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IMPROVED LAGRANGEAN DECOMPOSITION: AN APPLICATION TO THE GENERALIZED ASSIGNMENT PROBLEM

by

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ABSTRACT

Recently two new ways of obtaining improved Lagrangean bounds have been suggested : Lagrangean decomposition and bound improving sequences.

In this work we will obtain a Lagrangean approach combining the two ideas mentioned above.

We provide theoretical results about the sharpness of the bounds obtained by the combined approach for the general case and an application to the generalized assignment problem.

Computational experience is reported.

KEYWORDS

Integer programming , Lagrange multipliers , assignment .

1. Introduction

Recently two new ways of obtaining improved Lagrangean bounds have been suggested : Lagrangean decomposition ,Jörnsten Näsberg & Smeds (1985) and Guignard & Kim (1987) , and bound improving sequences, Barcia (1987) .

In this work we follow the path laid down in Barcia & Jörnsten (1986) to obtain a Lagrangean approach combining the two ideas mentioned above.

In section 2 we will briefly sketch the bound improving sequence technique which consists in building a sequence of Lagrangean duals that progressively reduces the duality gap and , if some conditions are met, converges in a finite number of steps to the optimal value of the original problem .

Section 3 will be devoted to the discussion of the main results concerning Lagrangean decomposition / Variable splitting which consists in duplicating the problem variables thus enabling the use of more than one structure within a Lagragean approach.In section 4 we shall , following the ideas laid down in Barcia & Jörnsten (1986) and Barcia (1988),study the combination of the two approaches mentioned above and provide some theoretical results about the bounds thus obtained for the general case.

Section 5 will be devoted to the application of these ideas

2. Bound improving sequences

We start by briefly sketching the basics of the bound improving sequence technique. We shall state only the main results and the reader is referred to Barcia(1985) , Barcia(1987) and Barcia & Holm(1988) for formal proofs and more details on the subject.

Consider the following integer linear programming problem:

$$(P) \qquad z = \min cx \quad Ax \leq b \quad x \in \mathcal{X}$$

where c is an integer n-vector, b an integer m-vector and A a matrix of integers of apropriate dimension . For simplicity we shall assume that $\mathcal{X} \subset \mathbb{Z}^n$ is bounded and that problem (P) always has a finite solution.

Suppose that a lower bound for z , $\ell_k \leq z$, is known. Introducing the set $\mathcal{X}_k = \{ x \in \mathcal{X} : cx \geq \ell_k \}$, problem (P) can be restated in the following equivalent way:

$$(Pk) \qquad z = \min cx \quad Ax \leq b \quad x \in \mathcal{X}_k$$

Let $u \in \mathbb{R}_+^m$ be a vector of non-negative multipliers and consider a Lagragean dual of (Pk) defined as follows:

$$(\text{LDk}) \qquad \ell_{k+1} = \max_{u \in \mathbb{R}_+^m} \ \min_{x \in \mathcal{X}_k} cx + u(Ax-b) \qquad k=0,1,\ldots$$

Note first that the usual Lagrangean bound can be obtained from (LDk) as $\ell=\ell_1$ by taking $\ell_0=-\infty$. For simplicity we shall assume that ℓ_0 was selected this way and we denote by ℓ the usual Lagrangean bound whenever we need to refer it.

Note also that, if $\mathcal{X} = \{0,1\}^n$, the inner minimization problem on the right hand side of the equality is a single 0-1 knapsack problem which , although in $\mathcal{NP}$, is one of the easiest non-polynomial problems ; see Walukiewicz & Dudzinsky(1987). Since the outer maximization problem can be solved using a subgradient algorithm , see Demianov & Vasilev(1985), (LDk) can be computed with moderate effort.

Note also that if $\mathcal{X}$ is somewhat more complicated , say $\{0,1\}^n$ with semi-assignment constraints , one gets multiple-choice knapsack problems, which are as easy as single 0-1 knapsack problems; see Walukiewicz & Dudzinsky (1987). So ,some structure may be kept in the $\mathcal{X}$ set without deterioration of the computational efficiency of the method.

One can easily prove the following :

Theorem 2.1 : $\ell_{k+1} \geq \ell_k$

Proof: See Barcia (1985). ∎

Some necessary conditions for equality to hold in theorem 2.1

are easily derived . For instance , one has :

Theorem 2.2 : If $\ell_{k+1}=\ell_k$ then u=0 is an optimal vector of multipliers in (LDk).

Proof: See Barcia (1985). ■

Of course the interesting question is to be able to characterize the situations in which the inequality in theorem 2.1 is strict . The answer to this is addressed in the following:

Theorem 2.3 : Consider the level sets of the objective function, $\mathbb{L}(y) = \{ x \in \mathcal{X} : cx = y \}$. Then $\ell_{k+1} > \ell_k$ if and only if $\text{Conv}\,\mathbb{L}(\ell_k) \cap \{ x \in \mathbb{R}^n : Ax \leq b \} = \emptyset$.

Proof: See Barcia (1987). ■

Note now that if the objective function cx is such that its level sets are singletons the first value of the sequence $\{\ell_k\}$ for which there will de no strict improvement will be z . In this case (LDk) provides a device to bridge the duality gap in a finite number of steps.

For a general objective function one must either supplement (LDk) with an enumeration scheme to deal with the cases for which $\ell_{k+1} = \ell_k$ or just use the above method as a device for improving Lagrangean bounds . Note that the more "complicated" the original $\mathcal{X}$ set is the more "likely" the level sets of cx are sigletons , so more complicated $\mathcal{X}$ sets will , eventually, provide better bounds without enumeration.

When equality occurs , one must search the hyperplane cx = ℓ_k

for a (P)-feasible point. If such a point is found it's optimal . If no such a point can be found we can take $\ell_{k+1} = \ell_k + 1$ and keep using (LDk) to generate a non-decreasing sequence of bounds converging to z.

This procedure is interesting only when enumeration is not performed very often because of its time consuming nature.

Barcia & Holm (1988) report a revised version of the basic algorithm we just sketched enabling some savings in the number of times that the enumeration scheme must be used.

In this note we shall look at (LDk) just as a device for improving available bounds , so no enumeration step will be needed.

Let us state now the final result of this section which, although not computationaly interesting ,will be very useful when trying to understand , from a primal standpoint , how (LDk) operates:

Theorem 2.4: Computing (LDk) is tantamount to convexifying the non relaxed constraints in problem (Pk),i.e.,we have :

$$\ell_{k+1} = \min cx$$
$$Ax \leq b$$
$$x \in \text{Conv } \mathcal{X}_k$$

Proof: See Barcia (1987). ∎

3. LAGRANGEAN DECOMPOSITION

In this section we recall and comment on the main results on the Variable splitting / Lagrangean decomposition approach. Our aim is to lay the ground for section 4 where we will use these results to show how a combined approach with bound improving sequences can provide bounds dominating those obtained by either of the two individual techniques.

Again , only the main ideas will be sketched . All proofs will be ommited . The reader is referred to Jörnsten, Näsberg & Smeds (1985) and Guignard & Kim (1987) for full proofs and a more detailed treatment of the subject. Some previous applications of the technique can be found in Ribeiro & Minoux (1985) , Jörnsten & Näsberg (1986) and Minoux (1987) , among others.

Let us consider a structured pure integer programming problem of the form

$$\text{(SP)} \qquad \begin{aligned} z = \min \;& cx \\ Ax &\leq b \\ Bx &\leq d \\ x &\in X \end{aligned}$$

in which the constraint matrix A is such that a problem containing only the constraints $\{ Ax \leq b \, , \, x \in X \}$ is easier to solve than a general integer programming problem , i.e., there exists some special purpose method for the problem $\{\min cx : Ax \leq b \, , \, x \in X \}$. We will also assume that the same is true for the problem in the second structure { min cx : $Bx \le d$, $x \in$ [illegible] . However problem (SP) in which both constraint sets are present is assumed to be much more difficult to solve.

Problems involving such usable substructures are often solved using a Lagrangean relaxation approach in which one of the constraint sets is relaxed, thus generating subproblems in the other substructure as follows :

$$\ell_A = \max_{u \ge 0} \quad \min_{\substack{Bx \le d \\ x \in X}} \quad cx + u\,(Ax - b)$$

Depending on the problem and on the constraint sets, one of the bounds ℓ_A or ℓ_B, where ℓ_B stands for the Lagrangean bound obtained by relaxing only the constraints $Bx \le d$, may be stronger than the other.

Recently a method that makes use of more than one structure has been suggested and named "variable splitting" by Jörnsten, Näsberg & Smeds (1985) and "Lagrangean decomposition" by Guignard & Kim (1987).

The idea behind the technique is to use a different "copy" of the original variables for each substructure and thus reformulate (SP) into a problem having twice as many variables, as in problem (SPR) below:

$$\text{(SPR)}\qquad \begin{aligned} z = \min(\ \alpha\, cx - \beta\, cy\ ) \\ Ax \quad \le b \\ By \le d \\ x = y \\ x \in \mathbb{X}\ ,\ y \in \mathbb{Y}, \end{aligned}$$

which is a valid reformulation for any $\mathbb{Y} \supset \mathbb{X}$ and any $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta = 1$.

If the constraints $x = y$ are relaxed the subproblem obtained will be separable , i.e., in order to solve the Lagrangean dual one has to solve two subproblems with known usable structures.

Consider then free Lagrangean multipliers $v \in \mathbb{R}^n$ for the equality constraints $x = y$. The Lagrangean dual of (SPR) will be:

$$\text{(LDR)}\qquad d = \max_{v \in \mathbb{R}^n}\ \left[\ \min_{\substack{Ax \le b \\ x \in \mathbb{X}}} (\alpha c - v)x \ + \min_{\substack{By \le d \\ y \in \mathbb{Y}}} (\beta c + v)y\ \right]$$

Note that we now have a x-subproblem and a y-subproblem taking full advantage of the two substructures exibited by the problem. Note also that , as we must only have $\mathbb{Y} \supset \mathbb{X}$, one may "forget" the integrality constraints in one of the subproblems.

So the new bound d is easy to compute . Of course the key

Theorem 3.1: $d \geq \max\{\ell_A, \ell_B\} \geq \ell$.

Proof: See Jörnsten, Näsberg & Smeds (1985). ■

So we know that the Lagrangean decomposition bound dominates the usual Lagrangean bounds .

Of course the interesting question is to be able to tell in which situations the above inequality degenerates into an equality . The answer for this is based on a property which is an analogue of Geoffrion's integrality property for the conventional Lagrangean technique:

Definition 3.2: The set $\{ x : Ax \leq b \}$ is said to be $\mathcal{X}$-convex if $\mathrm{Conv}\{ x : Ax \leq b , x \in \mathcal{X} \} = \{ x : Ax \leq b , x \in \mathrm{Conv}\, \mathcal{X} \}$

Using the above $\mathcal{X}$-convexity property we can now state a sufficient condition for equality to hold in theorem 3.1. In fact the following result holds:

Theorem 3.3: If the set $\{ x : Ax \leq b \}$ is $\mathcal{X}$-convex and if $\{ y : By \leq d , y \in \mathcal{Y} \}$ is compact, then $d = \ell_A$.

Proof: See Guignard & Kim (1987). ■

Note first that a similar result holds for the case of the equality $d = \ell_B$. It can be derived from theorem 3.3 simply by interchanging the roles of the two constraint sets.

Note also that the compactness hypothesis assumed above is very "mild" since it will always be satisfied if we take $\mathcal{Y} = \mathcal{X}$, as $\mathcal{X}$ is assumed to be a bounded subset of $\mathbb{Z}^n$ .This means that , for "practical" purposes the $\mathcal{X}$-convexity property implies the equality $\mathit{d} = \ell_{A}$.

We shall now terminate this section by stating a result, similar in nature to theorem 2.4, giving a nice primal interpretation of (LDR) that will be useful later on.

Theorem 3.4: Computing (LDR) is tantamount to minimizing cx over the intersection of the convex hulls of the two substructures in (SP), i.e., we have:

$$\mathit{d} = \{ \min cx : x \in \text{Conv}\{x \in \mathcal{X} : Ax \leq b\} \cap \text{Conv}\{x \in \mathcal{Y} : Bx \leq d\} \}$$

Proof: See Guignard & Kim (1987). ■

4. The combined approach

We shall now combine the ideas of Lagrangean decomposition with those of bound improving sequences in order to obtain a bound that dominates the one obtained by either of the two individual approaches.

As in section 3 we shall consider the specially structured pure integer programming problem (SP).

Suppose now that a lower bound for z , the optimal value of (SP) , is available and call it $\mathscr{d}_k$. As in section 2 let us now consider the set $\mathscr{X}_k = \{ x \in \mathscr{X} : cx \geq \mathscr{d}_k \}$.

We can now state the following problem

$$
\text{(SPRk)} \qquad
\begin{aligned}
z = \min(\alpha\, cx + \beta\, cy)& \\
Ax \qquad &\leq b \\
By &\leq d \\
x = y& \\
x \in \mathscr{X}_k ,\ y \in \mathscr{Y}&,
\end{aligned}
$$

which we know to be a valid reformulation of (SP) for any $\mathscr{Y} \supset \mathscr{X}_k$ and any $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta = 1$.

In order to simplify some proofs we shall assume that the starting lower bound used was the Lagrangean decomposition bound , i.e. , $\mathscr{d}_o = \mathscr{d}$. We shall further assume that we took $\alpha=1$ and $\beta=0$ and that $\mathscr{Y}$ was chosen in such a way that the set $\{ y \in \mathscr{Y} : By \leq d \}$ is a non-empty compact subset of $\mathbb{R}^n$.

Now take a Lagrangean dual of (SPRk) by relaxing the equality constraints $x = y$. One gets:

$$(\text{LDRk}) \quad d_{k+1} = \max_{v \in \mathbb{R}^n} \left[\min_{\substack{Ax \leq b \\ x \in \mathcal{X}_k}} (c-v)x + \min_{\substack{By \leq d \\ y \in \mathcal{Y}}} vy \right]$$

Note that (LDRk) defines , for $k=0,1,\ldots$, a sequence of bounds in the spirit of section 2 .Note also that d_{k+1} need not be more difficult to compute than the usual Lagrangean decomposition bound d, depending on the structure of the constraints $Ax \leq b$ and on the original $\mathcal{X}$ set : if these constraints are , for instance , semi-assignment constraints and $\mathcal{X}=\{0,1\}^n$ then the x-subproblem is a 0-1 multiple-choice knapsack problem which can be solved with moderate effort.An obvious result is the following:

Theorem 4.1: $d_{k+1} \geq \ell_{k+1}$.

Proof: Directly from theorem 3.1 . ∎

The question we address now is the following : how does d_{k+1} relate to the usual Lagrangean decomposition bound ? . The following result is easily proved:

Theorem 4.2: $d_{k+1} \geq d_k \geq d$

Proof: The first inequality is obtained directly from (LDRk)

So now we know that (LDRk) defines a bound improving sequence dominating the Lagrangean decomposition bound . The interesting issue is , of course , to be able to state a condition for strict improvement . Before addressing this question we must state a very simple preliminary lemma.

Lemma 4.3: Consider $\mathbb{S}$ and $\mathbb{T}$ as two subsets of $\mathbb{U}$ and let $\mathbb{S}\otimes\mathbb{T}$ denote their Cartesian product . Then we have :

$(u,u) \in$ Conv $\mathbb{S}\otimes\mathbb{T}$ if and only if $u \in$ Conv $\mathbb{S} \cap$ Conv $\mathbb{T}$.

Proof: The proof is trivial. ∎

We are now equiped to prove a result that characterizes the cases for which we have $d_{k+1} > d_k$.

Theorem 4.4 : We shall have the strict inequality $d_{k+1} > d_k$ if and only if the following condition holds:

Conv$\{ x \in \mathbb{X} : cx = d_k , Ax \leq b \} \cap$ Conv$\{ y \in \mathbb{Y} : By \leq d \} = \emptyset$

Proof: Note first that problem (SPRk) can be stated in the following equivalent way :

$$\{ \min cx : x = y , cx \geq d_k \text{ and } (x,y) \in \mathcal{X} \}$$

where $\mathcal{X} \subseteq \mathbb{X}\otimes\mathbb{Y}$ denotes the set $\mathcal{X}=\{ x\in\mathbb{X} , y\in\mathbb{Y} : Ax \leq b , By \leq d \}$.

Let's denote by $\mathcal{X}_k$ the set $\mathcal{X}_k = \{ (x,y) \in \mathcal{X} : cx \geq d_k \}$.

Now , if we relax the equality constraints $x = y$ we will obtain d_{k+1} as follows:

(i) $$d_{k+1} = \max_{v \in \mathbb{R}^n} \ \min_{(x,y) \in \mathcal{X}_k} cx + v(y-x)$$

Note that (i) defines a bound improving sequence , as in section 2 , so theorem 2.3 applies and then we know that we will have $d_{k+1} > d_k$ if and only if the level set $\mathbb{L}(d_k)$ is such that Conv $\mathbb{L}(d_k) \cap \{ (x,y) \in \mathbb{R}^{2n} : x = y \} = \emptyset$ .So in the convex hull of the level set $\mathbb{L}(d_k)$

$\mathbb{L}(d_k) = \{(x,y) \in \mathcal{X} : cx=d_k\} = \{x \in \mathcal{X} : cx=d_k, Ax \le b\} \otimes \{y \in \mathcal{Y} : By \le d \}$

there can not exist any point with equal coordinates $x = y$.

Then lemma 4.3 implies that we must have

$\mathrm{Conv}\{ x \in \mathcal{X} : cx = d_k , Ax \le b \} \cap \mathrm{Conv}\{ y \in \mathcal{Y} : By \le d \} = \emptyset$

and the proof is complete. ∎

Compare now the contents of theorems 3.4 and 4.4.

We have the following , because of theorem 3.4 :

$d = \{ \min cx : x \in \mathrm{Conv}\{ x \in \mathcal{X} : Ax \le b \} \cap \mathrm{Conv}\{ x \in \mathcal{Y} : Bx \le d \} \}$

and call x^* an optimal solution of the above problem . If $x^* \in \mathcal{X}$ then it's the optimal solution for the original problem and $d = z$.

Suppose now that $x^* \notin \mathcal{X}$ and let's examine the possibility of not being able to improve the current bound d by using the combined approach.

If this is so theorem 4.4 implies the following :

$\mathrm{Conv}\{ x \in \mathcal{X} : cx = d , Ax \le b \} \cap \mathrm{Conv}\{ x \in \mathcal{Y} : Bx \le d \} \ne \emptyset$

But this is only possible if there exist points $x_i \in \mathcal{X}$ such that $Ax_i \le b$, $cx_i = cx^* = d$ and $x^* \in \mathrm{Conv}\{ x_i\} \cap \mathrm{Conv}\{ x \in \mathcal{Y} : Bx \le d \}$. This is a very "unlikely" situation because it would imply that $cx = d$ supports a non zero-dimensional (because $x^* \notin \mathcal{X}$) face of the polytope $\mathrm{Conv}\{ x \in \mathcal{X} : Ax \le b \}$ having a non empty

Corollary 4.5: Suppose $d < z$ and the following nondegeneracy assumption holds : $cx = d$ does not support any non zero-dimensional face of the polytope Conv{ $x \in \mathbb{X}$: $Ax \leq b$ } having a non empty intersection with Conv{ $x \in \mathbb{Y}$:$Bx \leq d$ }. Then we have $d_1 > d$.

which tells us that , in almost every practical situation , the new combined approach will produce a strict improvement on the Lagrangean decomposition bound.

We shall now terminate this section by giving a primal interpretation of (LDRk).

Theorem 4.6: Computing d_{k+1} is tantamount to solving either of the two following modified primals:

$$d_{k+1} = \min_{x = y\ ,\ (x,y) \in \text{Conv}\ \mathcal{X}_k} cx = \min_{x \in \text{Conv}\ \mathbb{A}_k \cap \text{Conv}\ \mathbb{B}} cx$$

where $\mathcal{X}_k = \{ x \in \mathbb{X}\ ,\ y \in \mathbb{Y} : cx \geq d_k\ ,\ Ax \leq b\ ,\ By \leq d \}$,

$\mathbb{A}_k = \{ x \in \mathbb{X} : cx \geq d_k,\ Ax \leq b \}$ and $\mathbb{B} = \{ x \in \mathbb{Y} : Bx \leq d \}$.

Proof: The equality d_{k+1}={ min cx: $x = y$, $(x,y) \in$ Conv $\mathcal{X}_k$ } comes from using theorem 2.4 in formulation (SPRk) while the equality d_{k+1} = { min cx: $x \in$ Conv $\mathbb{A}_k \cap$ Conv $\mathbb{B}$ } is obtained by using theorem 3.4 in the same formulation.

Note finally that the equality of the two primal formulations above can also be obtained directly using lemma 4.3. ■

5. THE GENERALIZED ASSIGNMENT PROBLEM.

The generalized assignment problem , (GAP) as stated below,is one of the problems to which Lagrangean relaxation has been applied with relatively good results.

$$z = \min \sum_{i,j} c_{ij} x_{ij}$$

$$\text{(GAP)} \qquad \sum_j a_{ij} x_{ij} \le b_i \quad \text{for all } i$$

$$\sum_i x_{ij} = 1 \quad \text{for all } j$$

$$x_{ij} \in \{0,1\} \quad \text{for all } i \text{ and } j$$

An approach to the GAP based on the variable splitting technique has been tested in Jörnsten & Näsberg (1986) with results of a similar quality as the results one gets when relaxing the semi-assignment constraints in the usual Lagrangean way . We know now that this is due to the fact that the semi-assignment constraints have the $\mathcal{K}$-convexity property.

In our combined approach this problem is resolved since a knapsack constraint is appended to the semi-assignment subproblem.

Let α_k be an available lower bound for z. A valid reformulation for the (GAP) , allowing for the combination of Lagrangean decomposition and bound improving sequences ,is the

$$z = \min \sum_{i,j} c_{ij}x_{ij}$$

$$\sum_j a_{ij}y_{ij} \leq b_i \quad \text{for all } i$$

$$\sum_i x_{ij} = 1 \quad \text{for all } j$$

$$\sum_{i,j} c_{ij}x_{ij} \geq d_k$$

$$x_{ij} = y_{ij} \quad \text{for all } i \text{ and } j$$

$$x_{ij}, y_{ij} \in \{0,1\} \quad \text{for all } i \text{ and } j$$

Relaxing now the constraints $x_{ij} = y_{ij}$ one gets the following subproblems:

$$\min \sum_{i,j} (c_{ij}-v_{ij})x_{ij} \quad + \quad \min \sum_{i,j} v_{ij}y_{ij}$$

$$\sum_{i,j} c_{ij}x_{ij} \geq d_k \qquad \sum_j a_{ij}y_{ij} \leq b_i \quad \text{for all } i$$

$$\sum_i x_{ij} = 1 \quad \text{for all } j \qquad y_{ij} \in \{0,1\} \quad \text{for all } i \text{ and } j$$

$$x_{ij} \in \{0,1\} \quad \text{for all } i \text{ and } j$$

Thus a multiple-choice knapsack problem has to be solved in the x-variables and a set of ordinary 0-1 knapsack problems have to be solved for the y-variables. Note that those problems are easy to solve , see Walukiewicz & Dudzinsky (1987) , and the dual thus formulated is a strong one since the $\mathcal{K}$-convexity property is not satisfied for either subproblem.

We have implemented the combined approach using Fayard & Plateau (1982) code to solve the single 0-1 knapsacks and a simple branch and bound algorithm, using bounds from the linear relaxation as described in Walukiewicz & Dudzinsky (1987), to solve the 0-1 multiple-choice knapsack problems.

The "optimal" multipliers were found using an "ad hoc" multiplier adjustment formula , as described in Jörnsten & Näsberg(1986) , because it was found to be faster than standard subgradient optimization, as in Held Wolfe & Crowder(1974).

We coded the new algorithm in FORTRAN and run it on a IBM PS/2 Model 80-111 personal computer equiped with a 80387 coprocessor, using the IBM FORTRAN/2 compiler under DOS 3.3, for 20 randomly generated 4X25 generalized assignment problems.

These problems are 100 variable 0-1 LP's. The first 10 are the same ones used as test problems in Jörnsten & Näsberg (1986). The last 10 problems are , on the average , much harder.

Data for problem 20 are presented in table II . Data for problems 1 to 19 can be obtained from the authors.

We also ran Roy Marsten's ZOOM/XMP (1985) mixed integer programming code, as well as the Martello & Toth(1981) optimal GAP algorithm, using the same computer environement, on the 20 test problems.

The results are presented on table I below.

Column 1 contains the optimal value of the LP relaxation, column 2 displays the Lagrangean decomposition bound while column 3 shows the value obtained by the combined approach. The three bounds are presented as a fraction of the optimal value.

The last three columns ilustrate the combined approach performance in column 6, as compared to a standard MIP code such as Roy Marsten(1985) ZOOM/XMP in column 4, or to an optimal GAP algorithm such as Martello & Toth(1981) in column 5. Figures show the computer time , in seconds , if it was smaller than the time limit of 15 minutes.

The new Lagrangean decomposition / bound improving sequences combined approach appears to be a promising tool for the generalized assignment problem: On one hand it generaly improves the Lagrangean decomposition bound up to the optimal value, if such an improvement is needed. On the other hand , when the optimal value is found , the new approach appears to be faster than either the general purpose MIP code or the optimal GAP algorithm we used , with the possible exception of very "easy" problems such as problems 1 and 20.

Further improvements in speed can be expected if a faster code could be used for the 0-1 multiple-choice knapsack problem, instead of our simple branch and bound routine, and an analogue of the multiplier adjustment technique used by Fisher,Jaikumar & Wassenhove(1986) could be derived for this case, to replace the "ad hoc" procedure implemented up to now.

6. Conclusions

In this paper we have presented how two "new" methods for pure integer programming can be combined . It has been shown that the new method , consisting of Lagrangean decomposition and bound improving sequences ideas , has the ability to generate better bounds. For specially structured problems, such as the generalized assignment problem, this combined method appears to be efficient.

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TABLE I

Problem	Bounds			Time		
	(1) BLP	(2) BLD	(3) BILD	(4) TXMP	(5) TMT	(6) TILD
1	.989	1.000	1.000	92	1	6
2	.961	.998	1.000	150	44	8
3	.940	1.000	1.000	153	76	6
4	.961	1.000	1.000	398	61	8
5	.892	.961	.968	637	64	78
6	.955	1.000	1.000	555	27	13
7	.944	.985	1.000	604	431	75
8	.961	1.000	1.000	170	4	6
9	.967	.981	.991	*	694	206
10	.968	1.000	1.000	313	53	10
11	.958	.994	1.000	*	645	93
12	.939	.992	.999	480	11	97
13	.969	.993	1.000	577	*	56
14	.965	1.000	1.000	*	81	37
15	.933	.959	1.000	*	147	62
16	.893	.935	1.000	*	388	209
17	.981	.992	1.000	*	*	58
18	.927	.943	.978	*	234	317
19	.930	.972	1.000	178	8	70
20	.992	1.000	1.000	50	1	4

* : Time limit (more than 900 seconds)

BLP: Linear programming relaxation bound (as a fraction of the optimum)

BLD: Lagrangean decomposition bound (as a fraction of the optimum)

BILD: Improved Lagrangean decomposition bound (as a fraction of the optimum)

TXMP: Computer time in seconds for XMP

TABLE II

a_{ij}

i \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	b_i
1	69	56	57	44	24	37	13	42	46	60	27	26	23	30	10	61	13	16	39	52	44	27	24	42	39	295
2	34	28	35	6	14	4	48	62	74	38	68	58	2	30	61	32	49	67	32	27	45	30	20	14	31	214
3	37	32	18	70	33	37	4	31	36	44	14	48	10	18	0	47	38	12	62	32	11	17	62	48	68	301
4	32	15	19	1	32	61	34	11	38	23	22	53	49	27	66	2	44	16	58	65	7	70	60	37	73	286

c_{ij}

i \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	32	53	84	23	74	63	97	39	84	51	65	27	23	70	8	20	41	46	49	4	38	92	29	14	55
2	62	90	36	36	26	80	83	64	66	18	51	34	11	91	16	79	27	33	89	22	52	51	18	98	24
3	69	82	83	6	69	82	99	69	20	86	86	69	20	80	93	39	1	86	26	94	62	62	66	86	71
4	91	40	81	82	16	47	3	8	36	22	39	85	50	23	9	8	38	63	97	46	74	95	16	72	6