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OPTIMAL CONTROL OF NEWTONIAN FLUIDS IN A STOCHASTIC ENVIRONMENT*

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Abstract. We consider a velocity tracking problem for stochastic Navier–Stokes equations in a 2D-bounded domain. The control acts on the boundary through an injection-suction device with uncertainty, which acts in accordance with the nonhomogeneous Navier-slip boundary conditions. After establishing a suitable stability result for the solution of the stochastic state equation, we prove the well-posedness of the stochastic linearized state equation and show that the Gâteaux derivative of the control-to-state mapping corresponds to the unique solution of the linearized equation. Next, we study the stochastic backward adjoint equation and establish a duality relation between the solutions of the forward linearized equation and the backward adjoint equation. Finally, we derive the first-order optimality conditions.

Key words. stochastic Navier–Stokes equations, Navier-slip boundary conditions, stochastic backward equation, first-order optimality conditions

MSC codes. 60H15, 6D55, 93E20, 49K45

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1. Introduction.

Optimal control problems for fluid flows have been extensively studied in the literature and are of major importance in technology. Roughly speaking, the control is exercised either within the domain occupied by the fluid, by some distributed force (acting over the entire domain or over some specific region), or on the boundary of the domain.

Distributed optimal control problems for Newtonian fluids, described by the deterministic Navier–Stokes equations have been addressed in the literature, let us mention [20], [24], [37], [38], and references therein (see also the recent related model [34]). The deterministic case for distributed forces is quite well understood from the theoretical and numerical point of views. The boundary control problems are more singular ones but of great importance because they appear in many technological applications, namely where the control of the flow is implemented by an injection/suction device placed on the boundary of the domain (see, for instance, [5], [6], [14], [16], [21], [24], [26], [27]).

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On the other hand, real world physical systems always experience random fluctuations and inaccuracy of measurements, which are modelled by adding to the deterministic partial differential equation suitable random internal/external forces or by considering random initial data and boundary conditions. It is worth stressing that the optimal control of a fluid flow in a stochastic environment is much more involved that its deterministic analogous, and just few results are available in the literature. Let us mention [3], [4], [7], [29], [30], [32], [33], where the authors solved stochastic tracking control problems for Newtonian and non-Newtonian fluids in two dimensions and three dimensions. In these works the control variables act in the interior of the domain.

The authors in [39] studied a stochastic boundary control problem for the deterministic steady Navier–Stokes equations, where the stochastic control is imposed just on the boundary by a nonhomogeneous Dirichlet boundary condition. We recall that the solution of the Navier–Stokes supplemented with the Dirichlet boundary condition develops strong boundary layer for small values of the viscosity. Then in the last few decades a great attention has been developed to the study of deterministic Navier–Stokes equations supplemented with the Navier-slip boundary conditions. According to the studies [8], [9], [10], [11], [12], [13], [14], [15], [16], [18], the nonhomogeneous Navier-slip boundary conditions are compatible with the inviscid limit transition, which suggests that, comparing with the nonhomogeneous Dirichlet boundary conditions, the Navier-slip boundary conditions seems more appropriate to control the evolution of turbulent flows typically associated with high Reynolds number (or small viscosity).

The present work addresses an optimal boundary control problem for stochastic Navier–Stokes equation driven by a multiplicative Gaussian noise, on a bounded domain $\mathcal{O} \subset \mathbb{R}^2$. The dynamical law reads

$$
\begin{cases}
d\mathbf{y} = (\nu\Delta\mathbf{y} - (\mathbf{y}\cdot\nabla)\,\mathbf{y} - \nabla\pi)\,dt + \mathbf{G}(t,\mathbf{y})\,d\mathcal{W}_t, & \\
 & \text{in } \mathcal{O}_T = (0,T)\times\mathcal{O}, \\
\operatorname{div}\mathbf{y} = 0, & \\
\mathbf{y}\cdot\mathbf{n} = a, \quad [2D(\mathbf{y})\,\mathbf{n} + \alpha\mathbf{y}]\cdot\boldsymbol{\tau} = b & \text{on } \Gamma_T = (0,T)\times\Gamma, \\
\mathbf{y}(0,\mathbf{x}) = \mathbf{y}_0(\mathbf{x}) & \text{in } \mathcal{O},
\end{cases}
\tag{1.1}
$$

where $\mathbf{y} = \mathbf{y}(t,\mathbf{x})$ is the two-dimensional (2D)-velocity random field, $\pi = \pi(t,\mathbf{x})$ is the pressure, $\nu > 0$ is the viscosity, and $\mathbf{y}_0$ is the initial condition that verifies

$$
\operatorname{div}\mathbf{y}_0 = 0 \qquad \text{in } \mathcal{O}. \tag{1.2}
$$

Here

$$
D(\mathbf{y}) = \frac{1}{2}[\nabla\mathbf{y} + (\nabla\mathbf{y})^T]
$$

is the rate-of-strain tensor; $\mathbf{n}$ is the external unit normal to the boundary $\Gamma \in C^2$ of the domain $\mathcal{O}$ and $\boldsymbol{\tau}$ is the tangent unit vector to Γ, such that $(\mathbf{n},\boldsymbol{\tau})$ forms a standard orientation in $\mathbb{R}^2$. The positive constant α is the so-called friction coefficient. The quantity a corresponds to the inflow and outflow fluid through Γ, satisfying the compatibility condition

$$
\int_\Gamma a(t,\mathbf{x})\,d\gamma = 0 \qquad \text{for any } t \in [0,T]. \tag{1.3}
$$

This condition means that the quantity of inflow fluid should coincide with the quantity of outflow fluid. The boundary functions a and b will be considered as the control variables for the physical system (1.1). The term $\mathbf{G}(t,\mathbf{y})\,\mathcal{W}_t$ is a multiplicative white noise.

The main goal is to minimize the following cost functional:

$$J(a,b,\mathbf{y}) = \mathbb{E}\int_{\mathcal{O}_T} \frac{1}{2}|\mathbf{y}-\mathbf{y}_d|^2\,d\mathbf{x}dt + \mathbb{E}\int_{\Gamma_T}\left(\frac{\lambda_1}{2}|a|^2 + \frac{\lambda_2}{2}|b|^2\right)d\gamma dt,$$

constrained to the stochastic Navier–Stokes equation (1.1), where $\mathbf{y}_d \in L_2(\Omega\times\mathcal{O}_T)$ is a desired target field, $\lambda_1, \lambda_2 > 0$ and (a,b) will be taken is an appropriate space of admissible controls. From physical point of view, the variables a and b describe the quantity of the fluid crossing the boundary, and the vorticity component on the boundary (see [8, Corollary 1] and [9]), respectively, which can be prescribed by the operational controller of the physical system. In practice, the vorticity can be induced through some mechanisms such as rotating container walls, moving walls, jet injection/suction (tangentially to the wall), rough surfaces, electromagnetic forces (for conducting fluids), and many others. On the other hand, the induction of vorticity in fluid dynamics is crucial for optimizing certain industrial processes, enhancing mixing efficiency, and controlling flow patterns in various applications. By precisely manipulating boundary conditions to generate desired vorticity, engineers can improve the performance of systems such as chemical reactors, aerodynamic surfaces, heating-ventilation-air conditioning systems, etc. This control allows for efficient heat transfer, uniform mixing of reactants, and improved stability of fluid flows. Optimal vorticity induction is also essential also in environmental applications, such as pollutant dispersion and water treatment processes, ensuring effective and sustainable solutions.

The first stage towards the resolution of this optimal control problem has been already performed in the article [16], where the authors established the well-posedness of the state system (1.1), and showed the existence of an optimal solution (a,b) in a compact set of predictable stochastic processes verifying an exponential integrability condition.

The plan of the present paper is as follows. Section 2 introduces the appropriate functional spaces and recalls convenient results from [16] about the solvability of the stochastic Navier–Stokes equations supplemented with the nonhomogeneous Navier-slip boundary conditions. Section 4 presents a stability result for the solution of the stochastic state equation. Next, we formulate the control problem in section 3.

Section 5 improves the integrability properties of the state, which is crucial in section 6 to show the well-posedness of the linearized state equation. Then, we proceed by studying the Gâteaux differentiability of the control-to-state mapping in section 7. Section 8 is devoted to the analysis of the backward stochastic adjoint equation. Finally, section 9 is devoted to show a suitable duality relation between the solution of the linearized equation and the solution of the adjoint equation, which is the main ingredient to deduce the first-order-optimality conditions in section 10.

2. Functional setting and preliminary results to the state equation. First, let us introduce the notations and present some results, used in the article.

Let us consider a real Banach space X, endowed with the norm $\|\cdot\|_X$. The space of X-valued measurable p-integrable functions, defined on the time interval $[0,T]$, is denoted by $L_p(0,T;X)$ for $p \geqslant 1$.

The space $L_p(\Omega, L_r(0,T;X))$ for $p,r \geqslant 1$ of the processes $\mathbf{v} = \mathbf{v}(\omega,t)$ with values in X, defined on $\Omega\times[0,T]$ and adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, is endowed with the norms

$$\|\mathbf{v}\|_{L_p(\Omega,L_r(0,T;X))} = \left(\mathbb{E}\left(\int_0^T \|\mathbf{v}\|_X^r\, dt\right)^{\frac{p}{r}}\right)^{\frac{1}{p}}$$

and

$$\|\mathbf{v}\|_{L_p(\Omega,L_\infty(0,T;X))} = \left(\mathbb{E}\sup_{t\in[0,T]} \|\mathbf{v}\|_X^p\right)^{\frac{1}{p}} \quad \text{for } r=\infty,$$

where $\mathbb{E}$ is the mathematical expectation with respect to the probability measure P, defined on Ω. By default we will omit the dependence on the parameter $\omega\in\Omega$ in the notation for processes $\mathbf{v}=\mathbf{v}(\omega,t)$.

We use the standard notation for the Lebesgue spaces $L_p(\mathcal{O})$ and the Sobolev spaces $H^p(\mathcal{O})$ with the norms denoted by $\|\cdot\|_p$ and $\|\cdot\|_{H^p}$, respectively. The inner product in $L_2(\mathcal{O})$ is denoted as $(\cdot,\cdot)$ with the associated norm $\|\cdot\|_2$. Let us introduce the following divergence free spaces:

$$\begin{aligned} H &= \{\mathbf{v}\in L_2(\mathcal{O}) : \text{div } \mathbf{v}=0 \quad \text{in } \mathcal{D}'(\mathcal{O}),\ \ \mathbf{v}\cdot\mathbf{n}=0 \quad \text{in } H^{-1/2}(\Gamma)\}, \\ V &= \{\mathbf{v}\in H^1(\mathcal{O}) : \text{div } \mathbf{v}=0 \text{ a.e. in } \mathcal{O},\ \ \mathbf{v}\cdot\mathbf{n}=0 \quad \text{in } H^{1/2}(\Gamma)\}, \end{aligned}$$

where the following inner product

$$(\mathbf{v},\mathbf{z})_V = 2\,(D\mathbf{v},D\mathbf{z}) + \alpha\int_\Gamma \mathbf{v}\cdot\mathbf{z}\, d\gamma$$

is considered on the space V endowed with the norm $\|\mathbf{v}\|_V = \sqrt{(\mathbf{v},\mathbf{v})_V}$.

In what follows we will often use the results of continuous embedding

$$H^1(0,T)\subset C([0,T]), \qquad H^1(\mathcal{O})\subset L_2(\Gamma). \tag{2.1}$$

Let us present the results of [28, pp. 62 and 69], [31, pp. 125], and [35, pp. 16–20] in the next lemma.

Lemma 2.1. *Let us introduce the notation*

$$\mathbf{v}_{\mathcal{O}} = \int_{\mathcal{O}} \mathbf{v}\ d\mathbf{x}. \tag{2.2}$$

For $q\geqslant 2$ and $\mathbf{v}\in H^1(\mathcal{O})$ the Gagliano–Nirenberg–Sobolev inequality

$$||\mathbf{v}-\mathbf{v}_{\mathcal{O}}||_q \leqslant C||\mathbf{v}||_2^{2/q}||\nabla\mathbf{v}||_2^{1-2/q}, \tag{2.3}$$

and the trace interpolation inequality

$$||\mathbf{v}-\mathbf{v}_{\mathcal{O}}||_{L_q(\Gamma)} \leqslant C||\mathbf{v}||_2^{1/q}||\nabla\mathbf{v}||_2^{1-1/q} \tag{2.4}$$

are valid. Moreover, any $\mathbf{v}\in V$ satisfies the Korn inequality

$$\|\mathbf{v}\|_{H^1} \leqslant C\,\|\mathbf{v}\|_V\,. \tag{2.5}$$

Let us remark that

$$\mathbf{v}_{\mathcal{O}} = 0 \qquad \forall \mathbf{v}\in V,$$

since

$$\int_{\mathcal{O}} v_j \, d\mathbf{x} = \int_{\mathcal{O}} \text{div } (\mathbf{v}x_j) \, d\mathbf{x} = \int_{\Gamma} x_j (\mathbf{v}\cdot\mathbf{n}) \, d\gamma = 0 \qquad \text{for} \quad j=1,2.$$

Also for arbitrary $\mathbf{v}\in H^2(\mathcal{O})$, $\mathbf{z}\in V$ integrating by parts, we have

$$-\int_{\mathcal{O}} \triangle \mathbf{v}\cdot\mathbf{z}\, d\mathbf{x} = 2\int_{\mathcal{O}} D\mathbf{v}\cdot D\mathbf{z}\, d\mathbf{x} - \int_{\Gamma} 2(\mathbf{n}\cdot D\mathbf{v})\cdot\mathbf{z}\, d\gamma. \tag{2.6}$$

Therefore,

$$-\int_{\mathcal{O}} \triangle \mathbf{v}\cdot\mathbf{z}\, d\mathbf{x} = (\mathbf{v},\mathbf{z})_V - \int_{\Gamma} b(\mathbf{z}\cdot\boldsymbol{\tau})\, d\gamma \tag{2.7}$$

if the function $\mathbf{v}$ fulfills the Navier-slip boundary conditions (see the system (1.1)).

We will frequently apply in our considerations the Young inequality

$$uv \leqslant \frac{u^p}{p} + \frac{v^q}{q}, \qquad \frac{1}{p}+\frac{1}{q} = 1 \quad \forall p,\, q>1. \tag{2.8}$$

Let us define the norm and the absolute value of the inner product for a vector

$$\mathbf{h} = (\mathbf{h}_1,\ldots,\mathbf{h}_m)\in H^m = \overbrace{H\times\cdots\times H}^{m-times}$$

with a fixed $\mathbf{v}\in H$ as

$$\|\mathbf{h}\|_2 = \left(\sum_{k=1}^m \|\mathbf{h}_k\|_2^2\right)^{1/2} \quad \text{and} \quad |\,(\mathbf{h},\mathbf{v})\,| = \left(\sum_{k=1}^m (\mathbf{h}_k,\mathbf{v})^2\right)^{1/2}. \tag{2.9}$$

Let

$$\mathbf{G}(t,\mathbf{y}) : [0,T]\times H \to H^m \quad \text{with} \quad \mathbf{G}(t,\mathbf{y}) = (G^1(t,\mathbf{y}),\ldots,G^m(t,\mathbf{y}))$$

be Lipschitz on $\mathbf{y}$ and satisfy the linear growth

$$\begin{aligned} &\|\mathbf{G}(t,\mathbf{v})-\mathbf{G}(t,\mathbf{z})\|_2^2 \leqslant K\,\|\mathbf{v}-\mathbf{z}\|_2^2, \\ &\|\mathbf{G}(t,\mathbf{v})\|_2 \leqslant K\,(1+\|\mathbf{v}\|_2) \qquad \forall \mathbf{v},\mathbf{z}\in H,\ t\in[0,T], \end{aligned} \tag{2.10}$$

for some constant $K>0$. Let

$$\mathbf{G}(t,\mathbf{y})\, d\mathcal{W}_t = \sum_{k=1}^m G^k(t,\mathbf{y})\, d\mathcal{W}_t^k$$

be the stochastic noise, where $\mathcal{W}_t = (\mathcal{W}_t^1,\ldots,\mathcal{W}_t^m)$ is a standard $\mathbb{R}^m$-valued Wiener process, defined on a complete probability space $(\Omega,\mathcal{F},P)$ and endowed with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, such that $\mathcal{F}_0$ contains every P-null subset of Ω.

Let us introduce the space of functions

$$\mathcal{H}_p(\Gamma) = \left\{(a,b) : ||(a,b)||_{\mathcal{H}_p(\Gamma)} < +\infty\right\}$$

with the norm

$$||(a,b)||_{\mathcal{H}_p(\Gamma)} = ||a||_{W_p^{1-\frac{1}{p}}(\Gamma)} + ||\partial_t a||_{H^{-\frac{1}{2}}(\Gamma)} + \|b\|_{W_p^{-\frac{1}{p}}(\Gamma)} + \|b\|_{L_2(\Gamma)} + ||\partial_t b||_{H^{-\frac{1}{2}}(\Gamma)}.$$

Let $p\in(2,+\infty)$ be given and the data a,b, and $\mathbf{u}_0$ belong to the following Banach spaces:

$$(a,b)\in L_4(\Omega\times(0,T);\mathcal{H}_p(\Gamma)), \qquad \mathbf{u}_0\in L_4(\Omega;H), \tag{2.11}$$

assuming that (a,b) is a pair of predictable stochastic processes.

Let us formulate the auxiliary result, demonstrated in [16].

LEMMA 2.2. *Let* (a,b) *be a given pair of functions, satisfying* (2.11). *Then there exists a unique solution* $\mathbf{a}$ *of the Stokes problem with the nonhomogeneous Navier-slip boundary condition*

$$\begin{cases} -\Delta \mathbf{a} + \nabla \pi = 0, \quad \nabla \cdot \mathbf{a} = 0 \quad \text{in } \mathcal{O}, \\ \mathbf{a} \cdot \mathbf{n} = a, \quad [2D(\mathbf{a})\,\mathbf{n} + \alpha \mathbf{a}] \cdot \boldsymbol{\tau} = b \quad \text{on } \Gamma, \end{cases} \quad \text{a.e. in } \Omega \times (0,T), \tag{2.12}$$

such that

$$||\mathbf{a}||_{W_p^1(\mathcal{O})} + ||\partial_t \mathbf{a}||_{L_2(\mathcal{O})} \leqslant C||(a,b)||_{\mathcal{H}_p(\Gamma)} \quad \text{a.e. in } \Omega \times (0,T). \tag{2.13}$$

In particular, we have

$$\begin{aligned} &\mathbf{a} \in L_4(\Omega; C([0,T]; L_2(\mathcal{O}))) \cap L_4(\Omega \times (0,T); C(\overline{\mathcal{O}}) \cap W_p^1(\mathcal{O})), \\ &\partial_t \mathbf{a} \in L_4(\Omega \times (0,T); L_2(\mathcal{O})). \end{aligned} \tag{2.14}$$

Here and below positive constants C *depend on the data considered for our problem, such as the domain* $\mathcal{O}$, *the regularity of* Γ, *the physical constants* ν, α, *and a given time moment* T.

Let us recall the concept of the solution of the stochastic differential system (1.1) and the properties of this solution, studied in [16], that will be relevant in our study of the control problem. Using the formula (2.7) and the solution $\mathbf{a}$, founded in Lemma 2.2, we can introduce the concept of the strong (*in the stochastic sense*) solution to the system (1.1).

DEFINITION 2.3. *Assume that the data* (a,b) *and* $\mathbf{u}_0$ *satisfy the regularity* (2.11). *An adapted stochastic process* $\mathbf{y} = \mathbf{u} + \mathbf{a}$ *with* $\mathbf{u} \in C([0,T]; H) \cap L_2(0,T; V)$, P-*a.e. in* Ω, *is a strong solution of* (1.1) *with* $\mathbf{y}_0 = \mathbf{u}_0 + \mathbf{a}(0)$ *if*

$$\begin{aligned} (\mathbf{y}(t), \boldsymbol{\varphi}) = &\int_0^t \left[-\nu\,(\mathbf{y}, \boldsymbol{\varphi})_V + \int_\Gamma \nu b(\boldsymbol{\varphi} \cdot \boldsymbol{\tau})\, d\gamma - ((\mathbf{y} \cdot \nabla)\mathbf{y}, \boldsymbol{\varphi}) \right] ds + (\mathbf{y}_0, \boldsymbol{\varphi}) \\ &+ \int_0^t (\mathbf{G}(s, \mathbf{y}(s)), \boldsymbol{\varphi})\, d\mathcal{W}_s \qquad \text{for a.e. } (\omega, t) \in \Omega \times (0,T), \end{aligned} \tag{2.15}$$

holds for any $\boldsymbol{\varphi} \in V$.

We notice that (2.10), (2.14), and $\mathbf{u} \in C([0,T]; H)$, P-a.e. in Ω, yield

$$\int_0^T \|\mathbf{G}(s, \mathbf{y}(s))\|_2^2\, ds \leq K \int_0^T (1 + \|\mathbf{y}(s)\|_2^2)\, ds < \infty, \qquad P\text{-a.e. in } \Omega.$$

Then the stochastic integral

$$\int_0^t \mathbf{G}(s, \mathbf{y}(s))\, d\mathcal{W}_s, \quad t \in [0,T],$$

is well defined as a H-valued local martingale (cf. [19, pp. 99–100], and

$$\int_0^t (\mathbf{G}(s, \mathbf{y}(s)), \boldsymbol{\varphi})\, d\mathcal{W}_s = \sum_{k=1}^m \int_0^t \left(G^k(s, \mathbf{y}(s)), \boldsymbol{\varphi}\right) d\mathcal{W}_s^k \quad \text{for any } \boldsymbol{\varphi} \in V.$$

By the embedding result $(2.1)_2$ we note that the boundary condition $\mathbf{y} \cdot \mathbf{n} = a$ on Γ, a.e. in $\Omega \times (0,T)$, is well defined for $\mathbf{y} = \mathbf{u} + \mathbf{a}$ with $\mathbf{u} \in L_2(0,T; V)$.

The following result has been proved in Theorem 3.5 of [16].

THEOREM 2.4. *Assume that the data* (a,b) *and* $\mathbf{u}_0$ *satisfy* (2.11). *Then, a strong solution* $\mathbf{y}=\mathbf{u}+\mathbf{a}$ *to the system* (1.1) *exists, such that*

$$\mathbf{u}\in C([0,T];H)\cap L_4(0,T;V), \quad P\text{-a.e. in } \Omega,$$

and there exists a positive constant $\widehat{C}_0$, *such that for any* $t\in[0,T]$,

$$\mathbb{E}\sup_{s\in[0,t]}\xi_0^2(s)\left\|\mathbf{u}(s)\right\|_2^2+\nu\mathbb{E}\int_0^t\xi_0^2(s)\left\|\mathbf{u}(s)\right\|_V^2\,ds$$
$$\leqslant C\left(\mathbb{E}\left\|\mathbf{u}_0\right\|_2^2+\mathbb{E}\int_0^t\xi_0^2(||(a,b)||^2_{\mathcal{H}_p(\Gamma)}+1)\,ds\right),$$

$$\mathbb{E}\sup_{s\in[0,t]}\xi_0^4(s)\left\|\mathbf{u}(s)\right\|_2^4+\nu^2\mathbb{E}\left(\int_0^t\xi_0^2\left\|\mathbf{u}\right\|_V^2\,ds\right)^2$$
$$\leqslant C\left(\mathbb{E}\left\|\mathbf{y}_0\right\|_2^4+\mathbb{E}\int_0^t\xi_0^4(||(a,b)||^4_{\mathcal{H}_p(\Gamma)}+1)\,ds\right) \tag{2.16}$$

with the function

$$\xi_0(t)=e^{-\widehat{C}_0\int_0^t(1+||(a,b)||^2_{\mathcal{H}_p(\Gamma)})\,ds}, \qquad P\text{-a.e. in } \Omega. \tag{2.17}$$

Remark 2.5. It is worth mentioning that we do not know if the solution $\mathbf{y}$, provided by Theorem 2.4, is integrable with respect to the variable $\omega\in\Omega$, we just have integrability of its multiplication by the weight ξ_0, which verifies $0<\xi_0\leq 1$. On the other hand, to tackle the control problem, the cost functional J must be well defined, requiring the square integrability of $\mathbf{y}$. The square integrability of $\mathbf{y}$ holds under suitable integrability of the inverse of the weight. In section 3, we will impose additional assumptions on the boundary conditions a, b, which are necessary to deduce the first-order optimality condition, and also guarantee integrability of $(\xi_0)^{-1}$.

3. Lipschitz continuity of the control-to-state mapping. This section shows the Lipschitz continuity of the control-to-state mapping, which is the first step to study the Gateaux derivative of this mapping. Here, we denote by

$$\widehat{\varphi}=\varphi_1-\varphi_2$$

the difference of two given functions φ_1,φ_2.

THEOREM 3.1. *Under the assumption* (2.11), $(a_i,b_i)\in\mathcal{A}$, *and* $\mathbf{y}_{i,0}$, $i=1,2$, *let* $\mathbf{y}_1=\mathbf{u}_1+\mathbf{a}_1$, $\mathbf{y}_2=\mathbf{u}_2+\mathbf{a}_2$ *with*

$$\mathbf{u}_1,\mathbf{u}_2\in C([0,T];H)\cap L_4(0,T;V), \quad P\text{-a.e.in } \Omega,$$

be two solutions of (1.1) *in the sense of the variational equality* (2.15), *satisfying the estimates* (2.16) *with two corresponding boundary conditions* a_1, b_1, a_2, b_2, *and the initial conditions*

$$\mathbf{y}_{1,0}=\mathbf{u}_{1,0}+\mathbf{a}_1(0), \qquad \mathbf{y}_{2,0}=\mathbf{u}_{2,0}+\mathbf{a}_2(0).$$

Then there exists a constant $\widehat{C}_1$ *such that the following estimates hold:*

$$
\begin{aligned}
(3.1)\qquad & \mathbb{E}\sup_{s\in[0,t]}\xi_1^2(s)\,\|\widehat{\mathbf{u}}(s)\|_2^2+2\nu\mathbb{E}\int_0^t\xi_1^2\|\widehat{\mathbf{u}}\|_{H^1}^2\,ds\\
&\leqslant C\left\{\mathbb{E}\,\|\widehat{\mathbf{u}}_0\|_2^2+\left(\mathbb{E}\int_0^t||(\widehat{a},\widehat{b})||_{\mathcal{H}_p(\Gamma)}^4\,ds\right)^{1/2}\right\},
\end{aligned}
$$

$$
\begin{aligned}
(3.2)\qquad & \mathbb{E}\sup_{s\in[0,t]}\xi_1^4(s)\,\|\widehat{\mathbf{u}}(s)\|_2^4+2\nu^2\mathbb{E}\left(\int_0^t\xi_1^2\|\widehat{\mathbf{u}}\|_{H^1}^2\,ds\right)^2\\
&\leqslant C\left\{\mathbb{E}\,\|\widehat{\mathbf{u}}_0\|_2^4+\mathbb{E}\int_0^t||(\widehat{a},\widehat{b})||_{\mathcal{H}_p(\Gamma)}^4\,ds\right\},
\end{aligned}
$$

where the function ξ_1 *is defined as*

$$
\xi_1(t)=e^{-\int_0^t f_1(s)ds}, \tag{3.3}
$$

with $f_1\in L_1(\Omega\times(0,T))$ *given by*

$$
f_1(t)=\widehat{C}_1(\nu^{-1}+1)\left(\|(a_1,b_1)\|_{\mathcal{H}_p(\Gamma)}^2+\|(a_2,b_2)\|_{\mathcal{H}_p(\Gamma)}^2+\|\mathbf{u}_1\|_V^2+1\right). \tag{3.4}
$$

The constant $\widehat{C}_1$ *is defined by the relations* (3.6)–(3.7).

Proof. Let us denote $\widehat{\mathbf{a}}$ the solution of the Stokes problem (2.12) with $(a,b)=(\widehat{a},\widehat{b})$. Recalling that $\mathbf{y}_1$ and $\mathbf{y}_2$ verify the system (1.1) in the sense of Definition 2.3, we have

$$
\begin{aligned}
d\,(\widehat{\mathbf{y}}(t),\boldsymbol{\varphi})&=\left[-\nu\,(\widehat{\mathbf{y}},\boldsymbol{\varphi})_V+\nu\int_\Gamma\widehat{b}(\boldsymbol{\varphi}\cdot\boldsymbol{\tau})\,d\gamma-(\widehat{B},\boldsymbol{\varphi})\right]dt\\
&\quad+(\widehat{\mathbf{G}},\boldsymbol{\varphi})\,d\mathcal{W}_t\quad\forall t\in[0,T],\\
\widehat{\mathbf{y}}(0)&=\widehat{\mathbf{y}}_0\qquad\forall\boldsymbol{\varphi}\in V,\quad P\text{-a.e.in }\Omega,
\end{aligned}
$$

with

$$
\widehat{B}=(\mathbf{y}_1\cdot\nabla)\,\mathbf{y}_1-(\mathbf{y}_2\cdot\nabla)\,\mathbf{y}_2,\qquad\widehat{\mathbf{G}}(t)=\mathbf{G}(t,\mathbf{y}_1(t))-\mathbf{G}(t,\mathbf{y}_2(t)).
$$

Taking $\boldsymbol{\varphi}=\mathbf{e}_i$ for each $i\in\mathbb{N}$, we can verify that the process $\widehat{\mathbf{u}}=\widehat{\mathbf{y}}-\widehat{\mathbf{a}}$ satisfies the system

$$
\begin{aligned}
d\,(\widehat{\mathbf{u}}(t),\mathbf{e}_i)&=\left[-\nu\,(\widehat{\mathbf{u}}+\widehat{\mathbf{a}},\mathbf{e}_i)_V-((\mathbf{y}_2\cdot\nabla)\,\widehat{\mathbf{u}},\mathbf{e}_i)+\int_\Gamma\nu\widehat{b}(\mathbf{e}_i\cdot\boldsymbol{\tau})\,d\gamma+(\mathbf{U},\mathbf{e}_i)\right]dt\\
&\quad+\,(\widehat{\mathbf{G}},\mathbf{e}_i)\,d\mathcal{W}_t,\\
\widehat{\mathbf{u}}(0)&=\widehat{\mathbf{y}}_0-\widehat{\mathbf{a}}(0),
\end{aligned}
$$

with $\mathbf{U}=-\left[\partial_t\widehat{\mathbf{a}}+((\widehat{\mathbf{u}}+\widehat{\mathbf{a}})\cdot\nabla)\,\mathbf{y}_1+(\mathbf{y}_2\cdot\nabla)\,\widehat{\mathbf{a}}\right]$. Hence the Itô formula gives

$$
\begin{aligned}
d\left((\widehat{\mathbf{u}}(t),\mathbf{e}_i)^2\right)&=2\,(\widehat{\mathbf{u}}(t),\mathbf{e}_i)\,[-\nu\,(\widehat{\mathbf{u}}+\widehat{\mathbf{a}},\mathbf{e}_i)_V-((\mathbf{y}_2\cdot\nabla)\,\widehat{\mathbf{u}},\mathbf{e}_i)\\
&\quad+\int_\Gamma\nu\widehat{b}(\mathbf{e}_i\cdot\boldsymbol{\tau})\,d\gamma+(\mathbf{U},\mathbf{e}_i)\,]\,dt\\
&\quad+2\,(\widehat{\mathbf{u}}(t),\mathbf{e}_i)\,(\widehat{\mathbf{G}},\mathbf{e}_i)\,d\mathcal{W}_t+|(\widehat{\mathbf{G}},\mathbf{e}_i)|^2\,dt.
\end{aligned}
$$

Summing these equalities over $i\in\mathbb{N}$, we obtain

$$
d\left(\|\widehat{\mathbf{u}}\|_2^2\right)+2\nu\,\|\widehat{\mathbf{u}}\|_V^2\,dt=J\,dt\,+2(\widehat{\mathbf{G}},\widehat{\mathbf{u}})\,d\mathcal{W}_t, \tag{3.5}
$$

where we denote

$$\begin{aligned} J = & \int_\Gamma \left\{ -a_2(\widehat{\mathbf{u}}\cdot\boldsymbol{\tau})^2 + 2\nu\widehat{b}(\widehat{\mathbf{u}}\cdot\boldsymbol{\tau}) \right\} d\gamma - 2\left(\left[\partial_t\widehat{\mathbf{a}} + \left((\widehat{\mathbf{u}}+\widehat{\mathbf{a}})\cdot\nabla\right)\mathbf{y}_1\right], \widehat{\mathbf{u}}\right) \\ & - 2\left((\mathbf{y}_2\cdot\nabla)\,\widehat{\mathbf{a}}, \widehat{\mathbf{u}}\right) - 2\nu\left(\widehat{\mathbf{a}}, \widehat{\mathbf{u}}\right)_V + \|\widehat{\mathbf{G}}\|_2^2 \\ = & J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned}$$

We have

$$\begin{aligned} J_1 & \leqslant (\|a_2\|_{L_\infty(\Gamma)} + \nu)\|\widehat{\mathbf{u}}\|^2_{L_2(\Gamma)} + C\nu\|\widehat{b}\|^2_{L_2(\Gamma)} \\ & \leqslant h_1(t)||\widehat{\mathbf{u}}||_2^2 + \frac{\nu}{4}||\widehat{\mathbf{u}}||_V^2 + C\nu\|\widehat{b}\|^2_{L_2(\Gamma)} \end{aligned}$$

with

$$h_1(t) = \frac{C}{\nu}(\|(a_2,b_2)\|_{\mathcal{H}_p(\Gamma)} + \nu)^2 \in L_1(0,T), \quad P\text{-a.e. in } \Omega,$$

by (2.11). The term J_2 is estimated as follows:

$$\begin{aligned} J_2 & \leqslant 2\left(\|\partial_t\widehat{\mathbf{a}}\|_2 + \|\widehat{\mathbf{a}}\|_{C(\overline{\Omega})}\|\nabla\mathbf{y}_1\|_2\right)\|\widehat{\mathbf{u}}\|_2 + 2\,\|\nabla\mathbf{y}_1\|_2\,\|\widehat{\mathbf{u}}\|_4^2 \\ & \leqslant 2\left(\|\partial_t\widehat{\mathbf{a}}\|_2 + \|\widehat{\mathbf{a}}\|_{C(\overline{\Omega})}\right)(1+\|\nabla\mathbf{y}_1\|_2)\,\|\widehat{\mathbf{u}}\|_2 \\ & \quad + 2\|\nabla\mathbf{y}_1\|_2\|\widehat{\mathbf{u}}\|_2\|\nabla\widehat{\mathbf{u}}\|_2 \\ & \leqslant h_2(t)\|\widehat{\mathbf{u}}\|_2^2 + C\|(\widehat{a},\widehat{b})\|^2_{\mathcal{H}_p(\Gamma)} + \frac{\nu}{4}||\widehat{\mathbf{u}}||_V^2 \end{aligned}$$

with

$$h_2(t) = C\left(1 + (\nu^{-1}+1)\|\nabla\mathbf{y}_1\|_2^2\right) \in L_1(0,T), \quad P\text{-a.e. in } \Omega,$$

by (2.4) and (2.16). Using (2.3) with $q=4$, we have

$$\begin{aligned} J_3 & = 2\left[\left((\widehat{\mathbf{u}}\cdot\nabla)\,\widehat{\mathbf{a}}, \widehat{\mathbf{u}}\right) + \left((\widehat{\mathbf{a}}\cdot\nabla)\,\widehat{\mathbf{a}}, \widehat{\mathbf{u}}\right) - \left((\mathbf{y}_1\cdot\nabla)\,\widehat{\mathbf{a}}, \widehat{\mathbf{u}}\right)\right] \\ & \leqslant 2\left[\,\|\widehat{\mathbf{u}}\|_4^2||\nabla\widehat{\mathbf{a}}\|_2 + \|\widehat{\mathbf{a}}\|_{L_\infty}\|\nabla\widehat{\mathbf{a}}\|_2\|\widehat{\mathbf{u}}\|_2 + \|\mathbf{y}_1\|_4\|\nabla\widehat{\mathbf{a}}\|_2\|\widehat{\mathbf{u}}\|_4)\right] \\ & \leqslant C\|\widehat{\mathbf{u}}\|_2\|\widehat{\mathbf{u}}\|_V||\nabla\widehat{\mathbf{a}}\|_2 + C\|\nabla\widehat{\mathbf{a}}\|_2^2\|\widehat{\mathbf{u}}\|_2 \\ & \quad + C\|\mathbf{y}_1\|_2^{1/2}\|\mathbf{y}_1\|_{H^1}^{1/2}\|\nabla\widehat{\mathbf{a}}\|_2\|\widehat{\mathbf{u}}\|_2^{1/2}\|\widehat{\mathbf{u}}\|_V^{1/2} \\ & \leqslant C(\nu^{-1}+1)\|\nabla\widehat{\mathbf{a}}\|_2^2\|\widehat{\mathbf{u}}\|_2^2 + \frac{\nu}{8}\|\widehat{\mathbf{u}}\|_V^2 + \|\widehat{\mathbf{a}}\|_{L_\infty}^2 \\ & \quad + \frac{C}{\nu}\|\mathbf{y}_1\|_{H^1}^2\|\widehat{\mathbf{u}}\|_2^2 + \frac{\nu}{8}\|\widehat{\mathbf{u}}\|_V^2 + C\|\mathbf{y}_1\|_2\|\nabla\widehat{\mathbf{a}}\|_2^2 \\ & \leqslant h_3(t)\|\widehat{\mathbf{u}}\|_2^2 + \frac{\nu}{4}\|\widehat{\mathbf{u}}\|_V^2 + \|(\widehat{a},\widehat{b})\|^2_{\mathcal{H}_p(\Gamma)} + C\|\mathbf{y}_1\|_2\|(\widehat{a},\widehat{b})\|^2_{\mathcal{H}_p(\Gamma)} \end{aligned}$$

with

$$h_3(t) = C\left((\nu^{-1}+1)\|(\widehat{a},\widehat{b})\|^2_{\mathcal{H}_p(\Gamma)} + \nu^{-1}\|\mathbf{y}_1\|_{H^1}^2\right) \in L_1(0,T), \quad P\text{-a.e. in } \Omega.$$

Finally, we have

$$J_4 \leqslant C\nu\,\|\widehat{\mathbf{a}}\|_{H^1}^2 + \frac{\nu}{4}||\widehat{\mathbf{u}}||_V^2 \leqslant C\nu\,\|(\widehat{a},\widehat{b})\|^2_{\mathcal{H}_p(\Gamma)} + \frac{\nu}{4}||\widehat{\mathbf{u}}||_V^2$$

and

$$J_5 \leqslant K||\widehat{\mathbf{u}}||_2^2$$

by the assumption (2.10). Therefore, there exists a specific constant $\widehat{C}_1$, such that $\widehat{C}_1 \geqslant \widehat{C}_0$ with $\widehat{C}_0$ is introduced in (2.16)–(2.17) and

$$\frac{1}{2}\left(h_1(t)+h_2(t)+h_3(t)+K\right) \leqslant f_1(t), \tag{3.6}$$

where

$$f_1(t)=\widehat{C}_1(\nu^{-1}+1)\left(\|(a_1,b_1)\|^2_{\mathcal{H}_p(\Gamma)}+\|(a_2,b_2)\|^2_{\mathcal{H}_p(\Gamma)}+\|\mathbf{u}_1\|^2_V+1\right). \tag{3.7}$$

Combining the above deduced estimates for J_i, $i=1,\dots,5$, we get

$$\begin{aligned} J &\leqslant \left(h_1(t)+h_2(t)+h_3(t)+K\right)\|\widehat{\mathbf{u}}\|_2^2+\nu||\widehat{\mathbf{u}}||_V^2 \\ &\quad + C\left\{\nu\|\widehat{b}\|^2_{L_2(\Gamma)}+(1+\nu)\|(\widehat{a},\widehat{b})\|^2_{\mathcal{H}_p(\Gamma)}+\|\mathbf{y}_1\|_2||(\widehat{a},\widehat{b})||^2_{\mathcal{H}_p(\Gamma)}\right\} \\ &\leqslant 2f_1(t)\|\widehat{\mathbf{u}}\|_2^2+\nu||\widehat{\mathbf{u}}||_V^2+C\left(1+\|\mathbf{y}_1\|_2\right)\|(\widehat{a},\widehat{b})\|^2_{\mathcal{H}_p(\Gamma)}. \end{aligned} \tag{3.8}$$

Now introducing the function

$$\xi_1(t)=e^{-\int_0^t f_1(s)ds} \tag{3.9}$$

and applying Itô's formula to (3.5), we infer

$$\begin{aligned} d\left(\xi_1^2(t)\,\|\widehat{\mathbf{u}}\|_2^2\right)+2\nu\xi_1^2(t)\,\|\widehat{\mathbf{u}}\|_V^2\;dt&=\xi_1^2(t)J\,dt \\ &\quad +2\xi_1^2(t)(\widehat{\mathbf{G}},\widehat{\mathbf{u}})\,d\mathcal{W}_t-2f_1(t)\xi_1^2(t)||\widehat{\mathbf{u}}||_2^2\,dt. \end{aligned}$$

Integrating it over the interval $[0,t]$ and using the estimate (3.8), we derive

$$\begin{aligned} \xi_1^2(t)\,\|\widehat{\mathbf{u}}(t)\|_2^2+\nu\int_0^t\xi_1^2\|\widehat{\mathbf{u}}\|_V^2\,ds &\leqslant \|\widehat{\mathbf{u}}_0\|_2^2+C\int_0^t\xi_1^2\left(1+\|\mathbf{y}_1\|_2\right)||(\widehat{a},\widehat{b})||^2_{\mathcal{H}_p(\Gamma)}\,ds \\ &\quad +2\int_0^t\xi_1^2(\widehat{\mathbf{G}},\widehat{\mathbf{u}})\,d\mathcal{W}_s, \end{aligned} \tag{3.10}$$

which implies

$$\begin{aligned} \sup_{s\in[0,t]}\xi_1^2(s)\,\|\widehat{\mathbf{u}}(s)\|_2^2+\nu\int_0^t\xi_1^2\|\widehat{\mathbf{u}}\|_V^2\,ds &\\ \leqslant \|\widehat{\mathbf{u}}_0\|_2^2+C\int_0^t\xi_1^2\left(1+\|\mathbf{y}_1\|_2\right)||(\widehat{a},\widehat{b})||^2_{\mathcal{H}_p(\Gamma)}\,ds &\\ +2\sup_{s\in[0,t]}\int_0^s\xi_1^2(\widehat{\mathbf{G}},\widehat{\mathbf{u}})\,d\mathcal{W}_r. & \end{aligned}$$

Taking the expectation in this inequality, using the assumption (2.10), and the Burkholder–Davis–Gundy inequality,

$$\begin{aligned} \mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s\xi_1^2(r)(\widehat{\mathbf{G}},\widehat{\mathbf{u}})\;d\mathcal{W}_r\right| &\leqslant \mathbb{E}\left(\int_0^t\xi_1^4\left|(\mathbf{G}(s,\mathbf{y}_1)-\mathbf{G}(s,\mathbf{y}_2),\widehat{\mathbf{u}})\right|^2\,ds\right)^{\frac{1}{2}} \\ &\leqslant \frac{1}{2}\,\mathbb{E}\sup_{s\in[0,t]}\xi_1^2(s)\|\widehat{\mathbf{u}}(s)\|_2^2+C\mathbb{E}\int_0^t\xi_1^2(\|\widehat{\mathbf{u}}\|_2^2+\|\widehat{\mathbf{a}}\|_2^2)\,ds, \end{aligned}$$

we deduce

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\sup_{s\in[0,t]}\xi_1^2(s)\,\|\widehat{\mathbf{u}}(s)\|_2^2+\nu\mathbb{E}\int_0^t\xi_1^2\|\widehat{\mathbf{u}}\|_V^2\,ds\\
&\quad\leqslant C\mathbb{E}\int_0^t\sup_{s\in[0,t]}\xi_1^2(s)\|\widehat{\mathbf{u}}(s)\|_2^2\,ds+C\mathbb{E}\,\|\widehat{\mathbf{u}}_0\|_2^2\\
&\qquad+C\left(\mathbb{E}\int_0^t\xi_1^2\left(1+\|\mathbf{y}_1\|_2^2\right)\,ds\right)^{1/2}\left(\mathbb{E}\int_0^t\xi_1^2||(\widehat{a},\widehat{b})||_{\mathcal{H}_p(\Gamma)}^4\,ds\right)^{1/2}
\end{aligned}$$

by (2.13). Then the Grönwall inequality yields

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\xi_1^2(s)\,\|\widehat{\mathbf{u}}(s)\|_2^2+2\nu\mathbb{E}\int_0^t\xi_1^2\|\widehat{\mathbf{u}}\|_V^2\,ds\\
&\quad\leqslant C\left\{\mathbb{E}\,\|\widehat{\mathbf{u}}_0\|_2^2+\left(\mathbb{E}\int_0^t\xi_1^2\left(1+\|\mathbf{y}_1\|_2^2\right)\,ds\right)^{1/2}\left(\mathbb{E}\int_0^t\xi_1^2||(\widehat{a},\widehat{b})||_{\mathcal{H}_p(\Gamma)}^4\,ds\right)^{1/2}\right\},
\end{aligned}$$

which implies (3.1), applying (2.13), (2.16), (2.17), and the property that $\xi_1\leqslant 1$ a.e. in $\Omega\times[0,T]$.

On the other hand, taking the square of (3.10), we infer that

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\xi_1^4(s)\,\|\widehat{\mathbf{u}}(s)\|_2^4+\nu^2\mathbb{E}\left(\int_0^t\xi_1^2\|\widehat{\mathbf{u}}\|_V^2\,ds\right)^2\\
&\quad\leqslant C\mathbb{E}\,\|\widehat{\mathbf{u}}_0\|_2^4+C\mathbb{E}\left(\int_0^t\xi_1^2\left(1+\|\mathbf{y}_1\|_2\right)||(\widehat{a},\widehat{b})||_{\mathcal{H}_p(\Gamma)}^2\,ds\right)^2\\
&\qquad+C\mathbb{E}\sup_{s\in[0,t]}\left(\int_0^s\xi_1^2(r)(\widehat{\mathbf{G}},\widehat{\mathbf{u}})\,d\mathcal{W}_r\right)^2\\
&\quad\leqslant C\mathbb{E}\,\|\widehat{\mathbf{u}}_0\|_2^4+C\mathbb{E}\int_0^t\xi_1^2\left(1+\|\mathbf{y}_1\|_2^2\right)\,ds\times\mathbb{E}\int_0^t\xi_1^2||(\widehat{a},\widehat{b})||_{\mathcal{H}_p(\Gamma)}^4\,ds\\
&\qquad+C\mathbb{E}\sup_{s\in[0,t]}\left(\int_0^s\xi_1^2(\widehat{\mathbf{G}},\widehat{\mathbf{u}})\,d\mathcal{W}_r\right)^2.
\end{aligned}$$

The assumption (2.10) and the Burkholder–Davis–Gundy inequality,

$$\begin{aligned}
\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s\xi_1^2(r)(\widehat{\mathbf{G}},\widehat{\mathbf{u}})\,d\mathcal{W}_r\right|^2&\leqslant\mathbb{E}\int_0^t\xi_1^4(s)\left|(\mathbf{G}(s,\mathbf{y}_1)-\mathbf{G}(s,\mathbf{y}_2),\widehat{\mathbf{u}})\right|^2\,ds\\
&\leqslant\frac{1}{2}\,\mathbb{E}\sup_{s\in[0,t]}\xi_1^4(s)\|\widehat{\mathbf{u}}(s)\|_2^4+C\mathbb{E}\int_0^t\xi_1^4\left(\|\widehat{\mathbf{u}}\|_2^4+\|\widehat{\mathbf{a}}\|_2^4\right)\,ds,
\end{aligned}$$

allow one to deduce the following Grönwall inequality:

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\sup_{s\in[0,t]}\xi_1^4(s)\,\|\widehat{\mathbf{u}}(s)\|_2^4+\nu^2\mathbb{E}\left(\int_0^t\xi_1^2\|\widehat{\mathbf{u}}\|_V^2\,ds\right)^2\\
&\quad\leqslant C\mathbb{E}\,\|\widehat{\mathbf{u}}_0\|_2^4+C\mathbb{E}\int_0^t\xi_1^2\left(1+\|\mathbf{y}_1\|_2^2\right)\,ds\times\mathbb{E}\int_0^t\xi_1^2||(\widehat{a},\widehat{b})||_{\mathcal{H}_p(\Gamma)}^4\,ds\\
&\qquad+C\mathbb{E}\int_0^t\xi_1^4\,\|\widehat{\mathbf{a}}\|_2^4\,ds+C\mathbb{E}\int_0^t\sup_{r\in[0,s]}\xi_1^4(r)\,\|\widehat{\mathbf{u}}(r)\|_2^4\,ds,
\end{aligned}$$

implying the inequality

$$\mathbb{E}\sup_{s\in[0,t]}\xi_1^4(s)\,\|\widehat{\mathbf{u}}(s)\|_2^4+2\nu^2\mathbb{E}\left(\int_0^t\xi_1^2\|\widehat{\mathbf{u}}\|_V^2\,ds\right)^2$$
$$\leqslant C\left\{\mathbb{E}\,\|\widehat{\mathbf{u}}_0\|_2^4+\mathbb{E}\int_0^t\xi_1^2\left(1+\|\mathbf{y}_1\|_2^2\right)ds\times\mathbb{E}\int_0^t\xi_1^2||(\widehat{a},\widehat{b})||^4_{\mathcal{H}_p(\Gamma)}\,ds+C\mathbb{E}\int_0^t\xi_1^4\,\|\widehat{\mathbf{a}}\|_2^4\,ds\right\},$$

which gives (3.2), using (2.13), (2.16), (2.17), and $\xi_1\leqslant 1$ a.e. in $\Omega\times[0,T]$. ∎

4. Solvability of control problem. The main goal of this paper is to control the solution of the system (1.1) by boundary values (a,b). From now on, we assume these boundary values belong to the space of $\mathcal{H}_p(\Gamma)$-valued bounded stochastic processes (cf. [7], [29], [30]); namely we defined the space $\mathcal{A}$ of admissible controls as a bounded subset of $L_\infty(\Omega\times(0,T);\mathcal{H}_p(\Gamma))$, which is *compact* in $L_2(\Omega\times(0,T);\mathcal{H}_p(\Gamma))$.

The cost functional is given by

$$J(a,b,\mathbf{y})=\mathbb{E}\int_{\mathcal{O}_T}\frac{1}{2}|\mathbf{y}-\mathbf{y}_d|^2\,d\mathbf{x}dt+\mathbb{E}\int_{\Gamma_T}\left(\frac{\lambda_1}{2}|a|^2+\frac{\lambda_2}{2}|b|^2\right)d\gamma dt, \tag{4.1}$$

where $\mathbf{y}_d\in L_2(\Omega\times\mathcal{O}_T)$ is a desired target field and $\lambda_1,\lambda_2>0$. We aim to control the solution $\mathbf{y}$ through the minimization of the cost functional (4.1) over $\mathcal{A}$ and constrained to (1.1). More precisely, our goal is to solve the following problem:

$$(\mathcal{P})\begin{cases}\underset{(a,b)}{\text{minimize}}\{J(a,b,\mathbf{y}):\ (a,b)\in\mathcal{A}\quad\text{and}\\ \mathbf{y}\text{ is the solution of the system (1.1) for }(a,b)\in\mathcal{A}\}.\end{cases}$$

Let us notice that for $(a,b)\in\mathcal{A}$, the solution $\mathbf{y}=\mathbf{u}+\mathbf{a}$ of the state equation (2.15) satisfies

$$\mathbf{y}\in L_4(\Omega;C([0,T];L_2(\mathcal{O})))\subset L_2(\Omega\times\mathcal{O}_T),$$

ensuring that the cost functional (4.1) is well defined.

The existence result stated in Theorem 4.1 of [16] applies, and the optimal solution belongs to $\mathcal{A}$, namely we have the next theorem.

THEOREM 4.1. *Let* $\mathbf{y}_0$ *verify the assumptions* (2.11). *Then there exists at least one solution for the optimal control problem* $(\mathcal{P})$.

Remark 4.1. It is worth mentioning that the existence result established in [16] does not require the admissible set $\mathcal{A}$ to be bounded in $L_\infty(\Omega\times(0,T);\mathcal{H}_p(\Gamma))$. However, the deduction of first-order optimality conditions is more demanding, requiring integrability of the Gâteaux derivative of the control-to-state mapping. This will be achieved proving an exponential integrability condition for the state in Proposition 5.1 below. For that we need $\mathcal{A}$ to be bounded in $L_\infty(\Omega\times(0,T);\mathcal{H}_p(\Gamma))$. This requirement comes from the structure of the stochastic Navier–Stokes equation with multiplicative noise. Roughly speaking, this means that the control (in an optimal way) the stochastic dynamics inside the domain, through the control actions on the boundary, it is possible even if the control action have some randomness; however, this randomness should be bounded (see the assumption (vi) on p. 40 of [29] and the assumption on *bounded controls* given on p. 42 of [7]). A very particular case corresponds to deterministic control actions.

The next sections are devoted to establishing first-order optimality conditions.

5. Exponential integrability of the state. According to sections 2 and 3, let $\mathbf{y}=\mathbf{u}+\mathbf{a}$ be the unique solution of the stochastic differential equation (1.1) satisfying the estimates (2.16) and with data $(a,b),\mathbf{u}_0$ verifying (2.11). Hereafter, we assume the following additional assumptions on the data:

$$(a,b)\in\mathcal{A}, \qquad ||\mathbf{u}_0||_2\in L_\infty(\Omega), \tag{5.1}$$

in order to deduce a suitable exponential integrability condition for the stochastic process $\mathbf{y}$. We also introduce additional hypothesis on the diffusion operator $\mathbf{G}$, namely $\mathbf{G}$ is bounded by a positive constant L in the space H, such that

$$||\mathbf{G}\,(t,\mathbf{y})\,||_2^2\leqslant \frac{L}{1+||\mathbf{y}||_2^2} \qquad \text{for a.e.}\quad t\in[0,T] \quad \forall \mathbf{y}\in H. \tag{5.2}$$

Since $\mathcal{A}$ is a closed subset of $L_\infty(\Omega\times(0,T);\mathcal{H}_p(\Gamma))$, we can take the real number

$$r_*=\sup_{(a,b)\in\mathcal{A}} 2\widehat{C}_0\left(1+||(a,b)||^2_{L_\infty(\Omega_T;\mathcal{H}_p(\Gamma))}\right),$$

where the constant $\widehat{C}_0$ is introduced in (2.17) of Theorem 2.4, and define the time dependent functions $\lambda_*(t)$, $\beta_*(t)$ and the constants A_*, B_* by

$$\begin{aligned}\lambda_*(t)&=\frac{\nu e^{-r_*t}}{L}, \quad A_*=\frac{\nu^2e^{-2r_*T}}{2L},\\ \beta_*(t)&=\frac{\nu e^{-4(r_*+L)t}}{8L}, \quad B_*=\frac{\nu^2e^{-8(r_*+L)T}}{8L}.\end{aligned} \tag{5.3}$$

Let us note that there exists a constant $\widehat{C}$, depending only on $\mathcal{O}$, such that

$$||\mathbf{v}||_2^4\leqslant\widehat{C}||\mathbf{v}||_2^2||\mathbf{v}||_V^2 \qquad \forall\mathbf{v}\in V, \tag{5.4}$$

being a particular case of the inequality (2.3).

Let us mention that the main arguments to show the exponential integrability of the stochastic process $\mathbf{y}$ rely on the structure of the first equation in (1.1) for $\mathbf{y}$, and on the martingale property of the exponential process.

PROPOSITION 5.1. *Assume that the data* $(a,b)\in\mathcal{A}$, $\mathbf{u}_0$, *and* $\mathbf{G}$ *satisfy* (2.10), (2.11), (5.1), *and* (5.2). *Then there exist positive constants* C, *such that the following estimates are valid:*

$$\begin{gathered}\mathbb{E}\exp\left(\lambda_*(t)e^{-tr_*}\,\|\mathbf{u}(t)\|_2^2\right)\leqslant C, \quad \mathbb{E}\exp\left\{A_*\int_0^t\|\mathbf{u}\|_V^2\,ds\right\}\leqslant C,\\ \mathbb{E}\exp(\beta_*(t)e^{-4(r_*+L)t}\,\|\mathbf{u}(t)\|_2^4)\leqslant C, \quad \mathbb{E}\exp\left\{B_*\int_0^t\|\mathbf{u}\|_2^2\,\|\mathbf{u}\|_V^2\,ds\right\}\leqslant C,\\ \mathbb{E}\exp\left(\frac{B_*}{\widehat{C}}\int_0^t\|\mathbf{u}\|_2^4\,ds\right)\leqslant C \qquad \forall t\in[0,T],\end{gathered} \tag{5.5}$$

with λ_*, A_*, β_*,B_*, *and* $\widehat{C}$ *defined by* (5.3) *and* (5.4).

Proof. First step. Deduction of the estimates $(5.5)_{1,2}$. Since $\mathbf{y}$ is the solution of the state system (1.1), which exists by Theorem 2.4, then taking the test function $\boldsymbol{\varphi}=\mathbf{u}$ in (2.15) with $\mathbf{u}=\mathbf{y}-\mathbf{a}$, we obtain the inequality

$$\begin{aligned}d\left(\|\mathbf{u}\|_2^2\right)+\nu\,\|\mathbf{u}\|_V^2\,dt\leqslant 2\widehat{C}_0\left(||(a,b)||^2_{\mathcal{H}_p(\Gamma)}+1\right)(||\mathbf{u}\|_2^2+1)\,dt&\\ +\,2\,(\mathbf{G}(t,\mathbf{y}),\mathbf{u})\,d\mathcal{W}_t,&\end{aligned} \tag{5.6}$$

as it was done in the article [16] (see the deduction of the formula (3.11) in the article [16]). Integrating over the time interval $(0,t)$, we can write

$$\|\mathbf{u}(t)\|_2^2+\nu\int_0^t \|\mathbf{u}\|_V^2\,ds\leqslant\kappa\int_0^t\|\mathbf{u}\|_2^2\,ds+(C_\kappa+g(t)), \tag{5.7}$$

where

$$\kappa=2r_*\geq 2\widehat{C}_0\|1+||(a,b)||^2_{\mathcal{H}_p(\Gamma)}\|_{L_\infty(\Omega_T)},\qquad C_\kappa=\|\mathbf{u}_0\|_2^2+\kappa T,$$
$$g(t)=\int_0^t f(s)\,d\mathcal{W}_s,\qquad f(s)=2\left(\mathbf{G}(s,\mathbf{y}),\mathbf{u}\right).$$

The relation (5.7) corresponds to the following differential inequality:

$$z'\leqslant\kappa z+(C_\kappa+g(t))\qquad\text{for}\quad z(t)=\int_0^t\|\mathbf{u}\|_2^2\,ds, \tag{5.8}$$

which can be integrated by Grönwall's lemma. Hence using Fubini's theorem, we get

$$\begin{aligned}z(t)=\int_0^t\|\mathbf{u}\|_2^2\,ds&\leqslant C_\kappa\left(\frac{e^{\kappa t}-1}{\kappa}\right)+e^{\kappa t}\int_0^t e^{-\kappa s}g(s)\,ds\\&=C_\kappa\left(\frac{e^{\kappa t}-1}{\kappa}\right)+\int_0^t\left(\frac{e^{\kappa(t-s)}-1}{\kappa}\right)\,f(s)\,d\mathcal{W}_s.\end{aligned}$$

Substituting it into the right-hand side of (5.7), we deduce that

$$\|\mathbf{u}(t)\|_2^2+\nu\int_0^t\|\mathbf{u}\|_V^2\,ds\leqslant C_\kappa e^{\kappa t}+\int_0^t e^{\kappa(t-s)}\,f(s)\,d\mathcal{W}_s. \tag{5.9}$$

Setting $F(s)=e^{-\kappa s}f(s)$ and multiplying the inequality (5.9) by $\lambda e^{-\kappa t}$ with some $\lambda>0$, we infer that

$$\begin{aligned}\lambda e^{-\kappa t}\|\mathbf{u}(t)\|_2^2+\nu\lambda e^{-\kappa t}\int_0^t\|\mathbf{u}\|_V^2\,ds&\leqslant\lambda C_\kappa+\frac{\lambda^2}{2}\int_0^t|F(s)|^2\,ds\\&\quad+\left(\lambda\int_0^t F(s)\,d\mathcal{W}_s-\frac{\lambda^2}{2}\int_0^t|F(s)|^2\,ds\right).\end{aligned}$$

The boundedness (5.2) for $\mathbf{G}$ implies that

$$|F(t)|^2=e^{-2\kappa t}|\left(\mathbf{G}(t,\mathbf{y}),\mathbf{u}\right)|^2\leqslant L\,\|\mathbf{u}(t)\|_2^2\leqslant L\,\|\mathbf{u}(t)\|_V^2\qquad\forall t\in(0,T).$$

In addition, for $\lambda_*(t)=\frac{\nu e^{-\kappa t}}{L}$, we have

$$\max_\lambda\left(\nu\lambda e^{-\kappa t}-\frac{L\lambda^2}{2}\right)=\frac{L\lambda_*^2}{2}\geqslant\frac{\nu^2e^{-2\kappa T}}{2L}=A_*\quad\forall t\in[0,T].$$

Hence, we obtain

$$\lambda_*e^{-\kappa t}\|\mathbf{u}\|_2^2+A_*\int_0^t\|\mathbf{u}\|_V^2\,ds\leqslant\lambda_*C_\kappa+\left(\int_0^t(\lambda_*F)\,d\mathcal{W}_s-\frac{1}{2}\int_0^t(\lambda_*F)^2\,ds\right).$$

Taking the exponential function and the expectation in the last deduced inequality, we obtain

$$\begin{aligned}&\mathbb{E}\exp\left(\lambda_*e^{-\kappa t}\|\mathbf{u}(t)\|_2^2+A_*\int_0^t\|\mathbf{u}\|_V^2\,ds\right)\\&\qquad\leqslant H_1\times\exp\left\{\int_0^t(\lambda_*F)\,d\mathcal{W}_s-\frac{1}{2}\int_0^t(\lambda_*F)^2\,ds\right\}\end{aligned}$$

with $H_1 = \exp(\lambda_* C_\kappa) < \infty$ by the assumption (5.1). Since the expectation of the right-hand side is equal to 1 due to the Levy equality, we derive

$$\mathbb{E} \exp\left(\lambda_* e^{-\kappa t} \|\mathbf{u}(t)\|_2^2 + A_* \int_0^t \|\mathbf{u}\|_V^2 \, ds\right) \leqslant H_1,$$

which gives $(5.5)_{1,2}$

Second step. Deduction of the estimates $(5.5)_{3,4,5}$. Using the inequality (5.6) and applying the Itô formula, we ensure

$$\begin{aligned} \|\mathbf{u}\|_2^4 + 2\nu \int_0^t \|\mathbf{u}\|_2^2 \|\mathbf{u}\|_V^2 \, ds &\leqslant \|\mathbf{u}_0\|_2^4 \\ &+ 4 \int_0^t \widehat{C}_0(||(a,b)||^2_{\mathcal{H}_p(\Gamma)} + 1) \|\mathbf{u}\|_2^2 (||\mathbf{u}\|_2^2 + 1) \, ds \\ &+ 4 \int_0^t \|\mathbf{u}\|_2^2 (\mathbf{G}(t,\mathbf{y}), \mathbf{u}) \, d\mathcal{W}_s + \int_0^t 4 (\mathbf{G}(s,\mathbf{y}), \mathbf{u})^2 \, ds \\ \leqslant \|\mathbf{u}_0\|_2^4 &+ \int_0^t 4 \left(\left\{2\widehat{C}_0(||(a,b)||^2_{\mathcal{H}_p(\Gamma)} + 1)\right\} + L\right) ||\mathbf{u}\|_2^4 \, ds \\ &+ \int_0^t \left(\left\{2\widehat{C}_0(||(a,b)||)^2_{\mathcal{H}_p(\Gamma)} + 1\right\} + 1\right) dt \\ &+ 4 \int_0^t \|\mathbf{u}\|_2^2 (\mathbf{G}(s,\mathbf{y}), \mathbf{u}) \, d\mathcal{W}_s, \end{aligned}$$

by the inequality $a \leqslant a^2 + \frac{1}{4}$ and the assumption (5.2). Therefore,

$$\|\mathbf{u}(t)\|_2^4 + 2\nu \int_0^t \|\mathbf{u}\|_2^2 \|\mathbf{u}\|_V^2 \, ds \leqslant \kappa \int_0^t \|\mathbf{u}\|_2^4 \, ds + (C_\kappa + g(t)), \tag{5.10}$$

where

$$\kappa = 4(r_* + L), \qquad C_\kappa = \|\mathbf{u}_0\|_2^4 + (r_* + 1)T,$$
$$g(t) = \int_0^t f(s) \, d\mathcal{W}_s, \qquad f(s) = 4 \|\mathbf{u}\|_2^2 (\mathbf{G}(s,\mathbf{y}), \mathbf{u}).$$

The expression (5.10) can be written as the differential inequality (5.8) for

$$z(t) = \int_0^t \|\mathbf{u}\|_2^4 \, ds.$$

Hence, comparing with (5.9), Grönwall's inequality gives

$$\|\mathbf{u}(t)\|_2^4 + 2\nu \int_0^t \|\mathbf{u}\|_2^2 \|\mathbf{u}\|_V^2 \, ds \leqslant C_\kappa e^{\kappa t} + \int_0^t e^{\kappa(t-s)} f(s) \, d\mathcal{W}_s. \tag{5.11}$$

Let us denote by $F(s) = e^{-\kappa s} f(s)$ and multiply (5.11) by $\beta e^{-\kappa t}$ with $\beta > 0$, we obtain

$$\begin{aligned} &\beta e^{-\kappa t} \|\mathbf{u}(t)\|_2^4 + 2\nu\beta e^{-\kappa t} \int_0^t \|\mathbf{u}\|_2^2 \|\mathbf{u}\|_V^2 \, ds \\ &\quad \leqslant \beta C_\kappa + \frac{\beta^2}{2} \int_0^t |F(s)|^2 \, ds + \left(\beta \int_0^t F(s) \, d\mathcal{W}_s - \frac{\beta^2}{2} \int_0^t |F(s)|^2 \, ds\right). \end{aligned} \tag{5.12}$$

The boundedness (5.2) for $\mathbf{G}$ implies that

$$|F(t)|^2 \leqslant 16\,\|\mathbf{u}\|_2^4\,(\mathbf{G}(t,\mathbf{y}),\mathbf{y}-\mathbf{a})^2 \leqslant 16\,\|\mathbf{u}(t)\|_2^4\left(\frac{L\,\|\mathbf{y}(t)\|_2^2}{1+||\mathbf{y}||_2^2}+C\right)$$
$$\leqslant D_*\,\|\mathbf{u}(t)\|_2^2\,\|\mathbf{u}(t)\|_V^2 \quad \text{with } D_*=16L+C \quad \text{for a.e. } t\in(0,T).$$

On the other hand for $\beta_*(t)=\frac{2\nu e^{-\kappa t}}{D_*}$, we have

$$\max_{\beta}\left(2\nu\beta e^{-\kappa t}-\frac{D_*\beta^2}{2}\right)=\frac{D_*\beta_*^2}{2}\geqslant\frac{4\nu^2e^{-2\kappa T}}{2D_*}=B_* \quad \forall t\in[0,T].$$

Due to these relations, (5.12) yields

$$\beta_*e^{-\kappa t}\,\|\mathbf{u}(t)\|_2^4+B_*\int_0^t\|\mathbf{u}\|_2^2\,\|\mathbf{u}\|_V^2\,ds$$
$$\leqslant \beta_*C_\kappa+\left(\int_0^t(\beta_*F)\,d\mathcal{W}_s-\frac{1}{2}\int_0^t(\beta_*F)^2\,ds\right).$$

Taking the exponential function and the expectation in the last deduced inequality, we obtain

$$\mathbb{E}\exp\left(\beta_*e^{-\kappa t}\,\|\mathbf{u}(t)\|_2^4+B_*\int_0^t\|\mathbf{u}\|_2^2\,\|\mathbf{u}\|_V^2\,ds\right)$$
$$\leqslant H_2\times\mathbb{E}\exp\left(\int_0^t(\beta_*F)\,d\mathcal{W}_s-\frac{1}{2}\int_0^t(\beta_*F)^2\,ds\right)=H_2$$

with $H_2=\exp(\beta_*C_\kappa)<\infty$ by the assumption (5.1). Also, using the inequality (5.4), we conclude that

$$\mathbb{E}\exp\left(\frac{B_*}{\widehat{C}}\int_0^t\|\mathbf{u}(t)\|_2^4\,ds\right)\leqslant\mathbb{E}\exp\left(B_*\int_0^t\|\mathbf{u}\|_2^2\,\|\mathbf{u}\|_V^2\,ds\right)\leqslant H_2.$$

The last two inequalities are the estimates $(5.5)_{3,4,5}$. ☐

6. Linearized state equation. In this section we also assume that $\mathbf{G}(t,\mathbf{y})$ is Gâteaux differentiable in the variable $\mathbf{y}\in H$:

$$\lim_{s\to0}\frac{\mathbf{G}(t,\mathbf{y}+s\mathbf{v})-\mathbf{G}(t,\mathbf{y})}{s}=\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\mathbf{v} \quad \text{for each } t\in[0,T],$$

such that the function $\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})$ is continuous and bounded in the second variable $\mathbf{y}$, namely

$$||\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{x})-\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})||_2\to0 \qquad \text{when } ||\mathbf{x}-\mathbf{y}||_2\to0 \qquad \forall t\in[0,T],$$

$$\begin{aligned}&||\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\mathbf{v}||_2\leqslant C||\mathbf{v}||_2, &&\mathbf{v}\in H,\\ &||\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\mathbf{v}||_V\leqslant C||\mathbf{v}||_V, &&\mathbf{v}\in V,\end{aligned} \tag{6.1}$$

for some positive constant C. Due to Propositions A.2, A.3 of [1] we have that $\mathbf{G}(t,\mathbf{y})$ is Fréchet differentiable in the second variable $\mathbf{y}$:

$$\mathbf{G}(t,\mathbf{x}+\mathbf{y})-\mathbf{G}(t,\mathbf{y})-\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\mathbf{x}=o\left(t,\|\mathbf{x}\|_2\right), \quad \text{where } \lim_{s\to0}\frac{o\,(t,s)}{s}=0 \tag{6.2}$$

for any $\mathbf{x}$, $\mathbf{y}\in H$, $\quad t\in[0,T]$.

Let $\mathbf{y}$ be the solution of the state system (1.1). The corresponding linearized system for (1.1) can be written as the following Oseen's type system:

$$
\begin{cases}
d\mathbf{z} = (\nu\Delta\mathbf{z} - (\mathbf{z}\cdot\nabla)\,\mathbf{y} - (\mathbf{y}\cdot\nabla)\,\mathbf{z} - \nabla\pi)\,dt + \nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\mathbf{z}\,d\mathcal{W}_t & \text{in } \mathcal{O}_T, \\
\mathrm{div}\ \mathbf{z} = 0, & \\
\mathbf{z}\cdot\mathbf{n} = f, \qquad [2D(\mathbf{z})\,\mathbf{n} + \alpha\mathbf{z}]\cdot\boldsymbol{\tau} = g & \text{on } \Gamma_T, \\
\mathbf{z}(0) = 0 & \text{in } \mathcal{O}
\end{cases}
\tag{6.3}
$$

with the boundary data, satisfying the assumption

$$
(f,g) \in \mathcal{A}. \tag{6.4}
$$

Let $\mathbf{f}$ be the solution of the system (2.12) with the data (a,b) replaced by (f,g). Then the function $\mathbf{f}$ satisfies the estimates (2.13), namely

$$
||\mathbf{f}||_{W^1_p(\mathcal{O})} + ||\partial_t\mathbf{f}||_{L_2(\mathcal{O})} \leqslant C||(f,g)||_{\mathcal{H}_p(\Gamma)} \quad \text{a.e. in } \Omega\times(0,T),
$$

hence $\mathbf{f}$ has the following regularities:

$$
\begin{aligned}
&\mathbf{f} \in L_4(\Omega; C([0,T]; L_2(\mathcal{O}))) \cap L_4(\Omega\times(0,T); C(\overline{\mathcal{O}}) \cap H^1(\mathcal{O})), \\
&\partial_t\mathbf{f} \in L_4(\Omega\times(0,T); L_2(\mathcal{O})).
\end{aligned}
\tag{6.5}
$$

Definition 6.1. *A stochastic process* $\mathbf{z} = \tilde{\mathbf{z}} + \mathbf{f}$ *with* $\tilde{\mathbf{z}} \in L_2(0,T;V)$, P*-a.e. in* Ω, *is a strong solution of* (6.3) *with* $\mathbf{z}(0) = 0$ *if the following equation holds:*

$$
\begin{aligned}
(\mathbf{z}(t),\boldsymbol{\varphi}) = &\int_0^t \left[-\nu\,(\mathbf{z},\boldsymbol{\varphi})_V + \int_\Gamma \nu g(\boldsymbol{\varphi}\cdot\boldsymbol{\tau})\,d\gamma - ((\mathbf{z}\cdot\nabla)\mathbf{y} + ((\mathbf{y}\cdot\nabla)\mathbf{z},\boldsymbol{\varphi})) \right] ds \\
&+ \int_0^t (\nabla_{\mathbf{y}}\mathbf{G}(s,\mathbf{y})\mathbf{z},\boldsymbol{\varphi})\,d\mathcal{W}_s \qquad \forall\boldsymbol{\varphi}\in V, \quad a.e.\ in\ \Omega\times(0,T).
\end{aligned}
$$

In what follows we will establish the solvability of the system (6.3).

Proposition 6.2. *Let* $\mathbf{y}$ *be the solution of the state system* (1.1) *with the boundary data* $(a,b) \in \mathcal{A}$, *which was constructed in Theorem* 2.4. *Let* (f,g) *satisfy the assumption* (6.4). *Then there exists a unique solution* $\mathbf{z} = \tilde{\mathbf{z}} + \mathbf{f}$ *for the system* (6.3), *such that*

$$
\tilde{\mathbf{z}} \in C([0,T];H) \cap L_2\,(0,T;V)\,, \quad P\text{-}a.e.\ in\ \Omega.
$$

Moreover, there exists a positive constant $\widehat{C}_2$, *such that*

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]} \xi_2^2(s)\,\|\tilde{\mathbf{z}}(s)\|_2^2 + \nu\mathbb{E}\int_0^t \xi_2^2\,\|\tilde{\mathbf{z}}\|_V^2\,ds \\
&\qquad \leqslant C\mathbb{E}\int_0^t \xi_2^2\,(1+\nu)\,||(f,g)||^2_{\mathcal{H}_p(\Gamma)}\left(1+\|\mathbf{y}\|_2^2\right) ds
\end{aligned}
\tag{6.6}
$$

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]} \xi_2^4(s)\,\|\tilde{\mathbf{z}}(s)\|_2^4 + \nu^2\mathbb{E}\left(\int_0^t \xi_2^2\,\|\tilde{\mathbf{z}}\|_V^2\,ds\right)^2 \\
&\qquad \leqslant C\left(\mathbb{E}\int_0^t \xi_2^2\,(1+\nu)\,||(f,g)||^2_{\mathcal{H}_p(\Gamma)}\left(1+\|\mathbf{y}\|_2^2\right) ds\right)^2
\end{aligned}
\tag{6.7}
$$

with

$$\xi_2(t) = e^{-\int_0^t f_2(s)ds} \tag{6.8}$$

and

$$f_2(t) = \widehat{C}_2(\nu^{-1}+1)\left(1 + \|\mathbf{u}\|_V^2 + ||(a,b)||^2_{\mathcal{H}_p(\Gamma)}\right),$$

where $\widehat{C}_2$ *is defined according to the relations* (6.12)–(6.13).

Proof. The existence of a solution for the system (6.3) will be shown by Galerkin's method.

The injection operator $I : V \to H$ is a compact operator. Therefore, there exists a basis $\{\mathbf{e}_k\}_{k=1}^{\infty} \subset V$ of eigenfunctions

$$(\mathbf{v}, \mathbf{e}_k)_V = \lambda_k (\mathbf{v}, \mathbf{e}_k) \qquad \forall \mathbf{v} \in V,\ k \in \mathbb{N}, \tag{6.9}$$

which is an orthonormal basis for H, such that the sequence $\{\lambda_k\}_{k=1}^{\infty}$ of eigenvalues verifies the following properties:

$$\lambda_k > 0 \quad \forall k \in \mathbb{N} \quad \text{and} \quad \lambda_k \to \infty \quad \text{as } k \to \infty.$$

To justify it, we refer to a similar situation, which was considered in Lemma 2.2. of [17], Theorem 1 of [36], see also [22, pp. 297–307, Theorem 2, p. 300, and Theorem 5, p. 305].

The ellipticity of (6.9) and the regularity $\Gamma \in C^2$ imply that $\{\mathbf{e}_k\} \subset H^2(\mathcal{O}) \cap V$, hence the sequence $\{\mathbf{e}_k\}_{k=1}^{\infty}$ forms the eigenfunctions of the Stokes problem:

$$\begin{cases} -\Delta \mathbf{e}_k + \nabla \pi_k = \lambda_k \mathbf{e}_k, & \operatorname{div} \mathbf{e}_k = 0 \quad \text{in } \mathcal{O}, \\ \mathbf{e}_k \cdot \mathbf{n} = 0, & [2D(\mathbf{e}_k)\,\mathbf{n} + \alpha \mathbf{e}_k] \cdot \boldsymbol{\tau} = 0 \quad \text{on } \Gamma. \end{cases}$$

Let us fix an arbitrary $n \in \mathbb{N}$ and consider the finite-dimensional subspace $V_n = \operatorname{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of V. Let

$$\mathbf{z}_n = \tilde{\mathbf{z}}_n + \mathbf{f} \qquad \text{with} \qquad \tilde{\mathbf{z}}_n(t) = \sum_{k=1}^{n} r_k^{(n)}(t)\ \mathbf{e}_k \tilde{\mathbf{z}}_n \in V_n$$

be a solution of the following finite-dimensional problem:

$$\begin{cases} d(\mathbf{z}_n, \mathbf{e}_k) = \left[-\nu\,(\mathbf{z}_n, \mathbf{e}_k)_V + \nu \int_\Gamma g(\mathbf{e}_k \cdot \boldsymbol{\tau})\,d\gamma - ((\mathbf{z}_n \cdot \nabla)\mathbf{y} + ((\mathbf{y}\cdot\nabla)\mathbf{z}_n, \mathbf{e}_k))\right] dt \\ \qquad\qquad + (\nabla_{\mathbf{y}} \mathbf{G}(t,\mathbf{y})\mathbf{z}_n, \mathbf{e}_k)\ d\mathcal{W}_t \qquad \text{a.e. in } \Omega \times (0,T), \\ \tilde{\mathbf{z}}_n(0) = \tilde{\mathbf{z}}_{n,0}, \qquad k = 1, 2, \ldots, n, \end{cases} \tag{6.10}$$

where $\tilde{\mathbf{z}}_{n,0}$ is the orthogonal projections in H of $\tilde{\mathbf{z}}_0(\mathbf{x}) = -\mathbf{f}(0,\mathbf{x})$ onto the space V_n.

The problem (6.10) is a system of n-stochastic linear ordinary differential equations, which has a unique global-in-time solution $\tilde{\mathbf{z}}_n = \mathbf{z}_n - \mathbf{f}$, as an adapted process in the space $C([0,T]; V_n)$. We can write the equation in (6.10) as

$$\begin{aligned} d\,(\tilde{\mathbf{z}}_n, \mathbf{e}_i) = \Big[&- \nu\,(\tilde{\mathbf{z}}_n + \mathbf{f}, \mathbf{e}_i)_V + \nu \int_\Gamma g(\mathbf{e}_i \cdot \boldsymbol{\tau})\,d\gamma \\ &+ (-\partial_t \mathbf{f} - ((\tilde{\mathbf{z}}_n + \mathbf{f})\cdot\nabla)\,\mathbf{y} - (\mathbf{y}\cdot\nabla)\,(\tilde{\mathbf{z}}_n + \mathbf{f})\,, \mathbf{e}_i)\Big] dt \\ &+ (\nabla_{\mathbf{y}} \mathbf{G}(t,\mathbf{y})\,(\tilde{\mathbf{z}}_n + \mathbf{f})\,, \mathbf{e}_i)\ d\mathcal{W}_t. \end{aligned}$$

Step 1. *Deducing the estimate* (6.6)*.* The Itô formula gives

$$d\left((\tilde{\mathbf{z}}_n\,,\mathbf{e}_i)^2\right)=2\,(\tilde{\mathbf{z}}_n,\mathbf{e}_i)\bigg[-\nu\,(\tilde{\mathbf{z}}_n,\mathbf{e}_i)_V-\nu\,(\mathbf{f},\mathbf{e}_i)_V+\nu\int_\Gamma g(\mathbf{e}_i\cdot\boldsymbol{\tau})\,d\gamma$$
$$+(-\partial_t\mathbf{f}-((\tilde{\mathbf{z}}_n+\mathbf{f})\cdot\nabla)\,\mathbf{y}-(\mathbf{y}\cdot\nabla)\,(\tilde{\mathbf{z}}_n+\mathbf{f})\,,\mathbf{e}_i)\bigg]\,dt$$
$$+2\,(\tilde{\mathbf{z}}_n,\mathbf{e}_i)\,(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\,(\tilde{\mathbf{z}}_n+\mathbf{f})\,,\mathbf{e}_i)\;d\mathcal{W}_t$$
$$+\,|\,(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\,(\tilde{\mathbf{z}}_n+\mathbf{f})\,,\mathbf{e}_i)\,|^2\,dt,$$

where the module in the last term is defined by (2.9). Summing these equalities over $i=1,\dots,n$, we obtain

$$d\left(\|\tilde{\mathbf{z}}_n\|_2^2\right)+2\nu\,\|\tilde{\mathbf{z}}_n\|_V^2=\left[\int_\Gamma\left\{-a(\tilde{\mathbf{z}}_n\cdot\boldsymbol{\tau})^2+2\nu g(\tilde{\mathbf{z}}_n\cdot\boldsymbol{\tau})\right\}\,d\;\gamma\right]\,dt$$
$$-2\,(\partial_t\mathbf{f}+((\tilde{\mathbf{z}}_n+\mathbf{f})\cdot\nabla)\,\mathbf{y},\tilde{\mathbf{z}}_n)\;dt$$
$$-2\,((\mathbf{y}\cdot\nabla)\,\mathbf{f},\tilde{\mathbf{z}}_n)\,dt-2\nu\,(\mathbf{f},\tilde{\mathbf{z}}_n)_V\;dt$$
$$+\sum_{i=1}^n|\,(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\,(\tilde{\mathbf{z}}_n+\mathbf{f})\,,\mathbf{e}_i)\,|^2\,dt$$
$$+2\,(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\,(\tilde{\mathbf{z}}_n+\mathbf{f})\,,\tilde{\mathbf{z}}_n)\;d\mathcal{W}_t$$
$$=J\,dt+J_6\,d\mathcal{W}_t, \tag{6.11}$$

with $J=J_1+J_2+J_3+J_4+J_5$.

Let us estimate the terms J_i, $i=1,\dots,5$. First, we have

$$J_1\leqslant(\|a\|_{L_\infty(\Gamma)}+1)\|\tilde{\mathbf{z}}_n\|_{L_2(\Gamma)}^2+C\nu\|g\|_{L_2(\Gamma)}^2$$
$$\leqslant C\left(\|a\|_{W_p^{1-\frac{1}{p}}(\Gamma)}+1\right)||\tilde{\mathbf{z}}_n||_2||\nabla\tilde{\mathbf{z}}_n||_2+C\nu\|g\|_{L_2(\Gamma)}^2$$
$$\leqslant h_1(t)||\tilde{\mathbf{z}}_n||_2^2+\frac{\nu}{4}||\tilde{\mathbf{z}}_n||_V^2+C\nu\|g\|_{L_2(\Gamma)}^2$$

with

$$h_1(t)=C\nu^{-1}\left(\|a\|^2_{W_p^{1-\frac{1}{p}}(\Gamma)}+1\right)\in L_1(0,T),\quad P\text{-a.e. in }\Omega,$$

by the assumption (2.11) and

$$J_2\leqslant C\left(\|\partial_t\mathbf{f}\|_2+\|\mathbf{f}\|_{C(\overline{\mathcal{O}})}\|\nabla\mathbf{y}\|_2\right)\|\tilde{\mathbf{z}}_n\|_2+\|\nabla\mathbf{y}\|_2\,\|\tilde{\mathbf{z}}_n\|_4^2$$
$$\leqslant C\left(\|\partial_t\mathbf{f}\|_2+\|\mathbf{f}\|_{C(\overline{\mathcal{O}})}\right)(1+\|\nabla\mathbf{y}\|_2)\,\|\tilde{\mathbf{z}}_n\|_2$$
$$+\|\nabla\mathbf{y}\|_2\|\tilde{\mathbf{z}}_n\|_2\|\nabla\tilde{\mathbf{z}}_n\|_2$$
$$\leqslant h_2(t)\|\tilde{\mathbf{z}}_n\|_2^2+\frac{\nu}{4}||\tilde{\mathbf{z}}_n||_V^2+\|(f,g)\|_{\mathcal{H}_p(\Gamma)}^2$$

with

$$h_2(t)=C\max(\nu^{-1},1)\left(1+\|\nabla\mathbf{y}\|_2^2\right)\in L_1(0,T),\quad P\text{-a.e. in }\Omega,$$

by the estimates (2.16).

Reasoning as above and using (2.3) for $q=4$, we have

$$\|\mathbf{y}\|_4\leqslant C\left(||\mathbf{y}||_2^{1/2}||\nabla\mathbf{y}||_2^{1/2}+||\mathbf{y}||_2\right),$$

that implies

$$
\begin{aligned}
J_3 &\leqslant C\|\mathbf{y}\|_4||\nabla \mathbf{f}\|_2\|\tilde{\mathbf{z}}_n\|_4 \leqslant C||\nabla \mathbf{f}\|_2\|\mathbf{y}\|_4||\tilde{\mathbf{z}}_n||_2^{1/2}||\nabla \tilde{\mathbf{z}}_n||_2^{1/2} \\
&\leqslant C||\nabla \mathbf{f}\|_2^2\|\mathbf{y}\|_2 + \frac{C}{\nu}(\|\mathbf{y}\|_2^2 + \|\nabla \mathbf{y}\|_2^2) + \frac{\nu}{4}||\tilde{\mathbf{z}}_n||_V^2.
\end{aligned}
$$

Hence

$$
J_3 \leqslant C\|(f,g)\|^2_{\mathcal{H}_p(\Gamma)}\|\mathbf{y}\|_2 + h_3(t)\|\tilde{\mathbf{z}}_n\|_2^2 + \frac{\nu}{4}||\tilde{\mathbf{z}}_n||_V^2
$$

with

$$
h_3(t) = \frac{C}{\nu}\left(||\mathbf{y}||_2^2 + \|\nabla \mathbf{y}\|_2^2\right) \in L_1(0,T), \quad P\text{-a.e. in } \Omega,
$$

by the estimate $(2.16)_1$.

The terms J_4 and J_5 are estimated as

$$
\begin{aligned}
J_4 &\leqslant C\nu\,\|\mathbf{f}\|_V^2\, + ||\tilde{\mathbf{z}}_n||_V^2 + \frac{\nu}{4}||\tilde{\mathbf{z}}_n||_V^2 \\
&\leqslant C\nu\;\|(f,g)\|^2_{\mathcal{H}_p(\Gamma)} + \frac{\nu}{4}||\tilde{\mathbf{z}}_n||_V^2
\end{aligned}
$$

and

$$
\begin{aligned}
J_5 &= \sum_{i=1}^{n} |\left(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\,(\tilde{\mathbf{z}}_n + \mathbf{f})\,,\mathbf{e}_i\right)|^2 \leqslant C||\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\,(\tilde{\mathbf{z}}_n + \mathbf{f})\,||_2^2 \\
&\leqslant C||\tilde{\mathbf{z}}_n + \mathbf{f}||_2^2 \leqslant C||\tilde{\mathbf{z}}_n||_2^2 + C\|(f,g)\|^2_{\mathcal{H}_p(\Gamma)}
\end{aligned}
$$

by the assumption (6.1).

The above deduced estimates for the terms J_i, $i=1,\dots,5$, and (6.11) imply the existence of some positive constant $\widehat{C}_2$ such that

$$
J \leqslant 2f_2(t)\|\tilde{\mathbf{z}}_n\|_2^2 + C\left[(1+\nu)\,||(f,g)||^2_{\mathcal{H}_p(\Gamma)}\,(1+\|\mathbf{y}\|_2)\right] + \nu||\tilde{\mathbf{z}}_n||_V^2 \tag{6.12}
$$

with

$$
f_2(t) = \widehat{C}_2(\nu^{-1}+1)\left(1 + \|\mathbf{u}\|_V^2 + ||(a,b)||^2_{\mathcal{H}_p(\Gamma)}\right) \in L_1(0,T). \tag{6.13}
$$

Introducing the function

$$
\xi_2(t) = e^{-\int_0^t f_2(s)ds} \qquad \text{for } t \in [0,T],
$$

the Itô formula gives

$$
\begin{aligned}
&\xi_2^2(t)\,\|\tilde{\mathbf{z}}_n(t)\|_2^2 + \nu\int_0^t \xi_2^2\,\|\tilde{\mathbf{z}}_n\|_V^2\;ds \\
&\quad\leqslant C\int_0^t \xi_2^2\left[(1+\nu)\,||(f,g)||^2_{\mathcal{H}_p(\Gamma)}\left(1+\|\mathbf{y}\|_2^2\right)\right] ds \\
&\qquad + 2\int_0^t \xi_2^2(t)\left(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\,(\tilde{\mathbf{z}}_n + \mathbf{f})\,,\tilde{\mathbf{z}}_n\right) d\mathcal{W}_t.
\end{aligned} \tag{6.14}
$$

For each $n \in \mathbb{N}$, let us define the function

$$
d(t) = \xi_2^2(t)\,\|\tilde{\mathbf{z}}_n(t)\|_2^2 + \nu\int_0^t \xi_2^2\,\|\tilde{\mathbf{z}}_n\|_V^2\;ds, \quad \text{a.e. in } \Omega\times(0,T),
$$

and consider the sequence $\{\tau_N^n\}_{N\in\mathbb{N}}$ of the stopping times

$$\tau_N^n(\omega) = \inf\{t \geqslant 0 : d(t) \geqslant N\} \wedge T, \qquad P\text{-a.e. } \omega \in \Omega. \tag{6.15}$$

For $0 \leqslant s \leqslant \tau_N^n \wedge t$, we have

$$\begin{aligned} \xi_2^2(s)\,\|\tilde{\mathbf{z}}_n(s)\|_2^2 + \nu \int_0^s \xi_2^2(r)\,\|\tilde{\mathbf{z}}_n\|_V^2 \; dr \\ &\leqslant C \int_0^s \xi_2^2 \left[(1+\nu)\,||(f,g)||^2_{\mathcal{H}_p(\Gamma)}\left(1+\|\mathbf{y}\|_2^2\right)\right] dr \\ &+ 2\int_0^s \xi_2^2(r)\left(\nabla_{\mathbf{y}}\mathbf{G}(r,\mathbf{y})\left(\tilde{\mathbf{z}}_n + \mathbf{f}\right), \tilde{\mathbf{z}}_n\right) d\mathcal{W}_r. \end{aligned} \tag{6.16}$$

The Burkholder–Davis–Gundy inequality gives

$$\begin{aligned} &\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \left| \int_0^s \xi_2^2(r)\left(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\left(\tilde{\mathbf{z}}_n+\mathbf{f}\right),\tilde{\mathbf{z}}_n\right) d\mathcal{W}_r \right| \\ &\leqslant \mathbb{E}\left(\int_0^{\tau_N^n\wedge t} \xi_2^4(s)\left|\left(\nabla_{\mathbf{y}}\mathbf{G}(s,\mathbf{y})\left(\tilde{\mathbf{z}}_n+\mathbf{f}\right),\tilde{\mathbf{z}}_n\right)\right|^2 ds\right)^{\frac{1}{2}} \\ &\leqslant \mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \xi_2^2\,\|\tilde{\mathbf{z}}_n\|_2 \left(\int_0^{\tau_N^n\wedge t} \xi_2^2\,\|\tilde{\mathbf{z}}_n+\mathbf{f}\|_2^2\, ds\right)^{\frac{1}{2}} \\ &\leqslant \frac{1}{2}\,\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \xi_2^2(s)\,\|\tilde{\mathbf{z}}_n\|_2^2 + C\mathbb{E}\int_0^{\tau_N^n\wedge t} \xi_2^2\left(1 + ||\tilde{\mathbf{z}}_n||_2^2 + ||f||^2_{W_2^{1-\frac{1}{2}}(\Gamma)}\right) ds \end{aligned}$$

by the assumption (6.1). Substituting this inequality in (6.16), we derive

$$\begin{aligned} \frac{1}{2}\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \xi_2^2(s)\|\tilde{\mathbf{z}}_n(s)\|_2^2 + \nu\mathbb{E}\int_0^{\tau_N^n\wedge t} \xi_2^2\,\|\tilde{\mathbf{z}}_n\|_V^2\, ds \leqslant C\mathbb{E}\int_0^{\tau_N^n\wedge t} \xi_2^2\,\|\tilde{\mathbf{z}}_n\|_2^2\, ds \\ + C\mathbb{E}\int_0^{\tau_N^n\wedge t} \xi_2^2\left[(1+\nu)\,||(f,g)||^2_{\mathcal{H}_p(\Gamma)}\left(1+\|\mathbf{y}\|_2^2\right)\right] ds. \end{aligned}$$

Let $1_{[0,\tau_N^n]}$ be the characteristic function of the interval $[0,\tau_N^n]$. Then the function

$$f(t) = \mathbb{E}\sup_{s\in[0,t]} 1_{[0,\tau_N^n]}\xi_2^2(s)\|\tilde{\mathbf{z}}_n(s)\|_2^2$$

fulfills the Grönwall type inequality

$$\frac{1}{2}f(t) \leqslant C\int_0^t f(s)ds + C\mathbb{E}\int_0^t \xi_2^2\left[(1+\nu)\,||(f,g)||^2_{\mathcal{H}_p(\Gamma)}\left(1+\|\mathbf{y}\|_2^2\right)\right] dr,$$

which implies that

$$\begin{aligned} \mathbb{E}\sup_{s\in[0,\tau_N^n\wedge t]} \xi_2^2(s)\,\|\tilde{\mathbf{z}}_n(s)\|_2^2 + \nu\mathbb{E}\int_0^{\tau_N^n\wedge t} \xi_2^2\,\|\tilde{\mathbf{z}}_n\|_V^2\, ds \\ &\leqslant C\mathbb{E}\int_0^t \xi_2^2\left[(1+\nu)\,||(f,g)||^2_{\mathcal{H}_p(\Gamma)}\left(1+\|\mathbf{y}\|_2^2\right)\right] ds. \end{aligned} \tag{6.17}$$

Now let us justify the limit transition as $N\to\infty$ in the estimate (6.17). By (2.11), $(2.16)_1$, (6.4), and (6.17) we have

$$\mathbb{E}\sup_{s\in[0,\tau_N^n\wedge T]} d(s) \leqslant C$$

for some constant C being independent of N and n. Let us fix $n \in \mathbb{N}$. Since $\tilde{\mathbf{z}}_n \in C([0,T];V_n)$, then $d(\tau_N^n) \geqslant N$ and

$$\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge T]} d(s) \geqslant \mathbb{E}\left(\sup_{s\in[0,\tau_N^n\wedge T]} 1_{\{\tau_N^n<T\}} d(s)\right) = \mathbb{E}\left(1_{\{\tau_N^n<T\}}\, d(\tau_N^n)\right) \geqslant NP\left(\tau_N^n < T\right).$$

Hence $P\left(\tau_N^n < T\right) \to 0$, as $N\to\infty$, this means that $\tau_N^n \to T$ in probability as $N\to\infty$. Therefore, there exists a subsequence $\{\tau_{N_k}^n\}$ of $\{\tau_N^n\}$ (which may depend on n), such that

$$\tau_{N_k}^n(\omega) \to T \qquad \text{for a.e. } \ \omega\in\Omega \quad \text{as } k\to\infty.$$

So $\mathbf{z}_n = \tilde{\mathbf{z}}_n + \mathbf{f}$ is a global-in-time solution of the stochastic differential equation (6.10). In addition, the sequence $\{\tau_N^n\}$ of the stopping times is monotone on N for each fixed n, so we can apply the monotone convergence theorem in order to pass to the limit in the inequality (6.17) as $N\to\infty$, thereby deducing the estimate (6.6), which is valid for $\tilde{\mathbf{z}}_n$.

Step 2. *Deducing of the estimate (*6.7*).* For each $n\in\mathbb{N}$, let us consider the function

$$d(t) = \xi_2^4(t)\,\|\tilde{\mathbf{z}}_n(t)\|_2^4 + \nu^2\left(\int_0^t \xi_2^2\,\|\tilde{\mathbf{z}}_n\|_V^2\, ds\right)^2 \quad \text{a.e. in } \Omega\times(0,T),$$

and the sequence $\{\tau_N^n\}_{N\in\mathbb{N}}$ of the stopping times

$$\tau_N^n(\omega) = \inf\{t\geqslant 0 : d(t)\geqslant N\}\wedge T, \qquad P\text{ -a.e. } \omega\in\Omega.$$

Taking the square of (6.14), for $0\leqslant s\leqslant \tau_N^n\wedge t$ we infer that

$$\begin{aligned} &\xi_2^4(s)\,\|\tilde{\mathbf{z}}_n(s)\|_2^4 + \nu^2\left(\int_0^s \xi_2^2(r)\,\|\tilde{\mathbf{z}}_n\|_V^2\, dr\right)^2 \\ &\quad\leqslant C\left(\mathbb{E}\int_0^t \xi_2^2\left[(1+\nu)\,||(f,g)||^2_{\mathcal{H}_p(\Gamma)}\left(1+\|\mathbf{y}\|_2^2\right)\right] ds\right)^2 \\ &\qquad + 8\left(\int_0^s \xi_2^2(r)\left(\nabla_{\mathbf{y}}\mathbf{G}(r,\mathbf{y})\left(\tilde{\mathbf{z}}_n+\mathbf{f}\right),\tilde{\mathbf{z}}_n\right) d\mathcal{W}_r\right)^2. \end{aligned} \tag{6.18}$$

The Burkholder–Davis–Gundy inequality gives

$$\begin{aligned} &\mathbb{E}\sup_{s\in[0,\tau_N^n\wedge t]}\left|\int_0^s \xi_2^2(r)\left(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\left(\tilde{\mathbf{z}}_n+\mathbf{f}\right),\tilde{\mathbf{z}}_n\right) d\mathcal{W}_r\right|^2 \\ &\quad\leqslant \mathbb{E}\int_0^{\tau_N^n\wedge t} \xi_2^4(s)\left|\left(\nabla_{\mathbf{y}}\mathbf{G}(s,\mathbf{y})\left(\tilde{\mathbf{z}}_n+\mathbf{f}\right),\tilde{\mathbf{z}}_n\right)\right|^2 ds \\ &\quad\leqslant \frac{1}{2}\,\mathbb{E}\sup_{s\in[0,\tau_N^n\wedge t]} \xi_2^4(s)\,\|\tilde{\mathbf{z}}_n\|_2^4 \\ &\qquad + C\mathbb{E}\int_0^{\tau_N^n\wedge t} \xi_2^4\left(1+||\tilde{\mathbf{z}}_n||_2^4 + ||f||^4_{W_2^{1-\frac{1}{2}}(\Gamma)}\right) ds \end{aligned}$$

by the assumption (6.1). Using this inequality and (6.18), we derive

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\sup_{s\in[0,\tau_N^n\wedge t]}\xi_2^4(s)\|\tilde{\mathbf{z}}_n(s)\|_2^4+\nu^2\mathbb{E}\left(\int_0^{\tau_N^n\wedge t}\xi_2^2\,\|\tilde{\mathbf{z}}_n\|_V^2\,ds\right)^2\\
&\quad\leqslant C\mathbb{E}\int_0^{\tau_N^n\wedge t}\xi_2^4\,\|\tilde{\mathbf{z}}_n\|_2^4\,ds\\
&\qquad+C\left(\mathbb{E}\int_0^t\xi_2^2\left[(1+\nu)\,||(f,g)||_{\mathcal{H}_p(\Gamma)}^2\left(1+\|\mathbf{y}\|_2^2\right)\right]\,ds\right)^2.
\end{aligned}$$

Hence the function

$$f(t)=\mathbb{E}\sup_{s\in[0,t]}1_{[0,\tau_N^n]}\xi_2^4(s)\|\tilde{\mathbf{z}}_n(s)\|_2^4$$

fulfills the Grönwall type inequality

$$\frac{1}{2}f(t)\leqslant C\int_0^t f(s)ds+C\left(\mathbb{E}\int_0^t\xi_2^2\left[(1+\nu)\,||(f,g)||_{\mathcal{H}_p(\Gamma)}^2\left(1+\|\mathbf{y}\|_2^2\right)\right]\,ds\right)^2,$$

which implies that

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,\tau_N^n\wedge t]}\xi_2^4(s)\,\|\tilde{\mathbf{z}}_n(s)\|_2^4+\nu^2\left(\mathbb{E}\int_0^{\tau_N^n\wedge t}\xi_2^2\,\|\tilde{\mathbf{z}}_n\|_V^2\,ds\right)^2\\
&\quad\leqslant C\left(\mathbb{E}\int_0^t\xi_2^2\left[(1+\nu)\,||(f,g)||_{\mathcal{H}_p(\Gamma)}^2\left(1+\|\mathbf{y}\|_2^2\right)\right]\,ds\right)^2. \qquad (6.19)
\end{aligned}$$

Following the same approach as in Step 1, for each fixed n we can show that there exists a subsequence $\{\tau_{N_k}^n\}$ of $\{\tau_N^n\}$, such that

$$\tau_{N_k}^n(\omega)\to T\qquad\text{for a.e.}\quad\omega\in\Omega\quad\text{as } k\to\infty.$$

Accounting that the sequence $\{\tau_N^n\}$ is monotone on N and applying the monotone convergence theorem in the inequality (6.19) as $N\to\infty$, we deduce the estimate (6.7), which is valid for $\tilde{\mathbf{z}}_n$.

Step 3. *Passing to the limit in (*6.10*).*

Let us consider the function $\widetilde{h}=\widehat{C}_0(1+||(a,b)||_{\mathcal{H}_p(\Gamma)}^2)+f_2$ with the constant $\widehat{C}_0$ and the function f_2, defined by (2.17) and (6.13). Since

$$\int_0^T\widetilde{h}(s)\,ds\leqslant C(\omega)<+\infty\qquad\text{for a.e. }\omega\in\Omega$$

by (2.11) and (6.13), there exists a positive constant $K(\omega)$, which depends only on ω, satisfying

$$0<K(\omega)\leqslant\xi(t)=e^{-\int_0^t\widetilde{h}(s)\,ds}\leqslant 1\qquad\text{for a.e.}\quad(\omega,t)\in\Omega\times[0,T]. \qquad (6.20)$$

The estimate (6.6), written for $\tilde{\mathbf{z}}_n$, gives that

$$\mathbb{E}\sup_{t\in[0,T]}\|\xi(t)\tilde{\mathbf{z}}_n(t)\|_2^2+\nu\mathbb{E}\int_0^T\|\xi\tilde{\mathbf{z}}_n\|_V^2\,dt\leqslant C \qquad (6.21)$$

for some constant C, which is independent of the index n.

Therefore, (6.20)–(6.21) imply that there exist a suitable subsequence $\tilde{\mathbf{z}}_n$, which is indexed by the same index n (to simplify the notation), and a function $\tilde{\mathbf{z}}$, such that

$$\begin{aligned} &\xi\tilde{\mathbf{z}}_n \rightharpoonup \xi\tilde{\mathbf{z}} \qquad \text{weakly in} \quad L^2(\Omega\times(0,T);V), \\ &\xi\tilde{\mathbf{z}}_n \rightharpoonup \xi\tilde{\mathbf{z}} \qquad {}^*\text{-weakly in} \quad L^2(\Omega, L^\infty(0,T;H)). \end{aligned} \tag{6.22}$$

The limit function $\tilde{\mathbf{z}}$ satisfies the estimates (6.6) and (6.7) by the lower semicontinuity of integral in the L_2-space and (6.17).

Since $\mathbf{z}_n$ solves (6.10), then Itô's formula gives that for P-a.e. in Ω and any $t\in[0,T]$, we have

$$\begin{cases} d\left(\xi^2(\mathbf{z}_n,\boldsymbol{\varphi})\right) = \xi^2[-\nu\,(\mathbf{z}_n,\boldsymbol{\varphi})_V + \nu\int_\Gamma g(\boldsymbol{\varphi}\cdot\boldsymbol{\tau})\,d\gamma \\ \qquad\qquad -((\mathbf{z}_n\cdot\nabla)\mathbf{y},\boldsymbol{\varphi}) - ((\mathbf{y}\cdot\nabla)\mathbf{z}_n,\boldsymbol{\varphi}) - 2\widetilde{h}\,(\mathbf{z}_n,\boldsymbol{\varphi})]\,ds \\ \qquad\qquad +\xi^2\left(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\mathbf{z}_n,\boldsymbol{\varphi}\right)\,d\mathcal{W}_t \qquad \forall\boldsymbol{\varphi}\in V_n, \\ \mathbf{z}_n(0)=0, \end{cases}$$

that is

$$\begin{aligned} \left(\xi^2(t)\mathbf{z}_n(t),\boldsymbol{\varphi}\right) = &\int_0^t \xi^2(s)[-\nu\,(\mathbf{z}_n(s),\boldsymbol{\varphi})_V + \nu\int_\Gamma b(s)(\boldsymbol{\varphi}\cdot\boldsymbol{\tau})\,d\gamma \\ &-((\mathbf{z}_n\cdot\nabla)\mathbf{y},\boldsymbol{\varphi}) - ((\mathbf{y}\cdot\nabla)\mathbf{z}_n,\boldsymbol{\varphi}) - 2\widetilde{h}(s)\,(\mathbf{z}_n(s),\boldsymbol{\varphi})]\,ds \\ &+\int_0^t \xi^2\left(\nabla_{\mathbf{y}}\mathbf{G}(s,\mathbf{y})\mathbf{z}_n,\boldsymbol{\varphi}\right)\,d\mathcal{W}_s. \end{aligned}$$

Multiplying the last obtained equality by arbitrary fixed $\eta\in L^2(\Omega)$, and taking the expectation, we derive

$$\begin{aligned} \mathbb{E}\,\eta\left(\xi^2(t)\mathbf{z}_n(t),\boldsymbol{\varphi}\right) = \mathbb{E}\,\eta\Big\{&\int_0^t \xi^2[-\nu\,(\mathbf{z}_n,\boldsymbol{\varphi})_V + \nu\int_\Gamma b(\boldsymbol{\varphi}\cdot\boldsymbol{\tau})\,d\gamma \\ &-((\mathbf{z}_n\cdot\nabla)\mathbf{y},\boldsymbol{\varphi}) - ((\mathbf{y}\cdot\nabla)\mathbf{z}_n,\boldsymbol{\varphi}) - 2\widetilde{h}\,(\mathbf{z}_n,\boldsymbol{\varphi})]\,ds \\ &+\int_0^t \xi^2\left(\nabla_{\mathbf{y}}\mathbf{G}(s,\mathbf{y})\mathbf{z}_n,\boldsymbol{\varphi}\right)\,d\mathcal{W}_s\Big\}. \end{aligned}$$

Using that the right side of the last equation is continuous in the time variable $t\in[0,T]$ and applying (6.22), we pass to the limit $n\to\infty$ in this equality and easily deduce that

$$\begin{aligned} \mathbb{E}\,\eta\left(\xi^2(t)\mathbf{z}(t),\boldsymbol{\varphi}\right) = \mathbb{E}\,\eta\Big\{&\int_0^t \xi^2[-\nu\,(\mathbf{z},\boldsymbol{\varphi})_V + \nu\int_\Gamma b(\boldsymbol{\varphi}\cdot\boldsymbol{\tau})\,d\gamma \\ &-((\mathbf{z}\cdot\nabla)\mathbf{y},\boldsymbol{\varphi}) - ((\mathbf{y}\cdot\nabla)\mathbf{z},\boldsymbol{\varphi}) - 2\widetilde{h}\,(\mathbf{z},\boldsymbol{\varphi})]\,ds \\ &+\int_0^t \xi^2\left(\nabla_{\mathbf{y}}\mathbf{G}(s,\mathbf{y})\mathbf{z},\boldsymbol{\varphi}\right)\,d\mathcal{W}_s\Big\} \qquad \forall\boldsymbol{\varphi}\in V. \end{aligned}$$

Since $\eta\in L^2(\Omega)$ is arbitrary, then we have the validity of the equality

$$\begin{aligned} \left(\xi^2(t)\mathbf{z}(t),\boldsymbol{\varphi}\right) = &\int_0^t \xi^2\Big[-\nu\,(\mathbf{z},\boldsymbol{\varphi})_V + \nu\int_\Gamma b(\boldsymbol{\varphi}\cdot\boldsymbol{\tau})\,d\gamma \\ &-((\mathbf{z}\cdot\nabla)\mathbf{y},\boldsymbol{\varphi}) - ((\mathbf{y}\cdot\nabla)\mathbf{z},\boldsymbol{\varphi}) - 2\widetilde{h}\,(\mathbf{z},\boldsymbol{\varphi})\Big]\,ds \\ &+\int_0^t \xi^2\left(\nabla_{\mathbf{y}}\mathbf{G}(s,\mathbf{y})\mathbf{z},\boldsymbol{\varphi}\right)\,d\mathcal{W}_s, \end{aligned}$$

that is

$$
\begin{aligned}
d\left(\xi^2(\mathbf{z},\boldsymbol{\varphi})\right) = \xi^2 \Big[& -\nu\,(\mathbf{z},\boldsymbol{\varphi})_V + \nu \int_\Gamma g(\boldsymbol{\varphi}\cdot\boldsymbol{\tau})\,d\gamma \\
& - ((\mathbf{z}\cdot\nabla)\mathbf{y},\boldsymbol{\varphi}) - ((\mathbf{y}\cdot\nabla)\mathbf{z},\boldsymbol{\varphi}) - 2\widetilde{h}\,(\mathbf{z},\boldsymbol{\varphi}) \Big]\, ds \\
& + \xi^2\,(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\mathbf{z},\boldsymbol{\varphi})\; d\mathcal{W}_t.
\end{aligned}
$$

Moreover, if we use Itô's formula

$$
d\,(\mathbf{z},\boldsymbol{\varphi}) = d\left[\xi^{-2}\xi^2\,(\mathbf{z},\boldsymbol{\varphi})\right] = \xi^2\,(\mathbf{z},\boldsymbol{\varphi})\,d\left(\xi^{-2}\right) + \xi^{-2} d\left[\xi^2\,(\mathbf{z},\boldsymbol{\varphi})\right],
$$

we obtain that the limit function $\mathbf{z}$ in the form $\mathbf{z} = \tilde{\mathbf{z}} + \mathbf{f}$ fulfills the stochastic differential equation

$$
\left\{
\begin{aligned}
& d(\mathbf{z},\boldsymbol{\varphi}) = \left[-\nu\,(\mathbf{z},\boldsymbol{\varphi})_V + \nu \textstyle\int_\Gamma g(\boldsymbol{\varphi}\cdot\boldsymbol{\tau})\,d\gamma - ((\mathbf{z}\cdot\nabla)\mathbf{y},\boldsymbol{\varphi}) - ((\mathbf{y}\cdot\nabla)\mathbf{z},\boldsymbol{\varphi})\right] dt \\
& \qquad\qquad + (\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\mathbf{z},\boldsymbol{\varphi})\; d\mathcal{W}_t \qquad \forall \boldsymbol{\varphi}\in V \qquad \text{a.e. in } \Omega\times(0,T), \\
& \mathbf{z}(0) = 0.
\end{aligned}
\right. \tag{6.23}
$$

The uniqueness result follows from the linearity of this system by taking into account the estimates (6.6)–(6.7). ◻

7. Gâteaux differentiability of the control-to-state mapping. To derive the necessary conditions for first-order optimality, it is necessary to study differentiability of the Gateau cost functional, which is based on Gateau derivative mapping control to state.

To deduce the necessary first-order optimality conditions, it is necessary to study the Gâteaux differentiability of the cost functional J, which is based on the Gâteaux derivative of the control-to-state mapping. Introducing additional assumptions, in this section we show that the Gâteaux derivative of the control-to-state mapping $(a,b)\to\mathbf{y}$, at a point (a,b), in any direction (f,g), exists and is given by the solution of the linearized system (6.3).

PROPOSITION 7.1. *Assume*

$$
A_* \geqslant 32\max\left(\widehat{C}_1,\widehat{C}_2\right)\times\max\left(\nu^{-1},1\right) \tag{7.1}
$$

with A_* *given by* (5.3) *and* $\widehat{C}_1$, $\widehat{C}_2$ *as in Theorem* 3.1 *and Proposition* 6.2. *Let* (a,b), $(f,g)\in\mathcal{A}$, $\mathbf{y}_0$ *as in* (5.1) *and consider the data*

$$
a_\varepsilon = a+\varepsilon f,\quad b_\varepsilon = b+\varepsilon g \qquad \forall \varepsilon\in(0,1). \tag{7.2}
$$

Let $(\mathbf{y},q)$, $(\mathbf{y}_\varepsilon,q_\varepsilon)$ *be the solutions of* (1.1) *corresponding to* $(a,b,\mathbf{y}_0)$ *and* $(a_\varepsilon,b_\varepsilon,\mathbf{y}_0)$, *respectively. Then the following representation holds:*

$$
\mathbf{y}_\varepsilon = \mathbf{y} + \varepsilon\mathbf{z} + \varepsilon\,\boldsymbol{\delta}_\varepsilon \quad \text{with} \quad \lim_{\varepsilon\to0}\mathbb{E}\int_0^T \|\boldsymbol{\delta}_\varepsilon\|_V^2\,ds = 0, \tag{7.3}
$$

where

$$
\mathbf{z}\in C([0,T];H)\cap L_2(0,T;V), \quad P\text{-a.e.}\quad \omega\in\Omega,
$$

is the solution of (6.3) *satisfying the estimate* (6.6).

Proof. Let us define $\mathbf{z}_\varepsilon = \frac{\mathbf{y}_\varepsilon - \mathbf{y}}{\varepsilon}$ and $\pi_\varepsilon = \frac{q_\varepsilon - q}{\varepsilon}$. By direct calculations we can check that the pair $(\mathbf{z}_\varepsilon, \pi_\varepsilon)$ fulfills the following system in the distributional sense:

$$
\begin{cases}
d\mathbf{z}_\varepsilon = (\nu\Delta\mathbf{z}_\varepsilon - (\mathbf{y}\cdot\nabla)\,\mathbf{z}_\varepsilon - (\mathbf{z}_\varepsilon\cdot\nabla)\,\mathbf{y}_\varepsilon - \nabla\pi_\varepsilon)\,dt \\
\qquad + \dfrac{1}{\varepsilon}\left(\mathbf{G}(t,\mathbf{y}_\varepsilon) - \mathbf{G}(t,\mathbf{y})\right)\,d\mathcal{W}_t, \qquad \operatorname{div}\mathbf{z}_\varepsilon = 0 \quad \text{in} \quad \mathcal{O}_T, \\
\mathbf{z}_\varepsilon\cdot\mathbf{n} = f, \quad [2D(\mathbf{z}_\varepsilon)\,\mathbf{n} + \alpha\mathbf{z}_\varepsilon]\cdot\ \boldsymbol{\tau} = g \quad \text{on} \quad \Gamma_T, \\
\mathbf{z}_\varepsilon(0,\mathbf{x}) = 0 \quad \text{in} \quad \mathcal{O}
\end{cases}
\tag{7.4}
$$

and $\boldsymbol{\delta}_\varepsilon = \mathbf{z}_\varepsilon - \mathbf{z}$ satisfies the system

$$
\begin{cases}
d\boldsymbol{\delta}_\varepsilon = (\nu\Delta\boldsymbol{\delta}_\varepsilon - (\mathbf{y}\cdot\nabla)\,\boldsymbol{\delta}_\varepsilon - (\boldsymbol{\delta}_\varepsilon\cdot\nabla)\,\mathbf{y} - (\mathbf{z}_\varepsilon\cdot\nabla)\,(\mathbf{y}_\varepsilon - \mathbf{y}) - \nabla\,(\pi_\varepsilon - \pi))\,dt \\
\qquad + \mathbf{R}\,d\mathcal{W}_t, \qquad \operatorname{div}\boldsymbol{\delta}_\varepsilon = 0 \quad \text{in} \quad \mathcal{O}_T, \\
\boldsymbol{\delta}_\varepsilon\cdot\mathbf{n} = 0, \quad [2D(\boldsymbol{\delta}_\varepsilon)\,\mathbf{n} + \alpha\boldsymbol{\delta}_\varepsilon]\cdot\boldsymbol{\tau} = 0 \quad \text{on} \quad \Gamma_T, \\
\boldsymbol{\delta}_\varepsilon(0,\mathbf{x}) = 0 \quad \text{in} \quad \mathcal{O},
\end{cases}
\tag{7.5}
$$

where

$$
\begin{aligned}
\mathbf{R} &= \frac{1}{\varepsilon}\left[\mathbf{G}(t,\mathbf{y}_\varepsilon) - \mathbf{G}(t,\mathbf{y})\right] - \nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\mathbf{z} \\
&= \frac{1}{\varepsilon}\left[\mathbf{G}(t,\mathbf{y}+\varepsilon\mathbf{z}) - \mathbf{G}(t,\mathbf{y})\right] - \nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\mathbf{z} + \frac{1}{\varepsilon}\left[\mathbf{G}(t,\mathbf{y}_\varepsilon) - \mathbf{G}(t,\mathbf{y}+\varepsilon\mathbf{z})\right].
\end{aligned}
$$

The Itô formula, applied to the stochastic differential equation of (7.5), gives

$$
\begin{aligned}
d\left(||\boldsymbol{\delta}_\varepsilon||_2^2\right) + 2\nu||\boldsymbol{\delta}_\varepsilon||_V^2\,dt &= \left(-\int_\Gamma a(\boldsymbol{\delta}_\varepsilon\cdot\boldsymbol{\tau})^2\,d\gamma - (2\,(\boldsymbol{\delta}_\varepsilon\cdot\nabla)\,\mathbf{y},\boldsymbol{\delta}_\varepsilon)\right)\,dt' \\
&\quad + \frac{1}{\varepsilon}\left(-2\left((\mathbf{y}_\varepsilon - \mathbf{y})\cdot\nabla\right)(\mathbf{y}_\varepsilon - \mathbf{y}),\boldsymbol{\delta}_\varepsilon\right)\,dt \\
&\quad + \|\mathbf{R}\|_2^2\,dt + 2\,(\mathbf{R},\boldsymbol{\delta}_\varepsilon)\,d\mathcal{W}_t \\
&= (I_1 + I_2 + I_3 + I_4)\,dt + 2\,(\mathbf{R},\boldsymbol{\delta}_\varepsilon)\,d\mathcal{W}_t.
\end{aligned}
\tag{7.6}
$$

Applying the inequalities (2.3), (2.4), (2.8), and (2.13), the following estimates hold:

$$
\begin{aligned}
I_1 &\leqslant C\|a\|_{L_\infty(\Gamma)}\|\boldsymbol{\delta}_\varepsilon\|_{L_2(\Gamma)}^2 \leqslant C\nu^{-1}||a||^2_{W_p^{1-\frac{1}{p}}(\Gamma)}||\boldsymbol{\delta}_\varepsilon||_2^2 + \frac{\nu}{3}||\boldsymbol{\delta}_\varepsilon||_V^2, \\
I_2 &\leqslant C\|\mathbf{y}\|_{H^1}\|\boldsymbol{\delta}_\varepsilon\|_4^2 \leqslant C\nu^{-1}\left(\|\mathbf{u}\|_{H^1}^2 + ||(a,b)||^2_{\mathcal{H}_p(\Gamma)}\right)\|\boldsymbol{\delta}_\varepsilon\|_2^2 + \frac{\nu}{3}||\boldsymbol{\delta}_\varepsilon||_V^2,
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \frac{2}{\varepsilon}\left(\left((\mathbf{y}_\varepsilon - \mathbf{y})\cdot\nabla\right)\boldsymbol{\delta}_\varepsilon,(\mathbf{y}_\varepsilon - \mathbf{y})\right) - 2\int_\Gamma f\,(\mathbf{y}_\varepsilon - \mathbf{y})\cdot\boldsymbol{\delta}_\varepsilon d\gamma \\
&\leqslant \frac{2}{\varepsilon}\|\mathbf{y}_\varepsilon - \mathbf{y}\|_2\|\mathbf{y}_\varepsilon - \mathbf{y}\|_V\|\boldsymbol{\delta}_\varepsilon\|_V + \|f\|_{L_\infty(\Gamma)}\|\mathbf{y}_\varepsilon - \mathbf{y}\|_2^{1/2}\|\mathbf{y}_\varepsilon - \mathbf{y}\|_V^{1/2}\|\boldsymbol{\delta}_\varepsilon\|_V \\
&\leqslant \frac{C\nu^{-1}}{\varepsilon^2}\|\mathbf{y}_\varepsilon - \mathbf{y}\|_2^2\|\mathbf{y}_\varepsilon - \mathbf{y}\|_V^2 + C\nu^{-1}||f||_{W_p^{1-\frac{1}{p}}(\Gamma)}\|\mathbf{y}_\varepsilon - \mathbf{y}\|_2\|\mathbf{y}_\varepsilon - \mathbf{y}\|_V + \frac{\nu}{3}||\boldsymbol{\delta}_\varepsilon||_V^2.
\end{aligned}
$$

Due to (2.10), (6.1), and (6.2) we have

$$I_4 = \|\mathbf{R}\|_2^2 \leqslant \frac{2}{\varepsilon^2}\|o\left(t, \|\varepsilon\mathbf{z}\|_2\right)\|_2^2 + 2K\|\boldsymbol{\delta}_\varepsilon\|_2^2 \quad \text{a.e. in } [0,T]\times\Omega \tag{7.7}$$

with

$$\frac{2}{\varepsilon^2}\|o\left(t, \|\varepsilon\mathbf{z}\|_2\right)\|_2^2 = \|\mathbf{z}\|_2^2 \left\|\frac{o\left(t, \|\varepsilon\mathbf{z}\|_2\right)}{\|\varepsilon\mathbf{z}\|_2}\right\|_2^2 \longrightarrow 0 \text{ as } \varepsilon \to 0 \quad \text{a.e. in } [0,T]\times\Omega,$$

and

$$\frac{2}{\varepsilon^2}\|o\left(t, \|\varepsilon\mathbf{z}\|_2\right)\|_2^2 \leqslant C\|\mathbf{z}\|_2^2,$$

where C is a constant independent of ε.

Considering the above deduced estimates for the terms $I_i,$ $i=1,\dots,4$, we ensure that the following estimate holds:

$$\begin{aligned} I_1 + I_2 + I_3 + I_4 \leqslant \Bigg[& 2h(t)||\boldsymbol{\delta}_\varepsilon||_2^2 + \frac{C\nu^{-1}}{\varepsilon^2}\|\mathbf{y}_\varepsilon - \mathbf{y}\|_2^2\|\mathbf{y}_\varepsilon - \mathbf{y}\|_V^2 \\ & + C\nu^{-1}||f||_{W_p^{1-\frac{1}{p}}(\Gamma)}\|\mathbf{y}_\varepsilon - \mathbf{y}\|_2\|\mathbf{y}_\varepsilon - \mathbf{y}\|_V + \nu||\boldsymbol{\delta}_\varepsilon||_V^2 \\ & + \frac{2}{\varepsilon^2}\|o\left(t, \|\varepsilon\mathbf{z}\|_2\right)\|_2^2\Bigg] dt, \end{aligned} \tag{7.8}$$

where

$$h(t) = C(\nu^{-1}+1)\left(1 + \|\mathbf{u}\|_V^2 + ||(a,b)||_{\mathcal{H}_p(\Gamma)}^2\right) \tag{7.9}$$

for some positive constant C. We can choose C in (7.9) and $\widehat{C}_2$ in (6.13) in such a way that these constants are the same. Therefore, we can consider that the functions $h(t)$, $f_2(t)$ are equal.

Now let us introduce the function

$$\beta(t) = e^{-\int_0^t 2\max(f_1(s), f_2(s))\,ds}$$

with f_1 and f_2 defined in (3.4) and (6.13) for the data $(a_\varepsilon, b_\varepsilon)$, $\mathbf{y}_0$ and (a,b), $\mathbf{y}_0$, respectively. Considering the equality (7.6), the Itô formula yields

$$\begin{aligned} & \beta^2(t)\,\|\boldsymbol{\delta}_\varepsilon(t)\|_2^2 + \nu\int_0^t \beta^2\,\|\boldsymbol{\delta}_\varepsilon\|_V^2\,ds \\ & \leqslant C\int_0^t g_\varepsilon(s)\,ds + \int_0^t \beta^2\left(\frac{2}{\varepsilon^2}\|o\left(s, \|\varepsilon\mathbf{z}\|_2\right)\|_2^2\right)ds + 2\int_0^t \beta^2\left(\mathbf{R}, \boldsymbol{\delta}_\varepsilon\right)d\mathcal{W}_s, \end{aligned} \tag{7.10}$$

where

$$g_\varepsilon(t) = \beta^2(t)\nu^{-1}\left(\frac{1}{\varepsilon^2}\|\mathbf{y}_\varepsilon - \mathbf{y}\|_2^2\|\mathbf{y}_\varepsilon - \mathbf{y}\|_V^2 + ||f||_{W_p^{1-\frac{1}{p}}(\Gamma)}\|\mathbf{y}_\varepsilon - \mathbf{y}\|_2\|\mathbf{y}_\varepsilon - \mathbf{y}\|_V\right).$$

On the other hand, using the Hölder inequality and (3.1), (3.2), (6.4), and (7.2), we derive

$$\mathbb{E}\int_0^t g_\varepsilon(s)\,ds \leqslant C\varepsilon^2 \qquad \forall t \in [0,T],$$

and the Lebesgue dominated convergence theorem gives

$$\mathbb{E}\int_0^t \beta^2 \left(\frac{2}{\varepsilon^2}\|o(s,\|\varepsilon\mathbf{z}\|_2)\|_2^2\right) ds \to 0 \ \text{ as } \varepsilon \to 0.$$

Therefore, applying the mathematical expectation to (7.10) and taking the limit, as $\varepsilon \to 0$, we deduce

$$\lim_{\varepsilon\to 0} \mathbb{E}\beta^2(t)\,\|\boldsymbol{\delta}_\varepsilon(t)\|_2^2 = 0, \qquad \lim_{\varepsilon\to 0}\mathbb{E}\int_0^t \beta^2\,\|\boldsymbol{\delta}_\varepsilon\|_V^2\,ds = 0 \ \forall t\in[0,T]. \tag{7.11}$$

Taking into account $(5.5)_2$, (7.1), and the definition of β, we have $\mathbb{E}\left(\beta^{-8}(T)\right) < \infty$. Hence the Hölder inequality and the monotonicity of β give

$$\begin{aligned}
\mathbb{E}\int_0^T \|\boldsymbol{\delta}_\varepsilon(s)\|_V^2\,ds &\leqslant \mathbb{E}\left[\beta^{-2}(T)\int_0^T \beta^2(s)\,\|\boldsymbol{\delta}_\varepsilon(s)\|_V^2\,ds\right]\\
&\leqslant \left[\mathbb{E}\left(\int_0^T \beta^2(s)\,\|\boldsymbol{\delta}_\varepsilon(s)\|_V^2\,ds\right)\right]^{\frac12}\left[\mathbb{E}\left(\beta^{-8}(T)\right)\right]^{\frac14}\\
&\quad\times\left[\mathbb{E}\left(\int_0^T \beta^2(s)\,\|\boldsymbol{\delta}_\varepsilon(s)\|_V^2\,ds\right)^2\right]^{\frac14},
\end{aligned} \tag{7.12}$$

then using sequentially the relation in (7.3), (7.2), (3.2), and (6.7), we deduce the following uniform estimate with respect to ε:

$$\begin{aligned}
\mathbb{E}\left(\int_0^T \beta^2(s)\,\|\boldsymbol{\delta}_\varepsilon(s)\|_V^2\,ds\right)^2 &\leqslant 4\mathbb{E}\left(\int_0^T \xi_1^2(s)\left\|\frac{\mathbf{y}_\varepsilon-\mathbf{y}}{\varepsilon}\right\|_V^2 ds\right)^2\\
&\quad + 4\mathbb{E}\left(\int_0^T \xi_2^2(s)\|\mathbf{z}\|_V^2\,ds\right)^2 \leqslant C,
\end{aligned}$$

since $\beta \leqslant \xi_1, \xi_2$ for ξ_1 and ξ_2 defined by (3.3) and (6.8). Using this estimate together with (7.11), we see that the inequality (7.12) yields (7.3). ∎

8. Adjoint equation. This section is devoted to the study of the adjoint system.

Let $\mathbf{y}$ be the solution of the state equation (1.1) corresponding to the given data $(a,b,\mathbf{y}_0)$, then the adjoint system is given byIn

$$\begin{cases}
-d\mathbf{p} = [\nu\Delta\mathbf{p} + 2D(\mathbf{p})\mathbf{y} - \nabla\pi + \mathbf{U}]\,dt & \\
\qquad\qquad + \nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})^T\mathbf{q}\,dt - \mathbf{q}\,d\mathcal{W}_t, \qquad \operatorname{div}\mathbf{p} = 0, & \text{in } \mathcal{O}_T,\\
\mathbf{p}\cdot\mathbf{n} = 0, \qquad \left[2D(\mathbf{p})\mathbf{n} + (\alpha + \frac{a}{\nu})\mathbf{p}\right]\cdot\boldsymbol{\tau} = 0 & \text{on } \Gamma_T,\\
\mathbf{p}(T) = 0 & \text{in } \mathcal{O}.
\end{cases} \tag{8.1}$$

In next section 9, we will prove that the adjoint state $\mathbf{p}$ and the linearized state $\mathbf{z}$ are related through the duality relation (9.1). In order to give a meaning to boundary terms that will appear in that relation, it is necessary to have the H^2-regularity of the adjoint state $\mathbf{p}$, that we demonstrate in the following proposition.

PROPOSITION 8.1. *Let* (a,b), $\mathbf{u}_0$ *satisfy* (5.1), *and let* $\mathbf{y}$ *be the solution of the state system* (1.1), *constructed in Theorem* 2.4. *Assume that*

$$\mathbf{U} \in L_4(\Omega \times (0,T); L_2(\mathcal{O}))$$

is a predictable stochastic process. In addition, we assume that

$$\min\left(A_*, B_*, \frac{B_*}{\widehat{C}}\right) \geqslant 24 \max(\widetilde{C}_1, \widetilde{C}_2) \max(\nu^{-2}, 1) \tag{8.2}$$

with $A_*,\ B_*,\ \widehat{C}$ *defined by* (5.3) *and* $\widetilde{C}_1, \widetilde{C}_2$ *defined by the expressions* (8.9), (8.13) *below.*

Then, there exists a predictable stochastic process $(\mathbf{p}, \mathbf{q}, \pi)$, *such that*

$$\begin{aligned} &\mathbf{p} \in L_\infty(0,T; L_2(\Omega; V)) \cap L_2(\Omega \times (0,T); H^2(\mathcal{O})), \quad \mathbf{q} \in L_2(\Omega \times (0,T); V), \\ &\pi \in L_2(\Omega \times (0,T); H^1(\mathcal{O})), \end{aligned} \tag{8.3}$$

which fulfills the system (8.1) *for* P*-a.e. in* Ω. *Moreover, the following estimates hold:*

$$\begin{aligned} &\sup_{t\in[0,T]} \mathbb{E}\left(\|\mathbf{p}(t)\|_2^2\right) + \mathbb{E}\int_0^T (\nu \|\mathbf{p}\|_V^2 + \|\mathbf{q}\|_2^2)\, dt \leqslant C\left(\mathbb{E}\int_0^T ||\mathbf{U}||_2^4\, dt\right)^{\frac{1}{2}}, \\ &\sup_{t\in[0,T]} \left(\mathbb{E}\,\|\mathbf{p}(t)\|_V^2\right) + \mathbb{E}\int_0^T (\nu \|\mathbf{p}\|_{H^2}^2 + \|\nabla\pi\|_2 + \|\mathbf{q}\|_V^2)\, dt \leqslant C\left(\mathbb{E}\int_0^T ||\mathbf{U}||_2^4\, dt\right)^{\frac{1}{2}}. \end{aligned} \tag{8.4}$$

Proof. The proof is divided into three steps.

Step 1. *Finite-dimensional approximations.* The existence of a solution for the system (8.1) will be shown by Galerkin's method. Here, we introduce a basis $\{\mathbf{h}_k\} \subset H^2(\mathcal{O}) \cap V$ of eigenfunctions of the Stokes problem with the homogeneous Navier-slip boundary conditions

$$\begin{cases} -\Delta \mathbf{h}_k + \nabla \pi_k = \lambda_k \mathbf{h}_k, \qquad \operatorname{div} \mathbf{h}_k = 0 & \text{in } \mathcal{O}, \\ \mathbf{h}_k \cdot \mathbf{n} = 0, \quad \left[2D(\mathbf{h}_k)\,\mathbf{n} + \left(\alpha + \frac{a}{\nu}\right)\mathbf{h}_k\right] \cdot \boldsymbol{\tau} = 0 & \text{on } \Gamma. \end{cases} \tag{8.5}$$

Let

$$\mathbf{p}_n(t) = \sum_{j=1}^n s_j^{(n)}(t)\mathbf{h}_j \in C([0,T]; V_n) \quad \text{and} \quad \mathbf{q}_n = \sum_{j=1}^n \mathfrak{q}_j^{(n)}(t)\mathbf{h}_j$$

be the solution of the following backward stochastic system:

$$\begin{cases} -d\,(\mathbf{p}_n(t), \mathbf{h}_k) + \{\nu\,(\mathbf{p}_n, \mathbf{h}_k)_V - (2D(\mathbf{p}_n)\mathbf{y}, \mathbf{h}_k) \\ \qquad - (\mathbf{U}, \mathbf{h}_k) + \int_\Gamma a(\mathbf{p}_n \cdot \boldsymbol{\tau})(\mathbf{h}_k \cdot \boldsymbol{\tau})\, d\gamma\}\, dt \\ \qquad = (\nabla_{\mathbf{y}} \mathbf{G}(t,\mathbf{y})^T \mathbf{q}_n, \mathbf{h}_k)\, dt - (\mathbf{q}_n, \mathbf{h}_k)\, d\mathcal{W}_t, \\ \mathbf{p}_n(T) = \mathbf{0}, \qquad k = 1, 2, \ldots, n, \qquad P\text{-a.e. in } \Omega \quad \text{a.e. on } (0,T). \end{cases} \tag{8.6}$$

Following [23, Prop. 6.20], the system (8.6) of n linear stochastic differential equations has a unique global-in-time solution $(\mathbf{p}_n, \mathbf{q}_n)$, which is an adapted process verifying

$$(\mathbf{p}_n, \mathbf{q}_n) \in L_r(\Omega; C([0,T]; H_n)) \times L_r(\Omega_T; L_r(\mathcal{O})), \qquad 1 \leqslant r < \infty. \tag{8.7}$$

Step 2. *Uniform estimates.* Now, we show the first uniform estimates for $(\mathbf{p}_n, \mathbf{q}_n)$ with respect to n. The Itô formula gives

$$\begin{aligned} &- d\left((\mathbf{p}_n, \mathbf{h}_k)^2\right) + 2(\mathbf{p}_n, \mathbf{h}_k)\Big\{\nu(\mathbf{p}_n, \mathbf{h}_k)_V - (2D(\mathbf{p}_n)\mathbf{y}, \mathbf{h}_k) \\ &\quad - (\mathbf{U}, \mathbf{h}_k) + \int_\Gamma a(\mathbf{p}_n \cdot \boldsymbol{\tau})(\mathbf{h}_k \cdot \boldsymbol{\tau})\, d\gamma\Big\} dt \\ &= 2(\mathbf{p}_n, \mathbf{h}_k)\left(\nabla_{\mathbf{y}} \mathbf{G}(t, \mathbf{y})^T \mathbf{q}_n, \mathbf{h}_k\right) dt - 2(\mathbf{p}_n, \mathbf{h}_k)(\mathbf{q}_n, \mathbf{h}_k)\, d\mathcal{W}_t - |(\mathbf{q}_n, \mathbf{h}_k)|^2\, dt. \end{aligned}$$

Summing these equalities over $k = 1, \ldots, n$, we obtain

$$\begin{aligned} &- d\left(\|\mathbf{p}_n\|_2^2\right) + \left(2\nu\|\mathbf{p}_n\|_V^2 + \|\mathbf{q}_n\|_2^2\right) dt = \Big\{2\left(\left((\nabla \mathbf{p}_n)^T\right)\mathbf{y} + \mathbf{U}, \mathbf{p}_n\right) \\ &\quad - \int_\Gamma a(\mathbf{p}_n \cdot \boldsymbol{\tau})^2\, d\gamma + 2\left(\nabla_{\mathbf{y}} \mathbf{G}(t, \mathbf{y})^T \mathbf{q}_n, \mathbf{p}_n\right)\Big\} dt - 2(\mathbf{q}_n, \mathbf{p}_n)\, d\mathcal{W}_t \\ &= [J_1 + J_2 + J_3]\, dt - 2(\mathbf{q}_n, \mathbf{p}_n)\, d\mathcal{W}_t. \end{aligned} \tag{8.8}$$

Let us estimate the terms J_1, J_2, and J_3. Knowing that $div\, \mathbf{p}_n = 0$, applying the Gagliardo–Nirenberg–Sobolev inequality (2.3) with $q = 4$ and Young's inequality (2.8), we obtain

$$\begin{aligned} J_1 &= -2((\nabla \mathbf{y})\mathbf{p}_n, \mathbf{p}_n) + 2(\mathbf{U}, \mathbf{p}_n) \leqslant C\|\mathbf{y}\|_{H^1}\|\mathbf{p}_n\|_4^2 + C\|\mathbf{U}\|_2\|\mathbf{p}_n\|_2 \\ &\leqslant C\|\mathbf{y}\|_{H^1}\|\mathbf{p}_n\|_2\|\nabla \mathbf{p}_n\|_2 + C\|\mathbf{U}\|_2\|\mathbf{p}_n\|_2 \\ &\leqslant C(\nu^{-1}\|\mathbf{y}\|_{H^1}^2 + 1)\|\mathbf{p}_n\|_2^2 + \frac{\nu}{2}||\mathbf{p}_n||_V^2 + \|\mathbf{U}\|_2^2, \end{aligned}$$

and

$$\begin{aligned} J_2 &\leqslant C\|a\|_{L_\infty(\Gamma)}\|\mathbf{p}_n\|_{L_2(\Gamma)}^2 \leqslant C\|a\|_{W_p^{1-\frac{1}{p}}(\Gamma)}||\mathbf{p}_n||_{L_2(\Omega)}||\nabla \mathbf{p}_n||_{L_2(\Omega)} \\ &\leqslant C\nu^{-1}\|a\|_{W_p^{1-\frac{1}{p}}(\Gamma)}^2||\mathbf{p}_n||_{L_2(\Omega)}^2 + \frac{\nu}{2}||\mathbf{p}_n||_V^2. \end{aligned}$$

Also we have

$$\begin{aligned} J_3 &= 2|\left(\nabla_{\mathbf{y}} \mathbf{G}(t, \mathbf{y})^T \mathbf{q}_n, \mathbf{p}_n\right)| \leqslant C||\nabla_{\mathbf{y}} \mathbf{G}(t, \mathbf{y})^T \mathbf{q}_n||_2||\mathbf{p}_n||_2 \\ &\leqslant \frac{1}{2}||\mathbf{q}_n||_2^2 + C||\mathbf{p}_n||_2^2 \end{aligned}$$

by the assumption (6.1). Therefore, gathering the deduced estimates for the terms J_1, J_2, J_2, we obtain the existence of a positive constant $\widetilde{C}_1$, such that

$$J_1 + J_2 + J_3 \leqslant 2r_1(t)||\mathbf{p}_n||_2^2 + \|\mathbf{U}\|_2^2 + \frac{1}{2}||\mathbf{q}_n||_2^2 + \nu||\mathbf{p}_n||_V^2 \tag{8.9}$$

with

$$r_1(t) = \widetilde{C}_1 \max\left(\nu^{-1}, 1\right)\left(1 + ||(a,b)||_{\mathcal{H}_p(\Gamma)}^2 + \|\mathbf{u}\|_V^2\right) \in L_1(0,T), \quad P\text{-a.e. in } \Omega,$$

depending only on the data (2.11) of our problem by (2.16).

Let us consider the function

$$\beta_1(t) = e^{\int_0^t r_1(s)\,ds}$$

and apply the Itô formula

$$d\left(\beta_1^2 \|\mathbf{p}_n\|_2^2\right) = \beta_1^2 d\left(\|\mathbf{p}_n\|_2^2\right) + 2r_1\beta_1^2\|\mathbf{p}_n\|_2^2\,dt.$$

Using the equality (8.8), integrating over the time interval (t, T) and using the estimate (8.9), we deduce the estimate

$$\begin{aligned}\beta_1^2(t)\,\|\mathbf{p}_n(t)\|_2^2 + \int_t^T \beta_1^2\left(\nu\,\|\mathbf{p}_n\|_V^2 + \frac{1}{2}\|\mathbf{q}_n\|_2^2\right) ds \\ \leqslant \int_t^T \beta_1^2\|\mathbf{U}\|_2^2\,ds - 2\int_t^T \beta_1^2\left(\mathbf{q}_n, \mathbf{p}_n\right) d\mathcal{W}_s.\end{aligned} \tag{8.10}$$

Let us notice that the assumptions (8.2) imply

$$A_* \geqslant 16\widetilde{C}_1 \max\left(\nu^{-1}, 1\right),$$

which gives $\mathbb{E}\left[\beta_1^8(T)\right] < \infty$. Therefore, taking into accounting (8.7), the Hölder inequality yields

$$\begin{aligned}\mathbb{E}\int_0^T \left(\beta_1^2\left(\mathbf{q}_n, \mathbf{p}_n\right)\right)^2 ds \leqslant C\left(\mathbb{E}\int_0^T \beta_1^8\,ds\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T \|\mathbf{q}_n\|_2^8\,ds\right)^{\frac{1}{4}} \\ \times\left(\mathbb{E}\int_0^T \|\mathbf{p}_n\|_2^8\,ds\right)^{\frac{1}{4}} < C_n < \infty,\end{aligned}$$

which ensures that the stochastic process $M_t = \int_t^T \beta_1^2\left(\mathbf{q}_n, \mathbf{p}_n\right) d\mathcal{W}_s$ is a square integrable martingale. Hence, taking the expectation in (8.10), we infer that

$$\begin{aligned}\mathbb{E}\left(\beta_1^2(t)\,\|\mathbf{p}_n(t)\|_2^2\right) + \mathbb{E}\int_t^T \beta_1^2\left(\nu\,\|\mathbf{p}_n\|_V^2 + \frac{1}{2}\|\mathbf{q}_n\|_2^2\right) ds \\ \leqslant \mathbb{E}\int_t^T \beta_1^2\|\mathbf{U}\|_2^2\,ds \qquad \text{for } t \in (0, T).\end{aligned}$$

Since $\forall t \in [0, T]$, $\beta_1(t) \geqslant 1$, this deduced estimate implies

$$\begin{aligned}\sup_{t\in[0,T]} \mathbb{E}\left(\|\mathbf{p}_n(t)\|_2^2\right) + \mathbb{E}\int_0^T \left(\nu\,\|\mathbf{p}_n\|_V^2 + \frac{1}{2}\|\mathbf{q}_n\|_2^2\right) ds \\ \leqslant \mathbb{E}\int_0^T \beta_1^2\|\mathbf{U}\|_2^2\,ds \leqslant C\left(\mathbb{E}\left[\beta_1^4(T)\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T \|U\|_2^4\right)^{\frac{1}{2}},\end{aligned}$$

then the first estimate of (8.4) holds.

Step 3. *Improvement of the uniform estimates.* Let us show the second a priori estimate of (8.4) for $(\mathbf{p}_n, \mathbf{q}_n)$ with a better regularity. Let us consider Galerkin's approximation $\mathbf{p}_n$ that verifies (8.6). For each $n \in \mathbb{N}$, the function $\mathbf{p}_n(t, \cdot) \in H^2(\mathcal{O}) \cap V_n$ a.e. in Ω, being a linear combination of $\{\mathbf{h}_k\} \subset H^2(\mathcal{O}) \cap V$. The eigenfunctions

$\{\mathbf{h}_k\}_{k=1}^{\infty}$ of problem (8.5) fulfill the homogeneous Navier-slip boundary condition in the system (8.5); therefore, the integration by parts of (8.6) yields the equality

$$\begin{aligned}-d\left(\mathbf{p}_n(t),\mathbf{h}_k\right)=\Big\{&\left([\nu\Delta\mathbf{p}_n+2D(\mathbf{p}_n)\mathbf{y}+\mathbf{U}],\mathbf{h}_k\right)-\int_\Gamma a(\mathbf{p}_n\cdot\boldsymbol{\tau})(\mathbf{h}_k\cdot\boldsymbol{\tau})\,d\gamma\\&+\left(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})^T\mathbf{q}_n,\mathbf{h}_k\right)\Big\}\,dt-(\mathbf{q}_n,\mathbf{h}_k)\,d\mathcal{W}_t\end{aligned}\tag{8.11}$$

for $k=1,2,\ldots,n$.

Let us introduce the Helmholtz projector $\mathbb{P}_n : L_2(\mathcal{O})\longrightarrow V_n$ of V and define the function

$$A\mathbf{h}_k=\mathbb{P}_n\left(-\triangle\mathbf{h}_k\right)=-\triangle\mathbf{h}_k+\nabla\pi_k\in V_n\qquad\text{with}\qquad\pi_k\in H^1(\mathcal{O}),$$

defined in (8.5). Multiplying (8.11) by the eigenvalue λ_k of the problem (8.5), we have

$$\begin{aligned}-d\left(\mathbf{p}_n(t),\mathbf{h}_k\right)_V=\lambda_k\Big\{&\left([\nu\Delta\mathbf{p}_n+2D(\mathbf{p}_n)\mathbf{y}+\mathbf{U}],\mathbf{h}_k\right)-\int_\Gamma(a\mathbf{p}_n)\cdot\mathbf{h}_k\,d\gamma\\&+\left(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})^T\mathbf{q}_n,\mathbf{h}_k\right)\Big\}\,dt-(\mathbf{q}_n,\mathbf{h}_k)_V\,d\mathcal{W}_t,\end{aligned}$$

accounting that $\mathbf{p}_n$, $\mathbf{h}_k$ are tangent to the boundary Γ and

$$(\mathbf{p}_n,\lambda_k\mathbf{h}_k)=(\mathbf{p}_n,\mathbf{h}_k)_V=(A\mathbf{p}_n,\mathbf{h}_k),\qquad(\mathbf{q}_n,\lambda_k\mathbf{h}_k)=(\mathbf{q}_n,\mathbf{h}_k)_V\,.$$

The Itô formula gives

$$\begin{aligned}-d\left((\mathbf{p}_n,\mathbf{h}_k)_V^2\right)=2\lambda_k\left(A\mathbf{p}_n(t),\mathbf{h}_k\right)\Big\{&\left([-\nu A\mathbf{p}_n+2D(\mathbf{p}_n)\mathbf{y}+\mathbf{U}],\mathbf{h}_k\right)\\&-\int_\Gamma(a\mathbf{p}_n)\cdot\mathbf{h}_k\,d\gamma+\left(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})^T\mathbf{q}_n,\mathbf{h}_k\right)\Big\}\,dt\\&-2\lambda_k\left(A\mathbf{p}_n(t),\mathbf{h}_k\right)(\mathbf{q}_n,\mathbf{h}_k)\,d\mathcal{W}_t-(\mathbf{q}_n,\mathbf{h}_k)_V^2\,dt.\end{aligned}$$

Dividing this equality by λ_k and summing over $k=1,\ldots,n$, we obtain

$$\begin{aligned}&-d\left[\|\mathbf{p}_n\|_V^2\right]+\left(2\nu\|A\mathbf{p}_n\|_2^2+\|\mathbf{q}_n\|_V^2\right)dt=2\Big\{\left(2D(\mathbf{p}_n)\mathbf{y},A\mathbf{p}_n\right)+(\mathbf{U},A\mathbf{p}_n)\\&\quad-\int_\Gamma(a\mathbf{p}_n)\cdot A\mathbf{p}_n\,d\gamma+\left(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})^T\mathbf{q}_n,A\mathbf{p}_n\right)\Big\}dt\\&=[I_1+I_2+I_3+I_4]\,dt-2\left(\mathbf{q}_n,\mathbf{p}_n\right)_V\,d\mathcal{W}_t.\end{aligned}\tag{8.12}$$

Let us estimate the terms I_1, I_2, I_3, and I_4. We have

$$I_1\leqslant C\|\mathbf{y}\|_6\|D\mathbf{p}_n\|_3\,\|A\mathbf{p}_n\|_2.$$

Using the Gagliardo–Nirenberg–Sobolev inequality (2.3) with $q=6$ and with $q=3$, respectively,

$$\|\mathbf{y}\|_6\leqslant Cf(t)\equiv C(\|\mathbf{y}\|_2^{1/3}\|\nabla\mathbf{y}\|_2^{2/3}+\|\mathbf{y}\|_2),$$

and

$$\|D\mathbf{p}_n\|_3 \leqslant C\left(\|D\mathbf{p}_n\|_2^{2/3}\|\nabla(D\mathbf{p}_n)\|_2^{1/3} + \|D\mathbf{p}_n\|_2\right),$$

we get

$$\begin{aligned}I_1 &\leqslant Cf(t)\left(\|\mathbf{p}_n\|_V^{2/3}\|A\mathbf{p}_n\|_2^{1/3} + \|\mathbf{p}_n\|_V\right)\|A\mathbf{p}_n\|_2\\ &\leqslant 2C\max(\nu^{-2},1)\left(f^3(t)+f^2(t)\right)\|\mathbf{p}_n\|_V^2 + \frac{\nu}{3}\|A\mathbf{p}_n\|_2^2.\end{aligned}$$

Applying Young's inequality (2.8), we can show

$$\begin{aligned}f^3(t)+f^2(t) &\leqslant C(\|\mathbf{y}\|_2\|\nabla\mathbf{y}\|_2^2 + \|\mathbf{y}\|_2^3 + \|\nabla\mathbf{y}\|_2^2 + 1)\\ &\leqslant C(\|\mathbf{u}\|_2^2\|\mathbf{u}\|_V^2 + \|\mathbf{u}\|_2^4 + ||(a,b)||_{\mathcal{H}_p(\Gamma)}^4 + 1);\end{aligned}$$

therefore there exists a positive constant C, such that

$$I_1 \leqslant 2h(t)\|\mathbf{p}_n\|_V^2 + \frac{\nu}{3}\|A\mathbf{p}_n\|_2^2$$

with

$$h(t) = C\max(\nu^{-2},1)(1+||(a,b)||_{\mathcal{H}_p(\Gamma)}^4 + \|\mathbf{u}\|_2^2\|\mathbf{u}\|_V^2 + \|\mathbf{u}\|_2^4) \in L_1(0,T), \quad P\text{-a.e. in } \Omega,$$

by the estimates (2.16).

Also we have

$$I_2 = \int_{\mathcal{O}} \mathbf{U}\cdot A\mathbf{p}_n d\mathbf{x} \leqslant \|\mathbf{U}\|_2\|A\mathbf{p}_n\|_2 \leqslant C\nu^{-1}\|\mathbf{U}\|_2^2 + \frac{\nu}{3}\|A\mathbf{p}_n\|_2^2$$

and

$$\begin{aligned}I_3 &= -\int_\Gamma (a\mathbf{p}_n)\cdot A\mathbf{p}_n\, d\gamma \leqslant \|a\|_{L_\infty(\Gamma)}\|\mathbf{p}_n\|_{H^{1/2}(\mathcal{O})}\|A\mathbf{p}_n\|_{H^{-1/2}(\mathcal{O})}\\ &\leqslant C\|a\|_{W_p^{1-\frac{1}{p}}(\Gamma)}\|\mathbf{p}_n\|_{H^1}\|A\mathbf{p}_n\|_2 + \frac{\nu}{4}\|A\mathbf{p}_n\|_2^2\\ &\leqslant 2C\nu^{-1}\|a\|^2_{W_p^{1-\frac{1}{p}}(\Gamma)}\|\mathbf{p}_n\|_V^2 + \frac{\nu}{3}\|A\mathbf{p}_n\|_2^2\end{aligned}$$

by Theorem 1.5, p. 8 of [25].

The term I_4 is estimated as

$$\begin{aligned}I_4 &= 2|\left(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})^T\mathbf{q}_n,\mathbf{p}_n\right)_V| \leqslant C||\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})^T\mathbf{q}_n||_V||\mathbf{p}_n||_V\\ &\leqslant \frac{1}{2}||\mathbf{q}_n||_V^2 + 2C||\mathbf{p}_n||_V^2\end{aligned}$$

by (6.1).

Therefore, substituting the above deduced estimates for the terms I_1, I_2, I_3, I_4 in (8.12) we obtain that there exists a positive constant $\widetilde{C}_2$, such that

$$\begin{aligned}&-d\left(\|\mathbf{p}_n\|_V^2\right) + \left(\nu||A\mathbf{p}_n||_2^2 + \frac{1}{2}||\mathbf{q}_n||_V^2\right)dt\\ &\qquad \leqslant \left(2r_2(t)\|\mathbf{p}_n\|_V^2 + \|\mathbf{U}\|_2^2\right)dt - 2\left(\mathbf{q}_n,\mathbf{p}_n\right)_V\, d\mathcal{W}_t\end{aligned} \tag{8.13}$$

with

$$r_2(t) = \widetilde{C}_2 \max(\nu^{-2}, 1)(1 + ||(a,b)||^4_{\mathcal{H}_p(\Gamma)} + \|\mathbf{u}\|_2^2 \|\mathbf{u}\|_V^2 + \|\mathbf{u}\|_2^4) \in L_1(0,T),$$

P-a.e. in Ω, by (2.16).

Introducing the function

$$\beta_2(t) = e^{\int_0^t r_2(s)ds}$$

and applying the Itô formula

$$d\left(\beta_2^2 ||\mathbf{p}_n||_V^2\right) = \beta_2^2 d\left(||\mathbf{p}_n||_V^2\right) + 2r_2\beta_2^2 ||\mathbf{p}_n||_V^2 \, dt$$

to (8.13), we obtain that

$$\begin{aligned} &\beta_2^2(t)\,\|\mathbf{p}_n(t)\|_V^2 + \int_t^T \beta_2^2 \left(\nu||A\mathbf{p}_n||_2^2 + \frac{1}{2}||\mathbf{q}_n||_V^2\right) ds \\ &\leqslant \int_t^T \beta_2^2 \|\mathbf{U}\|_2^2 \, ds - 2\int_t^T \beta_2^2 \left(\mathbf{q}_n, \mathbf{p}_n\right)_V \, d\mathcal{W}_s \end{aligned} \tag{8.14}$$

by (8.13).

Let us notice that the assumptions (8.2) ensure that

$$\min\left(B_*, \frac{B_*}{\widehat{C}}\right) \geqslant 24\widetilde{C}_2 \max(\nu^{-2}, 1),$$

which imply $\mathbb{E}\left[\beta_2^8(T)\right] < \infty$. Then the regularity property (8.7) and the Hölder inequality give

$$\begin{aligned} \mathbb{E}\int_0^T \left(\beta_2^2\left(\mathbf{p}_n, \mathbf{q}_n\right)\right)^2 ds \leqslant C &\left(\mathbb{E}\int_0^T \beta_2^8 \, ds\right)^{\frac{1}{2}} \left(\mathbb{E}\int_0^T \|\mathbf{q}_n\|_2^8 \, ds\right)^{\frac{1}{4}} \\ &\times \left(\mathbb{E}\int_0^T \|\mathbf{p}_n\|_2^8 \, ds\right)^{\frac{1}{4}} < C_n < \infty, \end{aligned}$$

then the stochastic process $M_t = \int_t^T \beta_2^2\left(\mathbf{q}_n, \mathbf{p}_n\right) d\mathcal{W}_s$ is a square integrable martingale.

Thus, taking the expectation of (8.14), we derive

$$\begin{aligned} &\mathbb{E}\left(\beta_2^2(t)\,\|\mathbf{p}_n(t)\|_V^2\right) + \mathbb{E}\int_t^T \beta_2^2 \left(\nu\,\|A\mathbf{p}_n\|_2^2 + \frac{1}{2}\|\mathbf{q}_n\|_V^2\right) ds \\ &\leqslant \mathbb{E}\int_t^T \beta_2^2 \|\mathbf{U}\|_2^2 \, ds \\ &\leqslant C\left(\mathbb{E}\left[\beta_1^4(T)\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\int_0^T \|\mathbf{U}\|_2^4\right)^{\frac{1}{2}} \qquad \text{for } t \in (0,T), \end{aligned}$$

which implies

$$\begin{aligned} &\sup_{t\in[0,T]} \mathbb{E}\left(\|\mathbf{p}_n(t)\|_V^2\right) + \mathbb{E}\int_0^T \left(\nu\,\|A\mathbf{p}_n\|_2^2 + \frac{1}{2}\|\mathbf{q}_n\|_V^2\right) ds \\ &\leqslant C\left(\mathbb{E}\int_0^T \|\mathbf{U}\|_2^4\right)^{\frac{1}{2}}, \end{aligned}$$

since $\beta_2(t) \geqslant 1 \ \forall t \in [0,T]$.

Step 4. *Passage to the limit.* The following inequality:

$$\|\mathbf{p}_n\|_{H^2} \leqslant C(\|A\mathbf{p}_n\|_2 + \|a\mathbf{p}_n\|_{H^{1/2}(\Gamma)}) \tag{8.15}$$

holds due to the regular properties of the Stokes operator A (see Theorem 9 of [2] and Theorem 2 of [36]).

Therefore, the estimates (8.4) imply that there exists a suitable subsequence of $\{(\mathbf{p}_n, \mathbf{q}_n)\}$, such that

$$\begin{aligned} \mathbf{p}_n \rightharpoonup \mathbf{p} \qquad & *\text{- weakly in} \quad L_\infty(0,T;L_2(\Omega;H)), \\ & \text{weakly in} \quad L_2(\Omega \times (0,T);V), \\ \mathbf{q}_n \rightharpoonup \mathbf{q} \qquad & \text{weakly in} \quad L_2(\Omega \times (0,T);H) \end{aligned}$$

and

$$\begin{aligned} \mathbf{p}_n \rightharpoonup \mathbf{p} \qquad & *\text{- weakly in} \quad L_\infty(0,T;L_2(\Omega;V)), \\ & \text{weakly in} \quad L_2(\Omega \times (0,T);H^2(\mathcal{O})), \\ A\mathbf{p}_n \rightharpoonup A\mathbf{p} \qquad & \text{weakly in} \quad L_2(\Omega \times \mathcal{O}_T), \\ \mathbf{q}_n \rightharpoonup \mathbf{q} \qquad & \text{weakly in} \quad L_2(\Omega \times (0,T);V). \end{aligned}$$

Taking the limit transition in the system (8.6), written in the distributional sense, we derive that the limit pair $(\mathbf{p}, \mathbf{q})$ is the solution of the following integral system:

$$\begin{aligned} (\mathbf{p}(t), \boldsymbol{\phi}) + \int_t^T \left\{ -(2D(\mathbf{p})\mathbf{y}, \boldsymbol{\phi}) + \nu\,(\mathbf{p}, \boldsymbol{\phi})_V - (\mathbf{U}, \boldsymbol{\phi}) + \int_\Gamma a(\mathbf{p}\cdot\boldsymbol{\tau})(\boldsymbol{\phi}\cdot\boldsymbol{\tau})\,d\gamma \right\} ds \\ = \int_t^T \left(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})^T\mathbf{q}, \boldsymbol{\phi}\right) ds - \int_t^T (\boldsymbol{q}, \boldsymbol{\phi})\, d\mathcal{W}_s \qquad \text{a.e. in } \Omega, \end{aligned}$$

which is valid for all $t \in [0,T]$, $\boldsymbol{\phi} \in V$.

By the result given on the p. 208 of [35], we deduce the existence of the pressure $\pi \in H^{-1}(0,T;L_2(\mathcal{O}))$, a.e. in Ω. Moreover, reasoning as in Proposition 1.2, p. 182 of [35] and using the inequality

$$\|\mathbf{p}\|_{H^2} + \|\nabla\pi\|_2 \leqslant C\left(\|A\mathbf{p}\|_2 + \|a\mathbf{p}\|_{H^{1/2}(\Gamma)}\right), \tag{8.16}$$

by Theorem 9 of [2], we derive that $\pi \in L_2(0,T;H^1(\mathcal{O}))$.

In addition, it follows that the triple $(\mathbf{p}, \mathbf{q}, \pi)$ satisfies the estimates (8.4) by the lower semicontinuity of integral in the L_2-space. ∎

9. Duality property. In the next proposition we present the *duality* property for the solution $\mathbf{z}$ of the linearized equation (6.3) and the adjoint triplet $(\mathbf{p}, \mathbf{q}, \pi)$, being the solution of (8.1).

PROPOSITION 9.1. *For P-a.e. in Ω the solution $\mathbf{z}$ of the system* (6.3) *and the solution* $(\mathbf{p}, \pi)$ *of the adjoint system* (8.1) *verify the following duality relation:*

$$\begin{aligned} \int_0^T (\mathbf{z}, \mathbf{U})\,dt = \int_0^T \int_\Gamma \{\nu g(\mathbf{p}\cdot\boldsymbol{\tau}) + f\,[\pi - (\mathbf{p}\cdot\mathbf{y}) - 2\nu\,(D(\mathbf{p})\mathbf{n})\cdot\mathbf{n}]\}\;d\gamma dt \\ + \int_0^T [(\mathbf{q}, \mathbf{z}) + (\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\mathbf{z}, \mathbf{p})]\,d\mathcal{W}_t. \end{aligned} \tag{9.1}$$

Proof. Taking $\boldsymbol{\varphi}=\mathbf{p}\in L_2(0,T;V)$ in Definition 6.1, we obtain the equality

$$(d\mathbf{z},\mathbf{p})=\left[-\nu\,(\mathbf{z},\mathbf{p})_V\;+\nu\int_\Gamma g(\mathbf{p}\cdot\boldsymbol{\tau})\,d\gamma-((\mathbf{z}\cdot\nabla)\mathbf{y},\mathbf{p})-((\mathbf{y}\cdot\nabla)\mathbf{z},\mathbf{p})\right]dt$$
$$+\;(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\mathbf{z},\mathbf{p})\;d\mathcal{W}_t. \tag{9.2}$$

Multiplying (8.1) by $-\mathbf{z}$, we have

$$(d\mathbf{p},\mathbf{z})=[-(\nu\Delta\mathbf{p},\mathbf{z})-(2D(\mathbf{p})\mathbf{y},\mathbf{z})+(\nabla\pi,\mathbf{z})-(\mathbf{U},\mathbf{z})]\;dt$$
$$-\big(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})^T\mathbf{q},\mathbf{z}\big)\;dt+(\mathbf{q},\mathbf{z})\;d\mathcal{W}_t\qquad\text{a.e. in }\Omega\times(0,T). \tag{9.3}$$

Using that the functions $\mathbf{y}$, $\mathbf{z}$, and $\mathbf{p}$ are divergence free, the equality (2.6) and the integration by parts give the following three relations:

$$-\nu\,(\Delta\mathbf{p},\mathbf{z})=-\nu\int_\Gamma 2\,(D(\mathbf{p})\mathbf{n})\cdot\mathbf{z}\,d\gamma+2\nu\,(D\mathbf{p},D\mathbf{z}),$$

$$-\,(\mathbf{z},(2D(\mathbf{p})\mathbf{y}))=([(\mathbf{y}\cdot\nabla)\,\mathbf{z}+(\mathbf{z}\cdot\nabla)\,\mathbf{y}]\,,\mathbf{p})$$
$$-\int_\Gamma((\mathbf{y}\cdot\mathbf{n})\,(\mathbf{p}\cdot\mathbf{z})+(\mathbf{z}\cdot\mathbf{n})\,(\mathbf{p}\cdot\mathbf{y}))\;d\gamma$$

and

$$(\mathbf{z},\nabla\pi)=\int_\Gamma(\mathbf{z}\cdot\mathbf{n})\,\pi\,d\gamma.$$

Using these equalities in the right-hand side of (9.3), we derive

$$(d\mathbf{p},\mathbf{z})=\{([(\mathbf{y}\cdot\nabla)\,\mathbf{z}+(\mathbf{z}\cdot\nabla)\,\mathbf{y}]\,,\mathbf{p})+2\nu\;(D\mathbf{p},D\mathbf{z})-(\mathbf{U},\mathbf{z})\}\;dt$$
$$+\left(\int_\Gamma[(\mathbf{z}\cdot\mathbf{n})\,\pi-\{(\mathbf{y}\cdot\mathbf{n})\,(\mathbf{p}\cdot\mathbf{z})+(\mathbf{z}\cdot\mathbf{n})\,(\mathbf{p}\cdot\mathbf{y})+(2\nu D(\mathbf{p})\mathbf{n})\cdot\mathbf{z}\}]\;d\gamma\right)dt$$
$$-\big(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})^T\mathbf{q},\mathbf{z}\big)\;dt+(\mathbf{q},\mathbf{z})\;d\mathcal{W}_t. \tag{9.4}$$

Hence, applying the Itô formula

$$d\,(\mathbf{p},\mathbf{z})=(d\mathbf{p},\mathbf{z})+(\mathbf{p},d\mathbf{z})+(d\mathbf{p},d\mathbf{z}),$$

we deduce

$$d\,(\mathbf{z},\mathbf{p})=\Bigg\{-(\mathbf{U},\mathbf{z})+\nu\int_\Gamma[g(\mathbf{p}\cdot\boldsymbol{\tau})-\alpha(\mathbf{z}\cdot\boldsymbol{\tau})(\mathbf{p}\cdot\boldsymbol{\tau})+(\mathbf{z}\cdot\mathbf{n})\,\pi$$
$$-\{(\mathbf{y}\cdot\mathbf{n})\,(\mathbf{p}\cdot\mathbf{z})+(\mathbf{z}\cdot\mathbf{n})\,(\mathbf{p}\cdot\mathbf{y})+(2\nu D(\mathbf{p})\mathbf{n})\cdot\mathbf{z}\}\,]\;d\gamma\Bigg\}\,dt$$
$$+\,[(\mathbf{q},\mathbf{z})\;+(\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\mathbf{z},\mathbf{p})]\;d\mathcal{W}_t.$$

Due to the boundary conditions for $\mathbf{y}$, $\mathbf{z}$, and $\mathbf{p}$

$$\mathbf{y}\cdot\mathbf{n}=a,\qquad\mathbf{z}\cdot\mathbf{n}=f,\qquad\mathbf{p}\cdot\mathbf{n}=0,$$
$$(2\nu D(\mathbf{p})\mathbf{n})\cdot\boldsymbol{\tau}=-\nu\left(\frac{a}{\nu}+\alpha\right)(\mathbf{p}\cdot\boldsymbol{\tau}),$$

we obtain

$$
\begin{aligned}
d\left(\mathbf{z},\mathbf{p}\right) = \Bigg\{ & -(\mathbf{U},\mathbf{z}) + \int_\Gamma \nu g(\mathbf{p}\cdot\boldsymbol{\tau}) - \nu\alpha(\mathbf{p}\cdot\boldsymbol{\tau})(\mathbf{z}\cdot\boldsymbol{\tau}) + f\pi \\
& - \{a(\mathbf{p}\cdot\boldsymbol{\tau})(\mathbf{z}\cdot\boldsymbol{\tau}) + f(\mathbf{p}\cdot\mathbf{y}) + ((2\nu D(\mathbf{p})\mathbf{n})\cdot\mathbf{n})\,f \\
& - \nu\left(\frac{a}{\nu}+\alpha\right)(\mathbf{p}\cdot\boldsymbol{\tau})(\mathbf{z}\cdot\boldsymbol{\tau})\}\,d\gamma\Bigg\}\,dt \\
& + [(\mathbf{q},\mathbf{z}) + (\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\mathbf{z},\mathbf{p})]\,d\mathcal{W}_t \\
= & \left\{-(\mathbf{U},\mathbf{z}) + \int_\Gamma [\nu g(\mathbf{p}\cdot\boldsymbol{\tau}) + f\pi - \{f(\mathbf{p}\cdot\mathbf{y}) + ((2\nu D(\mathbf{p})\mathbf{n})\cdot\mathbf{n})\,f\}]\,d\gamma\right\} \\
& + [(\mathbf{q},\mathbf{z}) + (\nabla_{\mathbf{y}}\mathbf{G}(t,\mathbf{y})\mathbf{z},\mathbf{p})]\,d\mathcal{W}_t
\end{aligned}
$$

by (9.2), (9.4). Integrating this equality over the time interval $(0,T)$, we obtain the duality property (9.1). ∎

10. Main result. First-order optimality condition. Let us consider

$$\widehat{C}_{\max} = 24\max\left(\widehat{C}_1,\widehat{C}_2,\widetilde{C}_1,\widetilde{C}_2\right)\times\max\left(\nu^{-2},1\right)$$

with the constants $\widehat{C}_1,\widehat{C}_2,\widetilde{C}_1,\widetilde{C}_2$ defined in Theorem 3.1 and Propositions 6.2 and 8.1. In what follows we will assume that

$$\min\left(A_*,B_*,\frac{B_*}{\widehat{C}}\right) \geqslant \widehat{C}_{\max}, \tag{10.1}$$

where A_*, B_*, and $\widehat{C}$ defined by (5.3) and (5.4).

Let us point that the assumptions (5.1), Theorem 2.4, and Propositions 5.1, 6.2, 8.1 with the help of Hölder's inequality imply that the stochastic processes $\mathbf{y}$, $\mathbf{z}$ and $\mathbf{p}$, $\mathbf{q}$, π, being the solutions for the systems (1.1), (6.3), and (8.1), have the following regularity:

$$
\begin{aligned}
&\mathbf{y}\in L_4(\Omega;C([0,T];L_2(\mathcal{O})))\cap L_4(\Omega\times(0,T);H^1(\mathcal{O})),\\
&\mathbf{z}\in L_2(\Omega;C([0,T];L_2(\mathcal{O})))\cap L_2(\Omega\times(0,T);H^1(\mathcal{O}))
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{p}\in L_2(\Omega;C([0,T];V)\cap L_2(0,T;H^2(\mathcal{O}))), \qquad \mathbf{q}\in L_2(\Omega\times(0,T);V),\\
&\pi\in L_2(0,T;H^1(\mathcal{O})).
\end{aligned}
$$

Now, we demonstrate the main result of the article, establishing the first-order necessary optimality condition for the problem $(\mathcal{P})$.

THEOREM 10.1. *Under the assumptions of Theorem* 4.1, *let us assume that* $(\widehat{a},\widehat{b},\widehat{\mathbf{y}})$ *is an optimal solution of problem* $(\mathcal{P})$. *Then* P*-a.e. in* Ω *there exists a unique solution* $(\widehat{\mathbf{p}},\widehat{\mathbf{q}},\widehat{\pi})$ *of the adjoint system*

$$
\begin{cases}
-d\widehat{\mathbf{p}} = [\nu\Delta\widehat{\mathbf{p}} + 2D(\widehat{\mathbf{p}})\widehat{\mathbf{y}} - \nabla\widehat{\pi} + (\widehat{\mathbf{y}}-\mathbf{y}_d)]\,dt & \\
\qquad +\nabla_{\mathbf{y}}\mathbf{G}(t,\widehat{\mathbf{y}})^T\widehat{\mathbf{q}}\,dt - \widehat{\mathbf{q}}\,d\mathcal{W}_t, \qquad \operatorname{div}\widehat{\mathbf{p}}=0 & \text{in } \mathcal{O}_T,\\
\widehat{\mathbf{p}}\cdot\mathbf{n}=0, \qquad \left[2D(\widehat{\mathbf{p}})\mathbf{n} + (\alpha+\frac{\widehat{a}}{\nu})\widehat{\mathbf{p}}\right]\cdot\tau = 0 & \text{on } \Gamma_T,\\
\widehat{\mathbf{p}}(T)=0 & \text{in } \mathcal{O},
\end{cases}
\tag{10.2}
$$

verifying the optimality condition

$$\begin{aligned}&\mathbb{E}\int_{\Gamma_T}\{(f-\widehat{a})[\widehat{\pi}-(\widehat{\mathbf{p}}\cdot\widehat{\mathbf{y}})-(2\nu D(\widehat{\mathbf{p}})\mathbf{n})\cdot\mathbf{n}]+\nu(g-\widehat{b})(\widehat{\mathbf{p}}\cdot\boldsymbol{\tau})\\ &\qquad+\lambda_1\widehat{a}\,(f-\widehat{a})+\lambda_2\widehat{b}(g-\widehat{b})\}\,d\gamma dt\geqslant 0\end{aligned} \tag{10.3}$$

for all $(f,g)\in L_4(\Omega\times(0,T);\mathcal{H}_p(\Gamma))$.

Proof. Let $(\widehat{a},\widehat{b},\widehat{\mathbf{y}})$ be a solution of the problem $(\mathcal{P})$.

According to Theorem 2.4 and Proposition 7.1, for any $(a,b)\in L_4(\Omega\times(0,T);\mathcal{H}_p(\Gamma))$ the corresponding state system (1.1) has a unique solution $\mathbf{y}$ and the control-to-state mapping

$$(a,b)\to\mathbf{y}$$

is the Gâteaux differentiable at $(\widehat{a},\widehat{b})$.

For $\varepsilon\in(0,1)$ and $(f,g)\in L_4(\Omega\times(0,T);\mathcal{H}_p(\Gamma))$, let us set

$$a_\varepsilon=\widehat{a}+\varepsilon(f-\widehat{a}),\qquad b_\varepsilon=\widehat{b}+\varepsilon(g-\widehat{b}).$$

We consider the strong solution $\mathbf{y}_\varepsilon$ of (1.1) with the boundary conditions $(a_\varepsilon,b_\varepsilon)$ (see Definition 2.3). Since $(\widehat{a},\widehat{b},\widehat{\mathbf{y}})$ is the optimal solution and $(a_\varepsilon,b_\varepsilon,\mathbf{y}_\varepsilon)$ is admissible triple, using the convexity of the functional J, we have

$$\lim_{\varepsilon\to 0^+}\frac{J(a_\varepsilon,b_\varepsilon,\mathbf{y}_\varepsilon)-J(\widehat{a},\widehat{b},\widehat{\mathbf{y}})}{\varepsilon}\geqslant 0,$$

that, taking into account Proposition 7.1, gives the inequality

$$\mathbb{E}\left[\int_{\mathcal{O}_T}\widehat{\mathbf{z}}\cdot(\widehat{\mathbf{y}}-\mathbf{y}_d)\,d\mathbf{x}dt+\int_{\Gamma_T}\left\{\lambda_1\,\widehat{a}(f-\widehat{a})+\lambda_2\widehat{b}(g-\widehat{b})\right\}d\gamma dt\right]\geqslant 0, \tag{10.4}$$

where

$$\widehat{\mathbf{z}}=\lim_{\varepsilon\to 0^+}\frac{\widehat{\mathbf{y}}_\varepsilon-\widehat{\mathbf{y}}}{\varepsilon}$$

is the unique strong solution of the system

$$\begin{cases} d\widehat{\mathbf{z}}=(\nu\Delta\widehat{\mathbf{z}}-(\widehat{\mathbf{z}}\cdot\nabla)\,\widehat{\mathbf{y}}-(\widehat{\mathbf{y}}\cdot\nabla)\,\widehat{\mathbf{z}}-\nabla\pi)\,dt+\nabla_{\mathbf{y}}\mathbf{G}(t,\widehat{\mathbf{y}})\widehat{\mathbf{z}}\,d\mathcal{W}_t & \text{in } \mathcal{O}_T,\\ \operatorname{div}\,\widehat{\mathbf{z}}=0, & \\ \widehat{\mathbf{z}}\cdot\mathbf{n}=f-\widehat{a},\qquad [2D(\widehat{\mathbf{z}})\,\mathbf{n}+\alpha\widehat{\mathbf{z}}]\cdot\boldsymbol{\tau}=g-\widehat{b} & \text{on } \Gamma_T,\\ \widehat{\mathbf{z}}(0)=0 & \text{in } \mathcal{O}\end{cases}$$

in the sense of Definition 6.1.

On the other hand, if we consider the system (8.1) with

$$\mathbf{U}=\widehat{\mathbf{y}}-\mathbf{y}_d,\quad a=\widehat{a},\quad\text{and}\quad \mathbf{y}=\widehat{\mathbf{y}},$$

then Proposition 8.1 implies that there exists the adjoint state pair $(\widehat{\mathbf{p}},\widehat{\pi})$, which verifies (10.2), P-a.e. in Ω. Since the duality property (9.1) is valid for

$$\mathbf{z}=\widehat{\mathbf{z}},\quad \mathbf{U}=\widehat{\mathbf{y}}-\mathbf{y}_d,\quad\text{and}\quad(\mathbf{p},\pi)=(\widehat{\mathbf{p}},\widehat{\pi}),$$

we deduce the equality

$$\int_{\mathcal{O}_T} \widehat{\mathbf{z}} \cdot (\widehat{\mathbf{y}} - \mathbf{y}_d)\, d\mathbf{x}dt$$
$$= \int_{\Gamma_T} \left\{ \nu(g - \widehat{b})\,(\widehat{\mathbf{p}} \cdot \boldsymbol{\tau}) + (f - \widehat{a})\,[\pi - (\widehat{\mathbf{p}} \cdot \widehat{\mathbf{y}}) - 2\nu\,(\mathbf{n} \cdot D(\widehat{\mathbf{p}})) \cdot \mathbf{n}] \right\} d\gamma dt$$
$$+ \int_0^T [(\widehat{\mathbf{q}}, \widehat{\mathbf{z}}) + (\nabla_{\mathbf{y}} \mathbf{G}(t, \widehat{\mathbf{y}})\widehat{\mathbf{z}}, \widehat{\mathbf{p}})]\, d\mathcal{W}_t.$$

Substituting this equality into (10.4) and taking the expectation, we obtain the optimality condition (10.3). ∎

Remark 10.1. The condition (10.1) in Theorem 10.1 can be interpreted as follows:

(a) Under the presence of strong noise (corresponding to high values of L), the fluid behavior can be optimally controlled through the injection/suction device modelled by the Navier-slip boundary conditions if the fluid is very viscous.

(b) Under very small random disturbances (corresponding to small enough values of L), the condition (10.1) holds for high viscosity values ν, then the fluid behavior can be optimally controlled.

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