $i \in I$.

Output: The values of $\check{e}_i, \check{f}_i, \check{\varepsilon}_i$, and $\check{\varphi}_i$ on $x_1 \otimes x_2 \otimes \cdots \otimes x_m$.

Procedure:

- (1) Set $a := 0, b := 0, p := 0$, and $q := 0$.
- (2) For each $k = 1, 2, \dots, m$, do:

- (i) If $\ddot{\varepsilon}_i(x_k) = +\infty$, or $b > 0$ and $\ddot{\varepsilon}_i(x_k) > 0$, then set $a := -1$, $b := -1$, and break out of the loop.
 - (ii) If $\ddot{\varepsilon}_i(x_k) > 0$, set $a := a + \ddot{\varepsilon}_i(x_k)$ and $p := k$.
 - (iii) If $\ddot{\varphi}_i(x_k) > 0$, set $b := b + \ddot{\varphi}_i(x_k)$.
 - (iv) If $\ddot{\varphi}_i(x_k) > 0$ and $q = 0$, set $q := k$.
- (3) If $a = -1$, set $\ddot{\varepsilon}_i(x_1 \ddot{\otimes} x_2 \ddot{\otimes} \cdots \ddot{\otimes} x_m) := +\infty$ and $\ddot{\varphi}_i(x_1 \ddot{\otimes} x_2 \ddot{\otimes} \cdots \ddot{\otimes} x_m) := +\infty$. Otherwise, set $\ddot{\varepsilon}_i(x_1 \ddot{\otimes} x_2 \ddot{\otimes} \cdots \ddot{\otimes} x_m) := a$ and $\ddot{\varphi}_i(x_1 \ddot{\otimes} x_2 \ddot{\otimes} \cdots \ddot{\otimes} x_m) := b$.
- (4) If $a \geq 1$, set $\ddot{\varepsilon}_i(x_1 \ddot{\otimes} x_2 \ddot{\otimes} \cdots \ddot{\otimes} x_m) := x_1 \ddot{\otimes} \cdots \ddot{\otimes} x_{p-1} \ddot{\otimes} \ddot{\varepsilon}_i(x_p) \ddot{\otimes} x_{p+1} \ddot{\otimes} \cdots \ddot{\otimes} x_m$. Otherwise, set $\ddot{\varepsilon}_i(x_1 \ddot{\otimes} x_2 \ddot{\otimes} \cdots \ddot{\otimes} x_m) := \perp$.
- (5) If $b \geq 1$, set $\ddot{f}_i(x_1 \ddot{\otimes} x_2 \ddot{\otimes} \cdots \ddot{\otimes} x_m) := x_1 \ddot{\otimes} \cdots \ddot{\otimes} x_{q-1} \ddot{\otimes} \ddot{f}_i(x_q) \ddot{\otimes} x_{q+1} \ddot{\otimes} \cdots \ddot{\otimes} x_m$. Otherwise, set $\ddot{f}_i(x_1 \ddot{\otimes} x_2 \ddot{\otimes} \cdots \ddot{\otimes} x_m) := \perp$.
- (6) Return $\ddot{\varepsilon}_i(x_1 \ddot{\otimes} x_2 \ddot{\otimes} \cdots \ddot{\otimes} x_m)$, $\ddot{f}_i(x_1 \ddot{\otimes} x_2 \ddot{\otimes} \cdots \ddot{\otimes} x_m)$, $\ddot{\varepsilon}_i(x_1 \ddot{\otimes} x_2 \ddot{\otimes} \cdots \ddot{\otimes} x_m)$, and $\ddot{\varphi}_i(x_1 \ddot{\otimes} x_2 \ddot{\otimes} \cdots \ddot{\otimes} x_m)$.

CRYSTALIZING THE HYPOPLACTIC MONOID

In this chapter we present a general construction for the hypoplactic monoid analogous to the construction of the plactic monoid described in Chapter 5. To solve the problem showed in Remark 6.17, we introduced in Chapter 7 the theory of quasi-crystals in order to provide an algebraic framework, where a general construction of the hypoplactic monoid could be given, and the construction of the classical hypoplactic monoid described in [CM17] could be placed in context. We now prove that indeed quasi-crystals provide this algebraic framework.

In Section 8.1 we present the definition of quasi-crystal monoid, which is the basic concept for relating quasi-crystals and monoids. Then in Section 8.2, we introduce the notion of free quasi-crystal monoid over a seminormal quasi-crystal, and show that they satisfy a universal property. In Section 8.3, we present the definition of quasi-crystal monoid congruence, which allows to consider quotients of quasi-crystal monoids, and leads to the homomorphism theorems for quasi-crystal monoids. Then in Section 8.4, we introduce the hypoplactic congruence on a free quasi-crystal monoid, and study its properties. Finally, in Section 8.5 we show that the construction of the hypoplactic monoid of type $\mathfrak{A}_n$ is equivalent to the construction of the classical hypoplactic monoid given in [CM17].

At the end of this chapter, we have proven that our approach using quasi-crystals answers the problem described in Remark 6.17 without losing the construction of the classical hypoplactic monoid from the crystal graph of type $\mathfrak{A}_n$.

8.1 Quasi-crystal monoids and homomorphisms

In this section we introduce the notion of quasi-crystal monoid, and present some results showing how the quasi-crystal and monoid structures interact. In

particular, we describe an analogue of the signature rule, resulting in an algorithm to compute the quasi-crystal structure maps on a product based only on their values on each parcel. Finally, we introduce the notion of a homomorphism between quasi-crystal monoids, and obtain an algebraic framework that will be studied on the subsequent sections.

Definition 8.1. Let Φ be a root system with weight lattice Λ and index set I for the simple roots $(\alpha_i)_{i \in I}$. A $\ddot{\otimes}$ -quasi-crystal monoid $\mathcal{M}$ of type Φ consists of a set M together with maps $\text{wt} : M \rightarrow \Lambda$, $\ddot{e}_i, \ddot{f}_i : M \rightarrow M \sqcup \{\perp\}$, $\ddot{\varepsilon}_i, \ddot{\varphi}_i : M \rightarrow \mathbb{Z} \cup \{-\infty, +\infty\}$ ($i \in I$) and a binary operation $\cdot : M \times M \rightarrow M$ satisfying the following conditions:

- (i) M together with wt , $\ddot{e}_i$, $\ddot{f}_i$, $\ddot{\varepsilon}_i$ and $\ddot{\varphi}_i$ ($i \in I$) is a seminormal quasi-crystal;
- (ii) M together with $\cdot$ is a monoid;
- (iii) the map $M \ddot{\otimes} M \rightarrow M$, given by $x \ddot{\otimes} y \mapsto x \cdot y$ for $x, y \in M$, is a quasi-crystal homomorphism from $\mathcal{M} \ddot{\otimes} \mathcal{M}$ to $\mathcal{M}$.

Notice the similarities between the definition of quasi-crystal monoid and crystal monoid (Definition 5.1), as the conditions are essentially the same. If instead of $\ddot{\otimes}$ we considered a notion of tensor product $\otimes$ for quasi-crystals analogous to the one given in Definition 4.13 for crystals, then we would get a generalization of the definitions and results studied in Chapter 5. In particular, we would obtain the notion of a plactic monoid associated to a quasi-crystal. Since our goal is to introduce the notion of the hypoplactic monoid associated to a quasi-crystal, we will only consider quasi-crystal monoids with respect to the notion of quasi-tensor product $\ddot{\otimes}$ introduced in Definition 7.33. Thus, we will omit $\ddot{\otimes}$ and just say that $\mathcal{M}$ is a quasi-crystal monoid.

In a quasi-crystal monoid the interaction between the quasi-crystal structure maps and the binary operation satisfies rules similar to those satisfied by the quasi-crystal structure maps of a quasi-tensor product, as we show in the following result.

Lemma 8.2. *Let $\mathcal{M}$ be a quasi-crystal monoid. For $x, y \in M$, we have that*

$$\text{wt}(xy) = \text{wt}(x) + \text{wt}(y),$$

and for $i \in I$, if $\ddot{\varphi}_i(x), \ddot{\varepsilon}_i(y) > 0$, then

$$\ddot{e}_i(xy) = \ddot{f}_i(xy) = \perp \quad \text{and} \quad \ddot{\varepsilon}_i(xy) = \ddot{\varphi}_i(xy) = +\infty,$$

otherwise,

$$\begin{aligned} \ddot{e}_i(xy) &= \begin{cases} \ddot{e}_i(x) \cdot y & \text{if } \ddot{e}_i(y) = 0 \\ x \cdot \ddot{e}_i(y) & \text{if } \ddot{e}_i(y) > 0, \end{cases} & \ddot{f}_i(xy) &= \begin{cases} \ddot{f}_i(x) \cdot y & \text{if } \ddot{\varphi}_i(x) > 0 \\ x \cdot \ddot{f}_i(y) & \text{if } \ddot{\varphi}_i(x) = 0, \end{cases} \\ \ddot{e}_i(xy) &= \ddot{e}_i(x) + \ddot{e}_i(y), & \text{and} & \ddot{\varphi}_i(xy) = \ddot{\varphi}_i(x) + \ddot{\varphi}_i(y), \end{aligned}$$

where $x\perp = \perp y = \perp$.

Proof. Define $\psi : M \ddot{\otimes} M \rightarrow M$ by $\psi(x \ddot{\otimes} y) = xy$, for $x, y \in M$. From Definition 8.1(iii), ψ is a quasi-crystal homomorphism from $\mathcal{M} \ddot{\otimes} \mathcal{M}$ to $\mathcal{M}$. Let $x, y \in M$ and $i \in I$. By Definition 7.12(ii), we get that

$$\text{wt}(xy) = \text{wt}(\psi(x \ddot{\otimes} y)) = \text{wt}(x \ddot{\otimes} y) = \text{wt}(x) + \text{wt}(y).$$

and similarly, $\ddot{e}_i(xy) = \ddot{e}_i(x \ddot{\otimes} y)$ and $\ddot{\varphi}_i(xy) = \ddot{\varphi}_i(x \ddot{\otimes} y)$. From Definition 8.1(i), $\mathcal{M}$ is seminormal (Definition 7.9), and by Theorem 7.32, $\mathcal{M} \ddot{\otimes} \mathcal{M}$ is also seminormal. Since $\ddot{e}_i(xy) = \ddot{e}_i(x \ddot{\otimes} y)$, we have that $\ddot{e}_i$ is defined on xy if and only if $\ddot{e}_i$ is defined on $x \ddot{\otimes} y$, and since $\ddot{\varphi}_i(xy) = \ddot{\varphi}_i(x \ddot{\otimes} y)$, $\ddot{f}_i$ is defined on xy if and only if $\ddot{f}_i$ is defined on $x \ddot{\otimes} y$. Then, we obtain that $\ddot{e}_i(xy) = \psi(\ddot{e}_i(x \ddot{\otimes} y))$ and $\ddot{f}_i(xy) = \psi(\ddot{f}_i(x \ddot{\otimes} y))$, by Definition 7.12(iii) and (iv). Therefore, the result follows directly from Proposition 7.34. $\square$

The previous result can be generalized for getting the values of the quasi-crystal structure maps on an element of the form $x_1 \cdots x_m$ based only on their values on each x_k , $k = 1, \dots, m$. This leads to an analogue of Corollary 7.38.

Proposition 8.3. *Let $\mathcal{M}$ be a quasi-crystal monoid, and let $x_1, \dots, x_m \in M$. Then,*

$$\text{wt}(x_1 \cdots x_m) = \text{wt}(x_1) + \cdots + \text{wt}(x_m).$$

Also, for $i \in I$, by setting

$$\begin{aligned} p &= \max\{1 \leq k \leq m \mid \ddot{e}_i(x_k) > 0\} \text{ and} \\ q &= \min\{1 \leq l \leq m \mid \ddot{\varphi}_i(x_l) > 0\}, \end{aligned}$$

we have that

(i) if $p > q$ or $\ddot{e}_i(x_k) = +\infty$ for some $1 \leq k \leq m$, then

$$\ddot{e}_i(x_1 \cdots x_m) = \ddot{f}_i(x_1 \cdots x_m) = \perp \quad \text{and} \quad \ddot{e}_i(x_1 \cdots x_m) = \ddot{\varphi}_i(x_1 \cdots x_m) = +\infty;$$

(ii) otherwise,

$$\begin{aligned}\check{e}_i(x_1 \cdots x_m) &= x_1 \cdots x_{p-1} \cdot \check{e}_i(x_p) \cdot x_{p+1} \cdots x_m, \\ \check{f}_i(x_1 \cdots x_m) &= x_1 \cdots x_{q-1} \cdot \check{f}_i(x_q) \cdot x_{q+1} \cdots x_m,\end{aligned}$$

and

$$\begin{aligned}\ddot{e}_i(x_1 \cdots x_m) &= \ddot{e}_i(x_1) + \cdots + \ddot{e}_i(x_p), \\ \ddot{f}_i(x_1 \cdots x_m) &= \ddot{f}_i(x_q) + \cdots + \ddot{f}_i(x_m).\end{aligned}$$

Proof. We proceed by induction on m . If $m = 1$, then the result is trivial. If $m = 2$, then the result coincides with Lemma 8.2. Thus, assume as induction hypothesis (IH) that the result holds for any $x_1, \dots, x_k \in M$, $k \leq m$. Let $y_1, \dots, y_m, y_{m+1} \in M$. Since $\cdot$ is associative, we have that $y_1 \cdots y_m y_{m+1} = (y_1 \cdots y_m) y_{m+1}$, where

$$\text{wt}((y_1 \cdots y_m) y_{m+1}) = \text{wt}(y_1 \cdots y_m) + \text{wt}(y_{m+1}),$$

and as $\text{wt}(y_1 \cdots y_m) = \text{wt}(y_1) + \cdots + \text{wt}(y_m)$ by (IH), we obtain that

$$\text{wt}(y_1 \cdots y_m y_{m+1}) = \text{wt}(y_1) + \cdots + \text{wt}(y_m) + \text{wt}(y_{m+1}).$$

Let $i \in I$. If $\check{e}_i(y_k) = +\infty$, for some $k \in \{1, \dots, m\}$, then $\check{e}_i(y_1 \cdots y_m) = +\infty$, by (IH). Hence,

$$\check{e}_i(y_1 \cdots y_m y_{m+1}) = \check{e}_i((y_1 \cdots y_m) \check{\otimes} y_{m+1}) = +\infty,$$

whenever $\check{e}_i(y_k) = +\infty$, for some $k \in \{1, \dots, m, m+1\}$. So, assume that $\check{e}_i(y_k) \in \mathbb{Z}_{\geq 0}$, for each $k = 1, \dots, m+1$. Set

$$\begin{aligned}p &= \max\{1 \leq k \leq m+1 \mid \check{e}_i(x_k) > 0\} \text{ and} \\ q &= \min\{1 \leq l \leq m+1 \mid \check{f}_i(x_l) > 0\}.\end{aligned}$$

We consider the following cases

- Case 1: $p > q$. In particular, $p > 1$ and $q < m+1$, which implies that the sets where the maximum and minimum are taken are nonempty, and so, $\check{e}_i(y_p), \check{f}_i(y_q) > 0$. Take

$$p' = \min\{q+1 \leq k \leq p \mid \check{e}_i(y_k) > 0\}.$$

Thus, $\check{e}_i(y_k) = 0$, for $k = q+1, \dots, p'-1$. By (IH),

$$\check{f}_i(y_q \cdots y_{p'-1}) = \check{f}_i(y_q) + \cdots + \check{f}_i(y_{p'-1}) > 0,$$

and since $\ddot{\varepsilon}_i(y_{p'}) > 0$,

$$\ddot{\varepsilon}_i(y_q \cdots y_{p'}) = \ddot{\varepsilon}_i((y_q \cdots y_{p'-1})y_{p'}) = +\infty,$$

by Lemma 8.2. This implies that

$$\ddot{\varepsilon}_i(y_1 \cdots y_{p'}) = \ddot{\varepsilon}_i((y_1 \cdots y_{q-1})(y_q \cdots y_{p'})) = +\infty,$$

and finally,

$$\ddot{\varepsilon}_i(y_1 \cdots y_{m+1}) = \ddot{\varepsilon}_i((y_1 \cdots y_{p'}) (y_{p'+1} \cdots y_{m+1})) = +\infty.$$

Therefore, by Definition 7.1(vi), $\ddot{\varepsilon}_i(y_1 \cdots y_{m+1}) = \ddot{f}_i(y_1 \cdots y_{m+1}) = \perp$, and by Definition 7.1(i), $\ddot{\varphi}_i(y_1 \cdots y_{m+1}) = +\infty$.

- Case 2: $p \leq q$. Since $\ddot{\varphi}_i(y_k) = 0$, for $k = 1, \dots, q-1$, we have by (IH) that $\ddot{\varepsilon}_i(y_1 \cdots y_{p-1}) = \ddot{\varepsilon}_i(y_1) + \cdots + \ddot{\varepsilon}_i(y_{p-1})$ and $\ddot{\varphi}_i(y_1 \cdots y_{q-1}) = 0$. By Lemma 8.2, we get that

$$\ddot{\varepsilon}_i(y_1 \cdots y_{p-1}y_p) = \ddot{\varepsilon}_i(y_1 \cdots y_{p-1}) + \ddot{\varepsilon}_i(y_p) = \ddot{\varepsilon}_i(y_1) + \cdots + \ddot{\varepsilon}_i(y_p),$$

and if $\ddot{\varepsilon}_i(y_p) > 0$,

$$\ddot{\varepsilon}_i(y_1 \cdots y_{p-1}y_p) = y_1 \cdots y_{p-1} \cdot \ddot{\varepsilon}_i(y_p),$$

otherwise, $\ddot{\varepsilon}_i(y_p) = 0$ which by the definition of p implies that $p = 1$, and as $\mathcal{M}$ is seminormal, $\ddot{\varepsilon}_i(y_1) = \perp$. Since $\ddot{\varepsilon}_i(y_k) = 0$, for $k = p+1, \dots, m+1$, we have by (IH) that $\ddot{\varepsilon}_i(y_{p+1} \cdots y_{m+1}) = 0$. By Lemma 8.2, we obtain that

$$\begin{aligned} \ddot{\varepsilon}_i(y_1 \cdots y_p y_{p+1} \cdots y_{m+1}) &= \ddot{\varepsilon}_i(y_1 \cdots y_p) + \ddot{\varepsilon}_i(y_{p+1} \cdots y_{m+1}) \\ &= \ddot{\varepsilon}_i(y_1) + \cdots + \ddot{\varepsilon}_i(y_p) \end{aligned}$$

and

$$\begin{aligned} \ddot{\varepsilon}_i(y_1 \cdots y_p y_{p+1} \cdots y_{m+1}) &= \ddot{\varepsilon}_i(y_1 \cdots y_p) \cdot y_{p+1} \cdots y_{m+1} \\ &= y_1 \cdots y_{p-1} \cdot \ddot{\varepsilon}_i(y_p) \cdot y_{p+1} \cdots y_{m+1}. \end{aligned}$$

Analogously, we have that

$$\begin{aligned} \ddot{\varphi}_i(y_1 \cdots y_{m+1}) &= \ddot{\varphi}_i(y_1 \cdots y_{q-1}) + \ddot{\varphi}_i(y_q) + \ddot{\varphi}_i(y_{q+1} \cdots y_{m+1}) \\ &= \ddot{\varphi}_i(y_q) + \cdots + \ddot{\varphi}_i(y_{m+1}) \end{aligned}$$

and

$$\ddot{f}_i(y_1 \cdots y_{m+1}) = y_1 \cdots y_{q-1} \cdot \ddot{f}_i(y_q) \cdot y_{q+1} \cdots y_{m+1},$$

by (IH) and Lemma 8.2.

□

In the previous result we saw how the monoid operation interacts with the quasi-crystal structure maps. Now we show that the monoid identity 1 also satisfies some properties.

Proposition 8.4. *Let $\mathcal{M}$ be a quasi-crystal monoid where the monoid identity is denoted by 1. Then, $\text{wt}(1) = 0$, and for each $i \in I$, either $\check{\epsilon}_i(1) = \check{\varphi}_i(1) = 0$ or $\check{\epsilon}_i(x) = +\infty$, for all $x \in M$. Therefore, $\check{\epsilon}_i(1) = \check{f}_i(1) = \perp$, for all $i \in I$.*

Proof. We have that

$$\text{wt}(1) = \text{wt}(1 \cdot 1) = \text{wt}(1) + \text{wt}(1),$$

which implies that $\text{wt}(1) = 0$.

Let $i \in I$. Suppose there exists $x \in M$ such that $\check{\epsilon}_i(x) \neq +\infty$. Since $\mathcal{M}$ is seminormal, we have that $\check{\epsilon}_i(x), \check{\varphi}_i(x) \in \mathbb{Z}_{\geq 0}$ and so, $\check{\epsilon}_i(\check{e}_i^{\check{\epsilon}_i(x)}(x)) = \check{\varphi}_i(\check{f}_i^{\check{\varphi}_i(x)}(x)) = 0$. Set $y = \check{e}_i^{\check{\epsilon}_i(x)}(x)$ and $z = \check{f}_i^{\check{\varphi}_i(x)}(x)$. Since $\check{\epsilon}_i(1y) = \check{\epsilon}_i(y) = 0$, we get by Lemma 8.2 that

$$\check{\epsilon}_i(1) = \check{\epsilon}_i(1) + \check{\epsilon}_i(y) = \check{\epsilon}_i(1y) = \check{\epsilon}_i(y) = 0,$$

and analogously,

$$\check{\varphi}_i(1) = \check{\varphi}_i(1) + \check{\varphi}_i(z) = \check{\varphi}_i(1z) = \check{\varphi}_i(z) = 0.$$

Therefore, for any $i \in I$, we have that $\check{\epsilon}_i(1), \check{\varphi}_i(1) \in \{0, +\infty\}$, and so, $\check{e}_i$ and $\check{f}_i$ are undefined on 1, because $\mathcal{M}$ is a seminormal quasi-crystal. □

Note that by Definition 7.1(vi), if for some $i \in I$ we have $\check{\epsilon}_i(x) = +\infty$, for all $x \in M$, then the quasi-Kashiwara operators $\check{e}_i$ and $\check{f}_i$ are undefined on every element in M . Such a case have little interest to study in the context of this chapter, and we say that such a quasi-crystal monoid is *degenerate*. Thus, we say that a quasi-crystal monoid $\mathcal{M}$ is *nondegenerate* if $\check{\epsilon}_i(1) = \check{\varphi}_i(1) = 0$, for all $i \in I$.

Due to the interaction between the binary operation $\cdot$ of a quasi-crystal monoid $\mathcal{M}$ and the quasi-crystal structure of the quasi-tensor product $\mathcal{M} \tilde{\otimes} \mathcal{M}$ required by Definition 8.1(iii), we can extend the signature rule described in Section 7.4 to quasi-crystal monoids. Let $x, y \in M$ and $i \in I$. Since $\check{\epsilon}_i(xy) = \check{\epsilon}_i(x \tilde{\otimes} y)$ and $\check{\varphi}_i(xy) = \check{\varphi}_i(x \tilde{\otimes} y)$, we have that the i -signature of xy and $x \tilde{\otimes} y$ coincide. Hence, by Proposition 7.40,

$$\text{sgn}_i^{\tilde{\otimes}}(xy) = \text{sgn}_i^{\tilde{\otimes}}(x) \text{sgn}_i^{\tilde{\otimes}}(y).$$

Also, by Proposition 8.4, either $\text{sgn}_i^{\tilde{\otimes}}(1) = \epsilon$ or $\text{sgn}_i^{\tilde{\otimes}}(z) = 0$, for any $z \in M$. Therefore, we obtained the following result, which can be seen as an improvement of Proposition 7.40 for nondegenerate quasi-crystal monoids.

Proposition 8.5. *Let $\mathcal{M}$ be a quasi-crystal monoid, and let $i \in I$. Then,*

$$\text{sgn}_i^{\otimes}(xy) = \text{sgn}_i^{\otimes}(x) \text{sgn}_i^{\otimes}(y),$$

for any $x, y \in M$. Moreover, $\text{sgn}_i^{\otimes}$ is a monoid homomorphism from M to Z_0 if and only if $\check{\varepsilon}_i(x) \in \mathbb{Z}_{\geq 0}$, for some $x \in M$.

The signature rule for quasi-crystal monoids follows directly from the previous result and Proposition 8.3. Consider a quasi-crystal monoid $\mathcal{M}$. Let $x_1, \dots, x_m \in M$ and $i \in I$. Then, we can compute the i -signature of $x_1 \cdots x_m$ based only in the i -signature of each x_k , $k = 1, \dots, m$, because

$$\text{sgn}_i^{\otimes}(x_1 \cdots x_m) = \text{sgn}_i^{\otimes}(x_1) \cdots \text{sgn}_i^{\otimes}(x_m).$$

If $\text{sgn}_i^{\otimes}(x_1 \cdots x_m) = 0$, then $\check{\varepsilon}_i(x_1 \cdots x_m) = \check{\varphi}_i(x_1 \cdots x_m) = +\infty$ which implies that $\check{e}_i(x_1 \cdots x_m) = \check{f}_i(x_1 \cdots x_m) = \perp$. Otherwise,

$$\text{sgn}_i^{\otimes}(x_1 \cdots x_m) = -a + b$$

for some $a, b \in \mathbb{Z}_{\geq 0}$. Then, $\check{\varepsilon}_i(x_1 \cdots x_m) = a$ and $\check{\varphi}_i(x_1 \cdots x_m) = b$. The raising quasi-Kashiwara operator $\check{e}_i$ is defined on $x_1 \cdots x_m$ if and only if $a \geq 1$, in which case

$$\check{e}_i(x_1 \cdots x_m) = x_1 \cdots x_{p-1} \cdot \check{e}_i(x_p) \cdot x_{p+1} \cdots x_m,$$

where x_p originates the right-most $-$ in $\text{sgn}_i^{\otimes}(x_1 \cdots x_m)$. Similarly, the lowering quasi-Kashiwara operator $\check{f}_i$ is defined on $x_1 \cdots x_m$ if and only if $b \geq 1$, in which case

$$\check{f}_i(x_1 \cdots x_m) = x_1 \cdots x_{q-1} \cdot \check{f}_i(x_q) \cdot x_{q+1} \cdots x_m,$$

where x_q originates the left-most $+$ in $\text{sgn}_i^{\otimes}(x_1 \cdots x_m)$.

The signature rule for quasi-crystal monoids can be translated in an algorithm analogous to Algorithm 7.42.

Algorithm 8.6. *Input:* Elements $x_1, \dots, x_m \in M$ of a quasi-crystal monoid $\mathcal{M}$ and an index $i \in I$.

Output: The values of $\check{e}_i$, $\check{f}_i$, $\check{\varepsilon}_i$, and $\check{\varphi}_i$ on $x_1 \cdots x_m$.

Procedure:

- (1) Set $a := 0$, $b := 0$, $p := 0$, and $q := 0$.
- (2) For each $k = 1, 2, \dots, m$, do:
 - (i) If $\check{\varepsilon}_i(x_k) = +\infty$, or $b > 0$ and $\check{\varepsilon}_i(x_k) > 0$, then set $a := -1$, $b := -1$, and break the cycle.

- (ii) If $\ddot{\epsilon}_i(x_k) > 0$, then set $a := a + \ddot{\epsilon}_i(x_k)$ and $p := k$.
 - (iii) If $\ddot{\varphi}_i(x_k) > 0$, then set $b := b + \ddot{\varphi}_i(x_k)$.
 - (iv) If $\ddot{\varphi}_i(x_k) > 0$ and $q = 0$, then set $q := k$.
- (3) If $a = -1$, then set $\ddot{\epsilon}_i(x_1 \cdots x_m) := +\infty$, and $\ddot{\varphi}_i(x_1 \cdots x_m) := +\infty$. Otherwise, set $\ddot{\epsilon}_i(x_1 \cdots x_m) := a$ and $\ddot{\varphi}_i(x_1 \cdots x_m) := b$.
- (4) If $a \geq 1$, then set $\ddot{\epsilon}_i(x_1 \cdots x_m) := x_1 \cdots x_{p-1} \cdot \ddot{\epsilon}_i(x_p) \cdot x_{p+1} \cdots x_m$. Otherwise, set $\ddot{\epsilon}_i(x_1 \cdots x_m) := \perp$.
- (5) If $b \geq 1$, then set $\ddot{f}_i(x_1 \cdots x_m) := x_1 \cdots x_{q-1} \cdot \ddot{f}_i(x_q) \cdot x_{q+1} \cdots x_m$. Otherwise, set $\ddot{f}_i(x_1 \cdots x_m) := \perp$.
- (6) Return $\ddot{\epsilon}_i(x_1 \cdots x_m)$, $\ddot{f}_i(x_1 \cdots x_m)$, $\ddot{\epsilon}_i(x_1 \cdots x_m)$, and $\ddot{\varphi}_i(x_1 \cdots x_m)$.

Finally, we introduce the notion of a homomorphism between quasi-crystal monoids.

Definition 8.7. Let $\mathcal{M}$ and $\mathcal{M}'$ be quasi-crystal monoids of the same type. A *quasi-crystal monoid homomorphism* ψ from $\mathcal{M}$ to $\mathcal{M}'$, denoted by $\psi : \mathcal{M} \rightarrow \mathcal{M}'$, is a map $\psi : M \rightarrow M'$ that satisfies the following conditions:

- (i) ψ is a quasi-crystal homomorphism;
- (ii) ψ is a monoid homomorphism.

If ψ is also bijective, it is called a *quasi-crystal monoid isomorphism*.

Note that in the previous definition we only consider maps from M to M' , but as we observed after Remark 7.13, when we state that ψ is a quasi-crystal homomorphism in condition (i) we mean that the map $\psi' : M \sqcup \{\perp\} \rightarrow M' \sqcup \{\perp\}$, given by $\psi'(\perp) = \perp$ and $\psi'(x) = \psi(x)$, for each $x \in M$, is a quasi-crystal homomorphism.

Also, if $\psi : \mathcal{M} \rightarrow \mathcal{M}'$ is a quasi-crystal monoid isomorphism, then ψ is a quasi-crystal isomorphism, by Corollary 7.18, and a monoid isomorphism. The converse is immediate, because a monoid isomorphism implies the map to be bijective. Hence, a map $\psi : M \rightarrow M'$ is a quasi-crystal isomorphism if and only if ψ is a quasi-crystal isomorphism and a monoid isomorphism. This implies that if $\psi : \mathcal{M} \rightarrow \mathcal{M}'$ is a quasi-crystal monoid isomorphism, then ψ^{-1} is a quasi-crystal monoid isomorphism between $\mathcal{M}'$ and $\mathcal{M}$.

8.2 The free quasi-crystal monoid

In this section we first present a construction of a free quasi-crystal monoid. Then, we study some properties satisfied by the free quasi-crystal monoids. In particular, we show that free quasi-crystal monoids satisfy a universal property commonly associated to the term free in algebra. Finally, by this universal property, we present an alternative method of defining a free quasi-crystal monoid up to isomorphism.

Let $\mathbb{Q}$ be a seminormal quasi-crystal. For $k \geq 1$, set

$$\mathbb{Q}^{\otimes k} = \underbrace{\mathbb{Q} \otimes \cdots \otimes \mathbb{Q}}_{k \text{ times}}.$$

By Corollary 7.18 and Theorems 7.32 and 7.37, for $k, l \geq 1$ the map $\mathbb{Q}^{\otimes k} \otimes \mathbb{Q}^{\otimes l} \rightarrow \mathbb{Q}^{\otimes(k+l)}$, given by

$$(x_1 \otimes \cdots \otimes x_k) \otimes (y_1 \otimes \cdots \otimes y_l) \mapsto x_1 \otimes \cdots \otimes x_k \otimes y_1 \otimes \cdots \otimes y_l$$

for $x_1, \dots, x_k, y_1, \dots, y_l \in Q$, is a quasi-crystal isomorphism.

Let ϵ be an element that does not lie in Q . Set $\mathbb{Q}^{\otimes 0}$ to be the seminormal quasi-crystal of the same type as $\mathbb{Q}$ formed by the set $Q^{\otimes 0} = \{\epsilon\}$ and maps given by $\text{wt}(\epsilon) = 0$, $\check{e}_i(\epsilon) = \check{f}_i(\epsilon) = \perp$ and $\check{e}_i(\epsilon) = \check{\varphi}_i(\epsilon) = 0$ ($i \in I$). Note that $\mathbb{Q}^{\otimes 0} \otimes \mathbb{Q}^{\otimes 0}$ is quasi-crystal isomorphic to $\mathbb{Q}^{\otimes 0}$, as both quasi-crystals consist of a single element where the quasi-crystal structure maps coincide. For any $x \in Q$ and $i \in I$, by Theorem 7.32 and Proposition 7.34, we have that $\text{wt}(x \otimes \epsilon) = \text{wt}(x)$, $\check{e}_i(x \otimes \epsilon) = \check{e}_i(x)$ and $\check{\varphi}_i(x \otimes \epsilon) = \check{\varphi}_i(x)$. Also, by the signature rule, since $\text{sgn}_i^{\otimes}(\epsilon) = \epsilon$, it is immediate that $\check{e}_i$ (resp., $\check{f}_i$) is defined on $x \otimes \epsilon$ if and only if $\check{e}_i$ (resp., $\check{f}_i$) is defined on x . And if so, $\check{e}_i(x \otimes \epsilon) = \check{e}_i(x) \otimes \epsilon$ (resp., $\check{f}_i(x \otimes \epsilon) = \check{f}_i(x) \otimes \epsilon$). Therefore, the map $Q \otimes Q^{\otimes 0} \rightarrow Q$, given by $y \otimes \epsilon \mapsto y$ for each $y \in Q$, is a quasi-crystal isomorphism. Analogously, the map $Q^{\otimes 0} \otimes Q \rightarrow Q$, given by $\epsilon \otimes y \mapsto y$ for each $y \in Q$, is a quasi-crystal isomorphism. Since the quasi-tensor product of quasi-crystals is associative (Theorem 7.37), we get that $\mathbb{Q}^{\otimes k} \otimes \mathbb{Q}^{\otimes 0}$ and $\mathbb{Q}^{\otimes 0} \otimes \mathbb{Q}^{\otimes k}$ are isomorphic to $\mathbb{Q}^{\otimes k}$, for any $k \geq 0$.

The sets $Q^{\otimes k}$ and $Q^{\otimes l}$ are disjoint, whenever $k \neq l$. Thus, we can extend the maps wt , $\check{e}_i$, $\check{f}_i$, $\check{e}_i$ and $\check{\varphi}_i$ ($i \in I$) defined on each $Q^{\otimes k}$ to the set

$$M = \bigcup_{k \geq 0} Q^{\otimes k}$$

obtaining a seminormal quasi-crystal $\mathcal{M}$. By the quasi-crystal isomorphisms $\mathbb{Q}^{\otimes k} \otimes \mathbb{Q}^{\otimes l} \rightarrow \mathbb{Q}^{\otimes(k+l)}$ ($k, l \geq 0$) defined above, we get that $\mathcal{M}$ becomes a quasi-crystal

monoid with the binary operation $\cdot : M \times M \rightarrow M$ given by

$$(x_1 \ddot{\otimes} \cdots \ddot{\otimes} x_k) \cdot (y_1 \ddot{\otimes} \cdots \ddot{\otimes} y_l) = x_1 \ddot{\otimes} \cdots \ddot{\otimes} x_k \ddot{\otimes} y_1 \ddot{\otimes} \cdots \ddot{\otimes} y_l$$

and

$$(x_1 \ddot{\otimes} \cdots \ddot{\otimes} x_k) \cdot \epsilon = \epsilon \cdot (x_1 \ddot{\otimes} \cdots \ddot{\otimes} x_k) = x_1 \ddot{\otimes} \cdots \ddot{\otimes} x_k,$$

for $x_1, \dots, x_k, y_1, \dots, y_l \in Q$. If we identify an element of the form $x_1 \ddot{\otimes} \cdots \ddot{\otimes} x_k$ in M with the word $x_1 \dots x_k$ over the alphabet Q , we obtain a monoid isomorphism between M and the free monoid Q^* over Q , and through this identification we can also define a quasi-crystal structure on Q^* . Therefore, we constructed a quasi-crystal monoid that leads to the following definition.

Definition 8.8. Let $\mathbb{Q}$ be a seminormal quasi-crystal. The *free quasi-crystal monoid* $\mathbb{Q}^*$ over $\mathbb{Q}$ is a quasi-crystal monoid of the same type as $\mathbb{Q}$ consisting of the set Q^* of all words over Q , the usual concatenation of words, and maps defined as follows. For $i \in I$, set

$$\text{wt}(\epsilon) = 0, \quad \check{e}_i(\epsilon) = \check{f}_i(\epsilon) = \perp, \quad \check{e}_i(\epsilon) = \check{\varphi}_i(\epsilon) = 0,$$

where ϵ denotes the empty word, and for $u, v \in Q^*$, set

$$\text{wt}(uv) = \text{wt}(u) + \text{wt}(v)$$

and consider the following cases

(i) if $\check{\varphi}_i(u), \check{e}_i(v) > 0$, set

$$\check{e}_i(uv) = \check{f}_i(uv) = \perp \quad \check{e}_i(uv) = \check{\varphi}_i(uv) = +\infty;$$

(ii) otherwise, set

$$\begin{aligned} \check{e}_i(uv) &= \begin{cases} \check{e}_i(u)v & \text{if } \check{\varphi}_i(u) \geq \check{e}_i(v) \\ u\check{e}_i(v) & \text{if } \check{\varphi}_i(u) < \check{e}_i(v), \end{cases} \\ \check{f}_i(uv) &= \begin{cases} \check{f}_i(u)v & \text{if } \check{\varphi}_i(u) > \check{e}_i(v) \\ u\check{f}_i(v) & \text{if } \check{\varphi}_i(u) \leq \check{e}_i(v), \end{cases} \\ \check{e}_i(uv) &= \max\{\check{e}_i(u), \check{e}_i(v) - \langle \text{wt}(u), \alpha_i^\vee \rangle\}, \\ \check{\varphi}_i(uv) &= \max\{\check{\varphi}_i(u) + \langle \text{wt}(v), \alpha_i^\vee \rangle, \check{\varphi}_i(v)\}, \end{aligned}$$

where $u\perp = \perp v = \perp$.

As we constructed the free quasi-crystal monoid $\mathbb{Q}^*$ based on the quasi-tensor product of quasi-crystals (Definition 7.33), we described its quasi-crystal structure based on the definition of the quasi-crystal structure for a quasi-tensor product (Theorem 7.32). Notice that we explicitly gave the values of the quasi-crystal structure maps of $\mathbb{Q}^*$ on the empty word ϵ , for letters the values follow from the quasi-crystal structure of $\mathbb{Q}$, and for a word of the form uv they depend only on the values of the quasi-crystal structure maps on u and v . Thus, the definition of the quasi-crystal structure maps is not circular. Moreover, from Proposition 8.3 and Algorithm 8.6, we have how to get the values of the quasi-crystal structure maps on w based on their values on its letters, which implies the following result.

Proposition 8.9. *Let $\mathbb{Q}$ be a seminormal quasi-crystal. For any $w \in Q^*$ and $i \in I$, if $\check{e}_i(w) \in Q^*$ then $|\check{e}_i(w)| = |w|$, and if $\check{f}_i(w) \in Q^*$ then $|\check{f}_i(w)| = |w|$. Therefore, $|u| = |v|$, whenever u and v lie in the same connected component of $\mathbb{Q}^*$.*

Proof. Let $w \in Q^*$ and $i \in I$ such that $\check{f}_i$ is defined on w , then by Definition 8.8, we have that $w \neq \epsilon$, and so, $w = x_1 \dots x_m$, for some $x_1, \dots, x_m \in Q$ and $m \geq 1$. By Proposition 8.3, there exists $q \in \{1, \dots, m\}$ such that

$$\check{f}_i(w) = x_1 \dots x_{q-1}(\check{f}_i(x_q))x_{q+1} \dots x_m.$$

Since $x_q \in Q$, then $\check{f}_i(x_q) \in Q$, which implies that $|\check{f}_i(w)| = |w|$. Hence, for any $w \in Q^*$ and $i \in I$, $|\check{f}_i(w)| = |w|$, whenever $\check{f}_i(w) \in Q^*$. Similarly, for any $w \in Q^*$ and $i \in I$, if $\check{e}_i$ is defined on w , then $|\check{e}_i(w)| = |w|$.

Finally, let $u, v \in Q^*$ be such that u and v lie in the same connected component of $\mathbb{Q}^*$. Then, there exist $g_1, \dots, g_k \in \{\check{e}_i, \check{f}_i \mid i \in I\}$ such that $g_1 \dots g_k(u) = v$. By applying recursively what we have proven above, we obtain that $|g_1 \dots g_k(u)| = |u|$. $\square$

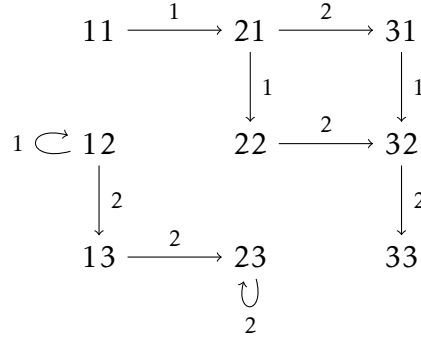
Example 8.10. Consider the standard quasi-crystal $\mathcal{A}_n$ of type $\mathfrak{A}_n$ as described in Example 7.4(i). Let $w \in A_n^*$. The weight of w is given by

$$\text{wt}(w) = |w|_1 \mathbf{e}_1 + |w|_2 \mathbf{e}_2 + \dots + |w|_n \mathbf{e}_n.$$

Let $i \in \{1, \dots, n-1\}$. If w has a decomposition of the form $w = w_1 i w_2 (i+1) w_3$, for some $w_1, w_2, w_3 \in A_n^*$, then $\check{e}_i(w) = \check{\varphi}_i(w) = +\infty$. Otherwise, $\check{e}_i(w) = |w|_{i+1}$ and $\check{\varphi}_i(w) = |w|_i$. The raising quasi-Kashiwara operator $\check{e}_i$ is defined on w if and only if w has a decomposition of the form $w = u_1 (i+1) u_2$, for some $u_1, u_2 \in A_n^*$ with $|u_1|_i = |u_2|_{i+1} = 0$, and if so, $\check{e}_i(w) = u_1 i u_2$. The lowering quasi-Kashiwara operator $\check{f}_i$ is defined on w if and only if w has a decomposition of the form $w = v_1 i v_2$, for some $v_1, v_2 \in A_n^*$ with $|v_1|_i = |v_2|_{i+1} = 0$, and if so, $\check{f}_i(w) = v_1 (i+1) v_2$. Informally,

provided that w does not have a decomposition of the form $w = w_1 i w_2 (i+1) w_3$, we have that $\check{e}_i(w)$ is obtained by replacing the right-most $i+1$ in w by i , and $\check{f}_i(w)$ is obtained by replacing the left-most i in w by $i+1$.

From Example 7.35, We have that the words of length 2 form the following subgraph of the quasi-crystal graph of $\mathcal{A}_3^*$.



We characterized the quasi-crystal monoid $\mathbb{Q}^*$ over a seminormal quasi-crystal $\mathbb{Q}$ as free, because of the following property.

Theorem 8.11. *Let $\mathbb{Q}$ be a seminormal quasi-crystal, and let $\mathcal{M}$ be a quasi-crystal monoid of the same type as $\mathbb{Q}$, where $\check{e}_i(1) = \check{\varphi}_i(1) = 0$, for all $i \in I$. Then, for each quasi-crystal homomorphism $\psi : \mathbb{Q} \rightarrow \mathcal{M}$ satisfying $\psi(Q) \subseteq M$, there exists a unique quasi-crystal monoid homomorphism $\hat{\psi} : \mathbb{Q}^* \rightarrow \mathcal{M}$ such that $\hat{\psi}(x) = \psi(x)$, for all $x \in Q$.*

Proof. Let $\psi : \mathbb{Q} \rightarrow \mathcal{M}$ be a quasi-crystal homomorphism such that $\psi(Q) \subseteq M$. So, we can consider ψ as a map from Q to M . It is well-known that there exists a unique monoid homomorphism $\hat{\psi} : Q^* \rightarrow M$ such that $\hat{\psi}(x) = \psi(x)$, for all $x \in Q$. We also have that $\hat{\psi}$ is given by

$$\hat{\psi}(\epsilon) = 1 \quad \text{and} \quad \hat{\psi}(x_1 \dots x_m) = \psi(x_1) \cdots \psi(x_m),$$

for $x_1, \dots, x_m \in Q$. It remains to show that $\hat{\psi}$ is a quasi-crystal homomorphism.

Let $i \in I$. By Proposition 8.4 and Definition 8.8, we have that $\text{wt}(\epsilon) = 0 = \text{wt}(1)$, $\check{e}_i(\epsilon) = \check{f}_i(\epsilon) = \perp = \check{e}_i(1) = \check{f}_i(1)$, and $\check{e}_i(\epsilon) = \check{\varphi}_i(\epsilon) = 0 = \check{e}_i(1) = \check{\varphi}_i(1)$. Let $x_1, \dots, x_m \in Q$, $m \geq 1$, and set $w = x_1 \dots x_m$. We have that $\hat{\psi}(w) = \psi(x_1) \cdots \psi(x_m)$, where $\text{wt}(\psi(x_k)) = \text{wt}(x_k)$, $\check{e}_i(\psi(x_k)) = \check{e}_i(x_k)$ and $\check{\varphi}_i(\psi(x_k)) = \check{\varphi}_i(x_k)$ for $k = 1, \dots, m$, because ψ is a quasi-crystal homomorphism. By Proposition 8.3, we obtain that

$$\text{wt}(\hat{\psi}(w)) = \text{wt}(\psi(x_1)) + \cdots + \text{wt}(\psi(x_m)) = \text{wt}(x_1) + \cdots + \text{wt}(x_m) = \text{wt}(w),$$

and since

$$p = \max\{1 \leq k \leq m \mid \check{e}_i(x_k) > 0\} = \max\{1 \leq k \leq m \mid \check{e}_i(\psi(x_k)) > 0\}$$

and

$$q = \min\{1 \leq l \leq m \mid \check{\varphi}_i(x_l) > 0\} = \min\{1 \leq l \leq m \mid \check{\varphi}_i(\psi(x_l)) > 0\},$$

we also get $\check{\varepsilon}_i(\hat{\psi}(w)) = \check{\varepsilon}_i(w)$ and $\check{\varphi}_i(\hat{\psi}(w)) = \check{\varphi}_i(w)$. By the same proposition, if $\check{\varepsilon}_i(w) \in Q^*$, then

$$\check{\varepsilon}_i(w) = x_1 \dots x_{p-1}(\check{\varepsilon}_i(x_p))x_{p+1} \dots x_m$$

where $\check{\varepsilon}_i(x_p) \in Q$, because $x_p \in Q$, and since ψ is a quasi-crystal homomorphism and $\psi(Q) \subseteq M$, we have that $\psi(x_p), \psi(\check{\varepsilon}_i(x_p)) \in M$, which implies that $\psi(\check{\varepsilon}_i(x_p)) = \check{\varepsilon}_i(\psi(x_p))$, and so,

$$\begin{aligned} \hat{\psi}(\check{\varepsilon}_i(w)) &= \psi(x_1) \cdots \psi(x_{p-1}) \cdot \psi(\check{\varepsilon}_i(x_p)) \cdot \psi(x_{p+1}) \cdots \psi(x_m) \\ &= \psi(x_1) \cdots \psi(x_{p-1}) \cdot \check{\varepsilon}_i(\psi(x_p)) \cdot \psi(x_{p+1}) \cdots \psi(x_m) = \check{\varepsilon}_i(\hat{\psi}(w)). \end{aligned}$$

Analogously, if $\check{f}_i(w) \in Q^*$, then $\hat{\psi}(\check{f}_i(w)) = \check{f}_i(\hat{\psi}(w))$. □

The property described in the previous result can be used to define the free quasi-crystal monoid up to isomorphism among nondegenerate quasi-crystal monoids.

Corollary 8.12. *Let $\mathbb{Q}$ be a seminormal quasi-crystal and $\mathcal{M}$ be a quasi-crystal monoid of the same type such that $Q \subseteq M$. If for each nondegenerate quasi-crystal monoid $\mathcal{M}'$, and each quasi-crystal homomorphism $\psi : \mathbb{Q} \rightarrow \mathcal{M}'$ satisfying $\psi(Q) \subseteq M'$, there exists a unique quasi-crystal monoid homomorphism $\hat{\psi} : \mathcal{M} \rightarrow \mathcal{M}'$ such that $\hat{\psi}(x) = \psi(x)$, for all $x \in Q$, then there exists a quasi-crystal monoid isomorphism between $\mathcal{M}$ and $\mathbb{Q}^*$.*

Proof. Define $\psi : Q \rightarrow Q^*$ by $\psi(x) = x$, for each $x \in Q$. As the quasi-crystal structure maps of $\mathbb{Q}^*$ agree on words of length 1 with the quasi-crystal structure maps of $\mathbb{Q}$, we have that ψ is a quasi-crystal homomorphism from $\mathbb{Q}$ to $\mathbb{Q}^*$. Thus, there exists a unique quasi-crystal monoid homomorphism $\psi_1 : \mathcal{M} \rightarrow \mathbb{Q}^*$ such that $\psi_1(x) = x$, for all $x \in Q$. In particular, by restricting ψ_1 to Q , we get that the quasi-crystal structure maps of $\mathcal{M}$ restricted to Q agree with the quasi-crystal structure maps of $\mathbb{Q}$, and so, the map $\psi' : Q \rightarrow M$, given by $\psi'(x) = x$ for each $x \in Q$, is a quasi-crystal homomorphism. By Theorem 8.11, there exists a unique quasi-crystal monoid homomorphism $\psi_2 : \mathbb{Q}^* \rightarrow \mathcal{M}$ such that $\psi_2(x) = x$, for all $x \in Q$. Then, $\psi_1 \circ \psi_2$ is a quasi-crystal monoid homomorphism from $\mathbb{Q}^*$ to $\mathbb{Q}^*$ such that $\psi_1 \psi_2(x) = x$, for any $x \in Q$, and by Theorem 8.11, we obtain that $\psi_1 \circ \psi_2$ must be the identity map on Q^* . Also, $\psi_2 \circ \psi_1$ is a quasi-crystal monoid homomorphism from $\mathcal{M}$ to $\mathcal{M}$ such that $\psi_2 \psi_1(x) = x$, for any $x \in Q$, and by uniqueness, we get that $\psi_2 \circ \psi_1$ must be the identity map on M . Hence, ψ_1 is a quasi-crystal monoid isomorphism. □

8.3 Congruences and quotients

In this section we introduce the notion of a congruence on a quasi-crystal monoid. This leads to the definition of a quotient quasi-crystal monoid. We then show how congruences relate with homomorphisms, and deduce a homomorphism theorem for quasi-crystal monoids.

Definition 8.13. Let $\mathcal{M}$ be a quasi-crystal monoid. A *quasi-crystal monoid congruence* on $\mathcal{M}$ is an equivalence relation $\theta \subseteq M \times M$ satisfying the following conditions:

- (i) if $(x, y) \in \theta$, then $\text{wt}(x) = \text{wt}(y)$, $\check{e}_i(x) = \check{e}_i(y)$ and $\check{\varphi}_i(x) = \check{\varphi}_i(y)$ for all $i \in I$;
- (ii) if $(x, y) \in \theta$ and $\check{e}_i(x) \in M$, then $(\check{e}_i(x), \check{e}_i(y)) \in \theta$;
- (iii) if $(x, y) \in \theta$ and $\check{f}_i(x) \in M$, then $(\check{f}_i(x), \check{f}_i(y)) \in \theta$;
- (iv) if $(x_1, y_1), (x_2, y_2) \in \theta$, then $(x_1 x_2, y_1 y_2) \in \theta$.

Let $\mathcal{M}$ be a quasi-crystal monoid. It is immediate from the definition that the equality relation $\Delta = \{(x, x) \mid x \in M\}$ is a quasi-crystal monoid congruence on $\mathcal{M}$. We have that $\Delta \subseteq \theta$, for any quasi-crystal monoid congruence θ on $\mathcal{M}$. Also, given a nonempty family Θ of quasi-crystal monoid congruences on $\mathcal{M}$, it is straightforward to show that $\bigcap \Theta$ is a quasi-crystal monoid congruence on $\mathcal{M}$. From this and the following result, we are able to show that the quasi-crystal monoid congruences on a quasi-crystal monoid form a lattice.

Lemma 8.14. Let $\mathcal{M}$ be a quasi-crystal monoid, and let $R \subseteq M \times M$ be a relation on M such that for any $(x, y) \in R$ and $i \in I$, the following conditions are satisfied:

- (i) $\text{wt}(x) = \text{wt}(y)$, $\check{e}_i(x) = \check{e}_i(y)$, and $\check{\varphi}_i(x) = \check{\varphi}_i(y)$;
- (ii) if $\check{e}_i(x) \in M$, then $(\check{e}_i(x), \check{e}_i(y)) \in R$; and
- (iii) if $\check{f}_i(x) \in M$, then $(\check{f}_i(x), \check{f}_i(y)) \in R$.

Then, the monoid congruence θ_R generated by R is a quasi-crystal monoid congruence on $\mathcal{M}$.

Proof. We check that in every step of constructing θ_R from R , properties (i) to (iii) are preserved. Set

$$R_1 = \{(ux, uy) \mid (x, y) \in R, u \in M\}.$$

For $(x, y) \in R$, $u \in M$ and $i \in I$, since $\text{wt}(x) = \text{wt}(y)$, $\check{e}_i(x) = \check{e}_i(y)$ and $\check{\varphi}_i(x) = \check{\varphi}_i(y)$, we get that $\text{wt}(ux) = \text{wt}(uy)$, $\check{e}_i(ux) = \check{e}_i(uy)$ and $\check{\varphi}_i(ux) = \check{\varphi}_i(uy)$, by Lemma 8.2.

As $\mathcal{M}$ is seminormal, we have that $\ddot{e}_i(ux) \in M$ if and only if $\ddot{e}_i(uy) \in M$. And if so, we obtain if $\ddot{\varphi}_i(x) = 0$ that $\ddot{e}_i(ux) = \ddot{e}_i(u) \cdot x$ and $\ddot{e}_i(uy) = \ddot{e}_i(u) \cdot y$ implying $(\ddot{e}_i(ux), \ddot{e}_i(uy)) \in R_1$, or otherwise, we obtain that $\ddot{e}_i(ux) = u \cdot \ddot{e}_i(x)$ and $\ddot{e}_i(uy) = u \cdot \ddot{e}_i(y)$ implying $(\ddot{e}_i(ux), \ddot{e}_i(uy)) \in R_1$, because $(\ddot{e}_i(x), \ddot{e}_i(y)) \in R$. Similarly, if $\ddot{f}_i(ux) \in M$, we have either $\ddot{f}_i(ux) = \ddot{f}_i(u) \cdot x$ and $\ddot{f}_i(uy) = \ddot{f}_i(u) \cdot y$, or $\ddot{f}_i(ux) = u \cdot \ddot{f}_i(x)$ and $\ddot{f}_i(uy) = u \cdot \ddot{f}_i(y)$, which results in $(\ddot{f}_i(ux), \ddot{f}_i(uy)) \in R_1$ in any case. Hence, R_1 satisfies conditions (i) to (iii).

Set

$$R_2 = \{(xv, yv) \mid (x, y) \in R_1, v \in M\}.$$

For $(x, y) \in R_1$, $v \in M$ and $i \in I$, we have that $\text{wt}(x) = \text{wt}(y)$, $\ddot{e}_i(x) = \ddot{e}_i(y)$ and $\ddot{\varphi}_i(x) = \ddot{\varphi}_i(y)$. As shown for R_1 , from Lemma 8.2 we can analogously deduce that R_2 satisfies conditions (i) to (iii).

It is immediate that the reflexive closure

$$R_3 = \{(x, y) \in M \times M \mid (x, y) \in R_2 \text{ or } x = y\}$$

of R_2 satisfies conditions (i) to (iii).

Set

$$R_4 = \{(x, y) \in M \times M \mid (x, y) \in R_3 \text{ or } (y, x) \in R_3\},$$

which correspondes to the symmetric closure of R_3 . Since R_3 satisfies condition (i), so it does R_4 . For $(x, y) \in R_3$, if $\ddot{e}_i(y) \in M$, then $\ddot{e}_i(x) \in M$, because $\ddot{e}_i(x) = \ddot{e}_i(y)$ and $\mathcal{M}$ is seminormal, and so, $(\ddot{e}_i(x), \ddot{e}_i(y)) \in R_3$. Similarly, if $\ddot{f}_i(y) \in M$, then $(\ddot{f}_i(x), \ddot{f}_i(y)) \in R_3$. Hence, R_4 satisfies conditions (ii) and (iii).

Finally, θ_R corresponds to the transitive closure of R_4 . For $(x, y) \in \theta_R$, there exist $x_0, x_1, \dots, x_m \in M$ such that $x = x_0$, $y = x_m$ and $(x_k, x_{k+1}) \in R_4$, for $k = 0, \dots, m-1$. For $i \in I$, since R_4 satisfies condition (i), we get that

$$\text{wt}(x) = \text{wt}(x_0) = \text{wt}(x_1) = \dots = \text{wt}(x_m) = \text{wt}(y),$$

and similarly, $\ddot{e}_i(x) = \ddot{e}_i(y)$ and $\ddot{\varphi}_i(x) = \ddot{\varphi}_i(y)$. If $\ddot{e}_i(x) \in M$, then for $k = 0, \dots, m-1$, since $\ddot{e}_i(x_k) = \ddot{e}_i(x)$ and $\mathcal{M}$ is seminormal, we have that $\ddot{e}_i(x_k) \in M$, and as R_4 satisfies condition (ii), $(\ddot{e}_i(x_k), \ddot{e}_i(x_{k+1})) \in R_4$. Hence, $(\ddot{e}_i(x), \ddot{e}_i(y)) = (\ddot{e}_i(x_0), \ddot{e}_i(x_m)) \in \theta_R$. Analogously, as R_4 satisfies condition (iii), if $\ddot{f}_i(x) \in M$, then $(\ddot{f}_i(x), \ddot{f}_i(y)) \in \theta_R$. Therefore, θ_R satisfies conditions (i) to (iii). Moreover, by construction θ_R is a monoid congruence on M , and so, θ_R is a quasi-crystal monoid congruence on $\mathcal{M}$. $\square$

Theorem 8.15. *Let $\mathcal{M}$ be a quasi-crystal monoid. Then, the quasi-crystal monoid congruences on $\mathcal{M}$ form a lattice with respect to the partial order $\subseteq$ of inclusion.*

Proof. Let Θ be the set of all quasi-crystal monoid congruences on $\mathcal{M}$. Since the equality relation Δ lies in Θ , we have that Θ is nonempty. Clearly, $\subseteq$ is a partial order on Θ . Let $\theta, \sigma \in \Theta$. Since $\theta \cap \sigma$ is a quasi-crystal monoid congruence and the largest set contained in θ and σ , we have that $\theta \cap \sigma$ is the infimum of θ and σ in Θ . Finally, the relation $R = \theta \cup \sigma$ satisfies conditions (i) to (iii) of Lemma 8.14, and so, the monoid congruence θ_R generated by R is a quasi-crystal monoid congruence on $\mathcal{M}$. Furthermore, θ_R is the smallest equivalence on M satisfying Definition 8.13(iv) and containing R . Hence, θ_R is the supremum of θ and σ in Θ . $\square$

Let θ be a quasi-crystal monoid congruence on a quasi-crystal monoid $\mathcal{M}$. From the conditions (i) to (iii) of Definition 8.13, it follows that the quasi-crystal structure maps wt , $\check{e}_i$, $\check{f}_i$, $\check{\varepsilon}_i$ and $\check{\varphi}_i$ ($i \in I$) of $\mathcal{M}$ can be defined on the set of all θ -equivalence classes M/θ in a natural way, because they do not depend on the equivalence class representative. If $x, y \in M$ are such that $[x] = [y]$, then $x \theta y$, and by Definition 8.13(i), $\text{wt}(x) = \text{wt}(y)$, $\check{e}_i(x) = \check{e}_i(y)$ and $\check{\varphi}_i(x) = \check{\varphi}_i(y)$. As $\mathcal{M}$ is seminormal and $\check{e}_i(x) = \check{e}_i(y)$, we have that $\check{e}_i$ is defined on x if and only if $\check{e}_i$ is defined on y . And if so, by Definition 8.13(ii), we have that $[\check{e}_i(x)] = [\check{e}_i(y)]$. Similarly, by Definition 8.13(iii), $[\check{f}_i(x)] = [\check{f}_i(y)]$, whenever $\check{f}_i$ is defined on x or y . Finally, if $x_1, x_2, y_1, y_2 \in M$ are such that $[x_1] = [y_1]$ and $[x_2] = [y_2]$, then $[x_1 x_2] = [y_1 y_2]$, by Definition 8.13(iv). Therefore, we obtain the following construction.

Definition 8.16. Let θ be a congruence on a quasi-crystal monoid $\mathcal{M}$. The *quotient quasi-crystal monoid* of $\mathcal{M}$ by θ is a quasi-crystal monoid $\mathcal{M}/\theta$ of the same type as $\mathcal{M}$ consisting of the set M/θ and maps given by

$$\begin{array}{lll} \text{wt}([x]) = \text{wt}(x) & \check{e}_i([x]) = [\check{e}_i(x)] & \check{f}_i([x]) = [\check{f}_i(x)] \\ \check{\varepsilon}_i([x]) = \check{\varepsilon}_i(x) & \check{\varphi}_i([x]) = \check{\varphi}_i(x) & [x] \cdot [y] = [x \cdot y], \end{array}$$

where $[\perp] = \perp$, for $x, y \in M$ and $i \in I$.

The following result follows directly from the previous definition.

Lemma 8.17. *Let θ be a congruence on a quasi-crystal monoid $\mathcal{M}$. Then, the map $\pi : M \rightarrow M/\theta$, given by $\pi(x) = [x]$ for each $x \in M$, is a surjective quasi-crystal monoid homomorphism from $\mathcal{M}$ to $\mathcal{M}/\theta$.*

This leads to the following result that relates congruences and homomorphisms on quasi-crystal monoids.

Theorem 8.18. *Let $\mathcal{M}$ be a quasi-crystal monoid, and let $\theta \subseteq M \times M$. Then, θ is a congruence on $\mathcal{M}$ if and only if there exist a quasi-crystal monoid $\mathcal{M}'$ and a quasi-crystal monoid homomorphism $\psi : \mathcal{M} \rightarrow \mathcal{M}'$ such that $\theta = \ker \psi$.*

Proof. If θ is a quasi-crystal monoid congruence on $\mathcal{M}$, set $\pi : \mathcal{M} \rightarrow \mathcal{M}/\theta$ to be the quasi-crystal monoid homomorphism defined in Lemma 8.17. Then, for $x, y \in M$, we have that $\pi(x) = \pi(y)$ if and only if $x \theta y$, and so, $\theta = \ker \pi$.

Conversely, let $\psi : \mathcal{M} \rightarrow \mathcal{M}'$ be a quasi-crystal monoid homomorphism. Clearly $\ker \psi$ is an equivalence on M . Let $(x, y) \in \ker \psi$ and $i \in I$. Since $\psi(x) = \psi(y)$, we have that

$$\text{wt}(x) = \text{wt}(\psi(x)) = \text{wt}(\psi(y)) = \text{wt}(y),$$

and similarly, $\check{e}_i(x) = \check{e}_i(y)$ and $\check{f}_i(x) = \check{f}_i(y)$. If $\check{e}_i(x) \in M$, then $\check{e}_i(y) \in M$, because $\mathcal{M}$ is seminormal and $\check{e}_i(x) = \check{e}_i(y)$. Also, since $\psi(M) \subseteq M'$, we get that $\psi(\check{e}_i(x)), \psi(\check{e}_i(y)) \in M'$, and since ψ is a quasi-crystal monoid homomorphism,

$$\psi(\check{e}_i(x)) = \check{e}_i(\psi(x)) = \check{e}_i(\psi(y)) = \psi(\check{e}_i(y)),$$

which implies that $(\check{e}_i(x), \check{e}_i(y)) \in \ker \psi$. Analogously, if $\check{f}_i(x) \in M$, then we deduce that $(\check{f}_i(x), \check{f}_i(y)) \in \ker \psi$. Finally, for $(x_1, y_1), (x_2, y_2) \in \ker \psi$, we have that

$$\psi(x_1 x_2) = \psi(x_1) \cdot \psi(x_2) = \psi(y_1) \cdot \psi(y_2) = \psi(y_1 y_2),$$

and thus, $(x_1 x_2, y_1 y_2) \in \ker \psi$. Hence, $\ker \psi$ is a quasi-crystal monoid congruence on $\mathcal{M}$. $\square$

Now, we introduce the homomorphism theorems for quasi-crystal monoids.

Theorem 8.19. *Let $\mathcal{M}$ and $\mathcal{M}'$ be quasi-crystal monoids of the same type, and let $\psi : \mathcal{M} \rightarrow \mathcal{M}'$ be a quasi-crystal monoid homomorphism. Then, for each quasi-crystal monoid congruence θ on $\mathcal{M}$ satisfying $\theta \subseteq \ker \psi$, there exists a unique quasi-crystal monoid homomorphism $\hat{\psi} : \mathcal{M}/\theta \rightarrow \mathcal{M}'$ such that $\hat{\psi}([x]_\theta) = \psi(x)$ for any $x \in M$.*

Furthermore, if ψ is surjective, then there exists a unique quasi-crystal monoid isomorphism $\hat{\psi} : \mathcal{M}/\ker \psi \rightarrow \mathcal{M}'$ such that $\hat{\psi}([x]) = \psi(x)$, for all $x \in M$.

Proof. Let θ be a quasi-crystal monoid congruence on $\mathcal{M}$ such that $\theta \subseteq \ker \psi$. For $x, y \in M$, if $[x]_\theta = [y]_\theta$, then $x \theta y$, which implies that $\psi(x) = \psi(y)$, because $\theta \subseteq \ker \psi$. Thus, we can define a map $\hat{\psi} : M/\theta \rightarrow M'$ by $\hat{\psi}([x]_\theta) = \psi(x)$, for each $x \in M$. Since ψ is a quasi-crystal monoid homomorphism, it is immediate from Definition 8.16 that $\hat{\psi}$ is a quasi-crystal monoid homomorphism from $\mathcal{M}/\theta$ to $\mathcal{M}'$.

Assume that $\theta = \ker \psi$ and that ψ is surjective. Then, given $x' \in M'$, there exists $x \in M$ such that $\psi(x) = x'$, and so, $\hat{\psi}([x]) = x'$. Also, if $y, z \in M$ are such that $\hat{\psi}([y]) =$

$\hat{\psi}([z])$, then $\psi(y) = \psi(z)$, or equivalently, $(y, z) \in \ker \psi$, which implies that $[y] = [z]$. Hence, $\hat{\psi}$ is bijective, and therefore, a quasi-crystal monoid isomorphism. $\square$

Theorem 8.20. *Let θ and σ be congruences on a quasi-crystal monoid $\mathcal{M}$ such that $\theta \subseteq \sigma$. Define a relation σ/θ on M/θ by*

$$\sigma/\theta = \{([x]_\theta, [y]_\theta) \mid (x, y) \in \sigma\}.$$

Then, σ/θ is a quasi-crystal monoid congruence on $\mathcal{M}/\theta$. Moreover, the map from $(M/\theta)/(\sigma/\theta)$ to M/σ , given by $[x]_\theta]_{\sigma/\theta} \mapsto [x]_\sigma$ for each $x \in M$, is a quasi-crystal monoid isomorphism between $(\mathcal{M}/\theta)/(\sigma/\theta)$ and $\mathcal{M}/\sigma$.

Proof. Define a map $\pi : M \rightarrow M/\sigma$ by $\pi(x) = [x]_\sigma$, for each $x \in M$. By Lemma 8.17, π is a quasi-crystal monoid homomorphism from $\mathcal{M}$ to $\mathcal{M}/\sigma$. By Theorem 8.19, since $\theta \subseteq \sigma = \ker \pi$, we have a quasi-crystal monoid homomorphism $\hat{\pi} : \mathcal{M}/\theta \rightarrow \mathcal{M}/\sigma$ given by $\hat{\pi}([x]_\theta) = [x]_\sigma$, for each $x \in M$. Clearly, $\hat{\pi}$ is surjective. For $x, y \in M$, we have that $\hat{\pi}([x]_\theta) = \hat{\pi}([y]_\theta)$ if and only if $(x, y) \in \sigma$. Hence, $\ker \hat{\pi} = \theta/\sigma$. By Theorem 8.19, we obtain that the map $(M/\theta)/(\sigma/\theta) \rightarrow M/\sigma$, given by $[x]_\theta]_{\sigma/\theta} \mapsto [x]_\sigma$ for each $x \in M$, is a quasi-crystal monoid isomorphism between $(\mathcal{M}/\theta)/(\sigma/\theta)$ and $\mathcal{M}/\sigma$. $\square$

8.4 The hypoplactic congruence

In this section we define the hypoplactic monoid associated to a seminormal quasi-crystal. This definition is based on the framework introduced on the previous sections, as the hypoplactic monoid will be obtained as a quotient quasi-crystal monoid of the free quasi-crystal monoid by the following congruence.

Definition 8.21. Let $\mathbb{Q}$ be a seminormal quasi-crystal. The *hypoplactic congruence* on $\mathbb{Q}^*$ is a relation $\sim$ on Q^* given as follows. For $u, v \in Q^*$, $u \sim v$ if and only if there exists a quasi-crystal isomorphism $\psi : \mathbb{Q}^*(u) \rightarrow \mathbb{Q}^*(v)$ such that $\psi(u) = v$.

To prove that $\sim$ is a congruence on $\mathbb{Q}^*$ we first show the following result.

Lemma 8.22. *Let $\mathbb{Q}$ be a seminormal quasi-crystal. For each $u, v \in Q^*$, the map from $(Q^*(u) \otimes Q^*(v))(u \otimes v)$ to $Q^*(uv)$, given by $x \otimes y \mapsto xy$, is a quasi-crystal isomorphism between $(\mathbb{Q}^*(u) \otimes \mathbb{Q}^*(v))(u \otimes v)$ and $\mathbb{Q}^*(uv)$.*

Proof. Let $u, v \in Q^*$. Since $\mathbb{Q}^*$ is a quasi-crystal monoid, the map $Q^* \otimes Q^* \rightarrow Q^*$, defined by $x \otimes y \mapsto xy$ for each $x, y \in Q^*$, is a quasi-crystal homomorphism (Definition 8.1(iii)). Then, by Proposition 7.30 we have a surjective quasi-crystal

homomorphism $\psi : (\mathbb{Q}^* \tilde{\otimes} \mathbb{Q}^*)(u \tilde{\otimes} v) \rightarrow \mathbb{Q}^*(uv)$ given by $\psi(x \tilde{\otimes} y) = xy$, for each $x \tilde{\otimes} y \in (\mathbb{Q}^* \tilde{\otimes} \mathbb{Q}^*)(u \tilde{\otimes} v)$.

By Definition 7.22, if $x \tilde{\otimes} y \in (\mathbb{Q}^* \tilde{\otimes} \mathbb{Q}^*)(u \tilde{\otimes} v)$, then there exist $g_1, \dots, g_m \in \{\check{e}_i, \check{f}_i \mid i \in I\}$ such that $x \tilde{\otimes} y = g_1 \cdots g_m(u \tilde{\otimes} v)$, and by Definition 7.33, $x = g'_1 \cdots g'_k(u)$ and $y = g''_1 \cdots g''_l(v)$ for some $g'_1, \dots, g'_k, g''_1, \dots, g''_l \in \{g_1, \dots, g_m\}$. Hence, $x \in \mathbb{Q}^*(u)$ and $y \in \mathbb{Q}^*(v)$, and so, $(\mathbb{Q}^* \tilde{\otimes} \mathbb{Q}^*)(u \tilde{\otimes} v) \subseteq \mathbb{Q}^*(u) \tilde{\otimes} \mathbb{Q}^*(v)$ which implies that $(\mathbb{Q}^* \tilde{\otimes} \mathbb{Q}^*)(u \tilde{\otimes} v) = (\mathbb{Q}^*(u) \tilde{\otimes} \mathbb{Q}^*(v))(u \tilde{\otimes} v)$. And therefore, $(\mathbb{Q}^* \tilde{\otimes} \mathbb{Q}^*)(u \tilde{\otimes} v)$ and $(\mathbb{Q}^*(u) \tilde{\otimes} \mathbb{Q}^*(v))(u \tilde{\otimes} v)$ correspond to the same quasi-crystal. In particular, ψ is a surjective quasi-crystal homomorphism from $(\mathbb{Q}^*(u) \tilde{\otimes} \mathbb{Q}^*(v))(u \tilde{\otimes} v)$ to $\mathbb{Q}^*(uv)$.

Finally, for $x_1 \tilde{\otimes} y_1, x_2 \tilde{\otimes} y_2 \in (\mathbb{Q}^*(u) \tilde{\otimes} \mathbb{Q}^*(v))(u \tilde{\otimes} v)$, we have by Proposition 8.9 that $|x_1| = |x_2|$ and $|y_1| = |y_2|$. Thus, if $x_1 y_1 = x_2 y_2$, then $x_1 = x_2$ and $y_1 = y_2$ implying $x_1 \tilde{\otimes} y_1 = x_2 \tilde{\otimes} y_2$. Hence, ψ is injective. By Corollary 7.18, ψ is a quasi-crystal isomorphism between $(\mathbb{Q}^*(u) \tilde{\otimes} \mathbb{Q}^*(v))(u \tilde{\otimes} v)$ and $\mathbb{Q}^*(uv)$. $\square$

Theorem 8.23. *Let $\mathbb{Q}$ be a seminormal quasi-crystal. Then, the hypoplactic congruence $\sim$ on $\mathbb{Q}^*$ is a quasi-crystal monoid congruence on $\mathbb{Q}^*$.*

Proof. Let $u, v, u', v' \in \mathbb{Q}^*$ with $u \sim v$ and $u' \sim v'$, and so, there exist quasi-crystal isomorphisms $\psi : \mathbb{Q}^*(u) \rightarrow \mathbb{Q}^*(v)$ and $\psi' : \mathbb{Q}^*(u') \rightarrow \mathbb{Q}^*(v')$ such that $\psi(u) = v$ and $\psi'(u') = v'$. We have that $u \sim u$ as the identity map on $\mathbb{Q}^*(u)$ is a quasi-crystal isomorphism on $\mathbb{Q}^*(u)$. We also have that $v \sim u$ since ψ^{-1} is a quasi-crystal isomorphism between $\mathbb{Q}^*(v)$ and $\mathbb{Q}^*(u)$ satisfying $\psi^{-1}(v) = u$. And finally, if $v = u'$, we have that $u \sim v'$, because $\psi' \circ \psi$ is a quasi-crystal isomorphism between $\mathbb{Q}^*(u)$ and $\mathbb{Q}^*(v')$ that satisfies $\psi'(\psi(u)) = v'$. Hence, $\sim$ is an equivalence on $\mathbb{Q}^*$.

Let $i \in I$. Since ψ is a quasi-crystal isomorphism, we get that

$$\text{wt}(u) = \text{wt}(\psi(u)) = \text{wt}(v),$$

and similarly, $\check{e}_i(u) = \check{e}_i(v)$ and $\check{f}_i(u) = \check{f}_i(v)$. If $\check{e}_i(u) \in \mathbb{Q}^*$, then

$$\psi(\check{e}_i(u)) = \check{e}_i(\psi(u)) = \check{e}_i(v),$$

and since $\mathbb{Q}^*(\check{e}_i(u)) = \mathbb{Q}^*(u)$ and $\mathbb{Q}^*(\check{e}_i(v)) = \mathbb{Q}^*(v)$, we obtain $\check{e}_i(u) \sim \check{e}_i(v)$. Analogously, if $\check{f}_i(u) \in \mathbb{Q}^*$, then $\check{f}_i(u) \sim \check{f}_i(v)$.

By Lemma 8.22, we have quasi-crystal isomorphisms $\psi_1 : \mathbb{Q}^*(uv) \rightarrow (\mathbb{Q}^*(u) \tilde{\otimes} \mathbb{Q}^*(v))(u \tilde{\otimes} v)$ and $\psi_2 : (\mathbb{Q}^*(u') \tilde{\otimes} \mathbb{Q}^*(v'))(u' \tilde{\otimes} v') \rightarrow \mathbb{Q}^*(u'v')$ such that $\psi_1(uv) = u \tilde{\otimes} v$ and $\psi_2(u' \tilde{\otimes} v') = u'v'$. By Theorem 7.39, we have a quasi-crystal isomorphism $\psi \tilde{\otimes} \psi'$ between $\mathbb{Q}^*(u) \tilde{\otimes} \mathbb{Q}^*(v)$ and $\mathbb{Q}^*(u') \tilde{\otimes} \mathbb{Q}^*(v')$ satisfying $(\psi \tilde{\otimes} \psi')(u \tilde{\otimes} v) = u' \tilde{\otimes} v'$. Set ψ_3 to be the restriction of $\psi \tilde{\otimes} \psi'$ to $(\mathbb{Q}^*(u) \tilde{\otimes} \mathbb{Q}^*(v))(u \tilde{\otimes} v)$. By Proposition 7.25, we get that ψ_3 is a quasi-crystal isomorphism between $(\mathbb{Q}^*(u) \tilde{\otimes} \mathbb{Q}^*(v))(u \tilde{\otimes} v)$ and

$(\mathbb{Q}^*(u') \ddot{\otimes} \mathbb{Q}^*(v'))(u' \ddot{\otimes} v')$. Then, $\psi_2 \circ \psi_3 \circ \psi_1$ is a quasi-crystal isomorphism between $\mathbb{Q}^*(uv)$ and $\mathbb{Q}^*(u'v')$ that satisfies

$$\psi_2 \psi_3 \psi_1(uv) = \psi_2 \psi_3(u \ddot{\otimes} v) = \psi_2(u' \ddot{\otimes} v') = u'v'.$$

Hence, $uv \sim u'v'$. Therefore, $\sim$ is a quasi-crystal monoid congruence on $\mathbb{Q}^*$. $\square$

We have set the framework to present the following definition.

Definition 8.24. Let $\mathbb{Q}$ be a seminormal quasi-crystal, and let $\sim$ be the hypoplactic congruence on $\mathbb{Q}^*$. The quotient quasi-crystal monoid $\mathbb{Q}^*/\sim$ is called the *hypoplactic quasi-crystal monoid*, or simply the *hypoplactic monoid*, associated to $\mathbb{Q}$, and is denoted by $\text{hypo}(\mathbb{Q})$.

Note that although $\text{hypo}(\mathbb{Q})$ is a quasi-crystal monoid, we are interested in studying its properties as a monoid, and thus, we just refer it as the hypoplactic monoid associated to $\mathbb{Q}$. However, we will be constantly considering its quasi-crystal structure, as it plays a fundamental role in the construction of $\text{hypo}(\mathbb{Q})$, and consequently, in its properties.

This terminology will make more sense in the following section, where we see how the classical hypoplactic monoid can be placed in context as the hypoplactic monoid associated to some quasi-crystal.

In a hypoplactic monoid the isolated elements (Definition 7.31) satisfy a commutative law, as described in the following results.

Theorem 8.25. Let $\mathbb{Q}$ be a seminormal quasi-crystal. If $u, v \in \mathbb{Q}^*$ are such that uv is an isolated element of $\mathbb{Q}^*$, then

$$uvw \sim u w v \sim w u v,$$

for any $w \in \mathbb{Q}^*$.

Proof. Let $u, v \in \mathbb{Q}^*$ such that uv is an isolated element of $\mathbb{Q}^*$. We only show that $uvw \sim u w v$, as $u w v \sim w u v$ follows analogously. Since uv is isolated, we have that $\ddot{e}_i(uv) = \ddot{f}_i(uv) = \perp$, for all $i \in I$. Then, for each $i \in I$, either $\ddot{e}_i(uv) = \ddot{\varphi}_i(uv) = 0$ or $\ddot{e}_i(uv) = \ddot{\varphi}_i(uv) = +\infty$, because $\mathbb{Q}^*$ is seminormal. Set $J = \{i \in I \mid \ddot{e}_i(uv) = 0\}$ and

$$X = \{w' \in \mathbb{Q}^* \mid w' = g_1 \cdots g_m(w), \text{ for some } g_1, \dots, g_m \in \{\ddot{e}_j, \ddot{f}_j \mid j \in J\}\}.$$

By Lemma 8.2, for any $i \in I \setminus J$, since $\ddot{e}_i(uv) = +\infty$, we have that $\ddot{e}_i(u) = +\infty$, $\ddot{e}_i(v) = +\infty$, or $\ddot{\varphi}_i(u), \ddot{e}_i(v) > 0$. This implies by Proposition 8.3 that for any $w' \in \mathbb{Q}^*$, the quasi-Kashiwara operators $\ddot{e}_i$ and $\ddot{f}_i$ are undefined on uvw' and on $uw'v$, and

$$\ddot{e}_i(uvw') = \ddot{\varphi}_i(uvw') = \ddot{e}_i(uw'v) \ddot{\varphi}_i(uw'v) = +\infty.$$

By Lemma 8.2, for any $j \in J$, we have that $\check{\epsilon}_j(u) = \check{\epsilon}_j(v) = \check{\varphi}_j(u) = \check{\varphi}_j(v) = 0$, because $\check{\epsilon}_j(uv) = \check{\varphi}_j(uv) = 0$. Thus, by Proposition 8.3, for $g_1, \dots, g_m \in \{\check{\epsilon}_j, \check{f}_j \mid j \in J\}$, we have that $g_1 \cdots g_m$ is defined on uvw or uwx if and only if $g_1 \cdots g_m$ is defined on w . And if so, $g_1 \cdots g_m(uvw) = uv(g_1 \cdots g_m(w))$ and $g_1 \cdots g_m(uwx) = u(g_1 \cdots g_m(w))v$, where

$$\check{\epsilon}_j(g_1 \cdots g_m(uvw)) = \check{\epsilon}_j(w) = \check{\epsilon}_j(g_1 \cdots g_m(uwx))$$

and

$$\check{\varphi}_j(g_1 \cdots g_m(uvw)) = \check{\varphi}_j(w) = \check{\varphi}_j(g_1 \cdots g_m(uwx)).$$

Therefore, we have that

$$Q^*(uvw) = \{uvw' \mid w' \in X\},$$

$$Q^*(uwx) = \{uw'v \mid w' \in X\},$$

and the map $\psi : Q^*(uvw) \rightarrow Q^*(uwx)$ given by $\psi(uvw') = uw'v$, for each $w' \in X$, is a bijective quasi-crystal homomorphism from $\mathbb{Q}^*(uvw)$ to $\mathbb{Q}^*(uwx)$. By Corollary 7.18, ψ is a quasi-crystal isomorphism between $\mathbb{Q}^*(uvw)$ and $\mathbb{Q}^*(uwx)$. And since $\psi(uvw) = uwv$, we obtain that $uvw \sim uwv$. $\square$

Theorem 8.26. *Let $\mathbb{Q}$ be a seminormal quasi-crystal. If $w \in Q^*$ is such that $wu \sim uw$, for any $u \in Q^*$, then w is an isolated element of $\mathbb{Q}^*$.*

Proof. Suppose there exists $w \in Q^*$ such that w is not an isolated element of $\mathbb{Q}^*$ and $wu \sim uw$, for any $u \in Q^*$. Since w is not isolated, take $i \in I$ such that $\check{\epsilon}_i$ or $\check{f}_i$ are defined on w . As $\mathbb{Q}^*$ is seminormal, if $\check{\epsilon}_i$ is defined on w , then $\check{\epsilon}_i(w), \check{\varphi}_i(w) \in \mathbb{Z}_{\geq 0}$, where $\check{\epsilon}_i(w) > 0$, and the element $u = \check{\epsilon}_i^{\check{\epsilon}_i(w)}(w)$ satisfies $\check{\epsilon}_i(u) = 0$ and $\check{\varphi}_i(u) \in \mathbb{Z}_{> 0}$. By Lemma 8.2, $\check{\epsilon}_i(wu) = \check{\epsilon}_i(w) \in \mathbb{Z}_{\geq 0}$ and $\check{\epsilon}_i(uw) = +\infty$, which contradicts the fact that $wu \sim uw$. Analogously, if we have that $\check{f}_i$ is defined on w , then the element $v = \check{f}_i^{\check{\varphi}_i(w)}(w)$ is such that $\check{\varphi}_i(wu) = +\infty$ and $\check{\varphi}_i(uw) = \check{\varphi}_i(w) \in \mathbb{Z}_{\geq 0}$. $\square$

8.5 The hypoplactic monoid of type $\mathfrak{A}_n$

In this section we show that the classical hypoplactic monoid hypo_n of rank n corresponds to the hypoplactic monoid $\text{hypo}(\mathcal{A}_n)$ associated to the standard quasi-crystal $\mathcal{A}_n$ of type $\mathfrak{A}_n$.

Recall that first notions of quasi-crystals and quasi-Kashiwara operators were introduced in Section 6.4 based on the work by Cain and Malheiro [CM17]. From this approach we obtained a construction of the classical hypoplactic monoid. We now show that from this construction we also obtain the hypoplactic monoid $\text{hypo}(\mathcal{A}_n)$, and therefore, the monoids hypo_n and $\text{hypo}(\mathcal{A}_n)$ are the same.

In this section, let $\sim$ be the hypoplactic congruence on $\mathcal{A}_n^*$. Also, consider the directed graph Γ' , and the partial maps $\dot{e}_i$ and $\dot{f}_i$ ($i \in \{1, \dots, n-1\}$) introduced in Section 6.4.

As we presented in Example 8.10, for $w \in A_n^*$ and $i \in \{1, \dots, n-1\}$, the quasi-Kashiwara operators $\ddot{e}_i$ and $\ddot{f}_i$ are not defined on w , if w has an i -inversion (Definition 6.11). On the other hand, if $\ddot{e}_i$ is defined on w , then $\ddot{e}_i(w)$ is obtained from w by replacing the right-most $i+1$ by i . If $\ddot{f}_i$ is defined on w , then $\ddot{f}_i(w)$ is obtained from w by replacing the left-most i by $i+1$. Therefore, $\ddot{e}_i$ and $\ddot{f}_i$ coincide with the partial maps $\dot{e}_i$ and $\dot{f}_i$, respectively.

Given $u, v \in A_n^*$ and $i \in \{1, \dots, n-1\}$, we have that $u \xrightarrow{i} v$ is an edge of Γ' if and only if $\dot{f}_i(u) = v$. Since $\dot{f}_i$ and $\ddot{f}_i$ coincide, then if $u \neq v$, then $u \xrightarrow{i} v$ is an edge of the quasi-crystal graph $\Gamma_{\mathcal{A}_n^*}$ of $\mathcal{A}_n^*$ if and only if $u \xrightarrow{i} v$ is an edge of Γ' . Hence, Γ' can be obtained from $\Gamma_{\mathcal{A}_n^*}$ by removing all loops. This leads to the following result.

Lemma 8.27. *Let $u, v \in A_n^*$. If $u \sim v$, then $u \sim_{\text{hypo}} v$.*

Proof. Let $\psi : \mathcal{A}_n^*(u) \rightarrow \mathcal{A}_n^*(v)$ be a quasi-crystal isomorphism such that $\psi(u) = v$. By Lemma 7.24, ψ is a graph isomorphism between $\Gamma_{\mathcal{A}_n^*}(u)$ and $\Gamma_{\mathcal{A}_n^*}(v)$ that preserves weights. Since $\Gamma'(u)$ and $\Gamma'(v)$ can be obtained from $\Gamma_{\mathcal{A}_n^*}(u)$ and $\Gamma_{\mathcal{A}_n^*}(v)$, respectively, by removing all loops, then ψ is also a graph isomorphism between $\Gamma'(u)$ and $\Gamma'(v)$. By Theorem 6.16, we obtain that $u \sim_{\text{hypo}} v$. $\square$

We now show that the converse of the previous result also holds, which implies that $\sim$ and $\sim_{\text{hypo}}$ are the same relation on A_n^* .

Lemma 8.28. *Let $u, v \in A_n^*$ be such that $u \sim_{\text{hypo}} v$. Then, for each $i \in \{1, \dots, n-1\}$, u has an i -inversion if and only if v has an i -inversion.*

Proof. Suppose there exist $u, v \in A_n^*$ and $i \in \{1, \dots, n-1\}$ such that $u \sim_{\text{hypo}} v$, u has an i -inversion, and v is i -inversion-free. By Theorem 6.16, there exists a weight-preserving graph isomorphism ψ between $\Gamma'(u)$ and $\Gamma'(v)$ such that $\psi(u) = v$. Since u has an i -inversion, we have that $|u|_i \geq 1$, and since ψ preserves weights, $\text{wt}(u) = \text{wt}(v)$, which implies that i occurs in v . As v is i -inversion-free, $\dot{f}_i$ is defined on v , and so, $v \xrightarrow{i} \dot{f}_i(v)$ is an edge of Γ' . Since ψ is a graph isomorphism and $\psi(u) = v$, we get that $u \xrightarrow{i} \psi^{-1}(\dot{f}_i(v))$ is an edge of Γ' . A contradiction, because u cannot be the start of an i -labelled edge in Γ' , as u has an i -inversion.

Note that as $\sim_{\text{hypo}}$ is an equivalence, we have that $u \sim_{\text{hypo}} v$ if and only if $v \sim_{\text{hypo}} u$. Thus, we would also obtain a contradiction if we suppose that u is i -inversion-free and v has an i -inversion. $\square$

Theorem 8.29. *We have that $\tilde{\sim} = \sim_{\text{hypo}}$.*

Proof. By Lemma 8.27, we have that $\tilde{\sim} \subseteq \sim_{\text{hypo}}$.

Let $u, v \in A_n^*$ such that $u \sim_{\text{hypo}} v$. By Theorem 6.16, there exists a weight-preserving graph isomorphism ψ between $\Gamma'(u)$ and $\Gamma'(v)$ such that $\psi(u) = v$. Since the vertex sets of $\Gamma'(u)$ and $\Gamma'(v)$ coincide with the vertex sets of $\Gamma_{\mathcal{A}_n^*}(u)$ and $\Gamma_{\mathcal{A}_n^*}(v)$, respectively, we get by Proposition 7.23 that ψ is a bijective map between $A_n^*(u)$ and $A_n^*(v)$. By Corollary 7.18, to show that ψ is a quasi-crystal isomorphism between $\mathcal{A}_n^*(u)$ and $\mathcal{A}_n^*(v)$, it suffices to show that ψ is a quasi-crystal homomorphism.

Let $w \in A_n^*(u)$. As ψ preserves weights, we obtain that $\text{wt}(\psi(w)) = \text{wt}(w)$. Let $i \in \{1, \dots, n-1\}$. If $\tilde{\epsilon}_i(w) = +\infty$, then w has an i -inversion (Example 8.10), and since $w \sim_{\text{hypo}} \psi(w)$ (Theorem 6.16), $\psi(w)$ also has an i -inversion, which implies that $\tilde{\epsilon}_i(\psi(w)) = +\infty$. Similarly, if $\tilde{\epsilon}_i(\psi(w)) = +\infty$, then $\tilde{\epsilon}_i(w) = +\infty$. Otherwise, $\tilde{\epsilon}_i(w), \tilde{\varphi}_i(w), \tilde{\epsilon}_i(\psi(w)), \tilde{\varphi}_i(\psi(w)) \in \mathbb{A}_{\geq 0}$, and by Lemma 7.26 and Proposition 7.27, we obtain that $\tilde{\epsilon}_i(w) = \tilde{\epsilon}_i(\psi(w))$ and $\tilde{\varphi}_i(w) = \tilde{\varphi}_i(\psi(w))$, because i -labelled paths in $\Gamma'(u)$ coincide with i -labelled paths in $\Gamma_{\mathcal{A}_n^*}(u)$, i -labelled paths in $\Gamma'(v)$ coincide with i -labelled paths in $\Gamma_{\mathcal{A}_n^*}(v)$, and ψ induces an one-to-one correspondence between i -labelled paths in $\Gamma'(u)$ and i -labelled paths in $\Gamma'(v)$. Finally, since $\tilde{f}_i$ and $\dot{f}_i$ coincide, if $\tilde{f}_i$ is defined on w , then $\dot{f}_i(w) = \tilde{f}_i(w)$, which implies that $w \xrightarrow{i} \dot{f}_i(w)$ is an edge in $\Gamma'(u)$, and since ψ is a graph isomorphism, $\psi(w) \xrightarrow{i} \psi(\dot{f}_i(w))$ is an edge in $\Gamma'(v)$, and so,

$$\tilde{f}_i(\psi(w)) = \dot{f}_i(\psi(w)) = \psi(\dot{f}_i(w)) = \psi(\tilde{f}_i(w)).$$

Analogously, since $\tilde{e}_i$ and $\dot{e}_i$ coincide, if $\tilde{e}_i$ is defined on w , then $\psi(\tilde{e}_i(w)) = \dot{e}_i(\psi(w))$. Hence ψ is a quasi-crystal homomorphism from $\mathcal{A}_n^*(u)$ to $\mathcal{A}_n^*(v)$.

By Corollary 7.18, ψ is a quasi-crystal isomorphism between $\mathcal{A}_n^*(u)$ and $\mathcal{A}_n^*(v)$, and $\psi(u) = v$. Hence, $u \tilde{\sim} v$. This implies that $\sim_{\text{hypo}} \subseteq \tilde{\sim}$. $\square$

From the previous result, we have that hypo_n and $\text{hypo}(\mathcal{A}_n)$ are the same monoid. This justifies the terminology used in Definitions 8.21 and 8.24. Moreover, we now have a process of crystalizing the hypoplactic monoid which gives a construction of the classical hypoplactic monoid from the standard crystal $\mathcal{A}_n$ (as a quasi-crystal) of type $\mathfrak{A}_n$, and allows a similar construction of a monoid based on any other seminormal crystal.

THE HYPOPLACTIC MONOID OF TYPE $\mathbb{C}_n$

In Chapter 8 we described a method of obtaining a monoid from a seminormal quasi-crystal. Then, in Section 8.5 we showed that this construction for the standard crystal $\mathcal{A}_n$ of type $\mathfrak{A}_n$ results in the classical hypoplactic monoid of rank n . A natural way of proceeding is to study the monoids that are obtained for other seminormal quasi-crystals. In particular, for the standard crystals associated to Cartan types other than $\mathfrak{A}_n$.

In this chapter we study the hypoplactic monoid $\text{hypo}(\mathcal{C}_n)$ associated to the standard crystal $\mathcal{C}_n$ of type $\mathbb{C}_n$. The chapter is split in two sections. In Section 9.1, we make a detailed study of the hypoplactic monoid $\text{hypo}(\mathcal{C}_2)$, where we show that it does not have a finite presentation and it satisfies identities. The results obtained for $\text{hypo}(\mathcal{C}_2)$ give an understanding of the properties we can expect $\text{hypo}(\mathcal{C}_n)$ to satisfy. Then, in Section 9.2, we study the hypoplactic monoid $\text{hypo}(\mathcal{C}_n)$ for $n \geq 3$, where we characterize its commutative elements, and show that it does not satisfy any identity. We also show that the hypoplactic monoid $\text{hypo}(\mathcal{C}_n)$ contains a copy of the classical hypoplactic monoid of rank $n - 1$.

9.1 The case $n = 2$

9.1.1 The definition of $\text{hypo}(\mathcal{C}_2)$

In this section we describe how to obtain the hypoplactic monoid $\text{hypo}(\mathcal{C}_2)$. This construction follows from the theory introduced in Chapters 7 and 8 applied to the standard quasi-crystal of type $\mathbb{C}_2$.

From Example 4.4(ii), we have that the standard quasi-crystal $\mathcal{C}_2$ of Cartan type $\mathbb{C}_n$ consists of an ordered set $C_2 = \{1 < 2 < \bar{2} < \bar{1}\}$ and a quasi-crystal structure

given as follows. The weight map $\text{wt} : C_2 \rightarrow \mathbb{Z}^2$ is defined by

$$\text{wt}(1) = \mathbf{e}_1, \quad \text{wt}(2) = \mathbf{e}_2, \quad \text{wt}(\bar{2}) = -\mathbf{e}_2 \quad \text{wt}(\bar{1}) = -\mathbf{e}_1.$$

The quasi-Kashiwara operators $\check{e}_1, \check{e}_2, \check{f}_1, \check{f}_2 : C_2 \rightarrow C_2 \sqcup \{\perp\}$ are defined for the following arguments

$$\begin{aligned} \check{e}_1(2) &= 1, & \check{e}_1(\bar{1}) &= \bar{2}, & \check{e}_2(\bar{2}) &= 2, \\ \check{f}_1(1) &= 2, & \check{f}_1(\bar{2}) &= \bar{1}, & \check{f}_2(2) &= \bar{2}. \end{aligned}$$

The maps $\check{\varepsilon}_1, \check{\varepsilon}_2, \check{\varphi}_1, \check{\varphi}_2 : C_2 \rightarrow \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ are given by

$$\begin{aligned} \check{\varepsilon}_1(x) &= \delta_{2,x} + \delta_{\bar{1},x}, & \check{\varepsilon}_2(x) &= \delta_{\bar{2},x}, \\ \check{\varphi}_1(x) &= \delta_{1,x} + \delta_{\bar{2},x}, & \check{\varphi}_2(x) &= \delta_{2,x}, \end{aligned}$$

where $\delta_{y,x} = 1$, if $x = y$, and $\delta_{y,x} = 0$, if $x \neq y$, for $x, y \in C_2$.

Notice that given $x, y \in C_2$, if $\check{\varphi}_1(x), \check{\varepsilon}_1(y) > 0$, then $x \in \{1, \bar{2}\}$ and $y \in \{2, \bar{1}\}$. If $\check{\varphi}_2(x), \check{\varepsilon}_2(y) > 0$, then $x = 2$ and $y = \bar{2}$. This leads to a generalization of Definition 6.11 for words over C_2 .

Definition 9.1. A word $w \in C_2^*$ is said to have a *1-inversion* if w admits a decomposition of the form $w = w_1 x w_2 y w_3$, for some $w_1, w_2, w_3 \in C_2^*$, $x \in \{1, \bar{2}\}$ and $y \in \{2, \bar{1}\}$.

A word $w \in C_2^*$ is said to have a *2-inversion* if w admits a decomposition of the form $w = w_1 2 w_2 \bar{2} w_3$, for some $w_1, w_2, w_3 \in C_2^*$.

For $i \in \{1, 2\}$, a word $w \in C_2^*$ is said to be *i-inversion-free* if w does not have an *i-inversion*.

By Definition 8.1 and Proposition 8.3, we obtain the following description of the free quasi-crystal monoid over $\mathfrak{C}_2$.

Definition 9.2. The free quasi-crystal monoid $\mathfrak{C}_2^*$ over $\mathfrak{C}_2$ consists of the set C_2^* of all words over C_2 , the concatenation of words, and a quasi-crystal structure given as follows. For $w \in C_2^*$, the weight of w is

$$\text{wt}(w) = (|w|_1 - |w|_{\bar{1}}, |w|_2 - |w|_{\bar{2}}).$$

If w has a 1-inversion, then

$$\check{\varepsilon}_1(w) = \check{\varphi}_1(w) = +\infty,$$

otherwise,

$$\check{\varepsilon}_1(w) = |w|_2 + |w|_{\bar{1}}, \quad \check{\varphi}_1(w) = |w|_1 + |w|_{\bar{2}}.$$

The raising quasi-Kashiwara operator $\check{e}_1$ is defined on w if $\check{e}_1(w) \in \mathbb{Z}_{>0}$, and if so, $\check{e}_1(w) = w_1 \check{e}_1(x) w_2$, where $w_1, w_2 \in C_2^*$ and $x \in \{2, \bar{1}\}$ are such that $w = w_1 x w_2$ and $|w_2|_2 = |w_2|_{\bar{1}} = 0$.

The lowering quasi-Kashiwara operator $\check{f}_1$ is defined on w if $\check{f}_1(w) \in \mathbb{Z}_{>0}$, and if so, $\check{f}_1(w) = w_1 \check{f}_1(x) w_2$, where $w_1, w_2 \in C_2^*$ and $x \in \{1, \bar{2}\}$ are such that $w = w_1 x w_2$ and $|w_1|_1 = |w_1|_{\bar{2}} = 0$.

If w has a 2-inversion, then

$$\check{e}_2(w) = \check{f}_2(w) = +\infty,$$

otherwise,

$$\check{e}_2(w) = |w|_{\bar{2}}, \quad \check{f}_2(w) = |w|_2.$$

The raising quasi-Kashiwara operator $\check{e}_2$ is defined on w if $\check{e}_2(w) \in \mathbb{Z}_{>0}$, and if so, $\check{e}_2(w) = w_1 2 w_2$, where $w_1, w_2 \in C_2^*$ are such that $w = w_1 \bar{2} w_2$ and $|w_2|_{\bar{2}} = 0$.

The lowering quasi-Kashiwara operator $\check{f}_2$ is defined on w if $\check{f}_2(w) \in \mathbb{Z}_{>0}$, and if so, $\check{f}_2(w) = w_1 \bar{2} w_2$, where $w_1, w_2 \in C_2^*$ are such that $w = w_1 2 w_2$ and $|w_1|_2 = 0$.

From the signature rule (Algorithm 8.6), the quasi-Kashiwara operators $\check{e}_1$ and $\check{f}_1$ can be computed on a 1-inversion-free word $w \in C_2^*$ as follows. If 2 or $\bar{1}$ occur in w , let x be the right-most 2 or $\bar{1}$ occurring in w , then $\check{e}_1(w)$ is obtained from w by replacing the right-most x by $\check{e}_1(x)$. If 1 or $\bar{2}$ occur in w , let y be the left-most 1 or $\bar{2}$ occurring in w , then $\check{f}_1(w)$ is obtained from w by replacing the left-most y by $\check{f}_1(y)$.

Also from the signature rule (Algorithm 8.6), the quasi-Kashiwara operators $\check{e}_2$ and $\check{f}_2$ can be computed on a 2-inversion-free word $w \in C_2^*$ as follows. If $\bar{2}$ occurs in w , then $\check{e}_2(w)$ is obtained from w by replacing the right-most $\bar{2}$ by 2. If 2 occurs in w , then $\check{f}_2(w)$ is obtained from w by replacing the left-most 2 by $\bar{2}$.

Example 9.3. In $\mathcal{C}_2^*$, we have that

$$\check{e}_1(221\bar{2}11) = 211\bar{2}11, \quad \check{f}_1(221\bar{2}11) = 222\bar{2}11,$$

and since $221\bar{2}11$ has a 2-inversion, $\check{e}_2$ and $\check{f}_2$ are undefined on it. Also,

$$\check{e}_2(1\bar{2}11\bar{2}1212) = 1121212, \quad \check{f}_1(11\bar{2}1212) = 11\bar{2}1\bar{2}12,$$

and as $11\bar{2}1212$ has a 1-inversion, $\check{e}_1$ and $\check{f}_1$ are undefined.

In Section 7.2, we showed that a seminormal quasi-crystal can be described by its quasi-crystal graph (Definition 7.19). Thus, in the sequel, we will frequently resort to the quasi-crystal graph $\Gamma_{\mathcal{C}_2^*}$ of $\mathcal{C}_2^*$ to study the hypoplactic monoid $\text{hypo}(\mathcal{C}_2)$. In the following example, we present part of $\Gamma_{\mathcal{C}_2^*}$.

Example 9.4. The empty word ϵ is an isolated vertex in $\Gamma_{\mathfrak{C}_2^*}$ without loops. Letters form the following connected component of $\Gamma_{\mathfrak{C}_2^*}$.

$$1 \xrightarrow{1} 2 \xrightarrow{2} \bar{2} \xrightarrow{1} \bar{1}$$

Words of length 2 form the following three connected components of $\Gamma_{\mathfrak{C}_2^*}$.

$$\begin{array}{ccccccc} 11 & \xrightarrow{1} & 21 & \xrightarrow{2} & \bar{2}1 & \xrightarrow{1} & \bar{1}1 \\ & & \downarrow 1 & & & & \downarrow 1 \\ 1 \circlearrowleft 12 & & 22 & \xrightarrow{2} & \bar{2}2 & \circlearrowright 1 & \bar{1}2 \\ & & \downarrow 2 & & \downarrow 2 & & \downarrow 2 \\ & & 1\bar{2} & \xrightarrow{1} & 2\bar{2} & \circlearrowleft 2 & \bar{2}\bar{2} & \xrightarrow{1} & \bar{1}\bar{2} \\ & & & & \downarrow 1 & & \downarrow 1 & & \downarrow 1 \\ 1 \circlearrowleft 1\bar{1} & & 2\bar{1} & \xrightarrow{2} & \bar{2}\bar{1} & \circlearrowright 1 & \bar{1}\bar{1} \end{array}$$

Note that $1\bar{1}$ is an isolated vertex.

The connected component $\Gamma_{\mathfrak{C}_2^*}(121)$ of $\Gamma_{\mathfrak{C}_2^*}$ containing 121 is the following.

$$\begin{array}{ccccccc} \overset{1}{\circlearrowleft} & & \overset{2}{\circlearrowleft} & & \overset{1}{\circlearrowleft} \\ 121 & \xrightarrow{2} & 1\bar{2}1 & \xrightarrow{1} & 2\bar{2}1 & \xrightarrow{1} & 2\bar{1}1 & \xrightarrow{2} & \bar{2}\bar{1}1 \\ & & & & \searrow 1 & & & & \\ & & & & 2\bar{1}2 & \xrightarrow{2} & \bar{2}\bar{1}2 & \xrightarrow{2} & \bar{2}\bar{1}\bar{2} \\ & & & & \downarrow 1 & & \downarrow 1 \end{array}$$

The connected component $\Gamma_{\mathfrak{C}_2^*}(212)$ of $\Gamma_{\mathfrak{C}_2^*}$ containing 212 is the following.

$$\begin{array}{ccccccc} \overset{1}{\circlearrowleft} & & \overset{1}{\circlearrowleft} \\ 212 & \xrightarrow{2} & \bar{2}12 & \xrightarrow{2} & \bar{2}1\bar{2} \\ & & \searrow 1 & & \\ \bar{1}12 & \xrightarrow{2} & \bar{1}\bar{1}2 & \xrightarrow{1} & \bar{1}\bar{2}\bar{2} & \xrightarrow{1} & \bar{1}\bar{2}\bar{1} & \xrightarrow{2} & \bar{1}\bar{2}\bar{1} \\ \downarrow 1 & & \downarrow 2 & & \downarrow 1 \end{array}$$

By Definitions 8.21 and 8.24, the hypoplactic monoid $\text{hypo}(\mathfrak{C}_2)$ is the quotient monoid of $\mathfrak{C}_2^*$ by the hypoplactic congruence $\sim$. Although we omit the weight map wt (Definition 9.2) on the examples, recall that the quasi-crystal graph $\Gamma_{\mathfrak{C}_2^*}$

is a weighted labelled graph, and so, we can obtain $\sim$ from $\Gamma_{\mathbb{C}_2^*}$, by Theorem 7.29. Thus, given two words $u, v \in C_2^*$, we have that $u \sim v$ if and only if there exists a graph isomorphism ψ between the connected components $\Gamma_{\mathbb{C}_2^*}(u)$ and $\Gamma_{\mathbb{C}_2^*}(v)$ of $\Gamma_{\mathbb{C}_2^*}$ such that $\psi(u) = v$.

In the following result we can see that the hypoplactic congruence $\sim$ respects inversions.

Proposition 9.5. *Let $u, v \in C_2^*$ with $u \sim v$, and let $i \in \{1, 2\}$. Then, u has an i -inversion if and only if v has an i -inversion.*

Proof. If u has an i -inversion, then $\check{\epsilon}_i(u) = +\infty$. Since $u \sim v$, then $\check{\epsilon}_i(v) = \check{\epsilon}_i(u) = +\infty$, which implies that v has an i -inversion.

The converse follows from the fact that $u \sim v$ implies $v \sim u$. $\square$

Like for the classical hypoplactic monoid, in the congruence classes of the hypoplactic monoid $\text{hypo}(\mathbb{C}_2)$ either all words have an i -inversion, or all words are i -inversion-free ($i \in \{1, 2\}$). In particular, we have that $1\bar{1} \not\sim \epsilon$ as expected by Remark 6.17. On the other hand, $1\bar{1}$ and ϵ correspond to the same element in the plactic monoid $\text{plac}(\mathbb{C}_2)$ associated to the standard crystal of type $\mathbb{C}_2$. Therefore, in contrast with the classical hypoplactic monoid, for type $\mathbb{C}_2$ the plactic congruence is not contained in the hypoplactic congruence.

9.1.2 Highest-weight words

In the study of plactic monoids for Cartan types [Lec02; Lec03], words of highest weight are extremely relevant as they are used to identify connected components of crystal graphs. An analogous relation was proven in [CM17] for the classical hypoplactic monoid. Thus, in this section we characterize the highest-weight words of $\mathbb{C}_2^*$, and check whether they satisfy properties similar to highest-words in the mentioned contexts.

From Definition 7.6(i), we have that a word $w \in C_2^*$ is of highest weight if $\check{\epsilon}_1$ and $\check{\epsilon}_2$ are undefined on w . Equivalently, w is of highest weight if the only edges in $\Gamma_{\mathbb{C}_2^*}$ ending on w are loops.

Proposition 9.6. *Let $w \in C_2^*$. Then, w is of highest weight if and only if w satisfies the following conditions:*

- (i) *if 2 or $\bar{1}$ occur in w , then w has a 1-inversion;*
- (ii) *if $\bar{2}$ occurs in w , then w has a 2-inversion.*

Proof. First, assume that w is of highest weight. If 2 or $\bar{1}$ occurs in w , by Definition 9.2, we have that $\check{e}_1(w) > 0$. Since w is of highest weight, $\check{e}_1$ is not defined on w , which implies that $\check{e}_1(w) = +\infty$. Hence, w has a 1-inversion. Analogously, if $\bar{2}$ occurs in w , then $\check{\varphi}_2(w) = +\infty$, and so, w has a 2-inversion.

Conversely, assume that w satisfies (i) and (ii). From Definition 9.2, we have that condition (i) implies $\check{e}_1(w) \in \{0, +\infty\}$, and so, $\check{e}_1$ is undefined on w . Similarly, condition (ii) implies $\check{\varphi}_2(w) \in \{0, +\infty\}$, and therefore, $\check{e}_2$ is undefined on w . Hence, w is of highest weight. $\square$

In Example 9.4, we can see that 212 and $\bar{1}12$ are highest-weight elements of $\mathfrak{C}_2^*$, which belong to the same connected component. We can also see that $\bar{2}11$ and $\bar{2}12$ are lowest-weight elements of $\mathfrak{C}_2^*$, which belong to the same connected component. These facts mark a difference when compared to the crystal graphs studied in [KN94], which lead to the construction of the plactic monoids of Cartan types [Lec02; Lec03], where each connected component has exactly one highest-weight element and exactly one lowest-weight element. Due to the results in [CM17] and their reinterpretation in Section 8.5, we also have that the connected components of the quasi-crystal monoid $\mathcal{A}_n^*$ have exactly one highest-weight element and exactly one lowest-weight element.

Although we have that a connected component of $\mathfrak{C}_2^*$ may have more than one highest-weight element or more than a lowest-weight element, we still can guarantee that it has at least one of each.

Proposition 9.7. *Each connected component of $\mathfrak{C}_2^*$ has at least one highest-weight word and at least one lowest-weight word.*

Proof. By Proposition 8.9, we have that words in the same connected component have the same length. Since C_2 is finite, there are finitely many words of a given length. Thus, connected components of $\mathfrak{C}_2^*$ have finitely many words.

Let $X \subseteq C_2^*$ be a connected component of $\mathfrak{C}_2^*$. Since $\mathbb{Z}^2$ is the weight lattice of type $\mathfrak{C}_2$ (Example 2.17) and X is finite, we can take a word $w \in X$ whose weight $\text{wt}(w)$ is maximal among weights of words in X with respect to the partial order in $\mathbb{Z}^2$ given in (2.3). If $\check{e}_i$ were defined on w , for some $i \in \{1, 2\}$, then $\check{e}_i(w) \in X$, and by Proposition 7.5, $\text{wt}(\check{e}_i(w)) > \text{wt}(w)$, contradicting the maximality of $\text{wt}(w)$. Hence, w is of highest weight.

Analogously, we can take a word $w' \in X$ whose weight $\text{wt}(w')$ is minimal among weights of words in X . If $\check{f}_i$ were defined on w' , for some $i \in \{1, 2\}$, then $\check{e}_i(w') \in X$, and by Proposition 7.5, $\text{wt}(w') > \text{wt}(\check{e}_i(w'))$, contradicting the minimality of $\text{wt}(w')$. And therefore, w is of lowest weight. $\square$

We have that we can take highest-weight words to index the connected components of $\mathcal{C}_2^*$, but this choice is not unique.

Due to the following result, highest weights (Definition 7.7(i)) have no meaning in $\mathcal{C}_2^*$.

Proposition 9.8. *Any weight $\lambda \in \mathbb{Z}^2$ is a highest weight in $\mathcal{C}_2^*$.*

Proof. Let $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1, \lambda_2 \in \mathbb{Z}$. For $i = 1, 2$, if $\lambda_i \geq 0$, set $a_i = \lambda_i$ and $b_i = 0$, otherwise, set $a_i = 0$ and $b_i = -\lambda_i$. Then,

$$\text{wt}(1^{a_1} 2^{a_2} \bar{2}^{b_2} \bar{1}^{b_1}) = (a_1 - b_1, a_2 - b_2) = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 = \lambda.$$

Set $w = 1^{a_1} 2^{a_2} \bar{2}^{b_2} \bar{1}^{b_1} 12\bar{2}\bar{1}$. Since $\text{wt}(12\bar{2}\bar{1}) = 0$, we have that $\text{wt}(w) = \lambda$. Also, w has a 1-inversion and a 2-inversion, which implies that $\check{e}_1$ and $\check{e}_2$ are undefined on w . $\square$

Finally, we show that highest-weight words and lowest-weight words are on one-to-one correspondence in $\mathcal{C}_2^*$. First, consider the following notation.

Definition 9.9. For the empty word $\epsilon \in C_2^*$ set $\bar{\epsilon} = \epsilon$. For $x \in \{1, 2\}$, set $\bar{\bar{x}} = x$. Given words $u, v \in C_2^*$, set $\overline{uv} = \bar{v}\bar{u}$.

Notice that for a word $w \in C_2^*$, the word $\bar{w}$ can be obtained by exchange each barred letter by its unbarred version and each unbarred letter by its barred version in w , and then reversing the resulting word. Thus, it follows that $\bar{\bar{w}} = w$. For example, $\overline{12\bar{1}\bar{2}} = 21\bar{2}\bar{1}$.

Lemma 9.10. *Let $w \in C_2^*$. Then, $\check{e}_1(w) = \check{\varphi}_1(\bar{w})$ and $\check{e}_2(w) = \check{\varphi}_2(\bar{w})$.*

Proof. If w has a 1-inversion, then there exist $w_1, w_2, w_3 \in C_2$, $x \in \{1, \bar{2}\}$ and $y \in \{2, \bar{1}\}$ such that $w = w_1 x w_2 y w_3$. Then, $\bar{w} = \overline{w_3 y w_2 x w_1}$, where $\bar{y} \in \{1, \bar{2}\}$ and $\bar{x} \in \{2, \bar{1}\}$, and so, $\bar{w}$ has a 1-inversion. Hence, $\check{e}_1(w) = \check{\varphi}_1(\bar{w}) = +\infty$.

Since $\bar{\bar{w}} = w$, if $\bar{w}$ has a 1-inversion, then as we checked above, w also has a 1-inversion, and thus, $\check{e}_1(w) = \check{\varphi}_1(w) = +\infty$.

Otherwise, w and $\bar{w}$ are 1-inversion-free. Then,

$$\check{e}_1(w) = |w|_2 + |w|_{\bar{1}} = |\bar{w}|_{\bar{2}} + |\bar{w}|_1 = \check{\varphi}_1(\bar{w}).$$

Analogously, we have that w has a 2-inversion if and only if $\bar{w}$ has a 2-inversion, and if so, $\check{e}_2(w) = \check{\varphi}_2(\bar{w}) = +\infty$. Otherwise, $\check{e}_2(w) = |w|_{\bar{2}} = |\bar{w}|_2 = \check{\varphi}_2(\bar{w})$. $\square$

Proposition 9.11. *Let $w \in C_2^*$. Then, w is of highest weight if and only if $\bar{w}$ is of lowest weight.*

Proof. If w is of highest weight, then $\check{e}_1$ and $\check{e}_2$ are undefined on w , which implies that $\check{e}_1(w), \check{e}_2(w) \in \{0, +\infty\}$. Then, by Lemma 9.10, we have that $\check{\varphi}_1(\bar{w}), \check{\varphi}_2(\bar{w}) \in \{0, +\infty\}$, which implies that $\check{f}_1$ and $\check{f}_2$ are undefined on $\bar{w}$. Hence, $\bar{w}$ is of lowest weight.

The converse implication is analogous, since if $\bar{w}$ is of lowest weight, we obtain that $\check{\varphi}_1(\bar{w}), \check{\varphi}_2(\bar{w}) \in \{0, +\infty\}$, and by Lemma 9.10, we deduce that w is of highest weight. $\square$

9.1.3 Isolated words

By Theorems 8.25 and 8.26 we have that the commutative elements of the hypoplactic monoid $\text{hypo}(\mathbb{C}_2)$ correspond to the isolated elements of $\mathbb{C}_2^*$ (Definition 7.31). In this section we identify the isolated elements, and consequently, obtain some equalities in $\text{hypo}(\mathbb{C}_2)$.

Theorem 9.12. *Let $w \in C_2^*$. Then, w is an isolated word in $\mathbb{C}_2^*$ if and only if w satisfies the following conditions:*

- (i) *if 1 or $\bar{1}$ occur in w , then w has a 1-inversion;*
- (ii) *if 2 or $\bar{2}$ occur in w , then w has a 1-inversion and a 2-inversion.*

Proof. Assume that w is an isolated word. If 2 occurs in w , then $\check{e}_1(w), \check{\varphi}_2(w) > 0$. Since w is isolated, $\check{e}_1$ and $\check{f}_2$ are undefined on w , which implies that $\check{e}_1(w) = \check{\varphi}_2(w) = +\infty$. Hence, w has a 1-inversion and a 2-inversion.

Analogously, if $\bar{2}$ occurs in w , then $\check{e}_2(w), \check{\varphi}_1(w) > 0$, and as $\check{e}_2$ and $\check{f}_1$ are undefined on w , then w has a 1-inversion and a 2-inversion.

If 1 or $\bar{1}$ occurs in w , then $\check{\varphi}_1(w) > 0$ or $\check{e}_1(w) > 0$. Since w is isolated, $\check{e}_1$ and $\check{f}_1$ are undefined on w , and so, w has a 1-inversion.

Conversely, assume that w satisfies conditions (i) and (ii). If 2 or $\bar{2}$ occur in w , then by (ii), $\check{e}_1, \check{e}_2, \check{f}_1$ and $\check{f}_2$ are undefined on w , because w has a 1-inversion and a 2-inversion. If 1 or $\bar{1}$ occur in w and neither 2 nor $\bar{2}$ occur, then $\check{e}_2$ and $\check{f}_2$ are undefined on w , as $\check{e}_2(w) = \check{\varphi}_2(w) = 0$, and $\check{e}_1$ and $\check{f}_1$ are undefined on w , because w has a 1-inversion, by (i). Finally note that the empty word is isolated. $\square$

Recall that any edges starting or ending on an isolated word in $\Gamma_{\mathbb{C}_2^*}$ are loops. Therefore, given two isolated words $u, v \in C_2^*$, we have that $u \sim v$ if and only if $\text{wt}(u) = \text{wt}(v)$, $\check{e}_1(u) = \check{e}_1(v)$ and $\check{e}_2(u) = \check{e}_2(v)$.

Corollary 9.13. *The central elements of $\sim(\mathbb{C}_2)$ other than the identity ϵ are of the form $1^{k_1}\bar{1}^{l_1}$ or $1^{k_1}2^{k_2}\bar{2}^{l_2}\bar{1}^{l_1}$, for some $k_1, k_2, l_1, l_2 > 0$.*

Proof. Let $w \in C_2^*$ be a commutative element in $\text{hypo}(\mathcal{C}_2)$. By Theorem 8.26, w is an isolated word in $\mathcal{C}_2^*$. By Theorem 9.12, if 2 or $\bar{2}$ occur in w , then $\varepsilon_1(w) = \varepsilon_2(w) = +\infty$. By taking $k_1 = |w|_1 + 1$, $k_2 = |w|_2 + 1$, $l_1 = |w|_{\bar{1}} + 1$, and $l_2 = |w|_{\bar{2}} + 1$, we get that

$$\text{wt}(1^{k_1} 2^{k_2} \bar{2}^{l_2} \bar{1}^{l_1}) = (k_1 - l_1, k_2 - l_2) = \text{wt}(w),$$

where $\varepsilon_1(1^{k_1} 2^{k_2} \bar{2}^{l_2} \bar{1}^{l_1}) = \varepsilon_2(1^{k_1} 2^{k_2} \bar{2}^{l_2} \bar{1}^{l_1}) = +\infty$. And therefore, $w \sim (1^{k_1} 2^{k_2} \bar{2}^{l_2} \bar{1}^{l_1})$.

If 1 or $\bar{1}$ occur in w and neither 2 nor $\bar{2}$, then $\varepsilon_1(w) = \check{\varphi}_1(w) = +\infty$ and $\varepsilon_2(w) = \check{\varphi}_2(w) = 0$. By taking $k_1 = |w|_1 + 1$ and $l_1 = |w|_{\bar{1}} + 1$, we have that

$$\text{wt}(1^{k_1} \bar{1}^{l_1}) = (k_1 - l_1, 0) = \text{wt}(w),$$

where $\varepsilon_1(1^{k_1} \bar{1}^{l_1}) = \check{\varphi}_1(1^{k_1} \bar{1}^{l_1}) = +\infty$ and $\varepsilon_2(1^{k_1} \bar{1}^{l_1}) = \check{\varphi}_2(1^{k_1} \bar{1}^{l_1}) = 0$. And thus, $w \sim 1^{k_1} \bar{1}^{l_1}$. $\square$

9.1.4 A presentation

In this section we first check some relations on $\text{hypo}(\mathcal{C}_2)$. This will lead to the conclusion that $\text{hypo}(\mathcal{C}_2)$ has no finite presentation. Then, we enumerate the connected components of $\text{hypo}(\mathcal{C}_2)$, which gives representatives for each hypoplactic congruence class.

Lemma 9.14. *Let $u \in \{1, 2\}^*$ and $v \in C_2^*$. If $u \sim v$, then $v \in \{1, 2\}^*$, and therefore, $|u|_1 = |v|_1$ and $|u|_2 = |v|_2$.*

Proof. We show that $|v|_{\bar{1}} = |v|_{\bar{2}} = 0$ which implies $v \in \{1, 2\}^*$. Since $\bar{2}$ does not occur in u , we have that u is 2-inversion-free, and by Definition 9.2, $\varepsilon_2(u) = |u|_{\bar{2}} = 0$. Since $u \sim v$, we have by Proposition 9.5 that v is also 2-inversion-free, and so,

$$|v|_{\bar{2}} = \varepsilon_2(v) = \varepsilon_2(u) = 0$$

and

$$|v|_2 = \check{\varphi}_2(v) = \check{\varphi}_2(u) = |u|_2.$$

The second formula implies that $\check{f}_2^{|u|_2}(u) \sim \check{f}_2^{|v|_2}(v)$. Note that the words $\check{f}_2^{|u|_2}(u)$ and $\check{f}_2^{|v|_2}(v)$ are respectively obtained from u and v by replacing each 2 by $\bar{2}$. In particular $\check{f}_2^{|u|_2}(u) \in \{1, \bar{2}\}^*$, and so, $\check{f}_2^{|u|_2}(u)$ is 1-inversion-free and $\varepsilon_1(\check{f}_2^{|u|_2}(u)) = 0$. Since $\check{f}_2^{|u|_2}(u) \sim \check{f}_2^{|v|_2}(v)$, we have that $\check{f}_2^{|v|_2}(v)$ is 1-inversion-free and

$$|\check{f}_2^{|v|_2}(v)|_2 + |\check{f}_2^{|v|_2}(v)|_{\bar{1}} = \varepsilon_1(\check{f}_2^{|v|_2}(v)) = \varepsilon_1(\check{f}_2^{|u|_2}(u)) = 0.$$

As $\check{f}_2^{|v|_2}(v) \in \{1, \bar{2}, \bar{1}\}^*$, we get that $|\check{f}_2^{|v|_2}(v)|_2 = 0$, and thus, $|\check{f}_2^{|v|_2}(v)|_{\bar{1}} = 0$. And as $|v|_{\bar{1}} = |\check{f}_2^{|v|_2}(v)|_{\bar{1}}$, we obtain that $|v|_{\bar{1}} = 0$. Finally, since $u \sim v$ where $u, v \in \{1, 2\}^*$, we have that $\text{wt}(u) = \text{wt}(v)$ which implies $|u|_1 = |v|_1$. $\square$

Lemma 9.15. *Let $u, v \in \{1, 2\}^*$ and $m \in \mathbb{Z}_{\geq 0}$. Then, $2^m 1u \not\sim 2^{m+1}v$.*

Proof. Suppose there exist $u, v \in \{1, 2\}^*$ and $m \in \mathbb{Z}_{\geq 0}$ such that $2^m 1u \sim 2^{m+1}v$. So, $\text{wt}(2^m 1u) = \text{wt}(2^{m+1}v)$ which implies that $|v|_1 = |u|_1 + 1$ and $|u|_2 = |v|_2 + 1$. In particular, 2 occurs in u . Since $\bar{2}$ does not occur in u and v , then $2^m 1u$ and $2^{m+1}v$ are 2-inversion-free, and so, $\check{\varphi}_2(2^m 1u) = |2^m 1u|_2 = |u|_2 + m$ and $\check{\varphi}_2(2^{m+1}v) = |2^{m+1}v|_2 = |v|_2 + m + 1$, where

$$\check{f}_2^{|u|_2+m}(2^m 1u) = \bar{2}^m 1\check{f}_2^{|u|_2}(u)$$

and

$$\check{f}_2^{|u|_2+m}(2^{m+1}v) = \bar{2}^{m+1}\check{f}_2^{|u|_2-1}(v) = \bar{2}^{m+1}\check{f}_2^{|v|_2}(v).$$

Note that $\check{f}_2^{|u|_2}(u)$ and $\check{f}_2^{|v|_2}(v)$ are respectively obtained from u and v by replacing each 2 by $\bar{2}$. As $\check{f}_2^{|u|_2}(u), \check{f}_2^{|v|_2}(v) \in \{1, \bar{2}\}^*$, $\bar{2}^m 1\check{f}_2^{|u|_2}(u)$ and $\bar{2}^{m+1}\check{f}_2^{|v|_2}(v)$ are 1-inversion-free, and so,

$$\check{f}_1^{m+1}(\bar{2}^m 1\check{f}_2^{|u|_2}(u)) = \bar{1}^m 2\check{f}_2^{|u|_2}(u)$$

and

$$\check{f}_1^{m+1}(\bar{2}^{m+1}\check{f}_2^{|v|_2}(v)) = \bar{1}^{m+1}\check{f}_2^{|v|_2}(v).$$

Since $|\check{f}_2^{|u|_2}(u)|_{\bar{2}} = |u|_2 = |v|_2 + 1$, $\bar{2}$ occurs in $\check{f}_2^{|u|_2}(u)$, which implies that $\bar{1}^m 2\check{f}_2^{|u|_2}(u)$ has a 2-inversion. And since 2 does not occur in $\check{f}_2^{|v|_2}(v)$, we have that $\bar{2}^{m+1}\check{f}_2^{|v|_2}(v)$ is 2-inversion-free. By Proposition 9.5, this is a contradiction, because

$$\bar{1}^m 2\check{f}_2^{|u|_2}(u) = \check{f}_1^{m+1}\check{f}_2^{|u|_2+m}(2^m 1u) \sim \check{f}_1^{m+1}\check{f}_2^{|u|_2+m}(2^{m+1}v) = \bar{2}^{m+1}\check{f}_2^{|v|_2}(v)$$

as $2^m 1u \sim 2^{m+1}v$. □

Lemma 9.16. *Let $u, v \in \{1, 2\}^*$ and $m \in \mathbb{Z}_{\geq 0}$. Then, $u21^m \not\sim v1^{m+1}$.*

Proof. Suppose there exist $u, v \in \{1, 2\}^*$ and $m \in \mathbb{Z}_{\geq 0}$ such that $u21^m \sim v1^{m+1}$. So, $\text{wt}(u21^m) = \text{wt}(v1^{m+1})$ which implies that $|u|_1 = |v|_1 + 1$ and $|v|_2 = |u|_2 + 1$. Since $\bar{2}$ does not occur in u and v , then $u21^m$ and $v1^{m+1}$ are 2-inversion-free, and so, $\check{\varphi}_2(u21^m) = |u21^m|_2 = |u|_2 + 1$ and $\check{\varphi}_2(v1^{m+1}) = |v1^{m+1}|_2 = |v|_2$, where

$$\check{f}_2^{|u|_2+1}(u21^m) = \check{f}_2^{|u|_2}(u)\bar{2}1^m$$

and

$$\check{f}_2^{|u|_2+1+m}(v1^{m+1}) = \check{f}_2^{|v|_2}(v1^{m+1}) = \check{f}_2^{|v|_2}(v)1^{m+1}.$$

Note that $\check{f}_2^{|u|_2}(u)$ and $\check{f}_2^{|v|_2}(v)$ are respectively obtained from u and v by replacing each 2 by $\bar{2}$. As $\check{f}_2^{|u|_2}(u), \check{f}_2^{|v|_2}(v) \in \{1, \bar{2}\}^*$, $\check{f}_2^{|u|_2}(u)\bar{2}1^m$ and $\check{f}_2^{|v|_2}(v)1^{m+1}$ are 1-inversion-free, and so,

$$\check{f}_1^{|u|}(\check{f}_2^{|u|_2}(u)\bar{2}1^m) = \check{f}_1^{|u|}\check{f}_2^{|u|_2}(u)\bar{2}1^m$$

and

$$\check{f}_1^{|u|}(\check{f}_2^{|v|_2}(v)1^{m+1}) = \check{f}_1^{|v|}(\check{f}_2^{|v|_2}(v)1^{m+1}) = \check{f}_1^{|v|}\check{f}_2^{|v|_2}(v)1^{m+1},$$

because $|u| = |u|_1 + |u|_2 = |v|_1 + 1 + |v|_2 - 1 = |v|$. Note that $\check{f}_1^{|u|}\check{f}_2^{|u|_2}(u)$ and $\check{f}_1^{|v|}\check{f}_2^{|v|_2}(v)$ are respectively obtained from u and v by replacing each 1 by 2 and each 2 by $\bar{1}$. Since $|\check{f}_1^{|u|}\check{f}_2^{|u|_2}(u)|_2 = |u|_1 = |v|_1 + 1$, 2 occurs in $\check{f}_1^{|u|}\check{f}_2^{|u|_2}(u)$, which implies that $\check{f}_1^{|u|}\check{f}_2^{|u|_2}(u)\bar{2}1^m$ has a 2-inversion. And since $\bar{2}$ does not occur in $\check{f}_1^{|v|}\check{f}_2^{|v|_2}(v)$, we have that $\check{f}_1^{|v|}\check{f}_2^{|v|_2}(v)1^{m+1}$ is 2-inversion-free. By Proposition 9.5, this is a contradiction, because

$$\check{f}_1^{|u|}(\check{f}_2^{|u|_2}(u)\bar{2}1^m) = \check{f}_1^{|u|}\check{f}_2^{|u|_2+1}(u21^m) \sim \check{f}_1^{|u|}\check{f}_2^{|u|_2+1}(v1^{m+1}) = \check{f}_1^{|v|}\check{f}_2^{|v|_2}(v)1^{m+1}$$

as $2^m 1u \sim 2^{m+1}v$. $\square$

Lemma 9.17. *Let $m_1, m_2, n_1, n_2, p_1, p_2 \in \mathbb{Z}_{\geq 0}$ with $n_1, n_2 \geq 1$. Then, $1^{m_1}2^{n_1}1^{p_1} \sim 1^{m_2}2^{n_2}1^{p_2}$ implies that $m_1 = m_2$, $n_1 = n_2$ and $p_1 = p_2$.*

Proof. Suppose that $1^{m_1}2^{n_1}1^{p_1} \sim 1^{m_2}2^{n_2}1^{p_2}$. Then, $\text{wt}(1^{m_1}2^{n_1}1^{p_1}) = \text{wt}(1^{m_2}2^{n_2}1^{p_2})$, which implies that $n_1 = n_2$ and $m_1 + p_1 = m_2 + p_2$. Suppose $m_1 \neq m_2$. Without loss of generality, assume $m_1 < m_2$. We have that

$$\check{f}_1^{m_2+n_1-1}\check{f}_2^{n_1}(1^{m_1}2^{n_1}1^{p_1}) = \check{f}_1^{m_2+n_1-1}(1^{m_1}\bar{2}^{n_1}1^{p_1}) = 2^{m_1}\bar{1}^{n_1}2^{m_2-m_1-1}1^{p_1-m_2+m_1+1}$$

and

$$\begin{aligned} \check{f}_1^{m_2+n_1-1}\check{f}_2^{n_1}(1^{m_2}2^{n_2}1^{p_2}) &= \check{f}_1^{m_2+n_2-1}\check{f}_2^{n_2}(1^{m_2}2^{n_2}1^{p_2}) \\ &= \check{f}_1^{m_2+n_2-1}(1^{m_2}\bar{2}^{n_2}1^{p_2}) = 2^{m_2}\bar{1}^{n_2-1}\bar{2}1^{p_2}, \end{aligned}$$

where $\check{\varphi}_2(2^{m_1}\bar{1}^{n_1}2^{m_2-m_1-1}1^{p_1-m_2+m_1+1}) = m_2 - 1$ and $\check{\varphi}_2(2^{m_2}\bar{1}^{n_2-1}\bar{2}1^{p_2}) = +\infty$. This is a contradiction, because

$$\begin{aligned} 2^{m_1}\bar{1}^{n_1}2^{m_2-m_1-1}1^{p_1-m_2+m_1+1} &= \check{f}_1^{m_2+n_1-1}\check{f}_2^{n_1}(1^{m_1}2^{n_1}1^{p_1}) \\ &\sim \check{f}_1^{m_2+n_1-1}\check{f}_2^{n_1}(1^{m_2}2^{n_1}1^{p_2}) = 2^{m_2}\bar{1}^{n_2-1}\bar{2}1^{p_2} \end{aligned}$$

as $1^{m_1}2^{n_1}1^{p_1} \sim 1^{m_2}2^{n_2}1^{p_2}$. $\square$

The hypoplactic congruence on C_2 is not as trivial as the lemmas above could suggest. In the following result, we present distinct words which represent the same element in $\text{hypo}(\mathcal{C}_2)$.

Lemma 9.18. *Let $m_1, m_2, n_1, n_2, p_1, p_2, q_1, q_2 \in \mathbb{Z}_{\geq 0}$ with $m_1, m_2, q_1, q_2 \geq 1$. Then, $1^{m_1}2^{n_1}1^{p_1}2^{q_1} \sim 1^{m_2}2^{n_2}1^{p_2}2^{q_2}$ if and only if $m_1 + p_1 = m_2 + p_2$ and $n_1 + q_1 = n_2 + q_2$.*

Proof. If $1^{m_1}2^{n_1}1^{p_1}2^{q_1} \sim 1^{m_2}2^{n_2}1^{p_2}2^{q_2}$ then $\text{wt}(1^{m_1}2^{n_1}1^{p_1}2^{q_1}) = \text{wt}(1^{m_2}2^{n_2}1^{p_2}2^{q_2})$, which implies that $m_1 + p_1 = m_2 + p_2$ and $n_1 + q_1 = n_2 + q_2$.

We now show that $12^k1^l2 \sim 1^{l+1}2^{k+1}$, for any $k, l \in \mathbb{Z}_{\geq 0}$. Each connected component $\Gamma_{\mathfrak{C}_2^*}(12^k1^l2)$ and $\Gamma_{\mathfrak{C}_2^*}(1^{l+1}2^{k+1})$ is a path with $2k + 2l + 5$ vertices. So, we can define a bijection $\psi : C_2^*(12^k1^l2) \rightarrow C_2^*(1^{l+1}2^{k+1})$ that maps each word $w \in C_2^*(12^k1^l2)$ to the word $\psi(w) \in C_2^*(1^{l+1}2^{k+1})$ such that the position of w in $\Gamma_{\mathfrak{C}_2^*}(12^k1^l2)$ is the same as $\psi(w)$ in $\Gamma_{\mathfrak{C}_2^*}(1^{l+1}2^{k+1})$.

The paths $\Gamma_{\mathfrak{C}_2^*}(12^k1^l2)$ and $\Gamma_{\mathfrak{C}_2^*}(1^{l+1}2^{k+1})$ start in 12^k1^l2 and $1^{l+1}2^{k+1}$, respectively, then followed by $k+1$ edges labelled by 2, followed by $k+l+2$ edges labelled by 1, and finally, $l+1$ edges labelled by 2. Also, in both paths there are loops labelled by 1 from position 1 to $k+1$ and from position $2k+l+5$ to $2k+2l+5$, and there are loops labelled by 2 from position $k+3$ to $2k+l+3$. Hence, $u \xrightarrow{i} v$ is an edge in $\Gamma_{\mathfrak{C}_2^*}(12^k1^l2)$ if and only if $\psi(u) \xrightarrow{i} \psi(v)$ is an edge of $\Gamma_{\mathfrak{C}_2^*}(1^{l+1}2^{k+1})$. And since $\psi(12^k1^l2) = 1^{l+1}2^{k+1}$, where

$$\text{wt}(12^k1^l2) = (l+1, k+1) = \text{wt}(1^{l+1}2^{k+1}),$$

we have that ψ preserves weights. Therefore, ψ is a quasi-crystal isomorphism, which implies that $12^k1^l2 \sim 1^{l+1}2^{k+1}$.

Finally, as $\sim$ is a monoid congruence, we get that $1^{m_1}2^{n_1}1^{p_1}2^{q_1} \sim 1^{m_1+p_1}2^{n_1+q_1}$ and $1^{m_2}2^{n_2}1^{p_2}2^{q_2} \sim 1^{m_2+p_2}2^{n_2+q_2}$. Since $m_1 + p_1 = m_2 + p_2$ and $n_1 + q_1 = n_2 + q_2$, we obtain that $1^{m_1}2^{n_1}1^{p_1}2^{q_1} \sim 1^{m_2}2^{n_2}1^{p_2}2^{q_2}$. $\square$

With the relations proven above, we can show that $\text{hypo}(\mathfrak{C}_2)$ does not admit a finite presentation.

Theorem 9.19. *The hypoplactic congruence $\sim$ on C_2^* is not finitely generated. Therefore, $\text{hypo}(\mathfrak{C}_2)$ has no finite presentation.*

Proof. Let R be a finite subset of $\sim$, and denote by $\sim_R$ the congruence on C_2^* generated by R . Thus, $\sim_R \subseteq \sim$. Let $m \in \mathbb{Z}_{\geq 0}$ be maximal among the lengths of words on either side of some pair in R .

First we show that for any subword u of $121^{m+1}2$ with length at most m and $v \in C_2^*$, if $u \sim v$, then $u = v$. So, let $u \sim v$ where $v \in C_2^*$ and u is a subword of $121^{m+1}2$ with length at most m , that is, $121^m2 = w_1uw_2$ for some $w_1, w_2 \in C : 2^*$ and $|u| \leq m$. By Lemma 9.14, $v \in \{1, 2\}^*$. By definition, u lies in one of the following cases.

- Case 1: $u = 1^p$ where $p \in \mathbb{Z}_{\geq 0}$. Since $u \sim v$, we have that $\text{wt}(u) = \text{wt}(v)$ where $u, v \in \{1, 2\}^*$, and so, $|v|_1 = |u|_1 = p$ and $|v|_2 = |u|_2 = 0$. Hence, $v = 1^p = u$.

- Case 2: $u = 21^p$ where $p \in \mathbb{Z}_{\geq 0}$. As in the previous case, we have that $|v|_1 = p$ and $|v|_2 = 1$. So, $v = 1^{n_1} 21^{n_2}$ for some $n_1, n_2 \in \mathbb{Z}_{\geq 0}$ such that $n_1 + n_2 = p$. By Lemma 9.15, we have that $1^{n_1} 21^{n_2} \sim 21^p$ implies $n_1 = 0$. Hence, $v = 21^p = u$.
- Case 3: $u = 1^{p_1} 21^{p_2}$ where $p_1, p_2 \in \mathbb{Z}_{\geq 0}$ with $p_1 \geq 1$. As in the previous case, we have that $v = 1^{n_1} 21^{n_2}$ where $n_1 + n_2 = p_1 + p_2$. By Lemma 9.14, $1^{n_1} 21^{n_2} \sim 1^{p_1} 21^{p_2}$ and $p_1 \geq 1$ imply $n_1 \geq 1$. And by Lemma 9.17, $1^{n_1} 21^{n_2} \sim 1^{p_1} 21^{p_2}$ and $n_1, p_1 \geq 1$ imply $n_1 = p_1$ and $n_2 = p_2$. Hence, $v = 1^{p_1} 21^{p_2} = u$.

Since $\sim_R \subseteq \sim$, we have that for any subwords u of $121^{m+1}2$ with length at most m and $v \in C_2^*$, if $u \sim_R v$, then $u = v$. As R generates $\sim_R$ and every pair $(u, v) \in R$ satisfies $|u|, |v| \leq m$, then $121^{m+1}2 \sim_R w$ if and only if $w = 121^{m+1}2$. On the other hand, $121^{m+1}2 \sim 1^{m+2}2^2$ by Lemma 9.18. Hence, $\sim_R \neq \sim$. And therefore, $\sim$ is not finitely generated. $\square$

Although, $\text{hypo}(\mathcal{C}_2)$ does not have a finite presentation, we are able to describe connected components of $\Gamma_{\mathcal{C}_2^*}$, and thus, find representatives for the hypoplactic congruence classes.

Lemma 9.20. *Let $u, v, w \in \{1, 2, \bar{1}\}^*$. Then, $2u\bar{1}v1w2 \sim 1^{m+1}2^{n+2}\bar{1}^{p+1}$, for some $m, n, p \in \mathbb{Z}_{\geq 0}$ where either $m = 0$ or $p = 0$.*

Proof. If $|u|_1 + |v|_1 + |w|_1 \geq |u|_{\bar{1}} + |v|_{\bar{1}} + |w|_{\bar{1}}$, set $m = |u|_1 + |v|_1 + |w|_1 - |u|_{\bar{1}} - |v|_{\bar{1}} - |w|_{\bar{1}}$ and $p = 0$. Otherwise, set $m = 0$ and $p = |u|_{\bar{1}} + |v|_{\bar{1}} + |w|_{\bar{1}} - |u|_1 - |v|_1 - |w|_1$. Also, set $n = |u|_2 + |v|_2 + |w|_2$. Since

$$|u|_1 + |v|_1 + |w|_1 - |u|_{\bar{1}} - |v|_{\bar{1}} - |w|_{\bar{1}} = m - p$$

and $\bar{2}$ does not occur in $2u\bar{1}v1w2$, we have that

$$\text{wt}(2u\bar{1}v1w2) = (m - p, n + 2) = \text{wt}(1^{m+1}2^{n+2}\bar{1}^{p+1}).$$

Note that $\varepsilon_1(w') = +\infty$, for any $w' \in C_2^*(2u\bar{1}v1w2)$, as $\check{f}_2^k(2u\bar{1}v1w2)$, with $1 \leq k \leq n + 2$, has a decomposition of the form $\bar{2}u'\bar{1}v'$, for some $u', v' \in C_2^*$. Since $\check{f}_2^k(1^{m+1}2^{n+2}\bar{1}^{p+1}) = 1^{m+1}\bar{2}^k 2^{n+2-k}\bar{1}^{p+1}$, for any $1 \leq k \leq n + 2$, then $\varepsilon_2(w'') = +\infty$, for all $w'' \in C_2^*(1^{m+1}2^{n+2}\bar{1}^{p+1})$. Also, $\varepsilon_2(2u\bar{1}v1w2) = 0 = \varepsilon_2(1^{m+1}2^{n+2}\bar{1}^{p+1})$ and $\check{\varphi}_2(2u\bar{1}v1w2) = n + 2 = \check{\varphi}_2(1^{m+1}2^{n+2}\bar{1}^{p+1})$. Thus, we have a quasi-crystal isomorphism $\mathcal{C}_2^*(2u\bar{1}v1w2) \rightarrow \mathcal{C}_2^*(1^{m+1}2^{n+2}\bar{1}^{p+1})$, given by

$$\check{f}_2^k(2u\bar{1}v1w2) \mapsto 1^{m+1}\bar{2}^k 2^{n+2-k}\bar{1}^{p+1},$$

for $k = 0, \dots, n + 2$. And therefore, $2u\bar{1}v1w2 \sim 1^{m+1}2^{n+2}\bar{1}^{p+1}$. $\square$

Theorem 9.21. *Any connected component of $\mathfrak{C}_2^*$ is quasi-crystal isomorphic to one and only one of the following*

- (i) $\mathfrak{C}_2^*(1^m)$, $m \geq 0$;
- (ii) $\mathfrak{C}_2^*(2^{m_1} 1^{m_2+1} 2^{m_3+1} 1^{m_4})$, $m_1, m_2, m_3, m_4 \geq 0$;
- (iii) $\mathfrak{C}_2^*(1^{m_1+1} 2^{m_2} \bar{1}^{m_3+1})$, $m_1, m_2, m_3 \geq 0$ with either $m_1 = 0$ or $m_3 = 0$;
- (iv) $\mathfrak{C}_2^*(1^{m_1+1} 2^{m_2+1} \bar{2}^{m_3+1} \bar{1}^{m_4+1})$, $m_1, m_2, m_3, m_4 \geq 0$ with either $m_1 = 0$ or $m_4 = 0$, and either $m_2 = 0$ or $m_3 = 0$.

Therefore, the elements in these connected components form a minimal set of representatives for the hypoplactic congruence $\sim$ on $\mathfrak{C}_2^*$.

Proof. By Proposition 9.7, any connected component of $\mathfrak{C}_2^*$ has at least a highest-weight element. Let w be a highest-weight element of $\mathfrak{C}_2^*$. If 2 , $\bar{2}$ and $\bar{1}$ do not occur in w , then $w = 1^m$, for some $m \geq 0$, and so, $\mathfrak{C}_2^*(w)$ coincides with $\mathfrak{C}_2^*(1^m)$. Thus, assume that 2 , $\bar{2}$ or $\bar{1}$ occur in w .

If $\bar{2}$ occurs in w , then w has a 2-inversion, because $\bar{e}_2$ is defined on $\bar{2}$ and w is of highest weight. This implies that 2 occurs in w , and as $\bar{e}_1$ is defined on 2 , w has a 1-inversion. Hence, $\bar{e}_1(w) = \bar{e}_2(w) = +\infty$, and so, w is an isolated word. If $|w|_1 \geq |w|_{\bar{1}}$, set $m_1 = |w|_1 - |w|_{\bar{1}}$ and $m_4 = 0$, otherwise, set $m_1 = 0$ and $m_4 = |w|_{\bar{1}} - |w|_1$. If $|w|_2 \geq |w|_{\bar{2}}$, set $m_2 = |w|_2 - |w|_{\bar{2}}$ and $m_3 = 0$, otherwise, set $m_2 = 0$ and $m_3 = |w|_{\bar{2}} - |w|_2$. Thus, $w \sim 1^{m_1+1} 2^{m_2+1} \bar{2}^{m_3+1} \bar{1}^{m_4+1}$. And therefore, in the following we assume that $\bar{2}$ does not occur in w .

If $\bar{1}$ does not occur in w , then 2 occurs in w , and so, w must have a 1-inversion. Thus,

$$w = 2^{k_0} 1^{l_1+1} 2^{k_1} 1^{l_2} \dots 2^{k_{n-1}} 1^{l_n} 2^{k_n+1} 1^{l_{n+1}},$$

for some $k_0, \dots, k_n, l_1, \dots, l_{n+1} \geq 0$. Set $m_1 = k_0$, $m_2 = l_1 + \dots + l_n$, $m_3 = k_1 + \dots + k_n$, and $m_4 = l_{n+1}$. By Lemma 9.18, we have that $w \sim 2^{m_1} 1^{m_2+1} 2^{m_3+1} 1^{m_4}$, that is, $\mathfrak{C}_2^*(w)$ is quasi-crystal isomorphic to $\mathfrak{C}_2^*(2^{m_1} 1^{m_2+1} 2^{m_3+1} 1^{m_4})$. Therefore, in the following we assume that $\bar{1}$ occurs in w .

Note that w has a decomposition of the form $w_1 1 w_2 \bar{1} w_3$ or $w_1 1 w_2 2 w_3$, for some $w_1, w_2, w_3 \in \{1, 2, \bar{1}\}^*$, because w is of highest weight. If w has a decomposition of the form $w_1 1 w_2 \bar{1} w_3$, for some $w_1, w_2, w_3 \in \{1, 2, \bar{1}\}^*$, then $w \sim 1^{|w|_1} 2^{|w|_2} \bar{1}^{|w|_{\bar{1}}}$, because $1\bar{1}$ is an isolated element (Theorem 8.25). If $|w|_1 \geq |w|_{\bar{1}}$, set $m_1 = |w|_1 - |w|_{\bar{1}}$ and $m_3 = 0$, otherwise, set $m_1 = 0$ and $m_3 = |w|_{\bar{1}} - |w|_1$. We have that $1^{|w|_1} \bar{1}^{|w|_{\bar{1}}} \sim 1^{M_1+1} \bar{1}^{m_3+1}$. Set $m_2 = |w|_2$. Thus, $w \sim 1^{m_1+1} 2^{m_2} \bar{1}^{m_3+1}$, that is, $\mathfrak{C}_2^*(w)$ is quasi-crystal isomorphic to $\mathfrak{C}_2^*(1^{m_1+1} 2^{m_2} \bar{1}^{m_3+1})$.

If w has a decomposition of the form $w_1 2 w_2 \bar{1} w_3 1 w_4 2 w_5$, for some words $w_1, w_2, w_3, w_4, w_5 \in \{1, 2, \bar{1}\}^*$, then $w \sim w_1 1^{k_1+1} 2^{k_2+2} \bar{1}^{k_3+1} w_5$, by Lemma 9.20. As in the previous case, there exists $m_1, m_2, m_3 \geq 0$ with either $m_1 = 0$ or $m_3 = 0$ such that $w_1 1^{k_1+1} 2^{k_2+2} \bar{1}^{k_3+1} w_5 \sim 1^{m_1+1} 2^{m_2} \bar{1}^{m_3+1}$. Hence, $\mathcal{C}_2^*(w)$ is quasi-crystal isomorphic to $\mathcal{C}_2^*(1^{m_1+1} 2^{m_2} \bar{1}^{m_3+1})$.

Finally, if w does not lie in one of the cases above, then w has a decomposition of the form $\bar{1}^k u$, for some $u \in \{1, 2\}^*$ and $k \geq 1$. Since $\check{f}_2^{|u|_2}(\bar{1}^k u) = \bar{1}^k \check{f}_2^{|u|_2}(u)$ is 1-inversion-free, we have that $\check{e}_1^k(\bar{1}^k \check{f}_2^{|u|_2}(u)) = \bar{2}^k \check{f}_2^{|u|_2}(u)$, which is 2-inversion-free, and so,

$$\check{e}_2^{|u|_2+k}(\bar{2}^k \check{f}_2^{|u|_2}(u)) = 2^k u.$$

Hence, the connected component of w is the same as $2^k u$. Since $2^k u$ is a highest-weight element where neither $\bar{1}$ nor $\bar{2}$ occur, we have that $2^k u \sim 2^{m_1} 1^{m_2+1} 2^{m_3+1} 1^{m_4}$, for some $m_1, m_2, m_3, m_4 \geq 0$, as in a previous case. And therefore, $\mathcal{C}_2^*(w)$ is quasi-crystal isomorphic to $\mathcal{C}_2^*(2^{m_1} 1^{m_2+1} 2^{m_3+1} 1^{m_4})$.

Note that every word lying on some connected component on (iv) has 1-inversions and 2-inversions. Any word lying on some component on (iii) has 1-inversions and is 2-inversion-free. The only element on $\{1, 2\}^*$ lying on a connected component on (ii) is its representative. From this facts and the Lemmas 9.15 and 9.16, it follows that there are no quasi-crystal isomorphic connected components among (i) to (iv). $\square$

By computing the elements in each connected component of the previous theorem, we obtain the following consequence.

Corollary 9.22. *Let $w \in C_2^*$. Then, one of the following statements holds:*

- (i) $w \sim \bar{1}^{m_1} \bar{2}^{m_2} 2^{m_3} 1^{m_4} 2^{m_6} 1^{m_7}$,
- (ii) $w \sim \bar{1}^{m_1} \bar{2}^{m_2} 1^{m_3} \bar{2}^{m_4} 2^{m_5} 1^{m_6}$,
- (iii) $w \sim \bar{1}^{m_1} 2^{m_2} 1^{m_3} \bar{2}^{m_4} 1^{m_5}$,
- (iv) $w \sim \bar{1}^{m_1} 2^{m_2} \bar{1}^{m_3} \bar{2}^{m_4} 1^{m_5}$,
- (v) $w \sim \bar{1}^{m_1} 2^{m_2} \bar{1}^{m_3} 2^{m_4} 1^{m_5}$,
- (vi) $w \sim \bar{1}^{m_1} \bar{2}^{m_2} 2^{m_3} \bar{1}^{m_4} 2^{m_5}$,
- (vii) $w \sim \bar{1}^{m_1} \bar{2}^{m_2} \bar{1}^{m_3} \bar{2}^{m_4} 2^{m_5}$,
- (viii) $w \sim 1^{m_1} \bar{2}^{m_2} 2^{m_3} \bar{1}^{m_4}$,
- (ix) $w \sim 1^{m_1} 2^{m_2} \bar{2}^{m_3} \bar{1}^{m_4}$,

for some $m_1, m_2, m_3, m_4, m_5, m_6, m_7 \in \mathbb{Z}_{\geq 0}$.

9.1.5 Identities

In this section we show that the hypoplactic monoid $\text{hypo}(\mathfrak{C}_2)$ satisfies some identities.

First, we check some properties of the identities of $\text{hypo}(\mathfrak{C}_2)$.

Theorem 9.23. *Let X be a finite alphabet, and let $u, v \in X^*$. If $\text{hypo}(\mathfrak{C}_2)$ satisfies the identity $u = v$, then the following conditions are satisfied.*

- (i) $|u|_x = |v|_x$, for all $x \in X$.
- (ii) Until the first occurrence of a letter $x \in X$ in u and v , each letter of X occurs exactly the same number of times in u and v , that is, if $u = u_1xu_2$, where $u_1, u_2 \in X^*$ and x does not occur in u_1 , then $v = v_1xv_2$, for some $v_1, v_2 \in C^*$ such that $|v|_y = |u|_y$, for all $y \in X$.
- (iii) After the last occurrence of a letter $x \in X$ in u and v , each letter of X occurs exactly the same number of times in u and v , that is, if $u = u_1xu_2$, where $u_1, u_2 \in X^*$ and x does not occur in u_2 , then $v = v_1xv_2$, for some $v_1, v_2 \in C^*$ such that $|v_2|_y = |u_2|_y$, for all $y \in X$.

Proof. Let $x \in X$.

(i) If we consider the map from X to C_2^* that sends x to 1 and each other letter of X to ϵ , we obtain that $1^{|u|_x} \sim 1^{|v|_x}$. So, $\text{wt}(1^{|u|_x}) = \text{wt}(1^{|v|_x})$ which implies that $|u|_x = |v|_x$.

(ii) If x occurs in u , then it also occurs in v , because $|u|_x = |v|_x$. And if so, there exist $u_1, u_2, v_1, v_2 \in X^*$ such that $u = u_1xu_2$, $v = v_1xv_2$ and x does not occur in u_1 and v_1 . Given $y \in X$, consider the monoid homomorphism $\psi : X^* \rightarrow C_2^*$ induced by $\psi(x) = 1$, $\psi(y) = 2$ and $\psi(z) = \epsilon$, for each $z \in X \setminus \{x, y\}$. Then,

$$2^{|u_1|_y} 1 \psi(u_2) = \psi(u) \sim \psi(v) = 2^{|v_1|_y} 1 \psi(v_2),$$

which implies that $|u_1|_y = |v_1|_y$, by Lemma 9.15.

(iii) If $u = u_1xu_2$, for some $u_1, u_2 \in X^*$ where x does not occur in u_2 , then there exist $v_1, v_2 \in X^*$ such that $v = v_1xv_2$ and x does not occur in v_2 . Given $y \in X$, consider the monoid homomorphism $\psi : X^* \rightarrow C_2^*$ induced by $\psi(x) = 2$, $\psi(y) = 1$ and $\psi(z) = \epsilon$, for each $z \in X \setminus \{x, y\}$. Then,

$$\psi(u_1) 2 1^{|u_2|_y} = \psi(u) \sim \psi(v) = \psi(v_1) 2 1^{|v_2|_y},$$

which implies that $|u_1|_y = |v_1|_y$, by Lemma 9.16. □

In a first approach to show that $\text{hypo}(\mathcal{C}_2)$ satisfies the identity $xyxyxy = xyxyxy$, we check in the following example that it holds when x and y are mapped to letters in C_2 .

Example 9.24. By Lemma 9.18, $\mathcal{C}_2^*(121212)$ and $\mathcal{C}_2^*(122112)$ are quasi-crystal isomorphic. This implies that

$$\begin{aligned} 121212 &\sim 122112, & 1\bar{2}1\bar{2}1\bar{2} &\sim 1\bar{2}211\bar{2}, \\ 2\bar{1}2\bar{1}2\bar{1} &\sim 2\bar{1}1\bar{2}2\bar{1}, & \overline{212121} &\sim \overline{211221}. \end{aligned}$$

Since $\mathcal{C}_2^*(212121)$ and $\mathcal{C}_2^*(211221)$ are quasi-crystal isomorphic, we have that

$$\begin{aligned} 212121 &\sim 211221, & \bar{2}1\bar{2}1\bar{2}1 &\sim \bar{2}11\bar{2}21, \\ \bar{1}2\bar{1}2\bar{1}2 &\sim \bar{1}22\bar{1}1\bar{2}, & \overline{121212} &\sim \overline{122112}. \end{aligned}$$

As $1\bar{1}$ is an isolated element, we have that $1\bar{1}1\bar{1}1\bar{1} \sim 1^3\bar{1}^3 \sim 1\bar{1}1\bar{1}1\bar{1}$ and $\bar{1}1\bar{1}1\bar{1}1 \sim 1^3\bar{1}^3 \sim \bar{1}1\bar{1}1\bar{1}1$. Similarly, as $2\bar{2}$ is an isolated element, we have that $2\bar{2}2\bar{2}2\bar{2} \sim 2\bar{2}2\bar{2}2\bar{2}$ and $\bar{2}2\bar{2}2\bar{2}2 \sim \bar{2}2\bar{2}2\bar{2}2$.

Theorem 9.25. *The hypoplactic monoid $\text{hypo}(\mathcal{C}_2)$ satisfies the identity $xyxyxy = xyxyxy$.*

Proof. Let $u, v \in C_2^*$. If $\check{\varepsilon}_i(uvuvuv) \in \mathbb{Z}_{>0}$, for some $i \in \{1, 2\}$, then $\check{\varepsilon}_i(uvuvuv) = 3\check{\varepsilon}_i(u) + 3\check{\varepsilon}_i(v)$, and

$$\check{\varepsilon}_i^{\check{\varepsilon}_i(uvuvuv)}(uvuvuv) = \check{\varepsilon}_i^{\check{\varepsilon}_i(u)}(u)\check{\varepsilon}_i^{\check{\varepsilon}_i(v)}(v)\check{\varepsilon}_i^{\check{\varepsilon}_i(u)}(u)\check{\varepsilon}_i^{\check{\varepsilon}_i(v)}(v)\check{\varepsilon}_i^{\check{\varepsilon}_i(u)}(u)\check{\varepsilon}_i^{\check{\varepsilon}_i(v)}(v)$$

which is also of the form $u'v'u'v'u'v'$. Thus, we may assume that $\check{\varepsilon}_i(uvuvuv) \in \{0, +\infty\}$, for $i = 1, 2$.

If $\bar{2}$ occurs in uv , then 2 also occurs in uv , because $\check{\varepsilon}_2(uvuvuv) \in \{0, +\infty\}$. So, both $uvuvuv$ and $uvvuuv$ have a decomposition of the form $w_1 2 w_2 \bar{2} w_3 2 w_4$, $w_1, w_2, w_3, w_4 \in C_2^*$. Hence,

$$uvuvuv \sim 1^{3|u|_1+3|v|_1+1} 2^{3|u|_2+3|v|_2} \bar{2}^{3|u|_{\bar{2}}+3|v|_{\bar{2}}} \bar{1}^{3|u|_{\bar{1}}+3|v|_{\bar{1}}+1} \sim uvvuuv.$$

And so, assume that $\bar{2}$ does not occur in uv .

If $\bar{1}$ occurs in uv , then 1 occurs in uv , because $\check{\varepsilon}_1(uvuvuv) \in \{0, +\infty\}$ and $\bar{2}$ does not occur in uv . So, both $uvuvuv$ and $uvvuuv$ have a decomposition of the form $w_1 1 w_2 \bar{1} w_3$, $w_1, w_2, w_3 \in \{1, 2, \bar{1}\}^*$, which implies that

$$uvuvuv \sim 1^{3|u|_1+3|v|_1} 2^{3|u|_2+3|v|_2} \bar{1}^{3|u|_{\bar{1}}+3|v|_{\bar{1}}} \sim uvvuuv.$$

And thus, assume that $\bar{1}$ does not occur in uv .

If 2 occurs in uv , then 1 occurs in uv , because $\tilde{\varepsilon}_1(uvuvuv) \in \{0, +\infty\}$ and $\bar{2}$ does not occur in uv . So, $uv = w_1 1 w_2$ and $uv = w_3 2 w_4$, for some $w_1, w_2, w_3, w_4 \in \{1, 2\}^*$. And since $u, v \in \{1, 2\}^*$, by Lemma 9.18, then

$$\begin{aligned} uvuvuv &= w_1 1 w_2 uv w_3 2 w_4 \sim w_1 1^{|w_2 uv w_3|_1 + 1} 2^{|w_2 uv w_3|_2 + 1} w_4 \\ &\sim w_1 1 w_2 v u w_3 2 w_4 = uvvuvuv. \end{aligned}$$

Finally, if also 2 does not occur in uv , then $u = 1^k$ and $v = 1^l$, for some $k, l \geq 0$, which implies that $uvuvuv = uvvuvuv$. $\square$

9.2 The case $n \geq 3$

In this section we study the hypoplactic monoid $\text{hypo}(\mathbb{C}_n)$ for $n \geq 3$. We generalize some results obtained in Section 9.1 for $\text{hypo}(\mathbb{C}_2)$. On the other hand, we present results that do not hold for $\text{hypo}(\mathbb{C}_n)$ when $n \geq 3$. In particular, we prove that $\text{hypo}(\mathbb{C}_n)$ does not satisfy any identity. Finally, we show that the hypoplactic monoid $\text{hypo}(\mathbb{C}_n)$ contains a copy of the classical hypoplactic monoid hypo_{n-1} of rank $n - 1$.

9.2.1 The definition of $\text{hypo}(\mathbb{C}_n)$

In this section we describe how to obtain the hypoplactic monoid $\text{hypo}(\mathbb{C}_n)$. This construction follows from the theory introduced in Chapters 7 and 8 applied to the standard quasi-crystal of type $\mathbb{C}_n$.

From Example 4.4(ii), we have that the standard quasi-crystal $\mathbb{C}_n$ of Cartan type $\mathbb{C}_n$ consists of an ordered set

$$C_n = \{1 < 2 < \dots < n < \bar{n} < \dots < \bar{2} < \bar{1}\},$$

and a quasi-crystal structure given as follows. The weight map $\text{wt} : C_n \rightarrow \mathbb{Z}^n$ is defined by

$$\text{wt}(x) = \mathbf{e}_x, \quad \text{wt}(\bar{x}) = -\mathbf{e}_x.$$

for $x \in \{1, \dots, n\}$. For $x \in C_n$ and $i \in \{1, \dots, n\}$, the quasi-Kashiwara operators $\tilde{e}_i, \tilde{f}_i : C_2 \rightarrow C_n \sqcup \{\perp\}$ are defined on x only in the following cases

$$\tilde{e}_i(x) = \begin{cases} i & \text{if } x = i + 1 \\ \overline{i + 1} & \text{if } x = \bar{i} \neq \bar{n} \\ n & \text{if } x = \bar{n} \text{ and } i = n, \end{cases}$$

and

$$\tilde{f}_i(x) = \begin{cases} i+1 & \text{if } x = i \neq n \\ \bar{i} & \text{if } x = \overline{i+1} \\ \bar{n} & \text{if } x = n \text{ and } i = n. \end{cases}$$

Also,

$$\tilde{\varepsilon}_i(x) = \delta_{x,i+1} + \delta_{x,\bar{i}}, \quad \tilde{\varphi}_i(x) = \delta_{x,i} + \delta_{x,\overline{i+1}}.$$

Notice that given $x, y \in C_n$ and $i \leq n-1$, if $\tilde{\varphi}_i(x), \tilde{\varepsilon}_i(y) > 0$, then $x \in \{i, \overline{i+1}\}$ and $y \in \{i+1, \bar{i}\}$. If $\tilde{\varphi}_n(x), \tilde{\varepsilon}_n(y) > 0$, then $x = n$ and $y = \bar{n}$. This leads to the notion of inversion on words over C_n .

Definition 9.26. Let $i \in \{1, \dots, n\}$. A word $w \in C_n^*$ is said to have an i -inversion if w admits a decomposition of the form $w = w_1 x w_2 y w_3$, for some $w_1, w_2, w_3 \in C_2^*$, $x \in \{i, \overline{i+1}\}$ and $y \in \{i+1, \bar{i}\}$.

A word $w \in C_n^*$ is said to be i -inversion-free if w does not have an i -inversion.

By Definition 8.1 and Proposition 8.3, we obtain the following description of the free quasi-crystal monoid over $\mathcal{C}_n$.

Definition 9.27. The free quasi-crystal monoid $\mathcal{C}_n^*$ over $\mathcal{C}_n$ consists of the set C_n^* of all words over C_n , the concatenation of words, and a quasi-crystal structure given as follows. For $w \in C_n^*$, the weight of w is

$$\text{wt}(w) = (|w|_1 - |w|_{\bar{1}}, |w|_2 - |w|_{\bar{2}}, \dots, |w|_n - |w|_{\bar{n}}).$$

For $i \in \{1, \dots, n\}$. If w has an i -inversion, then

$$\tilde{\varepsilon}_i(w) = \tilde{\varphi}_i(w) = +\infty,$$

otherwise,

$$\tilde{\varepsilon}_i(w) = |w|_{i+1} + |w|_{\bar{i}}, \quad \tilde{\varphi}_i(w) = |w|_i + |w|_{\overline{i+1}}.$$

The raising quasi-Kashiwara operator $\tilde{e}_i$ is defined on w if $\tilde{\varepsilon}_i(w) \in \mathbb{Z}_{>0}$, and if so, $\tilde{e}_i(w) = w_1 \tilde{e}_i(x) w_2$, where $w_1, w_2 \in C_n^*$ and $x \in \{i+1, \bar{i}\}$ are such that $w = w_1 x w_2$ and $|w_2|_{i+1} = |w_2|_{\bar{i}} = 0$.

The lowering quasi-Kashiwara operator $\tilde{f}_i$ is defined on w if $\tilde{\varphi}_i(w) \in \mathbb{Z}_{>0}$, and if so, $\tilde{f}_i(w) = w_1 \tilde{f}_i(x) w_2$, where $w_1, w_2 \in C_n^*$ and $x \in \{i, \overline{i+1}\}$ are such that $w = w_1 x w_2$ and $|w_1|_i = |w_1|_{\overline{i+1}} = 0$.

From the signature rule (Algorithm 8.6), the quasi-Kashiwara operators $\check{e}_i$ and $\check{f}_i$ can be computed on an i -inversion-free word $w \in C_n^*$ as follows. If $i + 1$ or $\bar{i}$ occur in w , let x be the right-most $i + 1$ or $\bar{i}$ occurring in w , then $\check{e}_i(w)$ is obtained from w by replacing the right-most x by $\check{e}_i(x)$. If i or $\overline{i + 1}$ occur in w , let y be the left-most i or $\overline{i + 1}$ occurring in w , then $\check{f}_i(w)$ is obtained from w by replacing the left-most y by $\check{f}_i(y)$.

As we did for the study of the hypoplactic monoid $\text{hypo}(\mathfrak{C}_2)$, we will frequently resort to the quasi-crystal graph $\Gamma_{\mathfrak{C}_n^*}^*$ of $\mathfrak{C}_n^*$ (Definition 7.19) to describe on the hypoplactic monoid $\text{hypo}(\mathfrak{C}_n)$. For the hypoplactic congruence $\sim$ on $\mathfrak{C}_n^*$ (Definition 8.21), we have that two words $u, v \in C_n^*$ are hypoplactic congruent $u \sim v$ if and only if there exists a graph isomorphism ψ between the connected components $\Gamma_{\mathfrak{C}_n^*}^*(u)$ and $\Gamma_{\mathfrak{C}_n^*}^*(v)$ of $\Gamma_{\mathfrak{C}_n^*}^*$ such that $\psi(u) = v$ (Theorem 7.29). Note that by graph isomorphism we require ψ to preserve weights.

The hypoplactic monoid $\text{hypo}(\mathfrak{C}_n)$ is the quotient monoid of C_n^* by the hypoplactic congruence $\sim$ (see Definition 8.24).

In the following result we can see that the hypoplactic congruence $\sim$ respects inversions.

Proposition 9.28. *Let $u, v \in C_n^*$ with $u \sim v$, and let $i \in \{1, \dots, n\}$. Then, u has an i -inversion if and only if v has an i -inversion.*

Proof. If u has an i -inversion, then $\check{e}_i(u) = +\infty$. Since $u \sim v$, then $\check{e}_i(v) = \check{e}_i(u) = +\infty$, which implies that v has an i -inversion.

The converse follows from the fact that $u \sim v$ implies $v \sim u$. □

This result is analogous to one obtained for the classical hypoplactic monoid in Lemma 8.28, where either all words in a congruence class of the hypoplactic congruence have an i -inversion, or all of them are i -inversion-free ($i \in \{1, \dots, n\}$).

Also, the quasi-crystal graph $\Gamma_{\mathfrak{C}_n^*}^*$ can be obtained from the crystal graph $\Gamma_{\mathfrak{C}_n^*}$ of the free crystal monoid $\mathfrak{C}_n^*$ as we constructed the quasi-crystal graph $\Gamma_{\mathfrak{A}_n^*}^*$ from the crystal graph $\Gamma_{\mathfrak{A}_n^*}$ (see Sections 6.4 and 8.5) By Propositions 5.2 and 8.3, when the quasi-crystal structure maps of $\mathfrak{C}_n^*$ are defined on a word w , then the crystal structure maps of $\mathfrak{C}_n^*$ are also defined on w , and both structures coincide. Therefore, for each $i \in \{1, \dots, n\}$, if we remove the i -labelled edges starting or ending on words with i -inversions from $\Gamma_{\mathfrak{C}_n^*}^*$, and then add i -labelled loops to each word with an i -inversion, the resulting graph is $\Gamma_{\mathfrak{C}_n^*}^*$.

Note that a word w in C_n^* may have unbarred numbers and barred numbers. For the sake of simplicity, we introduce the following notation.

Definition 9.29. Set $\bar{\epsilon} = \epsilon$, and for each $x \in \{1, 2, \dots, n\}$, $\bar{\bar{x}} = x$. Given a word $w = x_1 x_2 \dots x_m$, with $x_1, x_2, \dots, x_m \in C_n$, set $\bar{w} = \overline{x_m x_{m-1} \dots x_1}$.

This notation also preserves inversions on words.

Proposition 9.30. Let $w \in C_n^*$ and $i \in \{1, 2, \dots, n\}$. Then, w has an i -inversion if and only if $\bar{w}$ has an i -inversion.

Proof. If w has an i -inversion, then $w = w_1 x w_2 y w_3$, where $w_1, w_2, w_3 \in C_n^*$, $x \in \{i, \bar{i} + 1\}$ and $y \in \{i + 1, \bar{i}\}$. Thus, $\bar{w} = \overline{w_3 y w_2 x w_1}$, where $\bar{x} \in \{\bar{i}, i + 1\}$ and $\bar{y} \in \{\bar{i} + 1, i\}$, and so, $\bar{x}$ has an i -inversion.

Conversely, if $\bar{w}$ has an i -inversion, as $\bar{\bar{w}} = w$, we deduce that w has an i -inversion, by the proof of the direct implication. $\square$

In quasi-crystal monoid $\mathcal{C}_n^*$ we have the following relation between a word $w \in C_n^*$ and its barred version $\bar{w}$.

Proposition 9.31. Let $w \in C_n^*$. Then, $\text{wt}(\bar{w}) = -\text{wt}(w)$, and for $i \in \{1, 2, \dots, n\}$, $\bar{\epsilon}_i(\bar{w}) = \bar{\varphi}_i(w)$ and $\bar{\varphi}_i(\bar{w}) = \bar{\epsilon}_i(w)$. Furthermore, if $\bar{\epsilon}_i(w)$ is defined, then $\bar{\epsilon}_i(w) = \bar{\bar{f}}_i(\bar{w})$, and if $\bar{\bar{f}}_i(w)$ is defined, then $\bar{\bar{f}}_i(w) = \bar{\bar{\epsilon}}_i(\bar{w})$.

Proof. If $w = \epsilon$, then $\bar{w} = w$, $\text{wt}(w) = 0$, and $\bar{\epsilon}_i(w) = \bar{\varphi}_i(w) = 0$, for all $i \in \{1, 2, \dots, n\}$, and so, the result follows.

For letters the result is also immediate from the quasi-crystal structure of $\mathcal{C}_n$.

Let $w = x_1 \dots x_m$, for some $x_1, \dots, x_m \in C_n$, where

$$\text{wt}(\bar{w}) = \text{wt}(\overline{x_m}) + \dots + \text{wt}(\overline{x_1}) = -\text{wt}(x_m) - \dots - \text{wt}(x_1) = -\text{wt}(w),$$

and for $i \in \{1, 2, \dots, n\}$, from Definition 9.26 it follows that w has an i -inversion if and only if $\bar{w}$ has an i -inversion, and if so, $\bar{\varphi}_i(\bar{w}) = \bar{\epsilon}_i(w) = +\infty$. So, assume w and $\bar{w}$ are i -inversion-free. Then,

$$\bar{\varphi}_i(\bar{w}) = |\bar{w}|_i + |\bar{w}|_{\bar{i}+1} = |w|_{\bar{i}} + |w|_{i+1} = \bar{\epsilon}_i(w).$$

If $\bar{\epsilon}_i(w)$ is defined, then $\bar{\epsilon}_i(w) \in \mathbb{Z}_{>0}$ and $\bar{\epsilon}_i(w) = w_1 \bar{\epsilon}_i(x) w_2$, for $w_1, w_2 \in C_n^*$ and $x \in \{i + 1, \bar{i}\}$ are such that $w = w_1 x w_2$ and $|w_2|_{i+1} = |w_2|_{\bar{i}} = 0$. Thus, $\bar{\varphi}_i(\bar{w}) \in \mathbb{Z}_{>0}$, and since $\bar{w} = \overline{w_2 x w_1}$, where $\bar{x} \in \{\bar{i}, i + 1\}$ and $|\bar{w}_2|_i = |\bar{w}_2|_{\bar{i}+1} = 0$, we have that

$$\begin{aligned} \bar{\bar{f}}_i(\bar{w}) &= \overline{w_2 \bar{\bar{f}}_i(\bar{x}) w_1} = \overline{w_2 \bar{\epsilon}_i(x) w_1} \\ &= \overline{w_1 \bar{\epsilon}_i(x) w_2} = \bar{\bar{\epsilon}}_i(w). \end{aligned}$$

This implies $\bar{\bar{f}}_i(\bar{w}) = \bar{\bar{\epsilon}}_i(w)$.

Analogously, we deduce that $\bar{\epsilon}_i(\bar{w}) = \bar{\varphi}_i(w)$, and if $\bar{\bar{f}}_i(w)$ is defined, then $\bar{\bar{\epsilon}}_i(\bar{w}) = \bar{\bar{f}}_i(w)$. $\square$

From the previous result, given $u, v \in C_n^*$, we have an edge $u \xrightarrow{i} v$ in the quasi-crystal graph $\Gamma_{\mathfrak{C}_n^*}$ if and only if we have an edge $\bar{v} \xrightarrow{i} \bar{u}$.

9.2.2 Highest-weight words

Note that the arguments in the proof of Proposition 9.7 can be used to show that each connected component of $\mathfrak{C}_n^*$ have at least one highest-weight word and at least one lowest-weight word (Definition 7.6). Thus, in this section we characterize the highest-weight words of $\mathfrak{C}_n^*$.

Proposition 9.32. *Let $w \in C_n^*$. Then, w is of highest weight if and only if for each letter $z \geq 2$ occurring in w , there exists a decomposition $w = w_1 x w_2 y w_3$ with $w_1, w_2, w_3 \in C_n^*$ either $x = \bar{z}$ or $x = \check{e}_i(z)$, and either $y = z$ or $y = \check{f}_i(\bar{z})$, where $i \in \{1, 2, \dots, n\}$ is such that $z \in \{i+1, \bar{i}\}$.*

Proof. Let $z \geq 2$ be a letter occurring in w . Set $i \in \{1, 2, \dots, n\}$ such that $z \in \{i+1, \bar{i}\}$. Since w is highest-weight, then $\check{e}_i(w)$ is not defined, and since $\check{e}_i(w) \geq |w|_{i+1} + |w|_{\bar{i}} > 0$, we obtain that $\check{e}_i(w) = +\infty$. Hence, $w = w_1 x w_2 y w_3$, for some $w_1, w_2, w_3 \in C_n^*$, $x \in \{i, \bar{i}+1\}$ and $y \in \{i+1, \bar{i}\}$. If $z = i+1$, then $\check{e}_i(z) = i$ and $\check{f}_i(\bar{z}) = \bar{i}$, and so, $x \in \{\check{e}_i(z), \bar{z}\}$ and $y \in \{z, \check{f}_i(\bar{z})\}$. Otherwise, $z = \bar{i}$, and we have that $\check{e}_i(z) = \bar{i}+1$ and $\check{f}_i(\bar{z}) = i+1$, which implies that $x \in \{\check{e}_i(z), \bar{z}\}$ and $y \in \{z, \check{f}_i(\bar{z})\}$.

Conversely, let $i \in \{1, 2, \dots, n\}$. Suppose that $\check{e}_i(w)$ is defined. Then, $\check{e}_i(w) = u_1 \check{e}_i(z) u_2$, where $w = u_1 z u_2$, for some $u_1, u_2 \in C_n^*$ and $z \in C_n$. Since $\check{e}_i(z)$ is defined, we have that $z \geq 2$, and so, there exists $i \in \{1, 2, \dots, n\}$ such that $z \in \{i+1, \bar{i}\}$. By hypothesis, we have a decomposition $w = w_1 x w_2 y w_3$ with $w_1, w_2, w_3 \in C_n^*$, $x \in \{\check{e}_i(z), \bar{z}\}$ and $y \in \{z, \check{f}_i(\bar{z})\}$. As shown above, we have that $x \in \{i, \bar{i}+1\}$ and $y \in \{i+1, \bar{i}\}$, and thus, $\check{f}_i(x)$ and $\check{e}_i(y)$ are defined, which implies that $\check{e}_i(w) = +\infty$. A contradiction, because we supposed that $\check{e}_i(w)$ was defined. Therefore, w is highest-weight. $\square$

Example 9.33. The following words are highest weight: $1, 12, 1\bar{1}, 2\bar{2}\bar{1}, x\bar{x}x, \bar{x}x\bar{x}$

Note that the word $12\dots n\bar{n}\dots \bar{2}\bar{1}$ has weight 0 and i -inversions for each $i = 1, \dots, n$. So, we obtain the following result.

Proposition 9.34. *Any weight $\lambda \in \mathbb{Z}^n$ is a highest weight in $\mathfrak{C}_n^*$.*

Proof. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$. For each $i \in \{1, 2, \dots, n\}$, set

$$a_i = \begin{cases} \lambda_i & \text{if } \lambda_i \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$a_{\bar{i}} = \begin{cases} -\lambda_i & \text{if } \lambda_i \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The word $w = 1^{a_1+1} 2^{a_2+1} \dots n^{a_n+1} \overline{n}^{a_{\overline{n}}+1} (\overline{n-1})^{a_{\overline{n-1}}+1} \dots \overline{1}^{a_{\overline{1}}+1}$ is such that $\varepsilon_i(w) = +\infty$, for all $i \in \{1, 2, \dots, n\}$ and

$$\text{wt}(w) = (a_1 + 1 - a_{\overline{1}} - 1, \dots, a_n + 1 - a_{\overline{n}} - 1) = (\lambda_1, \dots, \lambda_n) = \lambda,$$

because $a_i - a_{\bar{i}} = \lambda_i$, for all $i = 1, 2, \dots, n$. □

The word $12 \dots n \overline{n} (\overline{n-1}) \dots \overline{1}$ is also lowest weight, that is, $\check{f}_i$ is undefined for all $i = 1, 2, \dots, n$. So, this word is an isolated element of $\mathcal{C}_n^*$. From a combinatorial point of view, it is an isolated vertex in the quasi-crystal graph $\Gamma_{\mathcal{C}_n^*}$.

9.2.3 Isolated words

To deduce some relations on the hypoplactic monoid $\text{hypo}(\mathcal{C}_n)$, we now look on isolated elements (Definition 7.31), also called isolated words, of $\mathcal{C}_n^*$. By Theorems 8.25 and 8.26 we have that the central elements of the hypoplactic monoid $\text{hypo}(\mathcal{C}_n)$ correspond to the isolated words of $\mathcal{C}_n^*$. In this section we identify the isolated words, and consequently, obtain some equalities in $\text{hypo}(\mathcal{C}_n)$. We also describe in Theorem 9.39 a bijection between the central elements of $\text{hypo}(\mathcal{C}_n)$ and a pair of vectors (λ, δ) . Furthermore, in the proof of this theorem we present an algorithm to compute all central elements of $\text{hypo}(\mathcal{C}_n)$.

An isolated word $w \in C_n^*$ corresponds to an isolated vertex in the quasi-crystal graph $\Gamma_{\mathcal{C}_n^*}$. In other words, w is of highest- and of lowest weight in the quasi-crystal $\mathcal{C}_n^*$, that is, $\check{e}_i$ and $\check{f}_i$ are undefined on w , for all $i \in \{1, \dots, n\}$. Thus, we get the following description.

Proposition 9.35. *Let $w \in C_n^*$. Then, w is an isolated word if and only if w has an $(i-1)$ -inversion and an i -inversion, for all $i \in \{1, \dots, n\}$ such that i or $\bar{i}$ occur in w .*

Proof. Let $i \in \{1, 2, \dots, n\}$. If i occurs in w , then $\check{\varphi}_i(w) > 0$, and if also $i \geq 2$, $\check{\varepsilon}_{i-1}(w) > 0$. Since w is an isolated word, $\check{f}_i$ and $\check{e}_{i-1}$ are undefined on w , which implies that $\check{\varphi}_i(w) = +\infty$ and $\check{\varepsilon}_{i-1}(w) = +\infty$. Therefore, w has an $(i-1)$ -inversion and an i -inversion.

If $\bar{i}$ occurs in w , then $\check{\varepsilon}_i(w) > 0$, and if also $i \geq 2$, $\check{\varphi}_{i-1}(w) > 0$. This implies that $\check{\varepsilon}_i(w) = +\infty$ and $\check{\varphi}_{i-1}(w) = +\infty$, because w is an isolated word. And so, w has an $(i-1)$ -inversion and an i -inversion.

Conversely, let $i \in \{1, 2, \dots, n\}$. If i , $i+1$, $\overline{i+1}$ or $\overline{i}$ occur in w , then w has an i -inversion, and so, $\check{e}_i$ and $\check{f}_i$ are undefined on w . Otherwise, we have that $\check{e}_i(w) = |w|_{i+1} + |w|_{\overline{i}} = 0$ and $\check{\varphi}_i(w) = |w|_i + |w|_{\overline{i+1}} = 0$, and so, $\check{e}_i$ and $\check{f}_i$ are undefined on w . This implies that $\check{e}_i$ and $\check{f}_i$ are undefined on w , for all $i \in \{1, \dots, n\}$. Hence, w is an isolated word. $\square$

Example 9.36. The following are isolated words: $1\overline{1}$, $12\overline{2}$, $22\overline{1}$, $3\overline{3}$, $33\overline{3}$.

Proposition 9.37. Let $w \in C_n^*$. Then, w is an isolated word if and only if $\overline{w}$ is an isolated word.

Proof. Given $i \in \{1, 2, \dots, n\}$, we have that i or $\overline{i}$ occur in w if and only if i or $\overline{i}$ occur in $\overline{w}$. Therefore, this result follows from Proposition 9.35. $\square$

For each word $w \in C_n^*$, define a vector $\text{inv}(w) = (\delta_1, \delta_2, \dots, \delta_n) \in \{0, 1\}^n$ where $\delta_i = 1$ if and only if w has an i -inversion. In other words,

$$\delta_i = \begin{cases} 0 & \text{if } \check{e}_i(w) \in \mathbb{Z} \\ 1 & \text{if } \check{e}_i(w) = +\infty. \end{cases}$$

Lemma 9.38. Let $u, v \in C_n^*$ be isolated words. Then, $u \sim v$ if and only if $\text{wt}(u) = \text{wt}(v)$ and $\text{inv}(u) = \text{inv}(v)$.

Proof. By Definition 8.21, $u \sim v$ if there exists a quasi-crystal isomorphism $\psi : \mathcal{C}_n^*(u) \rightarrow \mathcal{C}_n^*(v)$ such that $\psi(u) = v$. Since u and v are isolated words, we have that $C_n^*(u) = \{u\}$ and $C_n^*(v) = \{v\}$. Thus, there exists a quasi-crystal isomorphism between $\mathcal{C}_n^*(u)$ and $\mathcal{C}_n^*(v)$ if and only if $\text{wt}(u) = \text{wt}(v)$, $\check{e}_i(u) = \check{e}_i(v)$ and $\check{\varphi}_i(u) = \check{\varphi}_i(v)$, for all $i \in \{1, 2, \dots, n\}$. Let $\delta_1, \delta_2, \dots, \delta_n \in \{0, 1\}$ such that $\text{inv}(u) = (\delta_1, \delta_2, \dots, \delta_n)$. Since u is an isolated word, for each $i \in \{1, 2, \dots, n\}$, either $\check{e}_i(u) = \check{\varphi}_i(u) = 0$ and $\delta_i = 0$, or $\check{e}_i(u) = \check{\varphi}_i(u) = +\infty$ and $\delta_i = 1$. Similarly, $\check{e}_i(v) = \check{\varphi}_i(v) \in \{0, +\infty\}$, where the value of $\check{e}_i(v)$ is determined by $\text{inv}(v)$, $i \in \{1, 2, \dots, n\}$. Therefore, $u \sim v$ if and only if $\text{wt}(u) = \text{wt}(v)$ and $\text{inv}(u) = \text{inv}(v)$. $\square$

By Theorems 8.25 and 8.26, we have that central elements of $\text{hypo}(\mathcal{C}_n)$ correspond to isolated words in $\mathcal{C}_n^*$. Thus, we obtain the following result.

Theorem 9.39. The map that sends each isolated word $w \in C_n^*$ to $(\text{wt}(w), \text{inv}(w))$ is a bijection between the set of central elements of $\text{hypo}(\mathcal{C}_n)$ and the set of pairs (λ, δ) with $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$ and $\delta = (\delta_1, \delta_2, \dots, \delta_n) \in \{0, 1\}^n$ verifying

- (i) if $\lambda_i \neq 0$, with $i \in \{1, 2, \dots, n\}$, then $\delta_{i-1} = \delta_i = 1$; and
- (ii) if $\delta_i = 1$, with $i \in \{2, 3, \dots, n\}$, then $\delta_{i-1} = 1$ or $\delta_{i+1} = 1$.

Proof. Let $w \in C_n^*$ be an isolated word of $\mathcal{C}_n^*$. Consider $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{Z}$ and $\delta_1, \delta_2, \dots, \delta_n \in \{0, 1\}$ such that $\text{wt}(w) = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\text{inv}(w) = (\delta_1, \delta_2, \dots, \delta_n)$. If $\lambda_i \neq 0$, for some $i \in \{1, 2, \dots, n\}$, then i or $\bar{i}$ occur in w implying that w has an $(i-1)$ -inversion and an i -inversion, by Proposition 9.35, and so, $\delta_{i-1} = \delta_i = 1$.

If $\delta_i = 1$, for some $i \in \{2, 3, \dots, n\}$, then w has an i -inversion, and by Proposition 9.35, w has an $(i-1)$ -inversion or an $(i+1)$ -inversion, which implies that $\delta_{i-1} = 1$ or $\delta_{i+1} = 1$. Note that if $i = n$, then $\delta_{n-1} = 1$, as $n+1$ lies outside of the range $\{1, 2, \dots, n\}$.

Therefore, by Lemma 9.38, the map that sends each isolated word $u \in C_n^*$ to $(\text{wt}(u), \text{inv}(u))$ is well-defined on $\text{hypo}(\mathcal{C}_n)$, because if $u \sim v$, where u and v are isolated words, then $\text{wt}(u) = \text{wt}(v)$ and $\text{inv}(u) = \text{inv}(v)$. Also, by Lemma 9.38, we have that ψ is injective.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$ and $\delta = (\delta_1, \delta_2, \dots, \delta_n) \in \{0, 1\}^n$ satisfying conditions (i) and (ii). The following algorithm returns a word w such that $\text{wt}(w) = \lambda$ and $\text{inv}(w) = \delta$.

- (1) Set $w := \epsilon$, $i := 0$, $\delta_0 := 0$ and $\delta_{n+1} := 0$.
- (2) If $\delta_1 = 1$ and $\delta_2 = 0$, set $w := w1\bar{1}$ and $i := 1$;
- (3) While there exists $j \leq n$ such that $j > i$ and $\delta_j = 1$, do:
 - (a) Set j_1 and j_2 such that $i < j_1 < j_2 \leq n$, $\delta_{i+1} = \dots = \delta_{j_1-1} = 0$, $\delta_{j_1} = \dots = \delta_{j_2} = 1$ and $\delta_{j_2+1} = 0$;
 - (b) Set $w := w(\overline{j_1+1})(j_1+1)(j_1+2) \dots j_2 \overline{j_2}(j_2-1) \dots (\overline{j_1+1})(j_1+1)$ and $i := j_2$;
- (4) For $k = 1, 2, \dots, n$, do:
 - (a) If $\lambda_k > 0$, set $w := wk^{\lambda_k}$;
 - (b) If $\lambda_k < 0$, set $w := w\bar{k}^{-\lambda_k}$;
- (5) Return w .

Thus, w is an isolated vertex with $\text{wt}(w) = \lambda$ and $\text{inv}(w) = \delta$. □

9.2.4 Knuth relations on $\text{hypo}(\mathcal{C}_n)$

In this section we study whether the hypoplactic monoid satisfies Knuth relations $R^{\text{plac}}(\mathcal{A}_n) = R_1^{\text{plac}}(\mathcal{A}_n) \cup R_2^{\text{plac}}(\mathcal{A}_n)$ (Definition 3.6). In fact, we show that $R^{\text{plac}}(\mathcal{A}_n)$ induces identities which are not verified for almost any elements of $\text{hypo}(\mathcal{C}_n)$.

Lemma 9.40. *Let $u, v \in C_n^*$. Then, $u \sim v$ if and only if $\bar{u} \sim \bar{v}$.*

Proof. First, we show that for any $w \in C_n^*$, the map from $C_n^*(w)$ to $C_n^*(\overline{w})$, defined by $w' \mapsto \overline{w'}$, for each $w' \in C_n^*(w)$, is a bijection. Given $g \in \{\check{e}_i, \check{f}_i \mid i = 1, 2, \dots, n\}$, set $\overline{g} = \check{f}_i$, if $g = \check{e}_i$, or $\overline{g} = \check{e}_i$, if $g = \check{f}_i$, $i \in \{1, 2, \dots, n\}$. For any $w \in C_n^*$, By Proposition 9.31, we have that $g(w)$ is defined if and only if $\overline{g}(\overline{w})$ is defined. And if so, $\overline{g}(\overline{w}) = \overline{g(w)}$. Applying this recursively, for $g_1, \dots, g_m \in \{\check{e}_i, \check{f}_i \mid i = 1, 2, \dots, n\}$, we obtain that $g_1 \cdots g_m(w)$ is defined if and only if $\overline{g_1} \cdots \overline{g_m}(\overline{w})$ is defined. And if so,

$$\overline{g_1} \cdots \overline{g_m}(\overline{w}) = \overline{g_1 \cdots g_m(w)}.$$

Thus,

$$C_n^*(\overline{w}) = \{\overline{w'} \mid w' \in C_n^*(w)\}.$$

Also, as $\overline{\overline{w'}} = w'$, for any $w' \in C_n^*$, we get that the map from $C_n^*(w)$ to $C_n^*(\overline{w})$, defined by $w' \mapsto \overline{w'}$, for each $w' \in C_n^*(w)$, is a bijection.

Assume that $u \sim v$. Let $\psi : \mathcal{C}_n^*(u) \rightarrow \mathcal{C}_n^*(v)$ be a quasi-crystal isomorphism such that $\psi(u) = v$. Define $\psi' : C_n^*(\overline{u}) \rightarrow C_n^*(\overline{v})$ by $\psi'(w) = \overline{\psi(\overline{w})}$, for each $w \in C_n^*(\overline{u})$. Since ψ' can be obtained as the composition of bijections $C_n^*(\overline{u}) \rightarrow C_n^*(u) \rightarrow C_n^*(v) \rightarrow C_n^*(\overline{v})$, where $w \mapsto \overline{w} \mapsto \psi(\overline{w}) \mapsto \overline{\psi(\overline{w})} = \psi'(w)$, for each $w \in C_n^*(\overline{u})$, we have that ψ' is bijective. Let $w \in C_n^*(\overline{u})$. As ψ is a quasi-crystal isomorphism and by Proposition 9.31, we get that

$$\text{wt}(\psi'(w)) = \text{wt}(\overline{\psi(\overline{w})}) = -\text{wt}(\psi(\overline{w})) = -\text{wt}(\overline{w}) = \text{wt}(w),$$

and for $i \in \{1, 2, \dots, n\}$,

$$\check{e}_i(\psi'(w)) = \check{e}_i(\overline{\psi(\overline{w})}) = \check{\varphi}_i(\psi(\overline{w})) = \check{\varphi}_i(\overline{w}) = \check{e}_i(w)$$

and

$$\check{\varphi}_i(\psi'(w)) = \check{\varphi}_i(\overline{\psi(\overline{w})}) = \check{e}_i(\psi(\overline{w})) = \check{e}_i(\overline{w}) = \check{\varphi}_i(w).$$

Also, $\check{f}_i(w)$ is defined if and only if $\check{f}_i(\psi'(w))$, because $\check{\varphi}_i(\psi'(w)) = \check{\varphi}_i(w)$. And if so,

$$\psi'(\check{f}_i(w)) = \overline{\psi(\check{f}_i(\overline{w}))} = \overline{\psi(\check{e}_i(\overline{w}))} = \overline{\check{e}_i(\psi(\overline{w}))} = \check{f}_i(\overline{\psi(\overline{w})}) = \check{f}_i(\psi'(w)).$$

Analogously, $\psi'(\check{e}_i(w)) = \check{e}_i(\psi'(w))$, whenever $\check{e}_i(w)$ is defined. Therefore, ψ' is a quasi-crystal isomorphism between $\mathcal{C}_n^*(\overline{u})$ and $\mathcal{C}_n^*(\overline{v})$, where

$$\psi'(\overline{u}) = \overline{\psi(\overline{\overline{u}})} = \overline{\psi(u)} = \overline{v}.$$

Hence, $\overline{u} \sim \overline{v}$.

Conversely, if $\overline{u} \sim \overline{v}$, as $u = \overline{\overline{u}}$ and $v = \overline{\overline{v}}$, the result follows from the direct implication proved above. $\square$

Note that $1\bar{1}$ is an isolated word of $\mathcal{C}_n^*$. Thus, by Theorem 8.25, we have that $1\bar{1}x \sim 1x\bar{1} \sim x1\bar{1}$, for any $x \in C_n$. However, this is a particular case which does not hold in general, as shown in the following result.

Theorem 9.41. *For any $x, y \in C_n$ with $x \neq y$ and $(x, y) \neq (1, \bar{1})$, we have that*

(i) $xyx \not\sim xxy$;

(ii) $xyy \not\sim yxy$.

Proof. Let $x, y \in C_n$ with $x \neq y$ and $(x, y) \neq (1, \bar{1})$. Set $i, j \in \{1, 2, \dots, n\}$ such that $x \in \{i, \bar{i}\}$ and $y \in \{j, \bar{j}\}$.

(i) We consider the following cases.

- Case 1: $x < j$. So, $x = i$, as $j \leq n$. In the quasi-crystal graph, we have that

$$\begin{aligned} iji &\xrightarrow{j} i(j+1)i \xrightarrow{j+1} \dots \xrightarrow{n} i\bar{n}i \xrightarrow{n-1} \dots \xrightarrow{j} i\bar{j}i \xrightarrow{j-1} \dots \\ &\dots \xrightarrow{i+1} i(\bar{i}+1)i \xrightarrow{i} (i+1)(\bar{i}+1)i \xrightarrow{i} (i+1)\bar{i}i \end{aligned}$$

and

$$\begin{aligned} iij &\xrightarrow{j} ii(j+1) \xrightarrow{j+1} \dots \xrightarrow{n} ii\bar{n} \xrightarrow{n-1} \dots \xrightarrow{j} ii\bar{j} \xrightarrow{j-1} \dots \\ &\dots \xrightarrow{i+1} ii(\bar{i}+1) \xrightarrow{i} (i+1)i(\bar{i}+1) \xrightarrow{i} (i+1)(i+1)(\bar{i}+1). \end{aligned}$$

Since $\ddot{\varphi}_{i-1}((i+1)\bar{i}i) = +\infty$ and $\ddot{\varphi}_{i-1}((i+1)(i+1)(\bar{i}+1)) = 0$, then $(i+1)\bar{i}i \not\sim (i+1)(i+1)(\bar{i}+1)$, which implies that $iji \not\sim iij$ and $i\bar{j}i \not\sim ii\bar{j}$. As either $y = j$ or $y = \bar{j}$, we get that $xyx \not\sim xxy$.

- Case 2: $\bar{x} < j$. Thus, $x = \bar{i}$. In the quasi-crystal graph, we have that

$$\begin{aligned} \bar{i}\bar{j}\bar{i} &\xrightarrow{j} \bar{i}(j+1)\bar{i} \xrightarrow{j+1} \dots \xrightarrow{n-1} \bar{i}\bar{n}\bar{i} \xrightarrow{n} \bar{i}\bar{n}\bar{i} \xrightarrow{n-1} \dots \\ &\dots \xrightarrow{j} \bar{i}\bar{j}\bar{i} \xrightarrow{j-1} \dots \xrightarrow{i+1} \bar{i}(\bar{i}+1)\bar{i} \end{aligned}$$

and

$$\begin{aligned} \bar{i}\bar{i}\bar{j} &\xrightarrow{j} \bar{i}(j+1) \xrightarrow{j+1} \dots \xrightarrow{n-1} \bar{i}\bar{i}\bar{n} \xrightarrow{n} \bar{i}\bar{i}\bar{n} \xrightarrow{n-1} \dots \\ &\dots \xrightarrow{j} \bar{i}\bar{i}\bar{j} \xrightarrow{j-1} \dots \xrightarrow{i+1} \bar{i}\bar{i}(\bar{i}+1). \end{aligned}$$

Since $\ddot{\varphi}_i(\bar{i}(\bar{i}+1)\bar{i}) = +\infty$ and $\ddot{\varphi}_i(\bar{i}\bar{i}(\bar{i}+1)) = 1$, then $\bar{i}(\bar{i}+1)\bar{i} \not\sim \bar{i}\bar{i}(\bar{i}+1)$, which implies that $\bar{i}\bar{j}\bar{i} \not\sim \bar{i}\bar{i}\bar{j}$ and $\bar{i}\bar{j}\bar{i} \not\sim \bar{i}\bar{i}\bar{j}$. As either $y = j$ or $y = \bar{j}$, we get that $xyx \not\sim xxy$.

- Case 3: $x = \bar{y} \leq n$. So, $x = i$ and $y = \bar{i}$. As $(x, y) \neq (1, \bar{1})$, we have that $i \geq 2$. Since $\check{\varphi}_{i-1}(i\bar{i}i) = +\infty$ and $\check{\varphi}_{i-1}(i\bar{i}\bar{i}) = 1$, then $xyx = i\bar{i}i \not\sim i\bar{i}\bar{i} = xxy$.
- Case 4: $x = \bar{y} \geq n$. Thus, $y = j$ and $x = \bar{j}$. Since $\check{\varphi}_j(\bar{j}jj) = +\infty$ and $\check{\varphi}_j(j\bar{j}j) = 1$, then $xyx = \bar{j}jj \not\sim j\bar{j}j = xxy$.
- Case 5: $y < i$. So, $y = j$. In the quasi-crystal graph, we have that

$$\begin{aligned} (j+1)j(j+1) &\xrightarrow{j+1} (j+2)j(j+1) \xrightarrow{j+1} (j+2)j(j+2) \xrightarrow{j+2} \dots \\ \dots &\xrightarrow{i-2} (i-1)j(i-1) \xrightarrow{i-1} ij(i-1) \xrightarrow{i-1} iji \xrightarrow{i} \dots \\ \dots &\xrightarrow{n} \bar{n}jn \xrightarrow{n} \bar{n}j\bar{n} \xrightarrow{n-1} \dots \xrightarrow{i} \bar{i}j(\bar{i}+1) \xrightarrow{i} \bar{i}j\bar{i} \end{aligned}$$

and

$$\begin{aligned} (j+1)(j+1)j &\xrightarrow{j+1} (j+2)(j+1)j \xrightarrow{j+1} (j+2)(j+2)j \xrightarrow{j+2} \dots \\ \dots &\xrightarrow{i-2} (i-1)(i-1)j \xrightarrow{i-1} i(i-1)j \xrightarrow{i-1} iij \xrightarrow{i} \dots \\ \dots &\xrightarrow{n} \bar{n}nj \xrightarrow{n} \bar{n}n\bar{j} \xrightarrow{n-1} \dots \xrightarrow{i} \bar{i}(\bar{i}+1)j \xrightarrow{i} \bar{i}\bar{i}j. \end{aligned}$$

Since $\check{\varphi}_j((j+1)j(j+1)) = +\infty$ and $\check{\varphi}_j((j+1)(j+1)j) = 1$, then $(j+1)j(j+1) \not\sim (j+1)(j+1)j$, which implies that $iji \not\sim iij$ and $\bar{i}j\bar{i} \not\sim \bar{i}\bar{i}j$. As either $x = i$ or $x = \bar{i}$, we get that $xyx \not\sim xxy$.

- Case 6: $\bar{y} < i$. Thus, $y = \bar{j}$. In the quasi-crystal graph, we have that

$$\begin{aligned} (j+1)\bar{j}j &\xrightarrow{j} (j+1)\bar{j}(j+1) \xrightarrow{j+1} (j+2)\bar{j}(j+1) \xrightarrow{j+1} \dots \\ \dots &\xrightarrow{i-2} (i-1)\bar{j}(i-1) \xrightarrow{i-1} i\bar{j}(i-1) \xrightarrow{i-1} i\bar{j}i \xrightarrow{i} \dots \\ \dots &\xrightarrow{n} \bar{n}j\bar{n} \xrightarrow{n} \bar{n}\bar{j}\bar{n} \xrightarrow{n-1} \dots \xrightarrow{i} \bar{i}j(\bar{i}+1) \xrightarrow{i} \bar{i}j\bar{i} \end{aligned}$$

and

$$\begin{aligned} (j+1)(j+1)(\bar{j}+1) &\xrightarrow{j} (j+1)(j+1)\bar{j} \xrightarrow{j+1} (j+2)(j+1)\bar{j} \xrightarrow{j+1} \dots \\ \dots &\xrightarrow{i-2} (i-1)(i-1)\bar{j} \xrightarrow{i-1} i(i-1)\bar{j} \xrightarrow{i-1} i\bar{i}\bar{j} \xrightarrow{i} \dots \\ \dots &\xrightarrow{n} \bar{n}n\bar{j} \xrightarrow{n} \bar{n}\bar{n}\bar{j} \xrightarrow{n-1} \dots \xrightarrow{i} \bar{i}(\bar{i}+1)\bar{j} \xrightarrow{i} \bar{i}\bar{i}\bar{j}. \end{aligned}$$

Since $\check{\varphi}_{j+1}((j+1)\bar{j}j) = 1$ and $\check{\varphi}_{j+1}((j+1)(j+1)(\bar{j}+1)) = +\infty$, then $(j+1)\bar{j}j \not\sim (j+1)(j+1)(\bar{j}+1)$, which implies that $i\bar{j}i \not\sim i\bar{i}\bar{j}$ and $\bar{i}j\bar{i} \not\sim \bar{i}\bar{i}\bar{j}$. As either $x = i$ or $x = \bar{i}$, we get that $xyx \not\sim xxy$.

(ii) By Lemma 9.40, we have that $xyy \sim yxy$ if and only if $\overline{y}y\overline{x} \sim \overline{y}x\overline{y}$. Therefore, the result follows from (i). $\square$

Corollary 9.42. (i) For $m, n \geq 2$, there exists no injective monoid homomorphism $\psi : \text{hypo}(\mathcal{A}_m) \rightarrow \text{hypo}(\mathcal{C}_n)$ such that $\psi(x), \psi(y) \in C_n$, for some $x, y \in A_m$ with $x \neq y$.

(ii) For $m \geq 3$ and $n \geq 2$, no injective map from A_m to C_n can be extended to a monoid homomorphism from $\text{hypo}(\mathcal{A}_m)$ to $\text{hypo}(\mathcal{C}_n)$.

Proof. (i) Suppose $\psi : \text{hypo}(\mathcal{A}_m) \rightarrow \text{hypo}(\mathcal{C}_n)$ is an injective monoid homomorphism such that $\psi(x), \psi(y) \in C_n$, for some $x, y \in A_m$ with $x \neq y$. Without loss of generality assume $x < y$. If $\psi(x) = 1$ and $\psi(y) = \overline{1}$, then

$$\psi(xyxyxy) = 1\overline{1}1\overline{1}1\overline{1} = 1\overline{1}1\overline{1} = \psi(xyxy),$$

a contradiction as ψ is injective. Otherwise, we have that $(\psi(x), \psi(y)) \neq (1, \overline{1})$, and since $xyx = xxy$ in $\text{hypo}(\mathcal{A}_m)$, we get that $\psi(x)\psi(y)\psi(x) = \psi(x)\psi(x)\psi(y)$ in $\text{hypo}(\mathcal{C}_n)$, which is impossible by Theorem 9.41(i).

(ii) Suppose $\psi : \text{hypo}(\mathcal{A}_m) \rightarrow \text{hypo}(\mathcal{C}_n)$ is a monoid homomorphism where its restriction $\psi|_{A_m}$ is an injective map from A_m to C_n . Take $y \in \{2, 3\}$ such that $\psi(y) \neq \overline{1}$. Then, we have that $(\psi(1), \psi(y)) \neq (1, \overline{1})$. Since $1y1 = 11y$ in $\text{hypo}(\mathcal{A}_m)$, we obtain that $\psi(1)\psi(y)\psi(1) = \psi(1)\psi(1)\psi(y)$ in $\text{hypo}(\mathcal{C}_n)$, which is impossible by Theorem 9.41(i). $\square$

Theorem 9.43. For $x, y, z \in C_n$ with $x \neq z$, we have that

(i) $yzx \sim yxz$ if and only if $(y, x) = (1, \overline{1})$ or $(y, z) = (1, \overline{1})$;

(ii) $xzy \sim zxy$ if and only if $(x, y) = (1, \overline{1})$ or $(z, y) = (1, \overline{1})$;

Proof. Let $x, y, z \in C_n$. Without loss of generality, assume that $x < z$. Set $i, j, k \in \{1, \dots, n\}$ such that $x \in \{i, \overline{i}\}$, $y \in \{j, \overline{j}\}$ and $z \in \{k, \overline{k}\}$.

(i) Since $1\overline{1}$ is an isolated element, then $1\overline{1}u \sim 1u\overline{1}$, for all $u \in C_n^*$. So, assume that $(y, x) \neq (1, \overline{1})$ and $(y, z) \neq (1, \overline{1})$. We have the following cases.

- Case 1: $y = x$ or $y = z$. As $(y, x) \neq (1, \overline{1})$ and $(y, z) \neq (1, \overline{1})$, we have that $zyy \not\sim yyz$ and $yyx \not\sim yxy$, by Theorem 9.41(i).
- Case 2: $z \leq n$, $y \neq x$, and $j \leq i$ or $j > k$. So, $x = i$ and $z = k$. In the quasi-crystal graph, we have that

$$yzx \xrightarrow{i} yz(x+1) \xrightarrow{i+1} \dots \xrightarrow{k-2} yz(z-1)$$

and

$$yxz \xrightarrow{i} y(x+1)z \xrightarrow{i+1} \dots \xrightarrow{k-2} y(z-1)z.$$

Since $\ddot{\varphi}_{k-1}(yz(z-1)) = 1$ and $\ddot{\varphi}_{k-1}(y(z-1)z) = +\infty$, then $yz(z-1) \not\sim y(z-1)z$, which implies that $yzx \not\sim yxz$.

- Case 3: $x < \bar{z} \leq n$, $y \neq x$, and $j \leq i$ or $j > k$. Thus, $x = i$ and $z = \bar{k}$. In the quasi-crystal graph, we have that

$$yzx \xrightarrow{k-1} y(\overline{k-1})x \xrightarrow{k-2} \dots \xrightarrow{i+1} y(\overline{x+1})x \xrightarrow{i} y\bar{x}x$$

and

$$yxz \xrightarrow{k-1} yx(\overline{k-1}) \xrightarrow{k-2} \dots \xrightarrow{i+1} yx(\overline{x+1}) \xrightarrow{i} y(x+1)(\overline{x+1}).$$

Since $\ddot{\varphi}_{i+1}(y\bar{x}x) = \delta_{y, i+2} \in \{0, 1\}$ and $\ddot{\varphi}_{i+1}(y(x+1)(\overline{x+1})) = +\infty$, then $y\bar{x}x \not\sim y(x+1)(\overline{x+1})$, which implies that $yzx \not\sim yxz$.

- Case 4: $z = \bar{x}$, $y \neq x$ and $y \neq z$. As $x < z$, we have that $x = i$ and $z = \bar{i}$. If $y \neq \overline{x+1}$, we have that $\ddot{\varphi}_i(y\bar{x}x) = 1$ and $\ddot{\varphi}_i(yx\bar{x}) = +\infty$, which implies that $y\bar{x}x \not\sim yx\bar{x}$. Otherwise, in the quasi-crystal graph we have that

$$n\bar{x}x \xrightarrow{n} \bar{n}x\bar{x} \xrightarrow{n-1} \dots \xrightarrow{i+1} (\overline{x+1})\bar{x}x$$

and

$$nx\bar{x} \xrightarrow{n} \bar{n}x\bar{x} \xrightarrow{n-1} \dots \xrightarrow{i+1} (\overline{x+1})x\bar{x},$$

and since $n \neq \overline{x+1}$, we obtain that $n\bar{x}x \not\sim nx\bar{x}$, which implies that $(\overline{x+1})\bar{x}x \not\sim (\overline{x+1})x\bar{x}$.

- Case 5: $\bar{z} < x \leq n$, and $j > i$ or $j < k$. Thus, $x = i$ and $z = \bar{k}$. In the quasi-crystal graph, we have that

$$y(\overline{x-1})(x-1) \xrightarrow{i-1} y(\overline{x-1})x \xrightarrow{i-2} y(\overline{x-2})x \xrightarrow{i-3} \dots \xrightarrow{k} yzx$$

and

$$yx\bar{x} \xrightarrow{i-1} yx(\overline{x-1}) \xrightarrow{i-2} yx(\overline{x-2}) \xrightarrow{i-3} \dots \xrightarrow{k} yxz.$$

Since $\ddot{\varphi}_i(y(\overline{x-1})(x-1)) = \delta_{y, i+1} \in \{0, 1\}$ and $\ddot{\varphi}_i(yx\bar{x}) = +\infty$, then $y(\overline{x-1})(x-1) \not\sim yx\bar{x}$, which implies that $yzx \not\sim yxz$.

- Case 6: $x \geq \bar{n}$, $y \neq x$, and $j \geq i$ or $j < k$. So, $x = \bar{i}$ and $z = \bar{k}$. In the quasi-crystal graph, we have that

$$yzx \xrightarrow{i-1} y\bar{k}(\overline{i-1}) \xrightarrow{i-2} \dots \xrightarrow{k+1} y\bar{k}(\overline{k+1})$$

and

$$yxz \xrightarrow{i-1} y(\overline{i-1})\bar{k} \xrightarrow{i-2} \dots \xrightarrow{k+1} y(\overline{k+1})\bar{k}.$$

Since $\ddot{\varphi}_k(y\bar{k}(\overline{k+1})) = 1$ and $\ddot{\varphi}_k(y(\overline{k+1})\bar{k}) = +\infty$, then $y\bar{k}(\overline{k+1}) \not\sim y(\overline{k+1})\bar{k}$, which implies that $yzx \not\sim yxz$.

- Case 7: $x < y \leq k$ and $y \neq z$. Thus, $x = i$ and $y = j$. In the quasi-crystal graph, we have that

$$xzx \xrightarrow{i} (x+1)zx \xrightarrow{i+1} \dots \xrightarrow{j-1} yzx$$

and

$$xxz \xrightarrow{i} (x+1)xz \xrightarrow{i+1} \dots \xrightarrow{j-1} yxz.$$

Since $x < k$, where $z \in \{k, \bar{k}\}$, we get that $(x, z) \neq (1, \bar{1})$, and so, $xzx \not\sim xxz$, by Theorem 9.41(i). Hence, $yzx \not\sim yxz$.

- Case 8: $x < \bar{y} \leq k$. So, $x = i$ and $y = \bar{j}$. In the quasi-crystal graph, we have that

$$\begin{aligned} \bar{j}kx &\xrightarrow{k} \bar{j}(k+1)x \xrightarrow{k+1} \dots \xrightarrow{n-1} \bar{j}nx \xrightarrow{n} \bar{j}\bar{n}x \xrightarrow{n-1} \dots \\ &\dots \xrightarrow{k} \bar{j}\bar{k}x \xrightarrow{k-1} \dots \xrightarrow{j} \bar{j}\bar{j}x \end{aligned}$$

and

$$\begin{aligned} \bar{j}xk &\xrightarrow{k} \bar{j}x(k+1) \xrightarrow{k+1} \dots \xrightarrow{n-1} \bar{j}xn \xrightarrow{n} \bar{j}x\bar{n} \xrightarrow{n-1} \dots \\ &\dots \xrightarrow{k} \bar{j}x\bar{k} \xrightarrow{k-1} \dots \xrightarrow{j} \bar{j}x\bar{j}. \end{aligned}$$

Since $\bar{j}\bar{j}x \not\sim \bar{j}x\bar{j}$, by Theorem 9.41(i), then $\bar{j}kx \not\sim \bar{j}xk$ and $\bar{j}\bar{k}x \not\sim \bar{j}x\bar{k}$, which implies that $yzx \not\sim yxz$.

- Case 9: $\bar{z} \leq y < i$. Thus, $y = j$ and $z = \bar{k}$, where $k \leq j$. As $(y, z) \neq (1, \bar{1})$, we get that $y \geq 2$. In the quasi-crystal graph, we have that

$$(y-1)\bar{y}x \xrightarrow{j-1} y\bar{y}x \xrightarrow{j-1} y(\overline{y-1})x \xrightarrow{j-2} \dots \xrightarrow{k} yzx$$

and

$$(y-1)x\bar{y} \xrightarrow{j-1} yx\bar{y} \xrightarrow{j-1} yx(\overline{y-1}) \xrightarrow{j-2} \dots \xrightarrow{k} yzx.$$

By case 5, if $x = i$, or case 6, if $x = \bar{i}$, we obtain that $(y-1)\bar{y}x \not\sim (y-1)x\bar{y}$, which implies that $yzx \not\sim yxz$.

- Case 10: $\bar{z} < \bar{y} \leq i$. So, $y = \bar{j}$ and $z = \bar{k}$. In the quasi-crystal graph, we have that

$$\begin{aligned} \bar{j}zi &\xrightarrow{i} \bar{j}z(i+1) \xrightarrow{i+1} \dots \xrightarrow{n-1} \bar{j}zn \xrightarrow{n} \bar{j}z\bar{n} \xrightarrow{n-1} \dots \\ &\dots \xrightarrow{i} \bar{j}z\bar{i} \xrightarrow{i-1} \dots \xrightarrow{j} \bar{j}z\bar{j} \end{aligned}$$

and

$$\begin{aligned} \bar{j}iz &\xrightarrow{i} \bar{j}(i+1)z \xrightarrow{i+1} \dots \xrightarrow{n-1} \bar{j}nz \xrightarrow{n} \bar{j}\bar{n}z \xrightarrow{n-1} \dots \\ &\dots \xrightarrow{i} \bar{j}\bar{i}z \xrightarrow{i-1} \dots \xrightarrow{j} \bar{j}\bar{j}z. \end{aligned}$$

Since $\bar{j}z\bar{j} \not\sim \bar{j}\bar{j}z$, by Theorem 9.41(i), then $\bar{j}zi \not\sim \bar{j}iz$ and $\bar{j}zi \not\sim \bar{j}\bar{i}z$, which implies that $yzx \not\sim yxz$.

(ii) By Lemma 9.40, we have that $xzy \sim zxy$ if and only if $\overline{yzx} \sim \overline{yxz}$. Therefore, the result follows from (i). $\square$

9.2.5 From hypo_{n-1} to $\text{hypo}(\mathfrak{C}_n)$

In this section we show that the hypoplactic monoid $\text{hypo}(\mathfrak{C}_n)$ of type $\mathfrak{C}_n$ contains a copy of the classical hypoplactic monoid hypo_{n-1} of rank $n-1$. In fact, we give the explicit definition of an injective map $\psi : \text{hypo}_{n-1} \rightarrow \text{hypo}(\mathfrak{C}_n)$ that satisfies $\psi(uv) = \psi(u)\psi(v)$.

In Section 8.5, we showed that the classical hypoplactic monoid hypo_{n-1} corresponds to the hypoplactic monoid $\text{hypo}(\mathfrak{A}_{n-1})$ of type $\mathfrak{A}_{n-1}$.

Theorem 9.44. *Let $n \geq 3$. Define a map $\psi : \text{hypo}(\mathfrak{A}_{n-1}) \rightarrow \text{hypo}(\mathfrak{C}_n)$ by $\psi(w) = wn\bar{n}n\bar{n}$, for each $w \in A_{n-1}^*$. Then, ψ is injective and satisfies $\psi(uv) = \psi(u)\psi(v)$, for any $u, v \in A_{n-1}^*$.*

Proof. Denote the quasi-crystal structure of $\mathfrak{A}_{n-1}^*$ by $\text{wt}^{\mathfrak{A}}, \check{e}_i^{\mathfrak{A}}, \check{f}_i^{\mathfrak{A}}, \check{\varepsilon}_i^{\mathfrak{A}}$ and $\check{\varphi}_i^{\mathfrak{A}}$ ($i \in \{1, \dots, n-2\}$), and denote the quasi-crystal structure of $\mathfrak{C}_n^*$ by $\text{wt}^{\mathfrak{C}}, \check{e}_i^{\mathfrak{C}}, \check{f}_i^{\mathfrak{C}}, \check{\varepsilon}_i^{\mathfrak{C}}$ and $\check{\varphi}_i^{\mathfrak{C}}$ ($i \in \{1, \dots, n\}$).

Set $X = \{wn\bar{n}n\bar{n} \mid w \in A_{n-1}^*\}$. Let $w \in A_{n-1}^*$. Note that for $i \leq n-2$, w has an i -inversion if and only if $w = w_1 i w_2 (i+1) w_3$, for some $w_1, w_2, w_3 \in A_{n-1}^*$. From Example 8.10 and Definition 9.27, It follows that $g^{\mathfrak{A}}(w) = g^{\mathfrak{C}}(w)$, for all $g \in \{\text{wt}, \check{e}_i, \check{f}_i, \check{\varepsilon}_i, \check{\varphi}_i \mid i = 1, \dots, n-2\}$. And since $n\bar{n}n\bar{n}$ is an isolated word of $\mathfrak{C}_n^*$ with $\text{wt}^{\mathfrak{C}}(n\bar{n}n\bar{n}) = 0$, we have that

$$\text{wt}^{\mathfrak{C}}(wn\bar{n}n\bar{n}) = \text{wt}^{\mathfrak{C}}(w) = \text{wt}^{\mathfrak{A}}(w),$$

and for $i \in \{1, \dots, n-2\}$,

$$\begin{aligned} \check{e}_i^{\mathfrak{C}}(wn\bar{n}n\bar{n}) &= \check{e}_i^{\mathfrak{A}}(w)n\bar{n}n\bar{n}, & \check{\varepsilon}_i^{\mathfrak{C}}(wn\bar{n}n\bar{n}) &= \check{\varepsilon}_i^{\mathfrak{A}}(w), \\ \check{f}_i^{\mathfrak{C}}(wn\bar{n}n\bar{n}) &= \check{f}_i^{\mathfrak{A}}(w)n\bar{n}n\bar{n}, & \check{\varphi}_i^{\mathfrak{C}}(wn\bar{n}n\bar{n}) &= \check{\varphi}_i^{\mathfrak{A}}(w). \end{aligned}$$

Also, $\check{\varepsilon}_{n-1}^{\mathfrak{C}}(wn\bar{n}n\bar{n}) = \check{\varepsilon}_n^{\mathfrak{C}}(wn\bar{n}n\bar{n}) = +\infty$. In particular, $\check{f}_{n-1}^{\mathfrak{C}}$ and $\check{f}_n^{\mathfrak{C}}$ are undefined on $wn\bar{n}n\bar{n}$. And so, $\check{e}_j^{\mathfrak{C}}(w'), \check{f}_j^{\mathfrak{C}}(w') \in X \sqcup \{\perp\}$, for all $w' \in X$ and $j \in \{1, \dots, n\}$.

Furthermore, for $w \in A_{n-1}^*$, define $\psi_w : A_{n-1}^*(w) \rightarrow C_n^*(wn\bar{n}n\bar{n})$ by $\psi_w(u) = un\bar{n}n\bar{n}$, for each $u \in A_{n-1}^*(w)$. Note that ψ_w is injective. As we saw above, for $u \in A_{n-1}^*(w)$ and $i \in \{1, \dots, n-2\}$, we have that $\text{wt}^{\mathcal{C}}(\psi_w(u)) = \text{wt}^{\mathcal{A}}(u)$, $\check{\varepsilon}_i^{\mathcal{C}}(\psi_w(u)) = \check{\varepsilon}_i^{\mathcal{A}}(u)$ and $\check{\varphi}_i^{\mathcal{C}}(\psi_w(u)) = \check{\varphi}_i^{\mathcal{A}}(u)$. Since $\mathcal{A}_{n-1}^*$ and $\mathcal{C}_n^*$ are seminormal, we have that $\check{\varepsilon}_i^{\mathcal{A}}$ is defined on u if and only if $\check{\varepsilon}_i^{\mathcal{C}}$ is defined on $\psi_w(u)$, and if so, $\check{\varepsilon}_i^{\mathcal{C}}(\psi_w(u)) = \psi_w(\check{\varepsilon}_i^{\mathcal{A}}(u))$. Similarly, we have that $\check{f}_i^{\mathcal{A}}$ is defined on u if and only if $\check{f}_i^{\mathcal{C}}$ is defined on $\psi_w(u)$, and if so, $\check{f}_i^{\mathcal{C}}(\psi_w(u)) = \psi_w(\check{f}_i^{\mathcal{A}}(u))$.

Since $n\bar{n}n\bar{n}$ is isolated, we get that $C_n^*(wn\bar{n}n\bar{n}) \subseteq X$. Given $u' \in C_n^*(wn\bar{n}n\bar{n})$, we have that if $\check{\varepsilon}_j^{\mathcal{C}}$ or $\check{f}_j^{\mathcal{C}}$ are defined on u' , then $1 \leq j \leq n-2$. This implies that $u' = g_1^{\mathcal{C}} \cdots g_m^{\mathcal{C}}(wn\bar{n}n\bar{n})$, for some $g_1, \dots, g_m \in \{\check{\varepsilon}_j, \check{f}_j \mid 1 \leq j \leq n-2\}$. Therefore, $u' = \psi_w(v)$, for $v = g_1^{\mathcal{A}} \cdots g_m^{\mathcal{A}}(w)$. Hence, ψ_w is bijective.

By ψ_w , we have shown that the connected component $\Gamma_{\mathcal{C}_n^*}(wn\bar{n}n\bar{n})$ can be obtained from $\Gamma_{\mathcal{A}_{n-1}^*}(w)$ by adding $(n-1)$ -labelled and n -labelled loops to every vertex.

Let $u, v \in A_{n-1}^*$. If $\psi' : \mathcal{A}_{n-1}^*(u) \rightarrow \mathcal{A}_{n-1}^*(v)$ is a quasi-crystal isomorphism, we have that $\psi_v \circ \psi' \circ \psi_u^{-1}$ is a quasi-crystal isomorphism between $\mathcal{C}_n^*(un\bar{n}n\bar{n})$ and $\mathcal{C}_n^*(vn\bar{n}n\bar{n})$. Conversely, if $\psi'' : \mathcal{C}_n^*(un\bar{n}n\bar{n}) \rightarrow \mathcal{C}_n^*(vn\bar{n}n\bar{n})$ is a quasi-crystal isomorphism, then $\psi_v^{-1} \circ \psi'' \circ \psi_u$ is a quasi-crystal isomorphism between $\mathcal{A}_{n-1}^*(u)$ and $\mathcal{A}_{n-1}^*(v)$. Therefore, $u \sim_{\mathcal{A}_{n-1}} v$ if and only if $un\bar{n}n\bar{n} \sim_{\mathcal{C}_n} vn\bar{n}n\bar{n}$, which implies that ψ induces an injective map from $\text{hypo}(\mathcal{A}_{n-1})$ to $\text{hypo}(\mathcal{C}_n)$.

Finally, as $n\bar{n}n\bar{n}$ is an isolated element with $\text{wt}^{\mathcal{C}}(n\bar{n}n\bar{n}) = 0$, we have that

$$(un\bar{n}n\bar{n})(vn\bar{n}n\bar{n}) \sim_{\mathcal{C}_n} uv(n\bar{n}n\bar{n})^2 \sim_{\mathcal{C}_n} uvn\bar{n}n\bar{n},$$

that is, $\psi(uv) = \psi(u)\psi(v)$, for any $u, v \in A_{n-1}^*$. □

Note that in the previous result ψ is not a monoid homomorphism, because $\psi(\epsilon) = n\bar{n}n\bar{n}$, and since $\check{\varepsilon}_n^{\mathcal{C}}(\epsilon) = 0$ and $\check{\varepsilon}_n^{\mathcal{C}}(n\bar{n}n\bar{n}) = +\infty$, we have that $\epsilon \not\sim_{\mathcal{C}} n\bar{n}n\bar{n}$. Nonetheless, ψ is an injective semigroup homomorphism. And so, the semigroup formed by the words $wn\bar{n}n\bar{n}$, $w \in A_{n-1}^*$, consists of a copy of $\text{hypo}(\mathcal{A}_{n-1})$.

LITTELMANN PATH MODELS

A path model to study a Littlewood–Richardson rule for symmetrizable Kac–Moody algebras was introduced by Littelmann [Lit94; Lit95]. This model gives rise to a plactic algebra [Lit96], and was shown to yield a crystal structure [Kas95].

In this chapter we describe to models based on the Littelmann paths. In Section 10.1, we present the Littelmann path model for crystals, as it was introduced in [Lit96]. In Section 10.2, we show that the Littelmann paths also yield a quasi-crystal structure by restricting the root operators. Although the model for quasi-crystals is deduced from the model for crystals, many of the good properties that Littelmann paths satisfy are lost, as shown by Proposition 10.11. Due to this fact, the Littelmann path model does not have an impact for quasi-crystal similar to the impact that it has for crystals.

10.1 A model for crystals

In this section we construct a crystal monoid (Definition 5.1) based on the Littelmann path model. We make an approach similar to the one in [Lit95], but reinterpret with the definitions in [Lit96].

Given a root system Φ with weight lattice Λ , set $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ and $[0, 1]_{\mathbb{Q}} = \{t \in \mathbb{Q} \mid 0 \leq t \leq 1\}$.

Definition 10.1. Let Φ be a root system with weight lattice Λ . A *path* is a piecewise-linear, continuous map $\pi : [0, 1]_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$.

Let Π be the set of all paths π such that $\pi(0) = 0$ and $\pi(1) \in \Lambda$, modulo parameterization, i.e. $\pi = \pi \circ \phi$ for any piecewise-linear, continuous, nondecreasing, surjective map $\phi : [0, 1]_{\mathbb{Q}} \rightarrow [0, 1]_{\mathbb{Q}}$. A path $\pi \in \Pi$ is called a *Littelmann path*.

The Littelmann path model gives rise to a crystal monoid whose underlying set is Π , which we describe below.

The *concatenation of paths* $*$ is a binary operation on the set of all paths, which for paths π_1 and π_2 results in the path $\pi_1 * \pi_2$ given by

$$(\pi_1 * \pi_2)(t) = \begin{cases} \pi_1(2t) & \text{if } 0 \leq t \leq 1/2 \\ \pi_1(1) + \pi_2(2t - 1) & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

for $t \in [0, 1]_{\mathbb{Q}}$. By reparameterization, for any paths π_1 , π_2 and π_3 , we have that $(\pi_1 * \pi_2) * \pi_3$ and $\pi_1 * (\pi_2 * \pi_3)$ correspond to the same path, which we denote by $\pi_1 * \pi_2 * \pi_3$. Also, if we consider a path θ given by $\theta(t) = 0$, for $t \in [0, 1]_{\mathbb{Q}}$, then for any path π , the paths $\theta * \pi$ and $\pi * \theta$ correspond to π , by reparameterization. Therefore, the set of all paths together with $*$ form a monoid.

Note that if $\pi_1, \pi_2 \in \Pi$, then the concatenation $\pi = \pi_1 * \pi_2$ of π_1 and π_2 also lies in Π , because $\pi(0) = \pi_1(0) = 0$ and $\pi(1) = \pi_1(1) + \pi_2(1) \in \Lambda$. Also, the path θ given by $\theta(t) = 0$, for $t \in [0, 1]_{\mathbb{Q}}$, lies in Π . Hence, Π together with the concatenation of paths $*$ forms a monoid.

We now describe the crystal structure maps wt , $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{\varepsilon}_i$ and $\tilde{\varphi}_i$ ($i \in I$) of Π .

Definition 10.2. The *weight map* $\text{wt} : \Pi \rightarrow \Lambda$ is given by $\text{wt}(\pi) = \pi(1)$, for any $\pi \in \Pi$.

For each $i \in I$, the *simple reflection* s_i can be extended pointwise for paths as follows. For a path π , we have that $s_i(\pi)$ is the path given by

$$s_i(\pi)(t) = s_i(\pi(t)) = \pi(t) - \langle \pi(t), \alpha_i^\vee \rangle \alpha_i$$

for $t \in [0, 1]_{\mathbb{Q}}$. If $\pi \in \Pi$, then $s_i(\pi)(0) = 0$ and $s_i(\pi)(1) = \pi(1) - \langle \pi(1), \alpha_i^\vee \rangle \alpha_i \in \Lambda$, because $\langle \pi(1), \alpha_i^\vee \rangle \in \mathbb{Z}$ as $\pi(1) \in \Lambda$. This implies that $s_i(\pi) \in \Pi$, whenever $\pi \in \Pi$.

Let $\pi \in \Pi$ and $i \in I$. Define $h_i^\pi : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{Q}$ by

$$h_i^\pi(t) = \langle \pi(t), \alpha_i^\vee \rangle \quad \text{for } t \in [0, 1]_{\mathbb{Q}}.$$

Set $m_i^\pi = \min\{h_i^\pi(t) \mid t \in [0, 1]_{\mathbb{Q}}\}$. If $m_i^\pi \leq -1$, take t_1 minimal such that $h_i^\pi(t_1) = m_i^\pi$, and take t_0 maximal such that $h_i^\pi(t) \geq m_i^\pi + 1$ for $t \in [0, t_0]_{\mathbb{Q}}$. In particular, $t_0 < t_1$ and $h_i^\pi(t_0) = m_i^\pi + 1$. Let $t_0 = u_0 < u_1 < \dots < u_p = t_1$ be such that either

(LPe1) $h_i^\pi(u_{k-1}) = h_i^\pi(u_k)$ and $h_i^\pi(t) \geq h_i^\pi(u_{k-1})$ for all $t \in [u_{k-1}, u_k]_{\mathbb{Q}}$, or

(LPe2) h_i^π is strictly decreasing on $[u_{k-1}, u_k]_{\mathbb{Q}}$,

for $k = 1, \dots, p$. Consider $u_{-1} = 0$ and $u_{p+1} = 1$. For each $k = 0, \dots, p+1$, define a path π_k by

$$\pi_k(t) = \pi((u_k - u_{k-1})t + u_{k-1}) - \pi(u_{k-1}), \quad \text{for } t \in [0, 1]_{\mathbb{Q}}. \quad (10.1)$$

Thus, $\pi = \pi_0 * \pi_1 * \dots * \pi_p * \pi_{p+1}$.

Definition 10.3. The raising Kashiwara operators $\tilde{e}_i : \Pi \rightarrow \Pi \sqcup \{\perp\}$ ($i \in I$) are given as follows. For $\pi \in \Pi$ and $i \in I$, if $m_i^\pi > -1$ set $\tilde{e}_i(\pi) = \perp$. Otherwise, let $\pi = \pi_0 * \pi_1 * \cdots * \pi_p * \pi_{p+1}$ as described above, and set $\tilde{e}_i(\pi) = \pi_0 * \eta_1 * \cdots * \eta_p * \pi_{p+1}$, where $\eta_k = \pi_k$ if h_i^π satisfies condition (LPe1) on $[u_{k-1}, u_k]_{\mathbb{Q}}$, or $\eta_k = s_i(\pi_k)$ if h_i^π satisfies condition (LPe2) on $[u_{k-1}, u_k]_{\mathbb{Q}}$.

We proceed to define the lowering Kashiwara operators $\tilde{f}_i$ ($i \in I$). Let $\pi \in \Pi$ and $i \in I$. If $h_i^\pi(1) - m_i^\pi \geq 1$, take t_0 maximal such that $h_i^\pi(t_0) = m_i^\pi$, and take t_0 minimal such that $h_i^\pi(t) \geq m_i^\pi + 1$ for $t \in [t_1, 1]_{\mathbb{Q}}$. In particular, $t_0 < t_1$ and $h_i^\pi(t_1) = m_i^\pi + 1$. Let $t_0 = u_0 < u_1 < \cdots < u_q = t_1$ be such that either

(LPf1) $h_i^\pi(u_{k-1}) = h_i^\pi(u_k)$ and $h_i^\pi(t) \geq h_i^\pi(u_{k-1})$ for all $t \in [u_{k-1}, u_k]_{\mathbb{Q}}$, or

(LPf2) h_i^π is strictly increasing on $[u_{k-1}, u_k]_{\mathbb{Q}}$,

for $k = 1, \dots, q$. Let $u_{-1} = 0$, $u_{q+1} = 1$, and π_k as in (10.1), for $k = 0, 1, \dots, q+1$. Thus, $\pi = \pi_0 * \pi_1 * \cdots * \pi_q * \pi_{q+1}$.

Definition 10.4. The lowering Kashiwara operators $\tilde{f}_i : \Pi \rightarrow \Pi \sqcup \{\perp\}$ ($i \in I$) are given as follows. For $\pi \in \Pi$ and $i \in I$, if $h_i^\pi(1) - m_i^\pi < 1$ set $\tilde{f}_i(\pi) = \perp$. Otherwise, let $\pi = \pi_0 * \pi_1 * \cdots * \pi_q * \pi_{q+1}$ as described above, and set $\tilde{f}_i(\pi) = \pi_0 * \eta_1 * \cdots * \eta_q * \pi_{q+1}$, where $\eta_k = \pi_k$ if h_i^π satisfies condition (LPf1) on $[u_{k-1}, u_k]_{\mathbb{Q}}$, or $\eta_k = s_i(\pi_k)$ if h_i^π satisfies condition (LPf2) on $[u_{k-1}, u_k]_{\mathbb{Q}}$.

Note that the rising and lowering Kashiwara operators can be obtained from each other by path reversing. Given a path π , define the *reverse path* $\pi^\leftarrow$ of π by $\pi^\leftarrow(t) = \pi(1-t) - \pi(1)$, for $t \in [0, 1]_{\mathbb{Q}}$. Then, for $\pi \in \Pi$ and $i \in I$, we have that $\tilde{f}_i(\pi) = (\tilde{e}_i(\pi^\leftarrow))^\leftarrow$.

Finally, for each $i \in I$, define maps $\tilde{e}_i, \tilde{\varphi}_i : \Pi \rightarrow \mathbb{Z} \cup \{-\infty\}$ by

$$\tilde{e}_i(\pi) = -m_i^\pi \quad \text{and} \quad \tilde{\varphi}_i(\pi) = h_i^\pi(1) - m_i^\pi,$$

for $\pi \in \Pi$. Then, we have the following results from [Lit94; Lit95; Lit96]

Theorem 10.5. *The set of Littelmann paths Π together with the maps wt , $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{e}_i$ and $\tilde{\varphi}_i$ ($i \in I$) forms a seminormal crystal.*

Theorem 10.6. *The map $\Pi \otimes \Pi \rightarrow \Pi$ given by $\pi_1 \otimes \pi_2 \mapsto \pi_1 * \pi_2$, for $\pi_1, \pi_2 \in \Pi$, is a crystal homomorphism. And therefore, Π is a crystal monoid.*

Using the construction described in this section, we can obtain a crystal monoid from any root system. Due to this connection, we have that the crystal monoid reflects the properties of the root system. For example, we have that

a path $\pi \in \Pi$ is of highest weight if and only if π is contained in the dominant Weyl chamber. Moreover, P. Littelmann used this model to give a Littlewood–Richardson rule for symmetrizable Kac–Moody algebras [Lit94; Lit95], and a plactic algebra for semisimple Lie algebras [Lit96].

In the following section, the crystal monoid obtained above will be denoted by $\tilde{\Pi}$. Although the crystal monoid of Littelmann paths is usually denoted by Π , we need this change to avoid ambiguity, as we will describe another structure whose underlying set is also Π .

10.2 A model for quasi-crystals

In this section we show that the Littelmann path model gives rise to a quasi-crystal monoid (Definition 8.1). We construct this quasi-crystal monoid based on the crystal monoid presented in the previous section. This leads to a method of obtaining a quasi-crystal monoid for any root system, which can be regarded as a geometrical approach to quasi-crystals similar to the Littelmann path model for crystals.

The crystal monoid $\tilde{\Pi}$ of Littelmann paths over a root system gives rise to a quasi-crystal monoid as follows. The weight map wt remains as in Definition 10.2, while the maps $\tilde{e}_i, \tilde{f}_i, \tilde{\varepsilon}_i$ and $\tilde{\varphi}_i$ ($i \in I$) originate the following maps.

Definition 10.7. Let Π be the set of Littelmann paths for a root system Φ with weight lattice Λ and index set I for the simple roots $(\alpha_i)_{i \in I}$. For each $i \in I$, define maps $\check{e}_i, \check{f}_i : \Pi \rightarrow \Pi \sqcup \{\perp\}$ and $\check{\varepsilon}_i, \check{\varphi}_i : \Pi \rightarrow \mathbb{Z} \cup \{-\infty, +\infty\}$ as follows. For $\pi \in \Pi$, we have that

- (i) if there exist $\pi_1, \pi_2 \in \Pi$ such that $\pi = \pi_1 * \pi_2$ and $\tilde{f}_i(\pi_1), \tilde{e}_i(\pi_2) \in \Pi$, then set

$$\check{e}_i(\pi) = \check{f}_i(\pi) = \perp \quad \text{and} \quad \check{\varepsilon}_i(\pi) = \check{\varphi}_i(\pi) = +\infty;$$

- (ii) otherwise, set

$$\begin{aligned} \check{e}_i(\pi) &= \tilde{e}_i(\pi), & \check{f}_i(\pi) &= \tilde{f}_i(\pi), \\ \check{\varepsilon}_i(\pi) &= \tilde{\varepsilon}_i(\pi), & \text{and} & \quad \check{\varphi}_i(\pi) = \tilde{\varphi}_i(\pi). \end{aligned}$$

We now proceed to check that Π together with the maps $\text{wt}, \check{e}_i, \check{f}_i, \check{\varepsilon}_i$ and $\check{\varphi}_i$ ($i \in I$) forms a seminormal quasi-crystal (Definitions 7.1 and 7.9). First, in the following result we show an alternative description of Definition 10.7(i).

Proposition 10.8. *Let $\pi \in \Pi$ and $i \in I$. Then, $\ddot{\varphi}_i(\pi) = +\infty$ if and only if there exist $t_1, t_2, t_3 \in [0, 1]_{\mathbb{Q}}$ such that $t_1 \leq t_2 \leq t_3$, $\pi(t_2) \in \Lambda$, $h_i^\pi(t_2) - h_i^\pi(t_1) \geq 1$ and $h_i^\pi(t_3) - h_i^\pi(t_2) \leq -1$.*

Proof. $[\Rightarrow]$ Let $\pi_1, \pi_2 \in \Pi$ be such that $\pi = \pi_1 * \pi_2$ and $\tilde{f}_i(\pi_1), \tilde{e}_i(\pi_2) \in \Pi$. As $\pi = \pi_1 * \pi_2$, there exist $t_1 \leq t_2 \leq t_3$ such that $h_i^\pi(t_1) = m_i^{\pi_1}$, $\pi(t_2) = \pi_1(1)$, and $h_i^\pi(t_3) = h_i^{\pi_1}(1) + m_i^{\pi_2}$. We have that $\pi(t_2) = \pi_1(1) \in \Lambda$, because $\pi_1 \in \Pi$. Since $\tilde{f}_i(\pi_1) \in \Pi$, then

$$h_i^\pi(t_2) - h_i^\pi(t_1) = h_i^{\pi_1}(1) - m_i^{\pi_1} \geq 1.$$

Since $\tilde{e}_i(\pi_2) \in \Pi$, then

$$h_i^\pi(t_3) - h_i^\pi(t_2) = h_i^{\pi_1}(1) + m_i^{\pi_2} - h_i^{\pi_1}(1) = m_i^{\pi_2} \leq -1.$$

$[\Leftarrow]$ Let $t_1, t_2, t_3 \in [0, 1]_{\mathbb{Q}}$, where $t_1 \leq t_2 \leq t_3$, be such that $\pi(t_2) \in \Lambda$, $h_i^\pi(t_2) - h_i^\pi(t_1) \geq 1$ and $h_i^\pi(t_3) - h_i^\pi(t_2) \leq -1$. Define $\pi_1, \pi_2 : [0, 1]_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ by

$$\pi_1(t) = \pi(t_2 t) \quad \text{and} \quad \pi_2(t) = \pi((1 - t_2)t + t_2) - \pi(t_2),$$

for $t \in [0, 1]_{\mathbb{Q}}$. Clearly, $\pi = \pi_1 * \pi_2$ (up to reparametrization). We have that $\pi_1(0) = \pi_2(0) = 0$, and since $\pi(t_2), \pi(1) \in \Lambda$, we also have that $\pi_1(1) = \pi(t_2) \in \Lambda$ and $\pi_2(1) = \pi(1) - \pi(t_2) \in \Lambda$. Hence, $\pi_1, \pi_2 \in \Pi$. As $m_i^{\pi_1} \leq h_i^{\pi_1}(t_1/t_2) = h_i^\pi(t_1)$, then

$$h_i^{\pi_1}(1) - m_i^{\pi_1} \geq h_i^\pi(t_2) - h_i^\pi(t_1) \geq 1,$$

which implies that $\tilde{f}_i(\pi_1)$ is defined. Since

$$m_i^{\pi_2} \leq h_i^{\pi_2}((t_3 - t_2)/(1 - t_2)) = h_i^\pi(t_3) - h_i^\pi(t_2) \leq -1,$$

then $\tilde{e}_i(\pi_2)$ is defined. Therefore, $\ddot{\varphi}_i(\pi) = +\infty$. $\square$

Theorem 10.9. *The set of Littelmann paths Π together with the maps wt , $\ddot{e}_i$, $\ddot{f}_i$, $\ddot{\varepsilon}_i$ and $\ddot{\varphi}_i$ ($i \in I$) forms a seminormal quasi-crystal.*

Proof. Note that if for $i \in I$, a Littelmann path $\pi \in \Pi$ satisfies condition (i) of Definition 10.7, then all conditions in Definitions 7.1 and 7.9 hold for π and i . Otherwise, the maps $\ddot{e}_i$, $\ddot{f}_i$, $\ddot{\varepsilon}_i$ and $\ddot{\varphi}_i$ on π coincide with the maps $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{\varepsilon}_i$ and $\tilde{\varphi}_i$, respectively. By Theorem 10.5, $\tilde{\Pi}$ is a seminormal crystal, and so, to prove the statement it suffices to check that $\tilde{e}_i(\pi)$ and $\tilde{f}_i(\pi)$ do not satisfy Definition 10.7(i), whenever defined. Equivalently, we have to check that for $\pi \in \Pi$ and $i \in I$, if π has a decomposition of the form $\pi = \pi_1 * \pi_2$ with $\pi_1, \pi_2, \tilde{f}_i(\pi_1), \tilde{e}_i(\pi_2) \in \Pi$, so it does $\tilde{e}_i(\pi)$ and $\tilde{f}_i(\pi)$, whenever defined.

Let $\pi, \pi_1, \pi_2 \in \Pi$ and $i \in I$ be such that $\pi = \pi_1 * \pi_2$ and $\tilde{f}_i(\pi_1), \tilde{e}_i(\pi_2) \in \Pi$. First, suppose that $\tilde{e}_i(\pi) \in \Pi$. Since $\tilde{\Pi}$ is a crystal monoid (Theorem 10.6), we have that

$$\tilde{e}_i(\pi) = \begin{cases} \tilde{e}_i(\pi_1) * \pi_2 & \text{if } \tilde{\varphi}_i(\pi_1) \geq \tilde{\varepsilon}_i(\pi_2) \\ \pi_1 * \tilde{e}_i(\pi_2) & \text{if } \tilde{\varphi}_i(\pi_1) < \tilde{\varepsilon}_i(\pi_2). \end{cases}$$

If $\tilde{\varphi}_i(\pi_1) \geq \tilde{\varepsilon}_i(\pi_2)$, then $\tilde{e}_i(\pi) = \tilde{e}_i(\pi_1) * \pi_2$ where $\tilde{f}_i(\tilde{e}_i(\pi_1)) = \pi_1 \in \Pi$. Otherwise, we have $\tilde{e}_i(\pi) = \pi_1 * \tilde{e}_i(\pi_2)$ where $\tilde{e}_i^2(\pi_2) \in \Pi$, because $\tilde{\Pi}$ is a seminormal crystal and $\tilde{\varepsilon}_i(\pi_2) > \tilde{\varphi}_i(\pi_1) \geq 1$.

Finally, suppose that $\tilde{f}_i(\pi) \in \Pi$. Since $\tilde{\Pi}$ is a crystal monoid, we have that

$$\tilde{f}_i(\pi) = \begin{cases} \tilde{f}_i(\pi_1) * \pi_2 & \text{if } \tilde{\varphi}_i(\pi_1) > \tilde{\varepsilon}_i(\pi_2) \\ \pi_1 * \tilde{f}_i(\pi_2) & \text{if } \tilde{\varphi}_i(\pi_1) \leq \tilde{\varepsilon}_i(\pi_2). \end{cases}$$

If $\tilde{\varphi}_i(\pi_1) \leq \tilde{\varepsilon}_i(\pi_2)$, then $\tilde{f}_i(\pi) = \pi_1 * \tilde{f}_i(\pi_2)$ where $\tilde{e}_i(\tilde{f}_i(\pi_2)) = \pi_2 \in \Pi$. Otherwise, we have $\tilde{f}_i(\pi) = \tilde{f}_i(\pi_1) * \pi_2$ where $\tilde{f}_i^2(\pi_1) \in \Pi$, because $\tilde{\Pi}$ is a seminormal crystal and $\tilde{\varphi}_i(\pi_1) > \tilde{\varepsilon}_i(\pi_2) \geq 1$. $\square$

We now show that the seminormal quasi-crystal formed by Littelmann paths together with the concatenation of paths is a quasi-crystal monoid.

Theorem 10.10. *The set of Littelmann paths Π together with wt , $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{\varepsilon}_i$, $\tilde{\varphi}_i$ ($i \in I$), and the concatenation of paths $*$ is a quasi-crystal monoid.*

Proof. By Theorem 10.9, it remains to show that Definition 8.1(iii) is satisfied, that is, the concatenation of paths induces a quasi-crystal homomorphism from $\Pi \tilde{\otimes} \Pi$ to Π .

Let $\pi, \pi' \in \Pi$. Since Π together with wt , $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{\varepsilon}_i$, $\tilde{\varphi}_i$ ($i \in I$), and the concatenation of paths $*$ is a crystal monoid (Theorem 10.6), by Definition 10.7(ii), we just need to check the cases where $\tilde{\varphi}_i(\pi) = +\infty$, $\tilde{\varphi}_i(\pi') = +\infty$, $\tilde{\varphi}_i(\pi \tilde{\otimes} \pi') = +\infty$, or $\tilde{\varphi}_i(\pi * \pi') = +\infty$.

Let $i \in I$. We have the following cases.

- Case 1: $\tilde{\varphi}_i(\pi) = +\infty$. Then, $\tilde{\varphi}_i(\pi \tilde{\otimes} \pi') = +\infty$. By Proposition 10.8, there exists $t_1, t_2, t_3 \in [0, 1]_{\mathbb{Q}}$, $t_1 \leq t_2 \leq t_3$, such that $\pi(t_2) \in \Lambda$, $h_i^\pi(t_2) - h_i^\pi(t_1) \geq 1$ and $h_i^\pi(t_3) - h_i^\pi(t_2) \leq -1$. Set $u_1, u_2, u_3 \in [0, 1]_{\mathbb{Q}}$ such that $u_1 \leq u_2 \leq u_3$ and $(\pi * \pi')(u_k) = \pi(t_k)$, for $k = 1, 2, 3$. Then, $(\pi * \pi')(u_2) \in \Lambda$, $h_i^{\pi * \pi'}(u_2) - h_i^{\pi * \pi'}(u_1) \geq 1$ and $h_i^{\pi * \pi'}(u_3) - h_i^{\pi * \pi'}(u_2) \leq -1$. Hence, by Proposition 10.8, $\tilde{\varphi}_i(\pi * \pi') = +\infty$.
- Case 2: $\tilde{\varphi}_i(\pi') = +\infty$. So, $\tilde{\varphi}_i(\pi \tilde{\otimes} \pi') = +\infty$. By Proposition 10.8, there are $t_1, t_2, t_3 \in [0, 1]_{\mathbb{Q}}$, $t_1 \leq t_2 \leq t_3$, such that $\pi'(t_2) \in \Lambda$, $h_i^{\pi'}(t_2) - h_i^{\pi'}(t_1) \geq 1$ and

$h_i^{\pi'}(t_3) - h_i^{\pi'}(t_2) \leq -1$. Take $u_1, u_2, u_3 \in [0, 1]_{\mathbb{Q}}$ such that $u_1 \leq u_2 \leq u_3$ and $(\pi * \pi')(u_k) = \pi(1) + \pi(t_k)$, for $k = 1, 2, 3$. Then, $(\pi * \pi')(u_2) = \pi(1) + \pi'(t_2) \in \Lambda$, as $\pi(1), \pi'(t_2) \in \Lambda$,

$$h_i^{\pi * \pi'}(u_2) - h_i^{\pi * \pi'}(u_1) = h_i^{\pi}(1) + h_i^{\pi'}(t_2) - h_i^{\pi}(1) - h_i^{\pi'}(t_1) \geq 1$$

and

$$h_i^{\pi * \pi'}(u_3) - h_i^{\pi * \pi'}(u_2) = h_i^{\pi}(1) + h_i^{\pi'}(t_3) - h_i^{\pi}(1) - h_i^{\pi'}(t_2) \leq -1.$$

Thus, by Proposition 10.8, $\ddot{\varphi}_i(\pi * \pi') = +\infty$.

- Case 3: $\ddot{\varphi}_i(\pi \ddot{\otimes} \pi') = +\infty$, $\ddot{\varphi}_i(\pi) \neq +\infty$ and $\ddot{\varphi}_i(\pi') \neq +\infty$. By Theorem 7.32, we must have $\ddot{\varphi}_i(\pi), \ddot{e}_i(\pi') \in \mathbb{Z}_{>0}$, because $\ddot{\varphi}_i(\pi) \neq +\infty$ and $\ddot{\varphi}_i(\pi') \neq +\infty$. Since $\tilde{\Pi}$ is a seminormal crystal, then $\ddot{f}_i(\pi), \ddot{e}_i(\pi') \in \Pi$, and so, $\ddot{\varphi}_i(\pi * \pi') = +\infty$.
- Case 4: $\ddot{\varphi}_i(\pi * \pi') = +\infty$, $\ddot{\varphi}_i(\pi) \neq +\infty$ and $\ddot{\varphi}_i(\pi') \neq +\infty$. By Proposition 10.8, there exist $t_1, t_2, t_3 \in [0, 1]_{\mathbb{Q}}$, $t_1 \leq t_2 \leq t_3$, such that $(\pi * \pi')(t_2) \in \Lambda$, $h_i^{\pi * \pi'}(t_2) - h_i^{\pi * \pi'}(t_1) \geq 1$ and $h_i^{\pi * \pi'}(t_3) - h_i^{\pi * \pi'}(t_2) \leq -1$. Since $\ddot{\varphi}_i(\pi) \neq +\infty$, we must have $(\pi * \pi')(t_3) = \pi(1) + \pi'(u_3)$, for some $u_3 \in [0, 1]_{\mathbb{Q}}$. And as $\ddot{\varphi}_i(\pi') \neq +\infty$, we must have $(\pi * \pi')(t_1) = \pi(u_1)$, for some $u_1 \in [0, 1]_{\mathbb{Q}}$. We consider the following two subcases, depending of the form of $(\pi * \pi')(t_2)$.

- Subcase 4a: $(\pi * \pi')(t_2) = \pi(u_2)$, for some $u_2 \in [u_1, 1]_{\mathbb{Q}}$. Define paths $\pi_1, \pi_2 : [0, 1]_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ by

$$\pi_1(t) = \pi(u_2 t) \quad \text{and} \quad \pi_2(t) = \pi((1 - u_2)t + u_2) - \pi(u_2),$$

for $t \in [0, 1]_{\mathbb{Q}}$. Clearly, $\pi = \pi_1 * \pi_2$. We have that $\pi_1(0) = \pi_2(0) = 0$, $\pi_1(1) = \pi(u_2) = (\pi * \pi')(t_2) \in \Lambda$, and $\pi_2(1) = \pi(1) - (\pi * \pi')(t_2) \in \Lambda$, which implies that $\pi_1, \pi_2 \in \Pi$. Since

$$h_i^{\pi_1}(1) - m_i^{\pi_1} \geq h_i^{\pi_1}(1) - h_i^{\pi_1}(u_1/u_2) = h_i^{\pi * \pi'}(t_2) - h_i^{\pi * \pi'}(t_1) \geq 1,$$

then $\tilde{f}_i(\pi_1)$ is defined. As $\ddot{\varphi}_i(\pi) \neq +\infty$, we have that $\tilde{e}_i(\pi_2)$ is undefined, which implies that

$$h_i^{\pi}(1) - h_i^{\pi}(u_2) = h_i^{\pi_2}(1) \geq m_i^{\pi_2} > -1.$$

Since $\pi_2(1) \in \Lambda$, then $h_i^{\pi_2}(1) \in \mathbb{Z}$, and so, $h_i^{\pi}(1) - h_i^{\pi}(u_2) \geq 0$. We have that

$$\begin{aligned} h_i^{\pi}(1) - m_i^{\pi} &\geq h_i^{\pi}(1) - h_i^{\pi}(u_2) + h_i^{\pi}(u_2) - h_i^{\pi}(u_1) \\ &\geq h_i^{\pi}(u_2) - h_i^{\pi}(u_1) = h_i^{\pi * \pi'}(t_2) - h_i^{\pi * \pi'}(t_1) \geq 1, \end{aligned}$$

and thus, $\tilde{f}_i(\pi)$ is defined. Also, we have that

$$h_i^\pi(1) + h_i^{\pi'}(u_3) - h_i^\pi(u_2) = h_i^{\pi*\pi'}(t_3) - h_i^{\pi*\pi'}(t_2) \leq -1,$$

which implies that

$$m_i^{\pi'} \leq h_i^{\pi'}(u_3) \leq -1,$$

and so, $\tilde{e}_i(\pi')$ is defined. Hence, $\ddot{\varphi}_i(\pi) = \tilde{\varphi}_i(\pi) > 0$ and $\ddot{e}_i(\pi') = \tilde{e}_i(\pi') > 0$, which implies that $\ddot{\varphi}_i(\pi \ddot{\otimes} \pi') = +\infty$.

- Subcase 4b: $(\pi * \pi')(t_2) = \pi(1) + \pi'(u_2)$, for some $u_2 \in [0, u_3]_{\mathbb{Q}}$. Define paths $\pi'_1, \pi'_2 : [0, 1]_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ by

$$\pi'_1(t) = \pi'(u_2 t) \quad \text{and} \quad \pi'_2(t) = \pi'((1 - u_2)t + u_2) - \pi'(u_2),$$

for $t \in [0, 1]_{\mathbb{Q}}$. Clearly, $\pi' = \pi'_1 * \pi'_2$. We have that $\pi'_1(0) = \pi'_2(0) = 0$, $\pi'_1(1) = \pi'(u_2) = (\pi * \pi')(t_2) \in \Lambda$, and $\pi'_2(1) = \pi'(1) - (\pi * \pi')(t_2) \in \Lambda$, which implies that $\pi'_1, \pi'_2 \in \Pi$. Since

$$\begin{aligned} m_i^{\pi'_2} &\leq h_i^{\pi'_2}((u_3 - u_2)/(1 - u_2)) = h_i^{\pi'}(u_3) - h_i^{\pi'}(u_2) \\ &\leq h_i^{\pi*\pi'}(t_3) - h_i^{\pi*\pi'}(t_2) \leq -1, \end{aligned}$$

then $\tilde{e}_i(\pi'_2)$ is defined. As $\ddot{\varphi}_i(\pi') \neq +\infty$, we have that $\tilde{f}_i(\pi'_1)$ is undefined, which implies that

$$h_i^{\pi'}(u_2) = h_i^{\pi'_1}(1) \leq h_i^{\pi'_1}(1) - m_i^{\pi'_1} < 1,$$

as $m_i^{\pi'_1} \leq h_i^{\pi'_1}(0) = 0$. Since $\pi'_1(1) \in \Lambda$, then $h_i^{\pi'_1}(1) \in \mathbb{Z}$, and so, $h_i^{\pi'}(u_2) \leq 0$. We have that

$$h_i^\pi(1) + h_i^{\pi'}(u_2) - h_i^\pi(u_1) = h_i^{\pi*\pi'}(t_2) - h_i^{\pi*\pi'}(t_1) \geq 1,$$

which implies that

$$h_i^\pi(1) - m_i^\pi \geq h_i^\pi(1) - h_i^\pi(u_1) \geq 1 - h_i^{\pi'}(u_2) \geq 1,$$

and thus, $\tilde{f}_i(\pi)$ is defined. Also, we have that

$$h_i^{\pi'}(u_3) - h_i^{\pi'}(u_2) = h_i^{\pi*\pi'}(t_3) - h_i^{\pi*\pi'}(t_2) \leq -1,$$

which implies that

$$m_i^{\pi'} \leq h_i^{\pi'}(u_3) \leq -1 - h_i^{\pi'}(u_2) \leq -1,$$

and so, $\tilde{e}_i(\pi')$ is defined. Hence, $\ddot{\varphi}_i(\pi) = \tilde{\varphi}_i(\pi) > 0$ and $\ddot{e}_i(\pi') = \tilde{e}_i(\pi') > 0$, which implies that $\ddot{\varphi}_i(\pi \ddot{\otimes} \pi') = +\infty$.

Note that in each case we get $\ddot{\varphi}_i(\pi \otimes \pi') = \ddot{\varphi}_i(\pi * \pi') = +\infty$. $\square$

The quasi-crystal monoid described in this section will be denoted by $\ddot{\Pi}$.

Although $\ddot{\Pi}$ is constructed based on $\tilde{\Pi}$, many properties of $\tilde{\Pi}$ are not inherited by $\ddot{\Pi}$. As a consequence, some structural results obtained based on $\tilde{\Pi}$ do not have an analogue based on $\ddot{\Pi}$. For example, we have the following result on highest weights on $\ddot{\Pi}$ which contrast with highest weights on $\tilde{\Pi}$.

Proposition 10.11. *Any weight $\lambda \in \Lambda$ is an highest weight in $\ddot{\Pi}$.*

Proof. For each $i \in I$, define a path $\pi_i : [0, 1]_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ by $\pi_i(t) = t\alpha_i$, for $t \in [0, 1]_{\mathbb{Q}}$. Since $\pi_i(0) = 0$ and $\pi_i(1) = \alpha_i \in \Lambda$, we have that $\pi_i \in \Pi$. Note that $h_i^{\pi_i}(t) = 2t$, for $t \in [0, 1]_{\mathbb{Q}}$. Hence, $\ddot{\varphi}_i(\pi_i) = 2$, which implies that $\ddot{f}_i$ is defined on π_i , and $\ddot{f}_i$ is defined on $\ddot{f}_i(\pi_i)$. In fact, we have that $\ddot{f}_i^2(\pi)(t) = -t\alpha_i$, for $t \in [0, 1]_{\mathbb{Q}}$, and so, $\ddot{f}_i(\pi_i) = -\pi_i$. Therefore, we obtain that

$$\text{wt}(\pi_i * (-\pi_i)) = \pi_i(1) - \pi_i(1) = 0,$$

and $\ddot{\varepsilon}_i(\pi_i * (-\pi_i)) = +\infty$.

Define a path π by concatenating for each $i \in I$ the paths $\pi_i * (-\pi_i)$. Hence, $\text{wt}(\pi) = 0$, and for any $i \in I$, $\ddot{\varepsilon}_i(\pi) = +\infty$.

Given $\lambda \in \Lambda$, define a path $\eta : [0, 1]_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ by $\eta(t) = t\lambda$, for $t \in [0, 1]_{\mathbb{Q}}$. Note that $\eta \in \Pi$,

$$\text{wt}(\eta * \pi) = \text{wt}(\eta) + \text{wt}(\pi) = \lambda,$$

and for any $i \in I$, $\ddot{\varepsilon}_i(\eta * \pi) = +\infty$, which implies that $\ddot{\varepsilon}_i$ is not defined on $\eta * \pi$. Therefore, $\eta * \pi$ is an highest-weight element of $\ddot{\Pi}$, and λ a highest weight. $\square$

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