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"THREE NOTES ON SYMMETRIC GAMES WITH ASYMMETRIC EQUILIBRIA"

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Three Notes on Symmetric Games

With Asymmetric Equilibria

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Asymmetric Equilibria in Symmetric Games with a Large Number of Players

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Abstract

It is often the case in symmetric games in normal form that the only existing pure-strategy equilibria are asymmetric. If there is an asymmetric equilibrium for a model with N "equal" players, then there are multiple equilibria, only differing on the "name" of the players "assigned" to each one of the actions which together form an equilibrium. A natural question to ask is, then, how to select among these equilibria. In this note, we show that in symmetric games with a large number of players, an asymmetric pure-strategy equilibrium can be thought of as the approximate outcome of the play of a specific symmetric mixed-strategy equilibrium.

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1. Introduction

It is often the case in symmetric games in normal form (i.e. games with a symmetric payoff matrix), that the only existing pure-strategy equilibria are asymmetric. Examples of this are models of entry into an industry (cf Dixit and Shapiro (1986)), models of price-dispersion (cf Salop and Stiglitz (1976)), and models of information about prices (cf Grossman and Stiglitz (1976)).

If there is an asymmetric equilibrium for a model with N "equal" players, then there are multiple equilibria, only differing on the "name" of the players "assigned" to each one of the actions which together form an equilibrium. A natural question to ask is, then, how to select among these equilibria. If the number of players is small, one can assume that there is some communication and coordination mechanism which will lead the players to a specified equilibrium. (See for example Farrell (1987).) However, if the number of players is large, communication and coordination are not so simple, and another mechanism should be found.

In this note, we show that in symmetric games with a large number of players, an asymmetric pure-strategy equilibrium can be thought of as the approximate outcome of the play of a specific symmetric mixed-strategy equilibrium. In this mixed-strategy equilibrium, each player choses action a_i with probability close to the fraction of players choosing that action in the asymmetric pure-strategy equilibrium. The idea is that, with large numbers, ex-ante probability and ex-post frequency are approximately the same. Schmeidler (1973) presents a result (Theorem 2) similar to the one in this note, but for the case of nonatomic games, whereas we deal with games with a finite number of players.

2. The theorem

The result presented below is only valid for a specific class of models, which we will descibe next. While the assumptions made may seem too restrictive, they turn out to be satisfied by the examples refered to before. Furthermore, the result can be extended to a broader class of models, but at the cost of a more complicated proof.

We consider a game with complete information and a finite set of players N={1,...,N}. The set of pure-strategies of each player is given by S={0,1}. The payoff for choosing action i is assumed to depend only on the fraction of players choosing each action. Without loss of generality, we can write $\pi_i(p)$ (i=1,2) for the payoff of choosing action i given that a fraction p of players choose "1." We assume that π_0 is increasing, and π_1 decreasing with p. (Therefore $\Delta\pi\equiv\pi_0-\pi_1$ is increasing with p.) Define $\pi\equiv(\pi_0,\pi_1)$.

To simplify notation, we write "$x(N) \longrightarrow y$" to imply that $\lim_{N\to\infty} x(N)=y$ and "$x(N) \xrightarrow{P} y$" to imply that $\lim_{N\to\infty} P[|x(N)-y|>\epsilon]=0$ (convergence in probability). We can then state the following result.

<u>Theorem</u>. Suppose there is a sequence of games (N,S,π), $N=N_1,N_2,\ldots$,

$N \longrightarrow +\infty$, such that for each N_i there exists a unique (asymmetric) pure-strategy equilibrium with Np^* players choosing action "1" and $N(1-p^*)$ choosing action "0." Then,

(i) there exists a Nash mixed-strategy equilibrium in which each player chooses strategy "1" with probability $\hat{p}(N)$;

(ii) the outcome of each play is, with probability greater than $1-\delta(N)$, a Nash ϵ-equilibrium with $\epsilon=\epsilon(N)$ and a fraction $\bar{p}(N)$ of players choosing strategy "1";

(iii) $\hat{p}(N) \longrightarrow p^*$, $\bar{p}(N) \xrightarrow{P} p^*$, $\delta(N) \longrightarrow 0$, and $\epsilon(N) \longrightarrow 0$.

<u>Proof</u>. Suppose each player chooses a mixed-strategy in which he or she plays "1" with probability $\hat{p}$. A necessary and sufficient condition for $\hat{p}$ to be an optimal strategy is that the expected payoffs of playing "0" and "1" are the same:

$$E[\pi_0[\bar{p}(N-1)/N]] = E[\pi_1[(\bar{p}(N-1)+1)/N], \qquad (1)$$

where $\bar{p}$ is the fraction of <u>other</u> players <u>actually</u> choosing "1," given that each chooses "1" with probability $\hat{p}$. The number of <u>other</u> players choosing "1", n, has a binomial distribution:

$$n: B(N-1,\hat{p}) \quad (n=0,1,\ldots,N-1)$$

with $\quad E(n)=\hat{p}(N-1)$ and $V(n)=\hat{p}(1-\hat{p})(N-1)=\sigma(N-1)$, where $\sigma=\hat{p}(1-\hat{p})$. Therefore, $\bar{p}$ has also a binomial distribution, with

$$E(\bar{p})=\hat{p} \text{ and } V(\bar{p})=\sigma/(N-1).$$

An increase in $\hat{p}$ shifts the distribution of $\bar{p}$ to the right, in a continuous way, in the sense of first-order stochastic dominance. Given the monotonicity of $\pi_i(p)$, we conclude that $E[\pi_0(\bar{p})]$ is continuous and increasing, and $E[\pi_1(\bar{p})]$ continuous and decreasing with $\hat{p}$. Clearly, if $\hat{p}$ equals zero (one), so will $\bar{p}$ equal zero (one). Given the monotonicity of $\pi_i(p)$ and the fact that there is an interior Nash equilibrium, $\pi_0(0)\leq\pi_1(1/N)$ and $\pi_0((N-1)/N)\geq\pi_1(N)$. Together, these facts imply that there exists a unique solution to (1). Let us refer to it as $\hat{p}=\hat{p}(N)$. Also, define $\sigma(N)=\hat{p}(N)(1-\hat{p}(N))$.

Now suppose all players choose action "1" with probability $\hat{p}(N)$. Then, we can apply Chebyshev's inequality to get

$$P[\, p^L(N) \leq \bar{p}(N) \leq p^H(N)\,] \geq 1-1/r^2, \tag{2}$$

where

$$p^L(N)=\hat{p}(N)-r^2\sigma(N)/N$$

and $$p^H(N)=\hat{p}(N)+r^2\sigma(N)/N,$$

for any given positive r.

Define

$$\epsilon(N) = \max \{ \pi_1(p^L) - \pi_0(p^L) , \pi_0(p^H) - \pi_1(p^H) \} \tag{3}$$

$$\delta(N) = 1/r^2 \tag{4}$$

$$r = N^{\gamma}, \text{ where } 0 < \gamma < 1/2. \tag{5}$$

By (2) and definitions (3)-(4), the second part of the theorem follows. Clearly, $r \longrightarrow +\infty$, and thus $\delta(N) \longrightarrow 0$.

On the other hand, given (5) and the fact that $\sigma(N)$ is bounded, the order of magnitude of $r^2\sigma(N)/N$ is less than zero. Therefore,

$$p^L(N) \longrightarrow \hat{p}(N) \text{ and } p^H(N) \longrightarrow \hat{p}(N), \tag{6}$$

which shows that $\bar{p} \xrightarrow{p} \hat{p}$. Since payoffs are bounded and continuous, we have, by Slutsky's theorem [see Greenberg and Webster (1983, p. 8)],

$$E[\pi_0[\bar{p}(N)(N-1)/N]] \xrightarrow{p} \pi_0[\hat{p}(N)(N-1)/N] \longrightarrow \pi_0[\hat{p}(N)]$$

and $$E[\pi_1[(\bar{p}(N)(N-1)+1)/N] \xrightarrow{p} \pi_1[(\hat{p}(N)(N-1)+1)/N \longrightarrow \pi_1[(\hat{p}(N)],$$

which implies that

$$\Delta\pi[\hat{p}(N)]=\pi_1[(\hat{p}(N)]-\pi_0[\hat{p}(N)] \xrightarrow{p} 0. \tag{7}$$

Given that (N,p^*,π) is a Nash equilibrium,

$$\pi_0[p^*] \geq \pi_1[(p^*N+1)/N]$$

and $$\pi_1[p^*] \geq \pi_0[(p^*N-1)/N],$$

i.e. players choosing "0" or "1" have no incentive to deviate. Therefore,

$$\Delta\pi(p^*)=\pi_1(p^*)-\pi_0(p^*) \longrightarrow 0. \tag{8}$$

Putting together (7) and (8), and recalling that $\Delta\pi$ is monotonic, we conlude that

$$p(N)-p^* \xrightarrow{P} 0.$$

Finally, (3), (6) and (7) imply that $\epsilon(N) \longrightarrow 0$. ∎

3. Remarks

1. There are several alternative ways of interpreting the result presented above. One is the following. In general, mixed-strategy equilibria imply ex-post regret: after having observed the other players' actions, some players will regret the choice they have made. With a large number of players, however, the outcome of each play (of a mixed-strategy

equilibrium) is close to an ϵ-Nash equilibrium, and therefore it implies no ex-post regret (in an approximate sense).

2. A second alternative interpretation of the theorem is that, with a large number of players, mixed-strategy equilibria are close to correlated equilibria. Recall Harsanyi's "purification" argument: a mixed-strategy equilibrium can be interpreted as the reduced form of a game with incomplete information. Each player observes the realization of random shocks which affect their own payoffs, and which are each player's private information. If these disturbances are independent across players, then as the number of players increases players can "know" (in a statistical sense) the other players' random shocks, and act as if they were correlating their strategies.

3. Finally, note that with a large number of players, the sets of equilibrium payoffs of mixed- and pure-strategy equilibria are the same. This may have interesting implications in the area of repeated games (cf Abreu, Pearce, and Stacchetti (1986)).

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A Note on Bertrand Competition
With Capacity Precommitment

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Abstract

Kreps and Sheinkman (1983) claim that price competition with capacity precommitment is equivalent to quantity competition. Davidson and Deneckere (1986) showed that K-S's result depends crucially on the assumption of how excess demand is rationed. In general, they claim, the outcome of K-S's two-stage game is likely to be more competitive than the Cournot outcome predicted by K-S. This note shows that in the context of a finitely repeated game, D-D's assumption about demand rationing may imply an outcome which is less competitive than the one implied by K-S's assumption. The argument follows Benoit and Krishna's (1984) observation on collusion in finitely repeated games.

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The model of price competition with capacity constraints was first introduced by Edgeworth. More recent treatments are due to Kreps and Sheinkman (1983) and Davidson and Deneckere (1986). It is considered to be a more realistic model than the "extremes" of quantity competition (Cournot model) and price competition without capacity constrints (Bertrand model).

Kreps and Sheinkman (K-S) presented the following two-stage duopoly model. In the first stage, firms simultaneously choose their capacities, and then, in the second stage, they simultaneously choose prices. It is shown that, given certain assumptions, the outcome of this supergame is the same as the outcome of a one-shot Cournot game in capacities.

Davidson and Deneckere (D-D) pointed out that K-S's result depends crutially on the assumption of how demand is rationed when the low-price firm cannot meet market demand. If instead of the K-S rationing rule we adopt the one proposed by Beckmann, the outcome of the two-stage game may be quite different. In fact, D-D show that, in general, equilibria will be asymmetric, with one firm choosing a low capacity (approximately the one in the K-S equilibrium) and the other a large one.

Therefore, D-D conclude that assuming the Beckmann contingent demand curve, the outcome of the capacity precommitment game is far more competitive than was predicted by K-S.

The main point of this note is that D-D's equilibrium is a "two-edge sword." On the one hand, the average equilibrium price is lower than in K-S, which implies a more competitive outcome. On the other hand, equilibria are asymmetric, which may lead to a less competitive outcome. In fact, as was pointed out by Benoit and Krishna (1984), the existence of multiple equilibria in a one-shot game may generate "new" equilibria when the one-shot game is repeated a finite number of times.[1] If we consider D-D's two-stage game as Benoit and Krishna's one-shot game, and repeat it a finite number of times, the outcome may well be less competitive (on average) than in K-S.[2]

To see how this works, let us consider the simple case presented by D-D in Section 3 of their paper. Demand is given by $D(p)=1-p$ and there are no costs. D-D find a unique pair of asymmetric solutions, which involve firms choosing capacity levels roughly equal to .43 and .86, and gaining profits of .056 and .111, respectively. This contrasts with the Cournot outcome (the outcome of K-S's model), where capacity if fixed at the level of .33 and profits are .1089 for each firm.

Now consider a different game made up of N repetitions of the two-stage capacity precommitment game. In the case of K-S's model, given that there is only one equilibrium in the stage-game, the equilibrium of the N-times repeated game will be the repetition of the stage-game equilibrium. However, the same is not true for D-D's model. In fact, we can prove the following.

Claim. Consider the two-stage D-D model repeated N times. Then, if $N \geq 5$, there exists an equilibrium for which average profit (with no discounting) is

$$\frac{1}{N}\{.125\times(N-5)+.0835\times5\}.$$

The values of this expression for N=10, N=20, and N=100 are, respectively, .104, .115, and .123, which compares with .111 for the K-S model.

Proof. The equilibrium which yields the payoff given by the expression is the following. In every stage before the last five, both firms choose half the monopoly capacity and fix the monoply price. This yields a profit of .125 for each firm. At each of the last four stages, firms play the D-D asymmetric equilibrium, choosing randomly which one takes each position. This yields an average profit of .0835. Given that this is a Nash equilibrium in the stage-game, there is no problem of enforcing it in the repeated game. On the other hand, if any firm deviates in any of the previous stages, it will be punished by having to choose the low capacity in D-D's equilibrium for the remainder of the repeated game.[3]

We now have only to show that firms have no incentive to deviate from choosing half the monopoly capacity. If they do so, the maximum they can get in that stage is less than the monopoly profit. Therefore, the one-stage gain is less than half the monopoly profit, i.e. less than .125. On the other hand, they loose at least five times the difference between the average profit in D-D's equilibrium (.0835) and the lowest payoff in D-D's equilibrium (.055). One can easily check that losses outweight gains, which completes the proof. ∎

Notes

1. If, on the other hand, the one-shot game has only one equilibrium, the only equilibrium in the finitely repeated game is the repetition of the one-shot game.

2. It should be noted that this is a different type of repeated game than the one presented by Benoit and Krishna (1987). In the latter, capacity is fixed in the first stage and then a repeated price game is played. In our model, both capacity and prices are fixed in each period. Furthermore, our game is repeated only a finite number of times, as opposed to Benoit and Krishna's infinitely repeated game.

3. Note that this is subgame perfect.

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Asymmetric Equilibria in a Symmetric Cournot Duopoly With Technology Precommitment

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Abstract

We consider a symmetric two-stage Cournot duopoly in which firms must precommit to a given technology. If some conditions are satisfied, the only equilibrium solution consists of a firm choosing a "large-firm" technology and producing a large quantity, and the other firms choosing a "small-firm" technology and producing a small quantity. The conditions for this result essentially imply that there are non-convexities in the envelope average cost function.

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1. Introduction

Traditional oligopoly theory has in general considered symmetric oligopoly models which yield symmetric solutions. The rationale for this approach is that if we assume all firms have the same opportunities, then most likely their actions will also be the same.

Reality is quite different from the symmetric-setup/ symmetric-outcome oligopoly. In most oligopolies, large and small firms subsist together. How can this be be explained by oligopoly theory?

A first way of getting theory closer to reality is to depart from the assumption that the model is setup in a symmetric way. For example, there may be first-mover advantages to early entrants, or one of the firms may have exclusive access to a lower cost technology. If this is the case, one can easily derive an asymmetric outcome as being the theoretically reasonable one.

However, it is possible to find reasonable asymmetric outcomes even starting from a symmetric model. This has been the object of recent literature in industrial organization. For example, Katz and Shapiro (1985) have shown that in a duopoly for a product with "network" externalities there may be equilibria in which one of the firms has a much larger "network" than the other, or even a de facto monopoly. Davidson and Deneckere (1986) considered the case when two firms compete in prices after having commited to a certain capacity level. They show that in some cases an asymmetric solution, with a "large" firm and a "small" firm, may arise.

In this paper, we consider a very simple motive why asymmetric equilibria may arise in a symmetric Cournot duopoly. Besides choosing quantities, firms must choose one of two available technologies to produce it. One of the technologies has a low marginal cost but implies the payment of a high fixed cost; we call it the "large-firm" technology. The other technology has a low fixed cost, but marginal cost is higher; we call it the "small-firm" technology. One can think of retail sales as an example of this: retail stores may invest in having their own transportation force (a fixed cost), thus lowering the cost of acquisition of retail goods.

In some cases, the unique pure-strategy equilibrium market structure consists of a firm choosing the "large-firm" technology and producing a large quantity, and the other firm choosing the "small-firm" technology and producing a small quantity.

2. The model

The main feature of our model is that before playing a Cournot game each firm has to choose the technology it wants to employ in production. We assume that at each stage decisions are taken simultaneously, and restrict ourselves to pure strategies.

Each technology is characterized by a cost function $C(q)=F+cq$, or by the pair (F,c). We assume firms can choose between two technologies. By paying a fixed cost F_i, firms can produce at marginal cost c_i, $i=1,2$. To avoid trivialities, and without loss of generality, we assume $F_1<F_2$ and $c_1>c_2$. Therefore, technologies 1 and 2 are the "small-firm" and "large-firm" technologies, respectively.

The inverse market demand function is given by $p=A-Q$, where p is the market price and $Q=\Sigma q_i$.

We start by considering the solution of all subgames beginning at stage two, when technology choices have already been made. Each firm solves the usual profit maximization problem:

$$\max_{q_i} (A-q_j-q_i-c_i)q_i. \tag{1}$$

The first order condition to this problem is given by

$$q_i=(A-q_j-c_i)/2 \qquad (i,j=1,2;i\neq j) \tag{2}$$

Solving the system (2), we get

$$q_i = (A+c_j-2c_i)/3. \tag{3}$$

Finally, total profit is given by

$$\Pi_i = (A+c_j-2c_i)^2/3 - F_i. \tag{4}$$

We now turn to the solution of the complete two-stage game. The main result is that for a given set of values of c_i and F_i the solution is an asymmetric equilibrium.

Proposition. There exists a unique (up to relabeling of players) asymmetric equilibrium for the two-stage game of technology choice and Cournot competition if and only if

$$c_2 < A - \frac{3\,(F_2-F_1)}{4\,(c_1-c_2)} < c_1. \tag{5}$$

Proof. Equation (4) shows the subgame equilibrium payoffs given that firms chose technologies i and j. In order for (i,j) to be an equilibrium of the complete game, it has to be the case that none of the firms has an incentive to deviate from its technology choice.

For firm i, this implies

$$(A+c_j-2c_i)^2/3 - F_i > (A+c_j-2c_j)^2/3 - F_j \tag{6}$$

which is equivalent to

$$(A+c_j)^2 - 4(A+c_j)c_i + 4c_i^2 - 3F_i > (A+c_j)^2 - 4(A+c_j)c_j + 4c_j^2 - 3F_j,$$

$$3(F_j-F_i)/4 > (A+c_j)(c_i-c_j) + c_j^2 - c_i^2,$$

and $$3(F_j-F_i)/4 > (A+c_j-c_i-c_j)(c_i-c_j) = (A-c_i)(c_i-c_j). \quad (7)$$

Substituting 1 and 2 for i and j in (7), we get (5). ∎

3. Remarks

We have shown that in some circunstances a quantity-setting duopoly with technology precommitment may only have an asymmetric equilibrium (in pure strategies). How likely is an equilibrium like this to occur? By looking at (5), we can see that the farther apart c_1 and c_2 are, the more likely the conditions are met. On the other hand, F_1 and F_2 must be neither too close nor too far apart, such that both inequalities in (5) are satisfied. That is, the two technologies must be sufficiently differentiated, but at the same time competitive.

On the other hand, we must note that the assumption of there being only two available technologies is crucial for the results of our model. Alternatively, we can assume that there exist "medium-size" technologies, but these are dominated by the "large-firm" and "small-firm" technologies. In other words, it is essential for our purposes that the envelope average cost function be non-convex. Whether this is a reasonable assumption is a matter for empirical investigation.

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