International Journal of Approximate Reasoning 170 (2024) 109204

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

Triple perturbed consistent matrix and the efficiency of its principal right eigenvector

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ARTICLE INFO

Keywords:
Decision processes
Consistent matrix
Pairwise comparison matrix
Principal eigenvector efficiency
Strongly connected digraph

ABSTRACT

Let A be a pairwise comparison matrix obtained from a consistent one by perturbing three entries above the main diagonal, x, y, z, and the corresponding reciprocal entries, in a way that there is a submatrix of size 2 containing the three perturbed entries and not containing a diagonal entry. In this paper we describe the relations among x, y, z with which A always has its principal right eigenvector efficient. Previously, and only for a few cases of this problem, R. Fernandes and S. Furtado (2022) proved the efficiency of the principal right eigenvector of A. In this paper, we continue to use the strong connectivity of a certain digraph associated with A and its principal right eigenvector to characterize the vector efficiency. For completeness, we show that the existence of a sink in this digraph is equivalent to the inefficiency of the principal right eigenvector of A.

1. Introduction

The Analytic Hierarchy Process (AHP), [18,19], is one of the most used processes in decision making. Since 1977, several authors studied this process and introduced new concepts. The AHP is a method of multi-criteria decision making of pairwise comparison alternatives, [20], where Saaty proposed the construction of a pairwise comparison matrix A whose entry (i, j) indicates the strength which alternative i dominates alternative j relatively to a fixed criterion/alternative.

Formally, a matrix $A = [a_{ij}] \in M_n$ (the set of $n \times n$ real matrices) with positive entries is said to be a *pairwise comparison matrix* (*PC matrix*), or a *reciprocal matrix*, if $a_{ij} = 1/a_{ji}$ for all $i, j = 1, \ldots, n$. In particular, $a_{ii} = 1$ for all $i = 1, \ldots, n$. A PC matrix $A \in M_n$ is said to be *transitive* or *consistent* if $a_{ij}a_{jk} = a_{ik}$ for all $i, j, k = 1, \ldots, n$. It is well known that a consistent matrix $A \in M_n$ can be written as ww^{-1}, for some positive vector $w = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}^T$, with

$$w^{-1} = \begin{bmatrix} w_1^{-1} & \cdots & w_n^{-1} \end{bmatrix}. \tag{1}$$

Any $n \times n$ PC matrix is consistent if $n = 2$, but for $n > 2$ this is not generally true.

As many times a pairwise comparison matrix A is not consistent, Saaty proposed a unit positive right eigenvector associated with the largest eigenvalue of A to the vector that gives the priorities for the alternatives relatively to the fixed criterion/alternative.

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[1] The work of this author was supported in part by FCT - Fundação para a Ciência e Tecnologia, under project UIDB/00297/2020 (https://doi.org/10.54499/UIDB/00297/2020).

https://doi.org/10.1016/j.ijar.2024.109204
Received 26 January 2024; Received in revised form 12 April 2024; Accepted 12 April 2024

Available online 18 April 2024

Consequently, with this vector a consistent matrix for the problem can be constructed as we mentioned before. However, as he noted with his consistency ratio, sometimes this vector is not a good choice for the vector of priorities. More precisely, if the consistency ratio is less than or equal to 10% then the proposed vector is acceptable. Otherwise, we need to reestimate the elements of the matrix A to realize a better vector. Since Saaty's work, other methods to estimate and to analyze the vector of priorities appeared in the literature, [3,4,6,8,10,17]. One of these methods is the efficiency of the vector for A, see [7] and the references therein.

A positive vector $w = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}^T$ is said to be *efficient* for a PC matrix $A = [a_{ij}] \in M_n$ if there is no other vector $w' = \begin{bmatrix} w'_1 & \cdots & w'_n \end{bmatrix}^T$ such that

$$\left| a_{ij} - \frac{w'_i}{w'_j} \right| \leq \left| a_{ij} - \frac{w_i}{w_j} \right| \qquad \text{for all } 1 \leq i, j \leq n,$$

with the inequality being strict for at least one pair (i, j). Otherwise, we say that w is an inefficient vector for A. Note that if w is efficient for A then any positive multiple of w is also efficient.

In [5], a characterization of the efficiency of a vector w was presented in terms of a certain digraph associated with A and w, which we denote throughout by $G_{A,w}$ and describe in detail in the next section.

The principal right eigenvector of a PC matrix A is the right eigenvector associated with the largest eigenvalue (the Perron eigenvalue) having its first component equal to one. The study of the efficiency of the principal right eigenvector for a PC matrix has been done just in some special cases. In [1] and [2], simple and double perturbed consistent matrices were considered. In [11] a similar study has been made for some triple perturbed consistent matrices and examples for which the principal right eigenvector is inefficient have been done. On the other hand, the description of all efficient vectors have been done for simple, [9], double, [13], and some triple, [12], perturbed consistent matrices. More recently, in [15,14] the authors proved the efficiency of any Hadamard weighted geometric mean of the columns of a reciprocal matrix.

In this paper we continue the study of the efficiency of the principal right eigenvector of a PC matrix $A \in M_n$ obtained from a consistent matrix by perturbing three entries, called triple perturbed consistent matrices. More precisely, we consider the case in which three entries and their reciprocals are perturbed, as long as there is a 2×2 submatrix containing three of the perturbed entries and not containing a principal entry of A.

In the process, we obtain a result about the no existence of a source in $G_{A,w}$ when w is an inefficient principal right eigenvector of A.

Since the efficiency of a vector for a PC matrix A can be analyzed considering a permutation similarity of A followed by a similarity by a diagonal matrix with positive diagonal, [11], we will mainly focus on $n \times n$ PC matrices of the form

$$Z_n(x,y,z) := \begin{bmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 & y & x \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & z \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 \\ \frac{1}{y} & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 \\ \frac{1}{x} & \frac{1}{z} & 1 & \cdots & \cdots & 1 & 1 & 1 \end{bmatrix}, \tag{2}$$

with $x, y, z > 0$.

PC matrices as (2) can occur in real life. For example, consider some criterion C, like the weight or the price, and 4 alternatives, $x_1,\ x_2,\ x_3,\ x_4$, such that

$$C(x_1) > C(x_2) > C(x_3) > C(x_4),$$

where $C(x_i)$ denotes the value of the alternative x_i for the criterion C, with $i = 1, 2, 3, 4$. If the numbers $C(x_i)$, with $i = 1, 2, 3, 4$, are so close that we only consider the pairs

$$(C(x_1), C(x_3)),\ (C(x_1), C(x_4)),\ (C(x_2), C(x_4))$$

and their inverses having different coordinates then the PC matrix is the matrix in (2), with $n = 4$.

The paper is organized as follows. In Section 2, we introduce some definitions and known results in the literature related with the problem we study, which we also describe in detail. In Section 3, we state our main results and we present illustrative examples. In Section 4, we prove our main results. We conclude this paper with some final remarks and open questions in Section 5.

The calculations in this paper were made with the help of the software Maple and Octave.

2. Background

From Perron-Frobenius Theorem for positive matrices [16], it is known that a PC matrix $A = [a_{ij}] \in M_n$ has a positive eigenvalue r such that r is a simple eigenvalue and any associated right eigenvector has nonzero entries with constant sign. This eigenvalue

r is the largest eigenvalue of A and $r \geq n$. As usual, we call the eigenvalue r the *Perron eigenvalue* of A and the associated right eigenvector with the first component equal to 1 the *principal right eigenvector* of A.

In [5], a characterization of the efficiency of a vector $w = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}^T$ for a PC matrix $A \in M_n$ was presented using the following digraph $G_{A,w}$ associated with A and w: the vertex set of $G_{A,w} = (V, E)$ is $V = \{1, \ldots, n\}$ and the edge set is

$$E = \left\{ (i,j) : \frac{w_i}{w_j} \geq a_{ij},\ i \neq j \right\}.$$

It follows that if $\frac{w_i}{w_j} = a_{ij}$ then the edges (i,j) and (j,i) are in $G_{A,w}$.

Throughout this paper, sometimes we denote by $i \to j$ the edge (i,j).

In [5] the following result was given.

Theorem 1. *Let $A \in M_n$ be a PC matrix. A positive vector w is efficient for A if and only if $G_{A,w}$ is a strongly connected digraph.*

More recently, [12] verify that the last result is the same as the following.

Theorem 2. *Let $A \in M_n$, $n \geq 3$, be a PC matrix. A positive vector w is efficient for A if and only if there is a directed Hamiltonian cycle in $G_{A,w}$.*

The last two Theorems are the keys to proving the main results of this paper. Moreover, they have been used in many other papers on the efficiency of a positive vector for a PC matrix.

The study of the efficiency of the principal right eigenvector of a PC matrix obtained from a consistent one by perturbing one entry and its reciprocal entry was made in [1]. In [2], the authors studied the problem when two entries and their reciprocals were perturbed. In [11], the problem was considered when three entries and their reciprocals were perturbed in such a way that after a permutation similarity following by a similarity by a diagonal matrix we get the matrix in (2). That study was made with a PC matrix A having the three perturbed entries in positions (i,j), (i,k) and (ℓ,k), with i,j,k,ℓ being distinct, that is, are contained in the 2×2 submatrix of A determined by rows i, ℓ and columns j, k, which does not contain any diagonal entry of A. However, the main result obtained in [11] was the following and did not cover all cases of these kind of matrices.

Theorem 3. *Let $n \geq 4$ and $Z_n(x,y,z)$ be the matrix in (2), with $x,y,z > 0$. If either (i) $x \leq y, z \leq 1$, or (ii) $1 \leq y, z \leq x$, then the principal right eigenvector of $Z_n(x,y,z)$ is efficient.*

As mentioned in [11], other possible structures may occur for triple perturbed consistent matrices, which can be reduced by permutation similarity to the cases in which three of the perturbed entries lie in distinct off-diagonal positions (i_1, j_1), (i_2, j_2), (i_3, j_3) satisfying one of the following: (i) $i_1, i_2, i_3, j_1, j_2, j_3$ are pairwise distinct (which implies that the matrix has size at least 6), (ii) $i_1 = i_2 = i_3$ and i_1, j_1, j_2, j_3 are pairwise distinct (which implies that the matrix has size at least 4) and (iii) $i_1 = i_2$ and the number of distinct integers among i_1, i_3, j_1, j_2, j_3 is 5 (which implies that the matrix has size at least 5).

In this paper we describe the relations among x, y, z that imply the efficiency of the principal right eigenvector of the triple perturbed consistent matrix $Z_n(x,y,z)$, presented in (2).

3. Main results

We next give the main results of this paper. The proofs of these results are given in Section 4.

The first one focus in our problem when the size of the triple perturbed consistent matrix is 4.

Theorem 4. *Let $n = 4$ and $Z_4(x,y,z)$ be the matrix in (2), with $x,y,z > 0$. If neither*

(i) $\min\{1,x\} \leq \min\{y,z\} \leq \max\{1,x\} \leq \max\{y,z\}$, *nor*
(ii) $\min\{y,z\} \leq \min\{1,x\} \leq \max\{y,z\} \leq \max\{1,x\}$,

then the principal right eigenvector of $Z_4(x,y,z)$ is efficient.

In Table 1 we show with some examples that, for $n = 4$, the converse of Theorem 4 does not hold. More precisely, if $n = 4$,

- $\min\{1,x\} \leq \min\{y,z\} \leq \max\{1,x\} \leq \max\{y,z\}$, or
- $\min\{y,z\} \leq \min\{1,x\} \leq \max\{y,z\} \leq \max\{1,x\}$,

then the principal right eigenvector of $Z_4(x,y,z)$ may or may not be efficient. For this reason, we created Table 1 with the 8 cases that can occur. It is worth mentioning that the cases $x \leq z \leq 1 \leq y$ and $y \leq x \leq z \leq 1$ appeared in Table 2 of [11].

Table 1
Efficiency of the principal right eigenvector of $Z_4(x, y, z)$.

relations among $1, x, y, z$	x	y	z	efficiency
$1 \le y \le x \le z$	3	2	4.64 4.65	Yes No
$1 \le z \le x \le y$	3	5.27 5.28	2	Yes No
$y \le 1 \le z \le x$	3	0.47 0.48	2	Yes No
$z \le 1 \le y \le x$	3	2	0.54 0.55	No Yes
$x \le y \le 1 \le z$	0.2	0.5	8.26 8.27	No Yes
$x \le z \le 1 \le y$	0.2	5.51 5.52	0.5	No Yes
$y \le x \le z \le 1$	0.2	0.14 0.15	0.5	No Yes
$z \le x \le y \le 1$	0.2	0.5	0.13 0.12	Yes No

Table 2
Efficiency of the principal right eigenvector of $Z_4(2, 3, z)$ and of $Z_4(1/3, 1/2, 1/z)$.

z	$Z_4(2,3,z)$	$Z_4(2,3,z)^T$
4.64	efficient	efficient
4.65	inefficient	efficient
5.54	inefficient	inefficient
7.13	efficient	inefficient

As $Z_4(x, y, z)^T = Z_4(1/x, 1/y, 1/z)$ we could be led to believe that the efficiency of the principal right eigenvector of $Z_4(x, y, z)$ is the efficiency of the principal right eigenvector of $Z_4(1/x, 1/y, 1/z)$. In Table 2 we analyze in more detail the efficiency of the principal right eigenvector of $Z_4(3, 2, z)$ and of $Z_4(1/3, 1/2, 1/z)$ by considering several cases of z.

When the matrix $Z_n(x, y, z)$ has size at least 5, the relation among $1, x, y, z$, called the magnitude of x, y, z, that implies the efficiency of the principal right eigenvector is not so constrained as when the matrix has size 4.

Theorem 5. *Let $n \ge 5$ and $Z_n(x, y, z)$ be the matrix in (2), with $x, y, z > 0$. If neither*

(i) $1 \le z \le x \le y$, *nor*
(ii) $z \le 1 \le y \le x$, *nor*
(iii) $z \le x \le y \le 1$, *nor*
(iv) $x \le z \le 1 \le y$,

then the principal right eigenvector of $Z_n(x, y, z)$ is efficient.

In Table 3 we show with some examples that, for $n \ge 5$, the converse of Theorem 5 does not hold. More precisely, if $n = 5$,

- $1 \le z \le x \le y$, or
- $z \le 1 \le y \le x$, or
- $z \le x \le y \le 1$, or
- $x \le z \le 1 \le y$,

then the principal right eigenvector of $Z_5(x, y, z)$ may or may not be efficient.

As there are cases where the magnitude of x, y, z implies the efficiency of the principal right eigenvector of $Z_5(x, y, z)$ and does not imply the efficiency of the principal right eigenvector of $Z_4(x, y, z)$, we analyzed the case $n = 4$ and the matrix obtained from $Z_4(x, y, z)$ by adding a new row and a new column with all entries equal to one. For these cases, we studied the relations of the order of the weights of the corresponding components of the principal right eigenvectors in each of the matrices. In Table 4 we show with some examples from Table 1 that, for fixed x, y, z with inefficiency of the principal right eigenvector of $Z_4(x, y, z)$ and efficiency of the principal right eigenvector of $Z_5(x, y, z)$, the relations of the order of the weights of the corresponding components of the principal right eigenvectors of these two matrices may or may not change. Let

Table 3
Efficiency of the principal right eigenvector of $Z_5(x,y,z)$.

relations among $1,x,y,z$	x	y	z	efficiency
$1 \le z \le x \le y$	5	14.8 14.9	2	Yes No
$z \le 1 \le y \le x$	5	2	0.2 0.21	Yes No
$z \le x \le y \le 1$	0.2	0.5	0.07 0.06	Yes No
$x \le z \le 1 \le y$	0.16 0.15	2	0.5	No Yes

Table 4
Change in the relations of the order of the weights of the corresponding components of w in $Z_n(x,y,z)$, for $n=4,5$.

x	y	z	efficiency of w	relations among components of w	change in magnitude
3	2	4.65	in $Z_4(x,y,z)$, No in $Z_5(x,y,z)$, Yes	$w_2 > w_1 > w_3 > w_4$ $w_2 > w_1 > w_4 > w_5$	No
3	0.48	2	in $Z_4(x,y,z)$, No in $Z_5(x,y,z)$, Yes	$w_3 > w_1 > w_2 > w_4$ $w_4 > w_1 > w_2 > w_5$	No
0.2	0.5	8.26	in $Z_4(x,y,z)$, No in $Z_5(x,y,z)$, Yes	$w_2 > w_3 > w_4 > w_1$ $w_2 > w_5 > w_4 > w_1$	Yes
0.2	0.14	0.5	in $Z_4(x,y,z)$, No in $Z_5(x,y,z)$, Yes	$w_3 > w_4 > w_2 > w_1$ $w_4 > w_5 > w_2 > w_1$	No

$$w = \begin{bmatrix} w_1 & w_2 & \cdots & w_{n-1} & w_n \end{bmatrix}^T$$

be the principal right eigenvector of $Z_n(x,y,z)$, with $n=4$ and $n=5$. Note that, in our study, the correspondence of w_1, w_2, w_3 and w_4 in the vector w of $Z_4(x,y,z)$ are, respectively, w_1, w_2, w_4 and w_5 in the vector w of $Z_5(x,y,z)$.

We finish this section with a result about the characterization of the digraph $G_{Z_n(x,y,z),w}$ when w is the principal right eigenvector of $Z_n(x,y,z)$ and is inefficient.

Theorem 6. *Let $n \ge 4$, $Z_n(x,y,z)$ be the matrix in (2), with $x,y,z>0$, and w be the principal right eigenvector of $Z_n(x,y,z)$. Then the vector w is inefficient for $Z_n(x,y,z)$ if and only if the digraph $G_{Z_n(x,y,z),w}$ has a sink.*

Theorem 6 is equivalent to the next result where there is a unique coordinate of the principal right eigenvector that is undervalued relative to the product of any of the other coordinates with the respective entry of $Z_n(x,y,z)$.

Theorem 7. *Let $n \ge 4$, $Z_n(x,y,z) = [a_{ij}]$ be the matrix in (2), with $x,y,z>0$, and $w = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}^T$ be the principal right eigenvector of $Z_n(x,y,z)$. Then the vector w is inefficient for $Z_n(x,y,z)$ if and only if there is exactly one p, with $1 \le p \le n$, such that $\frac{w_p}{w_s} < a_{ps}$, for all $1 \le s \le n$, $s \ne p$.*

Next corollary is an easy consequence of Theorem 6.

Corollary 8. *There are no $n \ge 4$ and $x,y,z>0$ such that the digraph $G_{Z_n(x,y,z),w}$ has a source, where $Z_n(x,y,z)$ is the matrix in (2) and w is its principal right eigenvector.*

It is worth of mention that the propriety "the digraph $G_{A,w}$ has a sink" does not hold for any PC matrix A and its inefficient principal right eigenvector. For example, consider the PC matrix

$$A = \left[\begin{array}{c|c} Z_4(3,2,4.65) & Z_4(3,2,4.65) \\ \hline Z_4(3,2,4.65) & Z_4(3,2,4.65) \end{array}\right] = [a_{ij}].$$

Let r, r_1, r_2, r_3 be the eigenvalues of $Z_4(3,2,4.65)$ where r is the Perron eigenvalue of $Z_4(3,2,4.65)$. Then, the eigenvalues of A are $2r$, $2r_1$, $2r_2$, $2r_3$, 0, 0, 0, 0 and $2r$ is the Perron eigenvalue of A. By Table 1, the principal right eigenvector of $Z_4(3,2,4.65)$, $w = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \end{bmatrix}^T$ is inefficient. Using Theorem 6, the digraph $G_{Z_4(3,2,4.65),w}$ has a sink. Denote the sink as vertex k. So, we get $w' = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \end{bmatrix}^T$, with $w_5 = w_1$, $w_6 = w_2$, $w_7 = w_3$ and $w_8 = w_4$, is the principal right eigenvector of A. Since $\frac{w_i}{w_{i+4}} = 1 = a_{i,i+4}$, for $1 \le i \le 4$, we conclude there are arcs from the vertex i to the vertex $i+4$ and from the vertex $i+4$ to the vertex i, for $1 \le i \le 4$. So, there is no sink in $G_{A,w'}$. On the other hand, as k is the sink in $G_{Z_4(3,2,4.65),w}$, there is not a directed path from k to $k+1$ in $G_{A,w'}$ (note that $\frac{w_k}{w_j} < a_{kj}$ and $\frac{w_{k+4}}{w_j} < a_{k+4,j}$, for $j \ne k, k+4$ and $1 \le j \le 8$). Therefore, $G_{A,w'}$ is not a strongly connected digraph and w' is inefficient by Theorem 1.

4. Proof of the main results

In this section we prove Theorems 4, 5 and 6 presented in Section 3.

Suppose that $n \geq 4$. By r we denote the Perron eigenvalue of the matrix $Z_n(x,y,z)$, with $x,y,z>0$. A positive right eigenvector of $Z_n(x,y,z)$ associated with r is the vector

$$w = \left[\begin{array}{ccccc} w_1 & w_2 & \cdots & w_{n-1} & w_n \end{array}\right]^T.$$

Since $(Z_n(x,y,z) - rI_n)w = 0$, a calculation shows that

$$(1-r)w_1 + w_2 + \sum_{i=3}^{n-2} w_i + yw_{n-1} + xw_n = 0 \tag{3}$$

$$w_1 + (1-r)w_2 + \sum_{i=3}^{n-2} w_i + w_{n-1} + zw_n = 0 \tag{4}$$

$$w_1 + w_2 + \sum_{i=3, i\neq j}^{n-2} w_i + (1-r)w_j + w_{n-1} + w_n = 0 \tag{5}$$

$$\frac{1}{y}w_1 + w_2 + \sum_{i=3}^{n-2} w_i + (1-r)w_{n-1} + w_n = 0 \tag{6}$$

$$\frac{1}{x}w_1 + \frac{1}{z}w_2 + \sum_{i=3}^{n-2} w_i + w_{n-1} + (1-r)w_n = 0, \tag{7}$$

in which, if $n=4$ then $\sum_{i=3}^{n-2} w_i = 0$ and equation (5) is empty. Moreover, the equation (5) is for all j, with $3 \leq j \leq n-2$.

If $n>5$, subtracting both hand-sides of equation (5) with $j=3$ from those of equation (5) with $j \neq 3$ and $4 \leq j \leq n-2$, we get

$$w_j = w_3.$$

So, we can rewrite equations (3) to (7).

$$(1-r)w_1 + w_2 + (n-4)w_3 + yw_{n-1} + xw_n = 0 \tag{8}$$

$$w_1 + (1-r)w_2 + (n-4)w_3 + w_{n-1} + zw_n = 0 \tag{9}$$

$$w_1 + w_2 + (n-4-r)w_3 + w_{n-1} + w_n = 0 \tag{10}$$

$$\frac{1}{y}w_1 + w_2 + (n-4)w_3 + (1-r)w_{n-1} + w_n = 0 \tag{11}$$

$$\frac{1}{x}w_1 + \frac{1}{z}w_2 + (n-4)w_3 + w_{n-1} + (1-r)w_n = 0, \tag{12}$$

in which, if $n=4$, equation (10) is empty.

In the next equations the expression $(\ell) - s(j)$ means the result of subtracting both hand-sides of equation (ℓ) from those of equation (j), multiplied by s.

$$(8)-(9) \qquad r(w_2 - w_1) + (y-1)w_{n-1} + (x-z)w_n = 0 \tag{13}$$

$$(8)-(10) \qquad r(w_3 - w_1) + (y-1)w_{n-1} + (x-1)w_n = 0 \tag{14}$$

$$(8)-y(11) \qquad r(yw_{n-1} - w_1) + (1-y)(w_2 + (n-4)w_3) + (x-y)w_n = 0 \tag{15}$$

$$(8)-x(12) \qquad r(xw_n - w_1) + \left(1 - \frac{x}{z}\right)w_2 + (1-x)((n-4)w_3) + (y-x)w_{n-1} = 0 \tag{16}$$

$$(9)-(10) \qquad r(w_3 - w_2) + (z-1)w_n = 0 \tag{17}$$

$$(9)-(11) \qquad r(w_{n-1} - w_2) + \left(1 - \frac{1}{y}\right)w_1 + (z-1)w_n = 0 \tag{18}$$

$$(9)-z(12) \qquad r(zw_n - w_2) + \left(1 - \frac{z}{x}\right)w_1 + (1-z)((n-4)w_3 + w_{n-1}) = 0 \tag{19}$$

$$(10)-(11) \qquad r(w_{n-1} - w_3) + \left(1 - \frac{1}{y}\right)w_1 = 0 \tag{20}$$

$$(10)-(12) \qquad r(w_n - w_3) + \left(1 - \frac{1}{x}\right)w_1 + \left(1 - \frac{1}{z}\right)w_2 = 0 \tag{21}$$

$$(11)-(12) \qquad r(w_n - w_{n-1}) + \left(\frac{1}{y} - \frac{1}{x}\right)w_1 + \left(1 - \frac{1}{z}\right)w_2 = 0 \tag{22}$$

in which, if $n = 4$, equations (14), (17), (20) and (21) are empty.

We next give same lemmas from which our results easily follow.

Lemma 9. *Let $n \geq 4$ and $Z_n(x, y, z)$, with $x, y, z > 0$, be the matrix presented in (2). Let w be a positive right eigenvector associated with the Perron eigenvalue of $Z_n(x, y, z)$.*

1. *If $1 \leq y$ and $z \leq x$ then the edge $(1, 2)$ is in $G_{Z_n(x,y,z),w}$.*
2. *If $y \leq \min\{1, x\}$ then the edge $(1, n-1)$ is in $G_{Z_n(x,y,z),w}$.*
3. *If $n = 4$ and $x \leq \min\{y, z\}$ or $n \geq 5$ and $x \leq \min\{1, y, z\}$ then the edge $(1, n)$ is in $G_{Z_n(x,y,z),w}$.*
4. *If $1 \leq \min\{y, z\}$ then the edge $(2, n-1)$ is in $G_{Z_n(x,y,z),w}$.*
5. *If $z \leq \min\{1, x\}$ then the edge $(2, n)$ is in $G_{Z_n(x,y,z),w}$.*
6. *If $y \leq x$ and $1 \leq z$ then the edge $(n-1, n)$ is in $G_{Z_n(x,y,z),w}$.*

Proof. Let

$$w = \begin{bmatrix} w_1 & w_2 & \cdots & w_{n-1} & w_n \end{bmatrix}^T$$

and r be the Perron eigenvalue of $Z_n(x, y, z)$. Recall that $r \geq n > 0$ and w is a positive right eigenvector.

1. Since $1 \leq y$ and $z \leq x$, using equation (13) we conclude that $w_2 - w_1 \leq 0$. This implies $\frac{w_1}{w_2} \geq 1$ and consequently we get the result.
2. Since $y \leq \min\{1, x\}$, using equation (15) we conclude that $yw_{n-1} - w_1 \leq 0$. This implies $\frac{w_1}{w_{n-1}} \geq y$ and consequently we get the result.
3. If $n = 4$ and $x \leq \min\{y, z\}$ or $n \geq 5$ and $x \leq \min\{1, y, z\}$, using equation (16) we conclude that $xw_n - w_1 \leq 0$. This implies $\frac{w_1}{w_n} \geq x$ and consequently we get the result.
4. Since $1 \leq \min\{y, z\}$, using equation (18) we conclude that $w_{n-1} - w_2 \leq 0$. This implies $\frac{w_2}{w_{n-1}} \geq 1$ and consequently we get the result.
5. Since $z \leq \min\{1, x\}$, using equation (19) we conclude that $zw_n - w_2 \leq 0$. This implies $\frac{w_2}{w_n} \geq z$ and consequently we get the result.
6. Since $y \leq x$ and $1 \leq z$, using equation (22) we conclude that $w_n - w_{n-1} \leq 0$. This implies $\frac{w_{n-1}}{w_n} \geq 1$ and consequently we get the result. $\square$

With an analogous proof we have the next lemma.

Lemma 10. *Let $n \geq 4$ and $Z_n(x, y, z)$, with $x, y, z > 0$. Let w be a positive right eigenvector associated with the Perron eigenvalue of $Z_n(x, y, z)$.*

1. *If $y \leq 1$ and $x \leq z$ then the edge $(2, 1)$ is in $G_{Z_n(x,y,z),w}$.*
2. *If $\max\{1, x\} \leq y$ then the edge $(n-1, 1)$ is in $G_{Z_n(x,y,z),w}$.*
3. *If $n = 4$ and $\max\{y, z\} \leq x$ or $n \geq 5$ and $\max\{1, y, z\} \leq x$ then the edge $(n, 1)$ is in $G_{Z_n(x,y,z),w}$.*
4. *If $\max\{y, z\} \leq 1$ then the edge $(n-1, 2)$ is in $G_{Z_n(x,y,z),w}$.*
5. *If $\max\{1, x\} \leq z$ then the edge $(n, 2)$ is in $G_{Z_n(x,y,z),w}$.*
6. *If $x \leq y$ and $z \leq 1$ then the edge $(n, n-1)$ is in $G_{Z_n(x,y,z),w}$.*

For $n \geq 5$, we have two other lemmas.

Lemma 11. *Let $n \geq 5$ and $Z_n(x, y, z)$, with $x, y, z > 0$. Let w be a positive right eigenvector associated with the Perron eigenvalue of $Z_n(x, y, z)$.*

1. *If $1 \leq \min\{x, y\}$ then, for $3 \leq i \leq n-2$, the edge $(1, i)$ is in $G_{Z_n(x,y,z),w}$.*
2. *If $1 \leq z$ then, for $3 \leq i \leq n-2$, the edge $(2, i)$ is in $G_{Z_n(x,y,z),w}$.*
3. *If $y \leq 1$ then, for $3 \leq i \leq n-2$, the edge $(n-1, i)$ is in $G_{Z_n(x,y,z),w}$.*
4. *If $\max\{x, z\} \leq 1$ then, for $3 \leq i \leq n-2$, the edge (n, i) is in $G_{Z_n(x,y,z),w}$.*

Proof. Let

$$w = \begin{bmatrix} w_1 & w_2 & \cdots & w_{n-1} & w_n \end{bmatrix}^T$$

and r be the Perron eigenvalue of $Z_n(x, y, z)$. Recall that $r \geq n > 0$, w is a positive right eigenvector and $w_i = w_3$, for $3 \leq i \leq n-2$.

1. Since $1 \leq \min\{x, y\}$, using equation (14) we conclude that $w_3 - w_1 \leq 0$. This implies $\frac{w_1}{w_3} \geq 1$ and consequently we get the result.

2. Since $1 \le z$, using equation (17) we conclude that $w_3 - w_2 \le 0$. This implies $\frac{w_2}{w_3} \ge 1$ and consequently we get the result.
3. Since $y \le 1$, using equation (20) we conclude that $w_3 - w_{n-1} \le 0$. This implies $\frac{w_{n-1}}{w_3} \ge 1$ and consequently we get the result.
4. Since $\max\{x, z\} \le 1$, using equation (21) we conclude that $w_3 - w_n \le 0$. This implies $\frac{w_n}{w_3} \ge 1$ and consequently we get the result. $\square$

With an analogous proof we have the next lemma.

Lemma 12. *Let $n \ge 5$ and $Z_n(x, y, z)$, with $x, y, z > 0$. Let w be a positive right eigenvector associated with the Perron eigenvalue of $Z_n(x, y, z)$.*

1. *If $\max\{x, y\} \le 1$ then, for $3 \le i \le n-2$, the edge $(i, 1)$ is in $G_{Z_n(x,y,z),w}$.*
2. *If $z \le 1$ then, for $3 \le i \le n-2$, the edge $(i, 2)$ is in $G_{Z_n(x,y,z),w}$.*
3. *If $1 \le y$ then, for $3 \le i \le n-2$, the edge $(i, n-1)$ is in $G_{Z_n(x,y,z),w}$.*
4. *If $1 \le \min\{x, z\}$ then, for $3 \le i \le n-2$, the edge (i, n) is in $G_{Z_n(x,y,z),w}$.*

Proof of Theorem 4. Let

$$w = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \end{bmatrix}^T$$

be a positive right eigenvector associated with the Perron eigenvalue of $Z_4(x, y, z)$. If neither

(i) $\min\{1, x\} \le \min\{y, z\} \le \max\{1, x\} \le \max\{y, z\}$, nor
(ii) $\min\{y, z\} \le \min\{1, x\} \le \max\{y, z\} \le \max\{1, x\}$,

then the relations among $1, x, y, z$ verify one of the following cases:

Case 1 If $1 \le \min\{y, z\} \le \max\{y, z\} \le x$ or $x \le \min\{y, z\} \le \max\{y, z\} \le 1$ then by Theorem 3 we get the result.

Case 2 If $\max\{1, x\} \le \min\{y, z\}$ then by Lemmas 9 and 10 we conclude that the digraph $G_{Z_4(x,y,z),w}$ contains the directed Hamiltonian cycle

$$1 \to 4 \to 2 \to 3 \to 1.$$

Case 3 If $y \le \min\{1, x\} \le \max\{1, x\} \le z$ then by Lemmas 9 and 10 we conclude that the digraph $G_{Z_4(x,y,z),w}$ contains the directed Hamiltonian cycle

$$1 \to 3 \to 4 \to 2 \to 1.$$

Case 4 If $z \le \min\{1, x\} \le \max\{1, x\} \le y$ then by Lemmas 9 and 10 we conclude that the digraph $G_{Z_4(x,y,z),w}$ contains the directed Hamiltonian cycle

$$1 \to 2 \to 4 \to 3 \to 1.$$

Case 5 If $\max\{y, z\} \le \min\{1, x\}$ then by Lemmas 9 and 10 we conclude that the digraph $G_{Z_4(x,y,z),w}$ contains the directed Hamiltonian cycle

$$1 \to 3 \to 2 \to 4 \to 1.$$

Using Theorem 2, for Cases $2-5$ we get the result. $\square$

Proof of Theorem 5. Let

$$w = \begin{bmatrix} w_1 & w_2 & \cdots & w_{n-1} & w_n \end{bmatrix}^T$$

be a positive right eigenvector associated with the Perron eigenvalue r of $Z_n(x, y, z)$, with $n \ge 5$. Recall that if $n > 5$ then $w_i = w_3$, for $4 \le i \le n-2$, and $3 \to i$ and $i \to 3$ are edges of $Z_n(x, y, z)$. Therefore, if we find a directed cycle among the vertices $1, 2, 3, n-1, n$ of $Z_n(x, y, z)$ then we get a directed Hamiltonian cycle in $Z_n(x, y, z)$. If neither

(i) $1 \le z \le x \le y$, nor
(ii) $z \le 1 \le y \le x$, nor
(iii) $z \le x \le y \le 1$, nor
(iv) $x \le z \le 1 \le y$,

then the relations among $1, x, y, z$ verify one of the following cases:

Case 1 If $1 \leq \min\{y,z\} \leq \max\{y,z\} \leq x$ or $x \leq \min\{y,z\} \leq \max\{y,z\} \leq 1$ then by Theorem 3 we get the result.

Case 2 If $x \leq 1 \leq \min\{y,z\}$ then by Lemmas 9, 10, 11 and 12 we conclude that the digraph $G_{Z_n(x,y,z),w}$ contains the directed cycle

$$1 \to n \to 2 \to 3 \to n-1 \to 1.$$

Case 3 If $1 \leq x \leq \min\{y,z\}$ then by Lemmas 9, 10, 11 and 12 we conclude that the digraph $G_{Z_n(x,y,z),w}$ contains the directed cycle

$$1 \to 3 \to n \to 2 \to n-1 \to 1.$$

Case 4 If $y \leq 1 \leq x \leq z$ then by Lemmas 9, 10, 11 and 12 we conclude that the digraph $G_{Z_n(x,y,z),w}$ contains the directed cycle

$$1 \to n-1 \to 3 \to n \to 2 \to 1.$$

Case 5 If $y \leq x \leq 1 \leq z$ then by Lemmas 9, 10, 11 and 12 we conclude that the digraph $G_{Z_n(x,y,z),w}$ contains the directed cycle

$$1 \to n-1 \to n \to 2 \to 3 \to 1.$$

Case 6 If $z \leq 1 \leq x \leq y$ then by Lemmas 9, 10, 11 and 12 we conclude that the digraph $G_{Z_n(x,y,z),w}$ contains the directed cycle

$$1 \to 3 \to 2 \to n \to n-1 \to 1.$$

Case 7 If $z \leq x \leq 1 \leq y$ then by Lemmas 9, 10, 11 and 12 we conclude that the digraph $G_{Z_n(x,y,z),w}$ contains the directed cycle

$$1 \to 2 \to n \to 3 \to n-1 \to 1.$$

Case 8 If $\max\{y,z\} \leq 1 \leq x$ then by Lemmas 9, 10, 11 and 12 we conclude that the digraph $G_{Z_n(x,y,z),w}$ contains the directed cycle

$$1 \to n-1 \to 3 \to 2 \to n \to 1.$$

Case 9 If $\max\{y,z\} \leq x \leq 1$ then by Lemmas 9, 10, 11 and 12 we conclude that the digraph $G_{Z_n(x,y,z),w}$ contains the directed cycle

$$1 \to n-1 \to 2 \to n \to 3 \to 1.$$

Using Theorem 2 we get the result for Cases $2-9$.

Case 10 If $1 \leq y \leq x \leq z$ then by Lemmas 9, 10, 11 and 12 we conclude that the digraph $G_{Z_n(x,y,z),w}$ contains the directed cycle

$$3 \to n-1 \to n \to 2 \to 3 \text{ and the edge } 1 \to 3.$$

Suppose that the vertex 1 is a source in $G_{Z_n(x,y,z),w}$. This implies that $1 \to n-1$, $1 \to n$ and $1 \to 2$ are edges in $G_{Z_n(x,y,z),w}$. Therefore,

$$\frac{w_1}{w_{n-1}} > y, \quad \frac{w_1}{w_n} > x, \quad \frac{w_1}{w_2} > 1.$$

Since $w_1 > yw_{n-1}$, using equation (15) we conclude that $(x-y)w_n > (y-1)(w_2+(n-4)w_3) > (y-1)w_2$.

Since $w_1 > xw_n$, using equation (16) we conclude that $\left(1-\frac{x}{z}\right)w_2 > (x-1)((n-4)w_3)+(x-y)w_{n-1} \geq (x-y)w_{n-1}$.

Since $w_1 > w_2$, using equation (13) we conclude that $(y-1)w_{n-1} > (z-x)w_n$.

Consequently,

$$\frac{1}{z}(x-y)(z-x)w_n > \frac{1}{z}(y-1)(z-x)w_2 > (y-1)(x-y)w_{n-1} > (z-x)(x-y)w_n.$$

So, we get the contradiction $\frac{1}{z} > 1$. Therefore, 1 is not a source in $G_{Z_n(x,y,z),w}$ and by Theorem 1 we get the result.

Case 11 If $y \leq 1 \leq z \leq x$ then by Lemmas 9, 10, 11 and 12 we conclude that the digraph $G_{Z_n(x,y,z),w}$ contains the directed cycle

$$3 \to n \to 1 \to n-1 \to 3 \text{ and the edge } 2 \to 3.$$

Suppose that the vertex 2 is a source in $G_{Z_n(x,y,z),w}$. This implies that $2 \to n-1$, $2 \to n$ and $2 \to 1$ are edges in $G_{Z_n(x,y,z),w}$. Therefore,

$$\frac{w_2}{w_{n-1}} > 1, \quad \frac{w_2}{w_n} > z, \quad \frac{w_2}{w_1} > 1.$$

Since $w_2 > w_{n-1}$, using equation (18) we conclude that $(z-1)w_n > \left(\frac{1}{y}-1\right)w_1$.

Since $w_2 > zw_n$, using equation (19) we conclude that $\left(1-\frac{z}{x}\right)w_1 > (z-1)((n-4)w_3+w_{n-1}) > (z-1)w_{n-1}$.

Since $w_2 > w_1$, using equation (13) we conclude that $(1-y)w_{n-1} > (x-z)w_n$.

Consequently,

$$\frac{1}{x}(x-z)(z-1)w_n > \frac{1}{xy}(1-y)(x-z)w_1 > \frac{1}{y}(1-y)(z-1)w_{n-1}$$
$$> \frac{1}{y}(x-z)(z-1)w_n.$$

So, we get the contradiction $\frac{1}{x} > \frac{1}{y}$. Therefore, 2 is not a source in $G_{Z_n(x,y,z),w}$ and by Theorem 1 we get the result.

Case 12 If $y \le x \le z \le 1$ then by Lemmas 9, 10, 11 and 12 we conclude that the digraph $G_{Z_n(x,y,z),w}$ contains the directed cycle

$$3 \to 2 \to 1 \to n-1 \to 3 \text{ and the edge } n \to 3.$$

Suppose that the vertex n is a source in $G_{Z_n(x,y,z),w}$. This implies that $n \to n-1$, $n \to 2$ and $n \to 1$ are edges in $G_{Z_n(x,y,z),w}$. Therefore,

$$\frac{w_n}{w_{n-1}} > 1, \qquad \frac{w_n}{w_2} > \frac{1}{z}, \qquad \frac{w_n}{w_1} > \frac{1}{x}.$$

Since $w_n > w_{n-1}$, using equation (22) we conclude that $\left(\frac{1}{z}-1\right)w_2 > \left(\frac{1}{y}-\frac{1}{x}\right)w_1$.

Since $zw_n > w_2$, using equation (19) we conclude that $\left(\frac{z}{x}-1\right)w_1 > (1-z)((n-4)w_3 + w_{n-1}) > (1-z)w_{n-1}$.

Since $xw_n > w_1$, using equation (16) we conclude that $(x-y)w_{n-1} > \left(1-\frac{x}{z}\right)w_2 + (1-x)(n-4)w_3 > \left(1-\frac{x}{z}\right)w_2$.

Consequently,

$$\frac{1}{xz}(z-x)(1-z)w_2 > \frac{1}{x^2y}(x-y)(z-x)w_1 > \frac{1}{xy}(1-z)(x-y)w_{n-1}$$
$$> \frac{1}{xyz}(z-x)(1-z)w_2.$$

So, we get the contradiction $1 > \frac{1}{y}$. Therefore, n is not a source in $G_{Z_n(x,y,z),w}$ and by Theorem 1 we get the result.

Case 13 If $x \le y \le 1 \le z$ then by Lemmas 9, 10, 11 and 12 we conclude that the digraph $G_{Z_n(x,y,z),w}$ contains the directed cycle

$$3 \to 1 \to n \to 2 \to 3 \text{ and the edge } n-1 \to 3.$$

Suppose that the vertex $n-1$ is a source in $G_{Z_n(x,y,z),w}$. This implies that $n-1 \to n$, $n-1 \to 2$ and $n-1 \to 1$ are edges in $G_{Z_n(x,y,z),w}$. Therefore,

$$\frac{w_{n-1}}{w_n} > 1, \qquad \frac{w_{n-1}}{w_2} > 1, \qquad \frac{w_{n-1}}{w_1} > \frac{1}{y}.$$

Since $w_{n-1} > w_n$, using equation (22) we conclude that $\left(1-\frac{1}{z}\right)w_2 > \left(\frac{1}{x}-\frac{1}{y}\right)w_1$.

Since $w_{n-1} > w_2$, using equation (18) we conclude that $\left(\frac{1}{y}-1\right)w_1 > (z-1)w_n$.

Since $yw_{n-1} > w_1$, using equation (17) we conclude that $(y-x)w_n > (1-y)(w_2 + (n-4)w_3) > (1-y)w_2$.

Consequently,

$$\frac{1}{yz}(z-1)(1-y)w_2 > \frac{1}{xy^2}(y-x)(1-y)w_1 > \frac{1}{xy}(z-1)(y-x)w_n$$
$$> \frac{1}{xy}(z-1)(1-y)w_2.$$

So, we get the contradiction $\frac{1}{z} > \frac{1}{x}$. Therefore, $n-1$ is not a source in $G_{Z_n(x,y,z),w}$ and by Theorem 1 we get the result. □

Proof of Theorem 6. Let

$$w = \begin{bmatrix} w_1 & w_2 & \cdots & w_{n-1} & w_n \end{bmatrix}^T$$

be a positive right eigenvector associated with the Perron eigenvalue r of $Z_n(x,y,z)$, with $n \ge 4$. Recall that if $n > 5$ then $w_i = w_3$, for $4 \le i \le n-2$, and $3 \to i$ and $i \to 3$ are edges of $G_{Z_n(x,y,z)w}$.

Suppose that $G_{Z_n(x,y,z)w}$ has a sink. By Theorem 1, w is inefficient for $Z_n(x,y,z)$.

Conversely, suppose that w is inefficient for $Z_n(x,y,z)$. We will consider $n = 4$ and $n \ge 5$.

Suppose $n = 4$. By Theorem 4, w is inefficient for $Z_4(x,y,z)$ in the following cases:

Case 1 If $1 \le y \le x \le z$ or $z \le x \le y \le 1$ then by Lemmas 9 and 10, the subgraph of $G_{Z_4(x,y,z),w}$ generated by vertices 2, 3, 4 has a directed Hamiltonian cycle. Since w is not efficient for $Z_4(x,y,z)$, the vertex 1 of $G_{Z_4(x,y,z),w}$ is a source or a sink. Suppose that 1 is a source. This implies that $1 \to 2$, $1 \to 3$ and $1 \to 4$ are edges in $G_{Z_4(x,y,z),w}$. Therefore,

$$\frac{w_1}{w_2} > 1, \quad \frac{w_1}{w_3} > y, \quad \frac{w_1}{w_4} > x.$$

Since $w_1 > w_2$, using equation (13) we conclude that $(y-1)w_3 > (z-x)w_4$.

Since $w_1 > yw_3$, using equation (15) we conclude that $(x-y)w_4 > (y-1)w_2$.

Since $w_1 > xw_4$, using equation (16) we conclude that $\left(1 - \frac{x}{z}\right) w_2 > (x-y)w_3$.

Consequently, if $1 \le y \le x \le z$,

$$(y-1)(x-y)w_3 > (z-x)(x-y)w_4 > (z-x)(y-1)w_2 > z(x-y)(y-1)w_3$$

or if $z \le x \le y \le 1$,

$$(y-1)(x-y)w_3 < (z-x)(x-y)w_4 < (z-x)(y-1)w_2 < z(x-y)(y-1)w_3.$$

So, we get the contradiction $1 > z$ if $1 \le y \le x \le z$ or $1 < z$ if $z \le x \le y \le 1$. Therefore, 1 is not a source in $G_{Z_4(x,y,z),w}$ and by Theorem 1, the vertex 1 is a sink.

Case 2 If $1 \le z \le x \le y$ or $y \le x \le z \le 1$ then by Lemmas 9 and 10, the subgraph of $G_{Z_4(x,y,z),w}$ generated by vertices 1, 2, 3 has a directed Hamiltonian cycle. Since w is not efficient for $Z_4(x,y,z)$, the vertex 4 of $G_{Z_4(x,y,z),w}$ is a source or a sink. Suppose that 4 is a source. This implies that $4 \to 1$, $4 \to 2$ and $4 \to 3$ are edges in $G_{Z_4(x,y,z),w}$. Therefore,

$$\frac{w_4}{w_1} > \frac{1}{x}, \quad \frac{w_4}{w_2} > \frac{1}{z}, \quad \frac{w_4}{w_3} > 1.$$

Since $xw_4 > w_1$, using equation (16) we conclude that $\left(\frac{x}{z} - 1\right) w_2 > (y-x)w_3$.

Since $zw_4 > w_2$, using equation (19) we conclude that $(z-1)w_3 > \left(1 - \frac{z}{x}\right) w_1$.

Since $w_4 > w_3$, using equation (22) we conclude that $\left(\frac{1}{x} - \frac{1}{y}\right) w_1 > \left(1 - \frac{1}{z}\right) w_2$.

Consequently, if $1 \le z \le x \le y$,

$$\frac{1}{z}(x-z)(z-1)w_2 > (y-x)(z-1)w_3 > \frac{1}{x}(x-z)(y-x)w_1 > \frac{y}{z}(x-z)(z-1)w_2$$

or if $y \le x \le z \le 1$,

$$\frac{1}{z}(x-z)(z-1)w_2 < (y-x)(z-1)w_3 < \frac{1}{x}(x-z)(y-x)w_1 < \frac{y}{z}(x-z)(z-1)w_2.$$

So, we get the contradiction $1 > y$ if $1 \le z \le x \le y$ or $1 < y$ if $y \le x \le z \le 1$. Therefore, 4 is not a source in $G_{Z_4(x,y,z),w}$ and by Theorem 1, the vertex 4 is a sink.

Case 3 If $y \le 1 \le z \le x$ or $x \le z \le 1 \le y$ then by Lemmas 9 and 10, the subgraph of $G_{Z_4(x,y,z),w}$ generated by vertices 1, 3, 4 has a directed Hamiltonian cycle. Since w is not efficient for $Z_4(x,y,z)$, the vertex 2 of $G_{Z_4(x,y,z),w}$ is a source or a sink. Suppose that 2 is a source. This implies that $2 \to 1$, $2 \to 3$ and $2 \to 4$ are edges in $G_{Z_4(x,y,z),w}$. Therefore,

$$\frac{w_2}{w_1} > 1, \quad \frac{w_2}{w_3} > 1, \quad \frac{w_2}{w_4} > z.$$

Since $w_2 > w_1$, using equation (13) we conclude that $(1-y)w_3 > (x-z)w_4$.

Since $w_2 > w_3$, using equation (18) we conclude that $(z-1)w_4 > \left(\frac{1}{y} - 1\right) w_1$.

Since $w_2 > zw_4$, using equation (19) we conclude that $\left(1 - \frac{z}{x}\right) w_1 > (z-1)w_3$.

Consequently, if $y \le 1 \le z \le x$,

$$(1-y)(z-1)w_3 > (x-z)(z-1)w_4 > (x-z)\left(\frac{1}{y} - 1\right) w_1 > \frac{x}{y}(z-1)(1-y)w_3$$

or if $x \le z \le 1 \le y$,

$$(1-y)(z-1)w_3 < (x-z)(z-1)w_4 < (x-z)\left(\frac{1}{y} - 1\right) w_1 < \frac{x}{y}(z-1)(1-y)w_3.$$

So, we get the contradiction $1 > \frac{x}{y}$ if $y \le 1 \le z \le x$ or $1 < \frac{x}{y}$ if $x \le z \le 1 \le y$. Therefore, 2 is not a source in $G_{Z_4(x,y,z),w}$ and by Theorem 1, the vertex 2 is a sink.

Case 4 If $z \le 1 \le y \le x$ or $x \le y \le 1 \le z$ then by Lemmas 9 and 10, the subgraph of $G_{Z_4(x,y,z),w}$ generated by vertices 1, 2, 4 has a directed Hamiltonian cycle. Since w is not efficient for $Z_4(x,y,z)$, the vertex 3 of $G_{Z_4(x,y,z),w}$ is a source or a sink. Suppose that 3 is a source. This implies that $3 \to 1$, $3 \to 2$ and $3 \to 4$ are edges in $G_{Z_4(x,y,z),w}$. Therefore,

$$\frac{w_3}{w_1} > \frac{1}{y}, \quad \frac{w_3}{w_2} > 1, \quad \frac{w_3}{w_4} > 1.$$

Since $yw_3 > w_1$, using equation (15) we conclude that $(y-1)w_2 > (x-y)w_4$.

Since $w_3 > w_2$, using equation (18) we conclude that $(1-z)w_4 > \left(1-\frac{1}{y}\right)w_1$.

Since $w_3 > w_4$, using equation (22) we conclude that $\left(\frac{1}{y}-\frac{1}{x}\right)w_1 > \left(\frac{1}{z}-1\right)w_2$.

Consequently, if $z \le 1 \le y \le x$,

$$(y-1)(1-z)w_2 > (x-y)(1-z)w_4 > (x-y)\left(1-\frac{1}{y}\right)w_1 > xy\left(\frac{1}{z}-1\right)\left(1-\frac{1}{y}\right)w_2$$

or if $x \le y \le 1 \le z$,

$$(y-1)(1-z)w_2 < (x-y)(1-z)w_4 < (x-y)\left(1-\frac{1}{y}\right)w_1 < xy\left(\frac{1}{z}-1\right)\left(1-\frac{1}{y}\right)w_2.$$

So, we get the contradiction $1 > \frac{x}{z}$ if $z \le 1 \le y \le x$ or $1 < \frac{x}{z}$ if $x \le y \le 1 \le z$. Therefore, 3 is not a source in $G_{Z_4(x,y,z),w}$ and by Theorem 1, the vertex 3 is a sink.

Suppose $n \ge 5$. By Theorem 5, w is not efficient for $Z_n(x,y,z)$ in the following cases:

Case 1 If $1 \le z \le x \le y$ then by Lemmas 9, 10, 11 and 12 we conclude that the digraph $G_{Z_n(x,y,z),w}$ contains the directed cycle

$$3 \to n-1 \to 1 \to 2 \to 3 \text{ and the edge } 3 \to n.$$

By Theorem 1, the vertex n is a sink.

Case 2 If $z \le 1 \le y \le x$ then by Lemmas 9, 10, 11 and 12 we conclude that the digraph $G_{Z_n(x,y,z),w}$ contains the directed cycle

$$1 \to 3 \to 2 \to n \to 1 \text{ and the edge } 3 \to n-1.$$

By Theorem 1, the vertex $n-1$ is a sink.

Case 3 If $z \le x \le y \le 1$ then by Lemmas 9, 10, 11 and 12 we conclude that the digraph $G_{Z_n(x,y,z),w}$ contains the directed cycle

$$2 \to n \to n-1 \to 3 \to 2 \text{ and the edge } 3 \to 1.$$

By Theorem 1, the vertex 1 is a sink.

Case 4 If $x \le z \le 1 \le y$ then by Lemmas 9, 10, 11 and 12 we conclude that the digraph $G_{Z_n(x,y,z),w}$ contains the directed cycle

$$1 \to n \to 3 \to n-1 \to 1 \text{ and the edge } 3 \to 2.$$

By Theorem 1, the vertex 2 is a sink. $\square$

5. Conclusions

In this paper we focus on matrices obtained from consistent matrices by perturbing three entries and the corresponding reciprocal entries, so that there is a 2×2 submatrix containing three of the perturbed entries and not containing a diagonal entry. We prove for which magnitudes of the perturbed entries, the principal right eigenvector of this matrix is efficient. Moreover, we show that the inefficiency of the principal right eigenvector of our studied matrices is equivalent to the existence of a sink in the digraph associated with the matrix and the principal right eigenvector. We also present an example of a PC matrix A having its principal right eigenvector w inefficient such that the digraph $G_{A,w}$ does not have a sink.

When the magnitude of the three perturbed entries, x, y, z, of the matrix $Z_n(x,y,z)$, presented in (2), does not imply the efficiency of the principal right eigenvector of $Z_n(x,y,z)$, an open question is to find a process to approach the matrix $Z_n(x,y,z)$ by a matrix A having its principal right eigenvector efficient. Furthermore, the principal right eigenvectors of these two matrices, $Z_n(x,y,z)$ and A, must have corresponding components verifying the same relation among them.

CRediT authorship contribution statement

Rosário Fernandes: Writing – original draft, Investigation. **Susana Palheira:** Writing – original draft, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors would like to thank the referees for the valuable comments that helped improving the first version of this paper.

References

[1] K. Ábele-Nagy, S. Bozóki, Efficiency analysis of simple perturbed pairwise comparison matrices, Fundam. Inform. 144 (3–4) (2016) 279–289.
[2] K. Ábele-Nagy, S. Bozóki, Ö. Rebák, Efficiency analysis of double perturbed pairwise comparison matrices, J. Oper. Res. Soc. 69 (5) (2018) 707–713.
[3] M. Anholcer, J. Fülöp, Deriving priorities from inconsistent pcm using network algorithms, Ann. Oper. Res. 274 (2019) 57–74.
[4] N. Bebiano, R. Fernandes, S. Furtado, Reciprocal matrices: properties and approximation by a transitive matrix, Comput. Appl. Math. 39 (2) (2020) 50.
[5] R. Blanquero, E. Carrizosa, E. Conde, Inferring efficient weights from pairwise comparison matrices, Math. Methods Oper. Res. 64 (2006) 271–284.
[6] S. Bozóki, Solution of the least squares method problem of pairwise comparison matrices, Cent. Eur. J. Oper. Res. 16 (4) (2008) 345–358.
[7] S. Bozóki, J. Fülöp, Efficient weight vectors from pairwise comparison matrices, Eur. J. Oper. Res. 264 (2) (2018) 419–427.
[8] V. Čerňanová, W.W. Koczkodaj, J. Szybowski, Inconsistency of special cases of pairwise comparisons matrices, Int. J. Approx. Reason. 95 (2018) 36–45.
[9] H.F. da Cruz, R. Fernandes, S. Furtado, Efficient vectors for simple perturbed consistent matrices, Int. J. Approx. Reason. 139 (2021) 54–68.
[10] A. Farkas, P. Lancaster, P. Rózsa, Consistency adjustments for pairwise comparison matrices, Numer. Linear Algebra Appl. 10 (8) (2003) 689–700.
[11] R. Fernandes, S. Furtado, Efficiency of the principal eigenvector of some triple perturbed consistent matrices, Eur. J. Oper. Res. 298 (3) (2022) 1007–1015.
[12] R. Fernandes, R. Palma, Positive vectors, pairwise comparison matrices and directed Hamiltonian cycles, 2024, submitted for publication.
[13] S. Furtado, Efficient vectors for double perturbed consistent matrices, Optimization 72 (11) (2023) 2679–2701.
[14] S. Furtado, C. Johnson, Efficient vectors for block perturbed consistent matrices, SIAM J. Matrix Anal. Appl. 45 (1) (2024) 601–618.
[15] S. Furtado, C.R. Johnson, Efficient vectors in priority setting methodology, Ann. Oper. Res. (2024) 1–22.
[16] C.R. Johnson, R.A. Horn, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
[17] W.W. Koczkodaj, R. Urban, Axiomatization of inconsistency indicators for pairwise comparisons, Int. J. Approx. Reason. 94 (2018) 18–29.
[18] T.L. Saaty, A scaling method for priorities in hierarchical structures, J. Math. Psychol. 15 (3) (1977) 234–281.
[19] T.L. Saaty, The Analytic Hierarchy Process, McGraw-Hill, New York, 1980.
[20] J.J. Thakkar, Multi-Criteria Decision Making, vol. 336, Springer, 2021.