Intermediate Preferences and Behavioral Conformity in Large Games*

Guilherme Carmona†

Universidade Nova de Lisboa

November 9, 2007

Abstract

We consider games with a continuum of players and intermediate preferences. We show that any such game has a Nash equilibrium that induces a partition of the set of attributes into a bounded number of convex sets with the following property: all players with an attribute in the interior of the same element of the partition play the same action. Furthermore, if the game induces an absolutely continuous distribution (with respect to the Lebesgue measure) on the attribute space, then we can strengthen the conclusion by showing that all players with an attribute in the same element of the partition play the same action.

*I wish to thank Myrna Wooders for several comments and suggestions, especially the idea of exploring the implications of intermediate preferences for behavioral conformity. I also thank João Rosal for his comments and John Huffstot for editorial assistance. While they all improved the material in many ways, any errors that may remain are my own. Financial support from a Nova Fórum grant is gratefully acknowledged.

†Address: Universidade Nova de Lisboa, Faculdade de Economia, Campus de Campolide, 1099-032 Lisboa, Portugal; Phone: (351) 21 380 1672; Fax: (351) 21 387 0933; email: gcarmona@fe.unl.pt.

We then use these results to show that all sufficiently large, equicontinuous games with intermediate preferences have an approximate equilibrium with the same properties in both cases (for the stronger result, we require the attribute space to be a subset of the real line).

Our result on behavior conformity for large finite game generalizes Theorem 3 of Wooders, Cartwright, and Selten (2006) by allowing both a wider class of preferences and a more general attribute space.

1 Introduction

Consider an individual who lives in the suburbs of a large city and who can only reach her office in the city center by driving over one of two bridges. While the choice of which bridge to take depends on attributes such as her home and office locations and how pleasant either route may be, it also depends on the choice of all others driving to the center, namely through their influence on how crowded each bridge is. In this example, it is conceivable that equilibrium behavior displays individuals with similar attributes choosing the same bridge. Such equilibrium can be interpreted as displaying behavioral conformity in the sense that it induces a partition of the individuals into a small number of societies (two in the example) of similar individuals making the same choice.

Such form of conformity has been rationalized by Wooders, Cartwright, and Selten (2006). In particular, for games where each player's payoff depends only on his choice and on the average choice of the others, they have shown that if preferences depend linearly on individual attributes, then for all $\varepsilon > 0$, all sufficiently large games have an ε – equilibrium that induces a partition of players into a small number of societies (relative to the number of players), with an upper bound independent of ε.[1]

[1]Their Theorem 2 shows that by allowing the number of societies to depend on ε, a similar

In this paper, we extend the framework of Wooders et al. among several dimensions. We do this by allowing for preferences that do not satisfy the continuity and linearity assumptions, by considering a stronger notion of behavioral conformity, and by allowing for a continuum of players.

Our results rely on the notion of quasi-intermediate preferences, which is obtained by dropping the continuity requirement of intermediate preferences, introduced in Grandmont (1978). These preferences allow us to obtain the existence of a Nash equilibrium displaying strong behavioral conformity in dispersed games with a continuum of players in the following sense: all players in the same society, including those in its boundaries, play the same action. Furthermore, it allows us to obtain the existence of a Nash equilibrium displaying behavioral conformity (where all players in the interior of a society play the same action) when the game is not dispersed (i.e., when there is a positive measure of players with the same attribute). These two results are then used to obtain similar results for games with a large, but finite number of players.

An advantage of our approach is that our result on social conformity for large finite games generalizes Theorem 3 in Wooders, Cartwright, and Selten (2006). In fact, we show that the same conclusion can be reached under more general assumptions on the attribute space and on players' preferences: the attribute space can be any bounded, convex and (Borel) measurable subset of $\mathbb{R}^L$, and the function assigning preferences to players need only be measurable, quasi-intermediate and have an equicontinuous image.

Despite such generalization, the main advantage of focusing on games with a continuum of players with quasi-intermediate preferences is conceptual. Indeed, it clarifies the essential elements needed for behavioral conformity and allows for simple,

conclusion holds under more general preferences.

constructive proofs that can be used to construct equilibria displaying behavioral conformity in applications.

The paper is organized as follows: in Section 2 we introduce the notation and main definitions. In Section 3 we consider dispersed anonymous games (as in Mas-Colell (1984)) with quasi-intermediate preferences; the results in this section are not only interesting in their own right, but also serve as tools for several others in the paper. In Section 4, we generalize the notion of quasi-intermediate preferences and use it to obtain a stronger existence result in binary games (i.e., games with only two actions). Section 5 considers games with a continuum of players as defined in Schmeidler (1973), and Section 6 examines games with a large, but finite number of players. In Section 7 we consider the relationship between our results and Theorem 3 of Wooders, Cartwright, and Selten (2006). Some concluding remarks are in Section 8. Finally, the Appendix contains the proof of the lemmas and some other auxiliary results.

2 Notation and Definitions

In this section, we present the classes of games we consider. Then, we introduce the equilibrium concept that we use for each of them, as well as our behavioral conformity concepts. To simplify the presentation, we assume that there is a single crowding type. The (straightforward) extension to an arbitrary number of crowding types is discussed in Subsection 2.4 and in the remarks following the proof of Theorem 1.

2.1 Pregames, Induced Anonymous Games and Induced Games

We focus on games and anonymous games that are derived from pregames as in Wooders, Cartwright, and Selten (2006). In all cases, we assume that each player's preferences depend on her action and on the distribution of actions chosen by all players, which corresponds to the global interaction property of Wooders, Cartwright, and Selten (2006).[2] Throughout the paper, A denotes a finite set of actions, $\mathcal{M}$ denotes the set of probability measures on A and $\mathcal{U}$ the set of continuous functions $u : A \times \mathcal{M} \to \mathbb{R}$. The space $\mathcal{U}$ is endowed with the sup norm, which makes it a complete, separable metric space. Since A is finite, $\mathcal{M}$ can be identified with the standard unit simplex in $\mathbb{R}^{|A|}$, and so we endow it with the sup norm. Furthermore, we sometimes represent a probability measure $\mu \in \mathcal{M}$ by the vector $\mu = (\mu_1, \ldots, \mu_{|A|})$, where $A = \{a_1, \ldots, a_{|A|}\}$ and $\mu_i = \mu(\{a_i\})$ for all $1 \leq i \leq |A|$.

A pregame consists of a particular description of the primitive elements of games. Indeed, it is defined by an action space, by an attribute space and by a preference function, associating a payoff function to every attribute; hence, in order to obtain a game from a pregame, one has simply to associate an attribute to every player. Formally, a *pregame* G is described by a bounded, convex, Borel measurable attribute space $\Omega \subseteq \mathbb{R}^L$, $L \geq 1$, a finite action space A, and a Borel measurable preference function $U : \Omega \to \mathcal{U}$.[3] We denote such pregame by $G = (\Omega, U, A)$. Note that the measurability assumption on U generalizes the continuity in the attributes assumption of Wooders, Cartwright, and Selten (2006).

A class of games that can be obtained from pregames is that of anonymous games. The distinctive feature of an anonymous game is that there is no explicit reference to the players who will play the game; in fact, one is given only a statistical image of their preferences through a distribution on $\mathcal{U}$. A pregame induces an anonymous game simply by associating to it a probability measure λ defined on the measurable

[2] See Section 7 for the details.

[3] Throughout the paper, measurable will be understood to indicate Borel measurable.

subsets of Ω. This measure λ and U define a measure $\lambda \circ U^{-1}$ on $\mathcal{U}$ by

$$\lambda \circ U^{-1}(B) = \lambda(\{\omega \in \Omega : U(\omega) \in B\}) \tag{1}$$

for all measurable subsets B of $\mathcal{U}$. Thus, an *induced anonymous game* G is described by a probability measure λ on Ω and by a pregame $\tilde{G} = (\Omega, U, A)$, and so we denote it by $G = (\lambda, \Omega, U, A)$. The measure λ describes the demographics of the pregame $\tilde{G} = (\Omega, U, A)$, since roughly, for all measurable subsets C of Ω, $\lambda(C)$ is the fraction of the population with an attribute in C. Similarly, $\lambda \circ U^{-1}(B)$ is the fraction of the population with a payoff function in $B \subseteq \mathcal{U}$.

In contrast, a game makes explicit the set of players. As above, we can obtain a game from a given pregame. Since the pregame already lists the set of actions and the preferences associated with each attribute, a game induced by a pregame (an induced game, for short) is defined by any function assigning an attribute to every player. Formally, an *induced game* is described by a set T of players, a measure φ on T describing the relative weight of players, a measurable attribute function $\alpha : T \to \Omega$ assigning an attribute to each player and a pregame $\tilde{G} = (\Omega, U, A)$. Thus, we represent such a game by $G = (T, \varphi, \alpha, \Omega, U, A)$. Clearly, these elements allow us to define a game in the usual way, by listing its (measure space of) players (T, φ), their action space A, and their payoff function, here defined by the measurable function $U \circ \alpha$ that assigns payoff functions to players.

Both induced anonymous games and induced games can be classified according to the properties of their defining elements. A *binary* induced anonymous game (resp. induced game) is an induced anonymous game (resp. induced game) satisfying $|A| = 2$. An induced anonymous game G is *dispersed* if λ is absolutely continuous with respect to the Lebesgue measure ℓ. This condition means that $\lambda(B) = 0$ for all measurable sets B such that $\ell(B) = 0$.[4] In particular, $\lambda(\{\omega\}) = 0$ for all $\omega \in \Omega$, and so λ is nonatomic.

Regarding induced games, we use both the measure space of players and the attribute function to classify them. An induced game $G = (T, \varphi, \alpha, \Omega, U, A)$ is *injective* if α is an injective function and it is *dispersed* if the probability measure $\varphi \circ \alpha^{-1}$ is absolutely continuous with respect to the Lebesgue measure. An *induced game with a continuum of players* is an induced game $G = (T, \varphi, \alpha, \Omega, U, A)$ satisfying $T = [0, 1]$ and $\varphi = \ell$. That is, the set of players is the unit interval endowed with the Lebesgue measure.

Similarly, for all $n \in \mathbb{N}$, an *induced game with n players* is an induced game $G = (T, \varphi, \alpha, \Omega, U, A)$ where the set of players is a finite set with n elements endowed with the uniform measure. Formally, let $T_n = \{1, \ldots, n\}$ and ν_n be the uniform measure on T_n (i.e., ν_n satisfies $\nu_n(\{t\}) = 1/n$ for all $t \in T_n$). Then, G is an induced game with n players if $T = T_n$ and $\varphi = \nu_n$.

2.2 Equilibrium Concepts

We start by defining strategies for induced anonymous games and for induced games. For induced games, a strategy is a (measurable) function from players into actions (e.g., Schmeidler (1973)). Regarding induced anonymous games, we also define strategies as a function into actions, but this time defined on the attribute space. Essentially, it is as if players were named according to its attribute.[5]

Thus, a *strategy in an induced anonymous game* $G = (\lambda, \Omega, U, A)$ is a measurable

[4] Note that $\ell(\Omega) > 0$ since $\lambda(\Omega) = 1$. Hence, for example, λ defined by $\lambda(B) = \ell(B)/\ell(\Omega)$ for all measurable subsets B of Ω is a probability measure on Ω which is absolutely continuous with respect to the Lebesgue measure.

[5] Hence, it corresponds to a symmetric equilibrium (see Mas-Colell (1984)).

function $f : \Omega \to A$. A *strategy in an induced game* $G = (T, \varphi, \alpha, \Omega, U, A)$ is a measurable function $f : T \to A$. Of course, if T is finite, then the measurability requirement is trivially satisfied.[6]

The definition of equilibrium for induced games depends on whether an induced game has a continuum of players or not. The main difference lies in the fact that if there is a continuum of players, the action of a single one has no impact on the distribution of actions. Therefore, a strategy is a Nash equilibrium if all players are choosing an optimal action given the distribution induced by it on the set of actions. Formally, a *Nash equilibrium of an induced game* $G = ([0,1], \ell, \alpha, \Omega, U, A)$ *with a continuum of players* is a strategy f such that

$$U_{\alpha(t)}(f(t), \ell \circ f^{-1}) \geq U_{\alpha(t)}(a, \ell \circ f^{-1}) \tag{2}$$

for all $a \in A$ and all $t \in [0,1]$.

Note that a dispersed induced anonymous game is like an induced game with a continuum of players under the interpretation that the set of players is Ω. In fact, both $([0,1], \ell)$ and (Ω, λ) are nonatomic measure spaces. Therefore, the definition of a Nash equilibrium for induced anonymous games follows the one given above for induced games. A strategy f in a dispersed induced anonymous game $G = (\lambda, \Omega, U, A)$ is a *Nash equilibrium of* G if

$$U_\omega(f(\omega), \lambda \circ f^{-1}) \geq U_\omega(a, \lambda \circ f^{-1}) \tag{3}$$

for all $\omega \in \Omega$ and $a \in A$.

[6]If $A = \{a_1, \ldots, a_m\}$ has m elements, we can represent it by the standard unit basis $\{e_1, \ldots, e_m\}$ of $\mathbb{R}^m$ and represent a strategy $f : T \to A$ by a function $g : T \to \{e_1, \ldots, e_m\}$ defined by $g(t) = e_i$ if and only if $f(t) = a_i$. Then, regarding $\varphi \circ f^{-1}$ as a vector in $\mathbb{R}^m$, it follows that $\varphi \circ f^{-1} = \int_T g \mathrm{d}\varphi$ (see Rath (1995, Lemma 1)). With this remark in mind, it is appropriate to describe the class of games we consider as games where each player's payoff depends only on his choice and on the average choice of the others.

In contrast to games with a continuum of players, each player's choice affects the distribution of actions in a game with a finite number of players. Given a strategy f in an induced game G with n players, $a \in A$ and $\bar{t} \in T$, let $f \setminus_{\bar{t}} a$ be defined by:

$$f \setminus_{\bar{t}} a(t) = \begin{cases} a & \text{if } t = \bar{t}, \\ f(t) & \text{if } t \neq \bar{t}. \end{cases} \tag{4}$$

In words, $f \setminus_{\bar{t}} a$ is the strategy that results if player $\bar{t}$ changes his action from $f(\bar{t})$ to a, which in turn changes the distribution on A from $\nu_n \circ f^{-1}$ to $\nu_n \circ (f \setminus_{\bar{t}} a)^{-1}$.

Let $\varepsilon \geq 0$. A strategy f is an ε – *equilibrium of an induced game* $G = (T_n, \nu_n, \alpha, \Omega, U, A)$ *with* n *players* if

$$U_{\alpha(t)}(f(t), \nu_n \circ f^{-1}) \geq U_{\alpha(t)}(a, \nu_n \circ (f \setminus_t a)^{-1}) - \varepsilon$$

for all $t \in T_n$ and all $a \in A$.

2.3 Behavioral Conformity

The notion of behavioral conformity we use is the one considered by Wooders, Cartwright, and Selten (2006) in their Theorem 3. For every strategy f, let $A(f) = \{a \in A : f^{-1}(a) \neq \emptyset\}$ be the set of actions that are played by some player. A strategy f in an induced game G *displays behavioral conformity* if there exists a partition of Ω into $|A(f)|$ convex subsets $\{C_a\}_{a \in A(f)}$ such that $\alpha^{-1}(\text{int}(C_a)) \subseteq f^{-1}(a)$ for all $a \in A(f)$. That is, f displays behavioral conformity if $\alpha(t) \in \text{int}(C_a)$ implies $f(t) = a$ for all $t \in T$ and $a \in A(f)$.

This notion of behavioral conformity can be strengthened by requiring that all players in the same society play the same action. A strategy f in an induced game G *displays strong behavioral conformity* if there exists a partition of Ω into $|A(f)|$ convex subsets $\{C_a\}_{a \in A(f)}$ such that $f^{-1}(a) = \alpha^{-1}(C_a)$ for all $a \in A(f)$.

This notion can also be defined for induced anonymous games as follows. A strategy f in a induced anonymous game G *displays strong behavioral conformity* if there exists a partition of Ω into $|A(f)|$ convex subsets $\{C_a\}_{a\in A(f)}$ such that $f^{-1}(a) = C_a$ for all $a \in A(f)$.

2.4 Crowding Types

In this subsection, we consider the case where an attribute can be decomposed into a crowding type and a taste type, as in Wooders, Cartwright, and Selten (2006). As in their paper, assume the attribute space equals $C \times \Omega$, where $C = \{c_1, \dots, c_m\}$ is interpreted as the set of crowding types and Ω as the set of taste types.

In the context, players care about the distribution of actions for every crowding type, and so payoff functions are continuous functions $v : A \times \mathcal{M}^m \to \mathbb{R}$. Letting $\mathcal{V}$ be the space of all such functions, a pregame is now defined by the attribute space $C \times \Omega$, by a measurable preference function $U : C \times \Omega \to \mathcal{V}$ and by the action space A.

As before, a pregame and a probability measure λ on $C \times \Omega$ defines an induced anonymous game. Clearly, this measure can be decomposed into m measures $(\lambda_1, \dots, \lambda_m)$ such that λ_i is a measure on $\{c_i\} \times \Omega$. A strategy is now a measurable function $f : C \times \Omega \to A$, which again can be written as $(f_1, \dots, f_m)$ with $f_i : \{c_i\} \times \Omega \to A$. Finally, a strategy f is a Nash equilibrium if $U_{(c,\omega)}(f(c,\omega), \lambda_1 \circ f_1^{-1}, \dots, \lambda_1 \circ f_1^{-1}) \geq U_{(c,\omega)}(a, \lambda_1 \circ f_1^{-1}, \dots, \lambda_1 \circ f_1^{-1})$ for all $(c, \omega) \in C \times \Omega$ and $a \in A$. Clearly, a similar definition can be presented for induced games, both with a continuum and a finite number of players.

3 Quasi-Intermediate Preferences

In this section, we show that if players' preferences in a dispersed induced anonymous game satisfy some convexity properties, then every distribution over the set of actions induced by a Nash equilibrium can be obtained by a Nash equilibrium displaying strong behavioral conformity. Thus, every statistical image of the play of such a game — provided by the distribution over the actions — can be understood as a consequence of (strong) behavioral conformity by the players.

The notion of convexity of preferences needed for this result is (a generalization of) that of intermediate preferences, introduced by Grandmont (1978).

This notion is defined in the following way: The preference u *is between* u_1 *and* u_2 if for all $\mu \in \mathcal{M}$, a and $\bar{a} \in A$,

1. $u_1(a, \mu) \geq u_1(\bar{a}, \mu)$ and $u_2(a, \mu) \geq u_2(\bar{a}, \mu)$ implies $u(a, \mu) \geq u(\bar{a}, \mu)$;
2. $u_1(a, \mu) > u_1(\bar{a}, \mu)$ and $u_2(a, \mu) > u_2(\bar{a}, \mu)$ implies $u(a, \mu) > u(\bar{a}, \mu)$;
3. $[u_1(a, \mu) = u_1(\bar{a}, \mu)$ and $u_2(a, \mu) > u_2(\bar{a}, \mu)]$ or $[u_1(a, \mu) > u_1(\bar{a}, \mu)$ and $u_2(a, \mu) = u_2(\bar{a}, \mu)]$ implies $u(a, \mu) > u(\bar{a}, \mu)$.

Then, the family of preferences $U(\Omega)$ is *quasi-intermediate* if for all $\omega, \bar{\omega} \in \Omega$ and $\gamma \in (0, 1)$, $U_{\tilde{\omega}}$ is between U_ω and $U_{\bar{\omega}}$ where $\tilde{\omega} = \gamma\omega + (1 - \gamma)\bar{\omega}$.

Note that this definition generalizes Grandmont's notion of intermediate preferences, since it does not require the set $\{\omega \in \Omega : U_\omega(a, \mu) \geq U_\omega(\bar{a}, \mu)\}$ to be closed in Ω, for every a and $\bar{a}$ in A and all $\mu \in \mathcal{M}$. Furthermore, it follows from Example 3 in Grandmont (1978) that it also generalizes the linearity in the taste types assumption of Wooders, Cartwright, and Selten (2006).

The following theorem is our behavioral conformity result for this class of preferences in dispersed induced anonymous games.

Theorem 1 *Let $G = (\lambda, \Omega, U, A)$ be a dispersed induced anonymous game such that $U(\Omega)$ is quasi-intermediate. Then, for every Nash equilibrium f of G, there exists a Nash equilibrium h of G that displays strong behavioral conformity and satisfies $\lambda \circ h^{-1} = \lambda \circ f^{-1}$. In particular, there exists a Nash equilibrium that displays strong behavioral conformity.*

Theorem 1 shows that if players' preferences in a dispersed induced anonymous game are quasi-intermediate, then every distribution induced by a Nash equilibrium in A can also be induced by a Nash equilibrium displaying behavioral conformity. The existence of a Nash equilibrium for this class of games (Schmeidler (1973); see also Khan and Sun (2002, Theorem 2)), then implies the existence of a Nash equilibrium that displays strong behavioral conformity.[7]

The idea of the proof is illustrated by the following examples. The first example is of a trivial induced anonymous game in which all players are indifferent between all actions: Let $\Omega = [0,1]$, $\lambda = \ell$, $A = \{a_1, a_2\}$ and $U_\omega(a, \mu) = 0$ for all $\omega \in \Omega$, $a \in A$ and $\mu \in \mathcal{M}$. In this induced anonymous game, every strategy is a Nash equilibrium, and so, one easily constructs a Nash equilibrium f of G that does not display strong behavioral conformity. For example,

$$f(\omega) = \begin{cases} a_1 & \text{if } \omega \in [0, 1/3] \cup [2/3, 1], \\ a_2 & \text{otherwise.} \end{cases} \tag{5}$$

The distribution induced by f on A is $\ell \circ f^{-1}(\{a_1\}) = 2/3$ and $\ell \circ f^{-1}(\{a_2\}) = 1/3$. This distribution can be easily obtained by a Nash equilibrium h displaying strong

[7]Note that Theorem 2 in Khan and Sun (2002) guarantees the existence of a strategy such that almost all players best-reply. However, we can change it in a set of measure zero if necessary in order to guarantee that all players best-reply.

behavioral conformity. Indeed, let

$$h(\omega) = \begin{cases} a_1 & \text{if } \omega \in [0, 2/3], \\ a_2 & \text{otherwise.} \end{cases} \tag{6}$$

More generally, in order to deal with the case of several actions being indifferent for all players, we use the following lemma.

Lemma 1 *Let λ be absolutely continuous with respect to the Lebesgue measure, $C \subseteq \mathbb{R}^L$ be a bounded, measurable, convex set and $c_1, c_2, \ldots, c_k \in \mathbb{R}$ be such that $c_i \geq 0$ for all $1 \leq i \leq k$ and $\sum_{i=1}^{k} c_i = \lambda(C)$.*

Then, there exists a convex partition $\{C_i\}_{i=1}^{k}$ of C satisfying $\lambda(C_i) = c_i$ for all i.

We can use Lemma 1 in the above example by considering $C = \Omega$, $L = 1$, $c_1 = \ell \circ f^{-1}(\{a_1\}) = 2/3$ and $c_2 = \ell \circ f^{-1}(\{a_2\}) = 1/3$.

The second example we consider is an induced anonymous game in which players are inhabitants of a city with two bridges. Bridge 1 has a better view than that of bridge 2, and players' preferences depend on this fact and on how crowded each bridge is. For concreteness, let $\Omega = [0, 1] \times [0, 1]$, $\lambda = \ell$ and $A = \{b_1, b_2\}$. In order to define players' preferences, let $\mu_1 = \mu(\{b_1\})$ for all $\mu \in \mathcal{M}$ and $u : A \times \mathcal{M} \to \mathbb{R}^2$ be defined by

$$u(a, \mu) = \begin{cases} (1 - \mu_1, 1) & \text{if } a = b_1, \\ (\mu_1, 0) & \text{if } a = b_2. \end{cases} \tag{7}$$

and $U_\omega(a, \mu) = \omega \cdot u(a, \mu)$. The first component of u reflects the level of congestion of each bridge: if μ_1 is higher than $1/2$, it means that bridge 1 is more congested than bridge 2, and so the first component of u is higher for bridge 2 than for bridge 1. The second component simply expresses that bridge 1 has a better view than bridge 2. Finally, ω measures how much weight a player attributes to the view seen from the bridge and to its level of congestion. For example, player $\omega = (1, 0)$ cares only about the level of congestion, while player $\omega = (0, 1)$ cares only about the view seen from each bridge.

A Nash equilibrium of this induced anonymous game is

$$f(\omega) = \begin{cases} b_1 & \text{if } \omega_2 > \omega_1/2, \\ b_1 & \text{if } \omega_2 = \omega_1/2 \text{ and } \omega_1 \in [0, 1/3], \\ b_1 & \text{if } \omega_2 = \omega_1/2 \text{ and } \omega_1 \in [2/3, 1], \\ b_2 & \text{otherwise.} \end{cases} \tag{8}$$

In order to show that f is in fact a Nash equilibrium, we introduce the following notation. Let $BR_\omega(\tau) = \{a \in A : U_\omega(a, \tau) \geq U_\omega(\tilde{a}, \tau) \text{ for all } \tilde{a} \in A\}$ denote the set of best-replies of player ω when the distribution of actions is τ. Since the distribution induced by f on A is $\ell \circ f^{-1}(\{b_1\}) = 3/4$, one easily sees that

$$BR_\omega(\lambda \circ f^{-1}) = \begin{cases} \{b_1\} & \text{if } \omega_2 > \omega_1/2, \\ \{b_1, b_2\} & \text{if } \omega_2 = \omega_1/2, \\ \{b_2\} & \text{otherwise.} \end{cases} \tag{9}$$

Hence, all players are playing a best-reply.

However, f does not display strong behavioral conformity. Here, as it was in the case above, the failure of this property depends on the way f is specified in the set of players who are indifferent between the two actions. In this example, we can allocate all the indifferent players to bridge 1 since the set of those players has measure zero and so will not change the distribution over actions. Hence, the strategy h defined by

$$h(\omega) = \begin{cases} b_1 & \text{if } \omega_2 \geq \omega_1/2, \\ b_2 & \text{otherwise.} \end{cases} \tag{10}$$

is a Nash equilibrium of G that displays strong behavioral conformity and induces the same distribution on A as f.

In general, that this procedure of allocating all the indifferent players to a particular action can be used is guaranteed by the following lemma. It shows that if not all players are indifferent between two actions, the set of those that are indifferent is contained in a hyperplane, which has Lebesgue measure zero (and so measure zero for all measures that are absolutely continuous with respect to it).

Lemma 2 *Let $G = (\Omega, U, A)$ be a pregame such that $U(\Omega)$ is quasi-intermediate, $\mu \in \mathcal{M}$ and a and $\bar{a}$ be such that there exists $\omega \in \Omega$ satisfying $U_\omega(a, \mu) \neq U_\omega(\bar{a}, \mu)$.*

Then, there exist $q_{a,\bar{a}} \in \mathbb{R}^L$ and $c_{a,\bar{a}} \in \mathbb{R}$ such that $q_{a,\bar{a}} \neq 0$ and

$$\{\omega \in \Omega : U_\omega(a, \mu) = U_\omega(\bar{a}, \mu)\} \subseteq \{\omega \in \Omega : q_{a,\bar{a}} \cdot \omega = c_{a,\bar{a}}\}.$$

In the above example, the set $\{\omega \in \Omega : U_\omega(b_1, \ell \circ f^{-1}) = U_\omega(b_2, \ell \circ f^{-1})\}$ is contained in $\{\omega \in \Omega : (-1, 1/2) \cdot \omega = 0\}$.

We turn to the proof of Theorem 1.

Proof of Theorem 1. Let f be a Nash equilibrium of G. For convenience, let $\tau = \lambda \circ f^{-1}$. By definition, we have that $\tau(\{a\}) = \lambda(\{\omega \in \Omega : f(\omega) = a\})$. We will use f to define a strategy h displaying strong behavioral conformity such that $\lambda \circ h^{-1} = \tau$ and $h(\omega)$ solves $\max_{a \in A} U_\omega(a, \tau)$. Clearly, the two last properties imply that h is a Nash equilibrium.

Define the following equivalence relation in $A(f)$: $a \sim a'$ if $U_\omega(a, \tau) = U_\omega(a', \tau)$ for all $\omega \in \Omega$. Thus, action a is equivalent to a' under $\sim$ if all players are indifferent between a and a' when the distribution is τ. Let $A_i \subseteq A(f)$ be the set of actions $a \in A(f)$ for which there is $a' \in A(f)$, $a' \neq a$, such that $a \sim a'$ and, for every $a \in A(f)$, let $[a]$ denote the equivalence class of action a. Thus, A_i is the set of actions $a \in A(f)$ such that $[a]$ is not a singleton, or equivalently, $A(f) \setminus A_i$ is the set of actions $a \in A(f)$ such that $[a] = \{a\}$.

Enumerate A_i and $A(f) \setminus A_i$ and write $A(f) \setminus A_i = \{a_1, \ldots, a_k\}$ and $A_i = \{a_{k+1}, \ldots, a_M\}$.

For all $1 \leq i \leq k$, define

$$C_i = \left(\cap_{j=1}^{M}\{\omega \in \Omega : U_\omega(a_i, \tau) \geq U_\omega(a_j, \tau)\}\right) \cap \\ \cap \left(\cap_{j=1}^{i-1}\{\omega \in \Omega : U_\omega(a_i, \tau) > U_\omega(a_j, \tau)\}\right). \tag{11}$$

Since both $\{\omega \in \Omega : U_\omega(a_i, \tau) \geq U_\omega(a, \tau)\}$ and $\{\omega \in \Omega : U_\omega(a_i, \tau) > U_\omega(a, \tau)\}$ are convex for all $a \in A$, then C_i is convex. Let

$$B_i = \cap_{j=1}^{M}\{\omega \in \Omega : U_\omega(a_i, \tau) \geq U_\omega(a_j, \tau)\} \tag{12}$$

and

$$S_i = \cap_{j \neq i}\{\omega \in \Omega : U_\omega(a_i, \tau) > U_\omega(a_j, \tau)\}. \tag{13}$$

The set B_i (resp. S_i) is the set of players for whom a_i is a (resp. the unique) best-reply and so $\lambda(S_i) \leq \tau(\{a_i\}) \leq \lambda(B_i)$.

Since $B_i \setminus C_i \subseteq \cup_{j=1}^{i-1}\{\omega \in \Omega : U_\omega(a_i) = U_\omega(a_j)\}$, it follows by Lemma 2 that for all $1 \leq j \leq i-1$ there exist $q_{i,j} \in \mathbb{R}^L$ and $c_{i,j} \in \mathbb{R}$ such that $q_{i,j} \neq 0$ and

$$\{\omega \in \Omega : U_\omega(a_i, \tau) = U_\omega(a_j, \tau)\} \subseteq \{\omega \in \Omega : q_{i,j} \cdot \omega = c_{i,j}\}. \tag{14}$$

Hence, $\lambda(B_i \setminus C_i) = 0$ and so $\tau(\{a_i\}) \leq \lambda(B_i) = \lambda(C_i)$.

Similarly, $C_i \setminus S_i \subseteq \cup_{j=i+1}^{M}\{\omega \in \Omega : U_\omega(a_i) = U_\omega(a_j)\}$, and so $\lambda(C_i) = \lambda(S_i) = \tau(\{a_i\})$, as above.

In conclusion, we have shown that for all $1 \leq i \leq k$, C_i is convex and $\lambda(C_i) = \tau(\{a_i\})$, i.e., for all $a \in A(f) \setminus A_i$, there exists a convex set C_a such that $\lambda(C_a) = \tau(\{a\})$.

For all $k+1 \leq i \leq M$, let

$$E_i = \left(\cap_{j=1}^{M}\{\omega \in \Omega : U_\omega(a_i, \tau) \geq U_\omega(a_j, \tau)\}\right) \cap \\ \cap \left(\cap_{j=1}^{i-1}\{\omega \in \Omega : U_\omega(a_i, \tau) > U_\omega(a_j, \tau)\}\right). \tag{15}$$

Define $D_{k+1} = E_{k+1}$ and, for all $i > k+1$, define

$$D_i = \begin{cases} D_j & \text{if } a_i \sim a_j \text{ and } j < i, \\ E_i & \text{otherwise.} \end{cases} \tag{16}$$

Since E_j is convex for all $k+1 \leq j \leq M$, so is D_i.

Let $\bar{a} \in A_i$ and $i \in \{k+1, \ldots, M\}$ be such that $\bar{a} = a_i$. Define

$$D_{[\bar{a}]} = \cup_{j: a_j \in [\bar{a}]} D_j = D_i \tag{17}$$

since $D_j = D_i$ for all $k+1 \leq j \leq M$ such that $a_j \in [\bar{a}]$. Thus, $D_{[\bar{a}]}$ is convex.

Let

$$B_{[\bar{a}]} = \cap_{a \in A}\{\omega \in \Omega : U_\omega(\bar{a}, \tau) \geq U_\omega(a, \tau)\} \tag{18}$$

and

$$S_{[\bar{a}]} = \cap_{a \notin [\bar{a}]}\{\omega \in \Omega : U_\omega(\bar{a}, \tau) > U_\omega(a, \tau)\}. \tag{19}$$

As above, one has that $\lambda(D_{[\bar{a}]}) = \sum_{a \in [\bar{a}]} \tau(\{a\})$. Indeed, we have that $\lambda(S_{[\bar{a}]}) \leq \tau([\bar{a}]) \leq \lambda(B_{[\bar{a}]})$, both $B_{[\bar{a}]} \setminus D_{[\bar{a}]}$ and $D_{[\bar{a}]} \setminus S_{[\bar{a}]}$ are subsets of $\cup_{a \notin [\bar{a}]}\{\omega \in \Omega : U_\omega(\bar{a}, \tau) = U_\omega(a, \tau)\}$, which has λ – measure zero. Hence,

$$\lambda(D_{[\bar{a}]}) = \lambda(S_{[\bar{a}]}) \leq \tau([\bar{a}]) \leq \lambda(B_{[\bar{a}]}) = \lambda(D_{[\bar{a}]}). \tag{20}$$

Since $D_{[\bar{a}]} \subseteq \mathbb{R}^L$ is bounded, convex and satisfies $\lambda(D_{[\bar{a}]}) = \tau([\bar{a}]) = \sum_{a \in [\bar{a}]} \tau(\{a\})$, it follows by Lemma 1 that there exist disjoint convex subsets $\{C_a\}_{a \in [\bar{a}]}$ of $D_{[\bar{a}]}$ satisfying $\lambda(C_a) = \tau(\{a\})$ for all $a \in [\bar{a}]$ and $D_{[a]} = \cup_{a \in [\bar{a}]} C_a$.

Hence, for all $a \in A(f)$, there exists a convex subset C_a such that $\tau(\{a\}) = \lambda(C_a)$. Furthermore,

$$\Omega = \left(\cup_{a \in A(f) \setminus A_i} C_a\right) \cup \left(\cup_{a \in A_i} D_a\right) = \cup_{a \in A(f)} C_a. \tag{21}$$

and the family $\{C_a\}_{a \in A(f)}$ is disjoint. Thus, define $h : \Omega \to A$ by $h(\omega) = a$ if $\omega \in C_a$. Then h is a Nash equilibrium and induces a partition of Ω into $M = |A(f)| \leq |A|$ convex sets. ∎

We conclude this section with a remark on how to extend Theorem 1 to the case of several crowding types (see Subsection 2.4). Essentially, all it takes is to write $\Omega_i = \{c_i\} \times \Omega$ for all $1 \leq i \leq m$, $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $f = (f_1, \ldots, f_m)$, and then apply the approach used in its proof to each $i = 1, \ldots, m$ separately.

4 Generalized Quasi-Intermediate Preferences

The behavioral conformity result of Theorem 1 can be obtained under weaker restrictions on players' preferences. Essentially, the generalization that we consider dispenses with the property that indifference surfaces are hyperplanes, expressed in the statement of Lemma 2. However, our approach in this case requires that the set of action have only two actions.

Our notion of generalized quasi-intermediate preferences is as follows. A family of preferences $U(\Omega)$ is *generalized quasi-intermediate* if $\{\omega : U_\omega(a, \mu) \geq U_\omega(\bar{a}, \mu)\}$ and $\{\omega : U_\omega(a, \mu) > U_\omega(\bar{a}, \mu)\}$ are convex for all $a, \bar{a} \in A$ and $\mu \in \mathcal{M}$.

This notion generalizes the notion of quasi-intermediate preferences since $U_{\omega_1}(a, \mu) = U_{\omega_1}(\bar{a}, \mu)$ and $U_{\omega_2}(a, \mu) > U_{\omega_2}(\bar{a}, \mu)$ do not necessarily imply that $U_\omega(a, \mu) > U_\omega(\bar{a}, \mu)$ where $\omega = \gamma\omega_1 + (1 - \gamma)\omega_2$, with $\gamma \in (0, 1)$.

Note, however, that if $U(\Omega)$ is generalized quasi-intermediate, then the set $\{\omega \in \Omega : U_\omega(a, \mu) = U_\omega(\bar{a}, \mu)\}$ is convex for all $a, \bar{a} \in A$ and all $\mu \in \mathcal{M}$ since it equals $\{\omega \in \Omega : U_\omega(a, \mu) \geq U_\omega(\bar{a}, \mu)\} \cap \{\omega \in \Omega : U_\omega(\bar{a}, \mu) \geq U_\omega(a, \mu)\}$.

For binary dispersed induced anonymous games, we can obtain the conclusion of Theorem 1 assuming only that preferences are generalized quasi-intermediate.

Theorem 2 *Let $G = (\lambda, \Omega, U, A)$ be a binary dispersed induced anonymous game such that $U(\Omega)$ is generalized quasi-intermediate. Then, for every Nash equilibrium f of G, there exists a Nash equilibrium h of G that displays strong behavioral conformity* *and satisfies $\lambda \circ h^{-1} = \lambda \circ f^{-1}$. In particular, there exists a Nash equilibrium that displays strong behavioral conformity.*

The following example illustrates Theorem 2. It is again an example of a dispersed induced anonymous game where the players are inhabitants of a city with two bridges. In this example, there are three types of people: those who always prefer bridge 1, those who always prefer bridge 2 and those whose preferences depend on how crowded each bridge is.

Formally, for all $i = 1, 2, 3$, let $u_i : A \times \mathcal{M} \to \mathbb{R}$ be defined by

$$u_1(a, \mu) = \begin{cases} 1 & \text{if } a = b_1, \\ 0 & \text{if } a = b_2, \end{cases} \tag{22}$$

$$u_2(a, \mu) = \begin{cases} 0 & \text{if } a = b_1, \\ 1 & \text{if } a = b_2, \end{cases} \tag{23}$$

and

$$u_3(a, \mu) = \begin{cases} 1 - \mu_1 & \text{if } a = b_1, \\ \beta\mu_1 & \text{if } a = b_2. \end{cases} \tag{24}$$

We assume that $\beta \in (1, 3)$, and so players with preferences described by u_3 have a bias toward bridge 2.

Let $\Omega = [0, 1] \times [0, 1]$, $\lambda = \ell$ and $U : \Omega \to \mathcal{U}$ be defined by

$$U(\omega) = \begin{cases} u_1 & \text{if } \omega_2 > 2\omega_1, \\ u_2 & \text{if } \omega_2 < \omega_1/2, \\ u_3 & \text{otherwise.} \end{cases} \tag{25}$$

Clearly, players with preferences described by u_1 (resp. u_2) must play b_1 (resp. b_2) in a Nash equilibrium. Regarding players with preferences described by u_3, they are best-replying if $\mu(\{b_1\}) = 1/(1 + \beta)$, since in this case they are indifferent between the two bridges. Thus, if f is a strategy such that $\ell \circ f^{-1}(\{b_1\}) = 1/(1 + \beta)$, f equals

b_1 in the set $\Omega_1 := \{\omega \in \Omega : \omega_2 > 2\omega_1\}$ and it equals b_2 in $\Omega_2 := \{\omega \in \Omega : \omega_2 < \omega_1/2\}$, then f is a Nash equilibrium.

Given the above, it is easy to construct a Nash equilibrium that does not display strong behavioral conformity. As in Section 3, the failure of this property depends on how a strategy is specified in the set of players who are indifferent between the two actions. Note that, in contrast to the case in Section 3, the set of players who are indifferent between the two actions is no longer a set of measure zero.

Our approach to obtain a Nash equilibrium h displaying behavioral conformity and satisfying $\ell \circ h^{-1}(\{b_1\}) = 1/(1+\beta)$ is as follows. Let $H_1 = \{z \in \mathbb{R}^2 : (1/2, -1) \cdot z = 0\}$ and $H_2 = \{z \in \mathbb{R}^2 : (2, -1) \cdot z = 0\}$; then, H_1 is a hyperplane that separates the set of players that weakly prefer b_1 from the set of players that strictly prefer b_2, while H_2 separates the set of players that weakly prefer b_2 from the set of players that strictly prefer b_1. Note also that

$$\Omega \cap \{z \in \mathbb{R}^2 : (2, -1) \cdot z \geq 0\} = \{\omega \in \Omega : U_\omega(b_1, \ell \circ f^{-1}) \geq U_\omega(b_2, \ell \circ f^{-1})\}. \quad (26)$$

Our goal is to obtain a hyperplane $H = \{z : q^* \cdot z = d^*\}$ such that $\{z : q^* \cdot z \geq d^*\} \subseteq \{z \in \mathbb{R}^2 : (2, -1) \cdot z \geq 0\}$ and $\ell(\Omega \cap \{z : q^* \cdot z \geq d^*\}) = 1/(1+\beta)$. Then, defining

$$h(\omega) = \begin{cases} b_1 & \text{if } q^* \cdot \omega \geq d^*, \\ b_2 & \text{otherwise,} \end{cases} \quad (27)$$

the second property guarantees that $\ell \circ h^{-1}(\{b_1\}) = 1/(1+\beta)$, while the first guarantees that h equals b_1 in Ω_1 and b_2 in Ω_2.

In this example, it is easy to verify that $H = \{z \in \mathbb{R}^2 : ((1+\beta)/2, -1) \cdot z = 0\}$. In general, we use Lemma 3 and the intermediate value theorem to find the desired hyperplane.

Lemma 3 *Let λ be absolutely continuous with respect to the Lebesgue measure. Let $c, d \in \mathbb{R}$ and $p, q, w \in \mathbb{R}^L$ satisfy $p \neq 0$, $q \neq 0$, $p \cdot w = c$ and $q \cdot w = d$. For all $\eta \in [0,1]$, define $q_\eta = \eta p + (1-\eta) q$ and*

$$C_\eta = \Omega \cap \{z \in \mathbb{R}^L : q \cdot z \geq d\} \cap \{z \in \mathbb{R}^L : q_\eta \cdot z \leq q_\eta \cdot w\}.$$

Then, the function $\eta \mapsto \lambda(C_\eta)$ is continuous.

We turn next to the proof of Theorem 2.

Proof of Theorem 2. Let f be a Nash equilibrium of G and $\tau = \lambda \circ f^{-1}$. We can partition Ω in three sets Ω_1, Ω_2 and Ω_3, where:

$$\Omega_1 = \{\omega \in \Omega : U_\omega(a_1, \tau) > U_\omega(a_2, \tau)\}, \tag{28}$$

$$\Omega_2 = \{\omega \in \Omega : U_\omega(a_1, \tau) < U_\omega(a_2, \tau)\} \text{ and} \tag{29}$$

$$\Omega_3 = \{\omega \in \Omega : U_\omega(a_1, \tau) = U_\omega(a_2, \tau)\}. \tag{30}$$

By assumption, all these sets are convex. Also, note that $\lambda(\Omega_1) \leq \tau(\{a_1\}) \leq \lambda(\Omega_1 \cup \Omega_3)$. We will assume that all Ω_i, $i = 1, 2, 3$, have strictly positive measure, the other cases being analogous.

Define $h : \Omega \to A$ as follows:

$$h(\omega) = \begin{cases} a_1 \text{ if } \omega \in \tilde{\Omega}, \\ a_2 \text{ otherwise,} \end{cases} \tag{31}$$

where $\tilde{\Omega}$ satisfies

$$\Omega_1 \subseteq \tilde{\Omega} \subseteq \Omega_1 \cup \Omega_3, \tag{32}$$

$$\lambda(\tilde{\Omega}) = \tau(\{a_1\}) \text{ and} \tag{33}$$

$$\text{both } \tilde{\Omega} \text{ and } \Omega \setminus \tilde{\Omega} \text{ are convex.} \tag{34}$$

Thus, we have partitioned Ω into two convex sets. So, we are left to show that there exists a set $\tilde{\Omega}$ with the above properties.

Since $\Omega_1 \cup \Omega_3$ and Ω_2 are convex and disjoint, there is a hyperplane that separates them: i.e., there are $p \neq 0$ and c such that $p \cdot x \leq c \leq p \cdot y$ for all $x \in \Omega_1 \cup \Omega_3$ and all $y \in \Omega_2$. Let $H_1 = \{z \in \mathbb{R}^L : p \cdot z = c\}$.

Also, we can separate Ω_1 from $\Omega_2 \cup \Omega_3$: there exist $q \neq 0$, $q \in \mathbb{R}^L$ and $d \in \mathbb{R}$ such that $q \cdot x \leq d \leq q \cdot y$ for all $x \in \Omega_1$ and all $y \in \Omega_2 \cup \Omega_3$. Let $H_2 = \{z \in \mathbb{R}^L : q \cdot z = d\}$.

Let $M = \Omega \cap \{z \in \mathbb{R}^L : p \cdot z \leq c\} \cap \{z \in \mathbb{R}^L : q \cdot z \geq d\}$. Thus, $\Omega_3 \subseteq M$. If $H_1 = H_2$, then $\lambda(\Omega_3) = 0$ and we can let $\tilde{\Omega} = \Omega_1$. Hence, we may assume that $H_1 \neq H_2$. We will consider two cases.

Suppose first that H_1 is parallel to H_2. Then, there exists $w \in \mathbb{R}^L$ such that $H_1 + w = H_2$ and $e \in \mathbb{R}$ (simply let $e = p \cdot w + c$) such that $H_2 = \{z \in \mathbb{R}^L : p \cdot z = e\}$. That is, we may assume that $q = p$ and $c \neq d$. Also, $\Omega_3 = \Omega \cap \{z \in \mathbb{R}^L : d \leq p \cdot z \leq c\}$. For all $\eta \in [0, 1]$, consider $C_\eta = \Omega \cap \{z \in \mathbb{R}^L : d \leq p \cdot z \leq \eta c + (1 - \eta)d\}$. Clearly, $\lambda(C_0) = 0$ and $\lambda(C_1) \geq \lambda(\Omega_3)$. Furthermore, by Lemma 3, the function $\eta \mapsto \lambda(C_\eta)$ is continuous and so there is η^* such that $\lambda(C_{\eta^*}) = \tau(\{a_1\}) - \lambda(\Omega_1)$. Define $\tilde{\Omega} = \Omega_1 \cup C_{\eta^*}$. One easily sees that both $\tilde{\Omega}$ and $\Omega \setminus \tilde{\Omega}$ are convex.

Finally, suppose that H_1 is not parallel to H_2. This implies $L \geq 2$ and so, by Lemma 9, they intersect, say at w. For each $\eta \in [0, 1]$, we consider a hyperplane $H_\eta = \{z \in \mathbb{R}^L : q_\eta \cdot z = d_\eta\}$ defined by setting $q_\eta = \eta p + (1 - \eta)q$ and $d_\eta = q_\eta \cdot w$. Then, it follows that $w \in H_\eta$ and that

$$\Omega \cap \{z \in \mathbb{R}^L : q \cdot z \geq d\} \cap \{z \in \mathbb{R}^L : q_\eta \cdot z \leq d_\eta\} \subseteq M.$$

The last claim above can be established as follows: Let $z \in \Omega \cap \{z \in \mathbb{R}^L : q \cdot z \geq d\} \cap \{z \in \mathbb{R}^L : q_\eta \cdot z \leq d_\eta\}$. To show that $z \in M$, it is enough to show that $p \cdot z \leq c$. Since $q_\eta \cdot z \leq d_\eta$, $p \cdot w = c$ and $q \cdot w = d$ (since $w \in H_1 \cap H_2$), then

$$\eta p \cdot z + (1 - \eta) q \cdot z = q_\eta \cdot z \leq d_\eta = \eta p \cdot w + (1 - \eta) q \cdot w = \eta c + (1 - \eta) d.$$

Therefore,

$$\begin{aligned} p \cdot z &= \eta p \cdot z + (1-\eta) q \cdot z + (1-\eta)(p \cdot z - q \cdot z) \\ &\leq \eta c + (1-\eta) d + (1-\eta)(p \cdot z - q \cdot z) \\ &= (1-\eta) p \cdot z + \eta c + (1-\eta)(d - q \cdot z) \\ &\leq (1-\eta) p \cdot z + \eta c. \end{aligned} \tag{35}$$

Hence, $\eta p \cdot z \leq \eta c$ and so $p \cdot z \leq c$ since $\eta > 0$. It follows that $\Omega \cap \{z \in \mathbb{R}^L : q \cdot z \geq d\} \cap \{z \in \mathbb{R}^L : q_\eta \cdot z \leq d_\eta\} \subseteq M$.

Then, define

$$C_\eta = \Omega \cap \{z \in \mathbb{R}^L : q \cdot z \geq d\} \cap \{z \in \mathbb{R}^L : q_\eta \cdot z \leq d_\eta\}.$$

We have that $C_0 = \Omega \cap H_2$ and so $\lambda(C_0) = 0$. Also, $C_1 = M$ and so $\lambda(C_1) = \lambda(\Omega_3) > 0$. Furthermore, by Lemma 3, the function $\eta \mapsto \lambda(C_\eta)$ is continuous and so there is η^* such that $\lambda(C_{\eta^*}) = \tau(\{a_1\}) - \lambda(\Omega_1)$. Define $C = C_{\eta^*} \cap \Omega_3$. Then, $\lambda(C) = \lambda(C_{\eta^*})$. This follows because

$$\begin{aligned} C_{\eta^*} \setminus C &\subseteq (\Omega \setminus \Omega_3) \cap \{z : q \cdot z \geq d\} \cap \{z : p \cdot z \leq c\} \\ &\subseteq \{z : q \cdot z = d\} \cup \{z : p \cdot z = c\}, \end{aligned} \tag{36}$$

and $\lambda(\{z : q \cdot z = d\}) = \lambda(\{z : p \cdot z = c\}) = 0$.

Finally, define $\tilde{\Omega} = \Omega_1 \cup C$. Then, claims 1 and 2 below imply that both $\tilde{\Omega}$ and $\Omega \setminus \tilde{\Omega}$ are convex.

Claim 1 *The set $\tilde{\Omega}$ is convex.*

Proof. Since both Ω_1 and C are convex, it is enough to show that if $x \in \Omega_1$, $y \in C$ and $\alpha \in (0,1)$ then $z = \alpha x + (1-\alpha) y \in \tilde{\Omega}$. By the convexity of $\{\omega : U_\omega(a_1, \tau) \geq U_\omega(a_2, \tau)\}$, it follows that $z \in \Omega_1 \cup \Omega_3$. If $z \in \Omega_1$, then $z \in \tilde{\Omega}$. If $z \in \Omega_3$, then also $z \in \{w : q \cdot w \geq d\}$. Since $x \in \Omega_1$, then $p \cdot x \leq c$ and $q \cdot x \leq d$, and so $q_{\eta^*} \cdot x \leq d_{\eta^*}$. Since $q_{\eta^*} \cdot y \leq d_{\eta^*}$, it follows that $q_{\eta^*} \cdot z \leq d_{\eta^*}$ and so $z \in C \subseteq \tilde{\Omega}$. ■

Claim 2 *The set* $\Omega \setminus \tilde{\Omega}$ *is convex.*

Proof. Note first that

$$\Omega \setminus \tilde{\Omega} = \Omega_2 \cup (\Omega_3 \cap \{z : q \cdot z \geq d\} \cap \{z : q_{\eta^*} \cdot z > d_{\eta^*}\}).$$

This is shown as follows: let $D = \Omega_3 \cap \{z : q \cdot z \geq d\} \cap \{z : q_{\eta^*} \cdot z > d_{\eta^*}\}$. If $z \in \Omega \setminus \tilde{\Omega}$, then $z \in \Omega_2 \cup \Omega_3$. If $z \in \Omega_3$, then $q \cdot z \geq d$, and so $q_{\eta^*} \cdot z > d_{\eta^*}$, since $z \in \Omega \setminus \tilde{\Omega}$. Hence, $z \in D$. Conversely, if $z \in \Omega_2 \cup D$, then either $z \in \Omega_2$ or $z \in D$. In the first case, then $z \notin \Omega_1 \cup \Omega_3$ and so $z \notin \tilde{\Omega}$. In the second case, then $z \notin \Omega_1$ and also $z \notin C$.

Since both Ω_2 and D are convex, it is enough to show that if $x \in \Omega_2$, $y \in D$ and $\alpha \in (0,1)$ then $z = \alpha x + (1-\alpha)y \in \Omega_2 \cup D$. By the convexity of $\{\omega : U_\omega(a_2,\tau) \geq U_\omega(a_1,\tau)\}$, it follows that $z \in \Omega_2 \cup \Omega_3$. If $z \in \Omega_2$, then $z \in \Omega_2 \cup D$. If $z \in \Omega_3$, then also $z \in \{w : q \cdot w \geq d\}$. Since $x \in \Omega_2$, then $p \cdot x \geq c$ and $q \cdot x \geq d$, and so $q_{\eta^*} \cdot x \geq d_{\eta^*}$. Since $q_{\eta^*} \cdot y > d_{\eta^*}$, it follows that $q_{\eta^*} \cdot z > d_{\eta^*}$ and so $z \in D \subseteq \Omega_2 \cup D = \Omega \setminus \tilde{\Omega}$. ■

This completes the proof. ■

5 Behavioral Conformity in Induced Games with a Continuum of Players

So far we have been able to establish behavioral conformity in a strong sense by considering dispersed induced anonymous games. One can think of these games as games in which players are named according to their attribute. Furthermore, only a subset of (at most) measure zero of players have the same attribute, which makes it easier to partition the attribute set into societies.

The difficulty in establishing behavioral conformity in induced games is that a positive measure of players may have the same attribute. Therefore, it may be impossible to have them play the same action and to guarantee, at the same time, that the total measure of those playing that action is equal to a given value.

Nevertheless, it may happen that only a subset of measure zero of players have the same attribute. In this case, we obtain the following existence result as a consequence of Theorem 1.

Corollary 1 *Let $G = ([0,1], \ell, \alpha, \Omega, U, A)$ be a dispersed induced game with a continuum of players such that $U(\Omega)$ is quasi-intermediate. Then, there exists a Nash equilibrium of G displaying strong behavioral conformity.*

Proof. Let $\lambda = \ell \circ \alpha^{-1}$ and let $h : \Omega \to A$ be a Nash equilibrium of $\tilde{G} = (\lambda, \Omega, U, A)$ displaying strong behavioral conformity. Let $\{C_k\}_{k=1}^{K}$ be such that $h^{-1}(a_k) = C_k$, where $\{a_1, \ldots, a_K\} = \{a \in A : h^{-1}(a) \neq \emptyset\}$.

Define $f : [0,1] \to A$ by $f = h \circ \alpha$. Then, $f^{-1}(a_k) = \alpha^{-1}(h^{-1}(a_k)) = \alpha^{-1}(C_k)$ for all $k = 1, \ldots, K$ and so f displays strong behavioral conformity.

Furthermore, f is a Nash equilibrium. Indeed, let $t \in [0,1]$ and $\omega \in \Omega$ be such that $\alpha(t) = \omega$. Since, $\lambda \circ h^{-1} = (\ell \circ \alpha^{-1}) \circ h^{-1} = \ell \circ f^{-1}$ and $U_\omega(h(\omega), \lambda \circ h^{-1}) \geq U_\omega(a, \lambda \circ h^{-1})$ for all $a \in A$, then

$$U_{\alpha(t)}(f(t), \ell \circ f^{-1}) = U_\omega(h(\omega), \lambda \circ h^{-1}) \geq U_\omega(a, \lambda \circ h^{-1}) = U_{\alpha(t)}(a, \ell \circ f^{-1}) \tag{37}$$

for all $a \in A$, and so, f is a Nash equilibrium. ■

When $\ell \circ \alpha^{-1}$ is not absolutely continuous with respect to the Lebesgue measure, there may fail to exist a Nash equilibrium of G displaying strong behavioral conformity. A simple example is obtained by letting $\Omega = [0,2]$, $\alpha(t) = 1$ for all $t \in [0,1]$, $A = \{b_1, b_2\}$ and

$$U_\omega(a, \mu) = \begin{cases} 1 - \mu_1 & \text{if } a = b_1, \\ \omega \mu_1 & \text{if } a = b_2 \end{cases} \tag{38}$$

for all $\omega \in [0,2]$. Note that if f is a Nash equilibrium of G, then $\ell \circ f^{-1}(\{b_1\}) = 1/2$. Hence, it is impossible to partition Ω into two sets C_1 and C_2 in such a way that $f^{-1}(b_1) = \alpha^{-1}(C_1)$. In fact, if $1 \in C_1$, then $f^{-1}(b_1) = [0,1]$ and $\ell \circ f^{-1}(\{b_1\}) = 1 \neq 1/2$, while if $1 \in C_2$, then $f^{-1}(b_1) = \emptyset$ and $\ell \circ f^{-1}(\{b_1\}) = 0 \neq 1/2$.

Nevertheless, in this game there is a Nash equilibrium displaying behavioral conformity. In fact, letting $C_1 = [0,1]$ and $C_2 = (1,2]$, then 1 belongs to neither $\mathrm{int}(C_1)$ nor to $\mathrm{int}(C_2)$, implying that all strategies f satisfying $\ell \circ f^{-1}(\{b_1\}) = 1/2$ are Nash equilibria displaying behavioral conformity.

The above conclusion can be extended beyond this example, as shown by the following result.

Theorem 3 *Let $G = ([0,1], \ell, \alpha, \Omega, U, A)$ be an induced game with a continuum of players such that $U(\Omega)$ is quasi-intermediate. Then, for every Nash equilibrium g of G, there exists a Nash equilibrium f of G that displays behavioral conformity and satisfies $\ell \circ f^{-1} = \ell \circ g^{-1}$. In particular, there exists a Nash equilibrium that displays behavioral conformity.*

Theorem 3 shows that players' preferences being quasi-intermediate is a sufficient condition for the existence of Nash equilibria displaying behavioral conformity in games with a continuum of players. In fact, every Nash equilibrium of such a game can be modified in order to produce another Nash equilibrium that displays behavioral conformity and induces the same distribution over actions. Hence, despite the fact that behavioral conformity is weaker here than in dispersed induced anonymous games with quasi-intermediate preferences, it is still the case that every distribution on A induced by a Nash equilibrium is consistent with behavioral conformity.

The idea of the proof is illustrated by the following examples. The first consists of the example given above. In this example, $C_1 = [0,1] = \{\omega \in \Omega : U_\omega(b_1, \ell \circ f^{-1}) \geq$

$U_\omega(b_2, \ell \circ g^{-1})\} = [0, 1]$ and $C_2 = (1, 2] = \{\omega \in \Omega : U_\omega(b_2, \ell \circ f^{-1}) > U_\omega(b_1, \ell \circ g^{-1})\}$. Hence, one easily concludes that $\text{int}(C_i) \subseteq \{\omega \in \Omega : U_\omega(b_i, \ell \circ g^{-1}) > U_\omega(b_j, \ell \circ g^{-1})\}$ for all $i, j = 1, 2$ and $j \neq i$. Hence, defining $f(t) = a_i$ if $\alpha(t) \in \text{int}(C_i)$, we only need to divide the remaining players (i.e., those in $[0, 1] \setminus \alpha^{-1}(\cup_{i=1}^2 \text{int}(C_i))$) between the two actions in such a way that $\ell \circ f^{-1} = 1/2$.

The conclusion that $\text{int}(C_i) \subseteq \{\omega \in \Omega : U_\omega(b_i, \ell \circ g^{-1}) > U_\omega(b_j, \ell \circ g^{-1})\}$ for all $i, j = 1, 2$ and $j \neq i$ can be established in general using Lemma 4 below. It states that for any pair of actions such that, for some attributes one action is preferred, while for others they are indifferent, we can easily determine whether a player with an attribute in the interior of Ω strictly prefers one of them or whether he is indifferent between them. In fact, we simply need to determine the position of his attribute relative to a hyperplane. Despite the fact that we assume that preferences are merely quasi-intermediate and the attribute space is merely convex, this lemma can be proven using the same argument as in the proof of the Proposition in Grandmont (1978). However, since we do not assume that Ω is open, our conclusions apply only to points in the interior of Ω.

Lemma 4 *Let $a, \bar{a} \in A$, $\mu \in \mathcal{M}$ and suppose that $\Omega \subseteq \mathbb{R}^L$ is convex, $U(\Omega)$ is quasi-intermediate and there exist $\tilde{\omega}, \hat{\omega} \in \Omega$ such that $U_{\tilde{\omega}}(a, \mu) > U_{\tilde{\omega}}(\bar{a}, \mu)$ and $U_{\hat{\omega}}(a, \mu) = U_{\hat{\omega}}(\bar{a}, \mu)$.*

Then, there exist $q \in \mathbb{R}^L$, $q \neq 0$, and $c \in \mathbb{R}$ such that for all $\omega \in \text{int}(\Omega)$, $q \cdot \omega > c$ when $U_\omega(a, \mu) > U_\omega(\bar{a}, \mu)$, $q \cdot \omega = c$ when $U_\omega(a, \mu) = U_\omega(\bar{a}, \mu)$, and $q \cdot \omega < c$ when $U_\omega(a, \mu) < U_\omega(\bar{a}, \mu)$.

The second example we consider is again of a trivial induced game in which all players are indifferent between all actions. As in Section 3, the pregame is defined by $\Omega = [0, 1]$, $A = \{a_1, a_2\}$ and $U_\omega(a, \mu) = 0$ for all $\omega \in \Omega$, $a \in A$ and $\mu \in \mathcal{M}$. Finally, define

$$\alpha(t) = \begin{cases} 0 & \text{if } t \in [0, 1/4], \\ 8/10 & \text{if } t \in (1/4, 1/2), \\ 9/10 & \text{if } t \in [1/2, 3/4) \text{ and} \\ 1 & \text{if } t \in [3/4, 1] \end{cases} \tag{39}$$

Suppose that g is a Nash equilibrium of this game such that $\ell \circ g^{-1}(\{a_1\}) = 2/3$. Then, we can define $C_1 = [0, 9/10]$ and $C_2 = (9/10, 1]$. The important property guaranteed by this choice is that $\ell \circ \alpha^{-1}(\text{int}(C_1)) = \ell((1/4, 1/2)) = 1/4 < \ell \circ g^{-1}(\{a_1\})$ and $\ell \circ \alpha^{-1}(\text{int}(C_2)) = \ell(\emptyset) = 0 < \ell \circ g^{-1}(\{a_2\})$. Hence, defining $f(t) = a_i$ if $\alpha(t) \in \text{int}(C_i)$ for all $i = 1, 2$, we still leave f undefined in the set $[0, 1/4] \cup [1/2, 1]$. This set of players can then be divided in a way that those in it playing a_1 have a measure of $5/12 = \ell \circ g^{-1}(\{a_1\}) - \ell \circ \alpha^{-1}(\text{int}(C_1))$. For example, we can define $f(t) = a_1$ for all $t \in [7/12, 1]$ and $f(t) = a_2$ for all $t \in [0, 1/4] \cup [1/2, 7/12]$.

More generally, we use the following lemma when there are several actions which are indifferent for all players. Lemma 5 is similar to Lemma 1 and considers the case when λ may fail to be absolutely continuous with respect to the Lebesgue measure, and, in fact, may have atoms.

Lemma 5 *Let $C \subseteq \mathbb{R}^L$ be a bounded, measurable, convex set and $c_1, c_2, \ldots, c_k \in \mathbb{R}$ be such that $c_i \geq 0$ for all $1 \leq i \leq k$ and $\lambda(\text{int}(C)) \leq \sum_{i=1}^{k} c_i \leq \lambda(C)$.*

Then, there exists a convex partition $\{C_i\}_{i=1}^{k}$ of C satisfying $\lambda(\text{int}(C_i)) \leq c_i$ for all $i = 1, \ldots, k$.

We can use Lemma 5 in the above example by considering $C = \Omega$, $L = 1$, $c_1 = \ell \circ g^{-1}(\{a_1\}) = 2/3$ and $c_2 = \ell \circ g^{-1}(\{a_2\}) = 1/3$.

We now turn to the proof of Theorem 3.

Proof of Theorem 3. Let g be a Nash equilibrium of G. Let $\tau = \ell \circ g^{-1}$ and consider the partition $\{C_a\}_{a\in A(g)\setminus A_i} \cup \{D_{[a]}\}_{a\in A_i}$ of Ω defined in the proof of Theorem 1.

We start by establishing a property of points in the interior of C_a and $D_{[a]}$, for $a \in A(g)\setminus A_i$ and $a \in A_i$ respectively.

Let $a \in A(g)\setminus A_i$ and $\omega \in \text{int}(C_a)$. Then, $U_\omega(a,\tau) > U_\omega(\bar{a},\tau)$ for all $\bar{a} \in A$, $\bar{a} \neq a$, i.e., $\text{int}(C_a) \subseteq S_a$. In fact, suppose, in order to reach a contradiction, that $U_\omega(a_i,\tau) = U_\omega(a_j,\tau)$ for some $j > i$. Since $a_i \not\sim a_j$, then there exists $\hat{\omega} \in \Omega$ such that either $U_{\hat{\omega}}(a_i,\tau) > U_{\hat{\omega}}(a_j,\tau)$ or $U_{\hat{\omega}}(a_j,\tau) > U_{\hat{\omega}}(a_i,\tau)$. In both cases, it follows from Lemma 4 that there exist $q \in \mathbb{R}^L$, $q \neq 0$ and $c \in \mathbb{R}$ such that for all $\bar{\omega} \in \text{int}(\Omega)$, $q \cdot \bar{\omega} > c$ if $U_{\bar{\omega}}(a_i,\tau) > U_{\bar{\omega}}(a_j,\tau)$, $q \cdot \bar{\omega} = c$ if $U_{\bar{\omega}}(a_i,\tau) = U_{\bar{\omega}}(a_j,\tau)$ and $q \cdot \bar{\omega} < c$ if $U_{\bar{\omega}}(a_i,\tau) < U_{\bar{\omega}}(a_j,\tau)$. Since $\omega \in \text{int}(C_i) \subseteq \text{int}(\Omega)$, then $q \cdot \omega = c$ and so there exists $\bar{\omega} \in \text{int}(C_i)$ such that $q \cdot \bar{\omega} < c$. This implies that $U_{\bar{\omega}}(a_i,\tau) < U_{\bar{\omega}}(a_j,\tau)$, a contradiction to $\bar{\omega} \in C_i$.

In this case, define

$$f(t) = a \quad \text{if} \quad \alpha(t) \in \text{int}(C_a). \tag{40}$$

Since $\text{int}(C_a) \subseteq S_a$, then $\ell \circ \alpha^{-1}(\text{int}(C_a)) \leq \ell \circ g^{-1}(\{a\})$.

We now turn to actions in A_i. Let $\bar{a} \in A_i$. A similar argument to the one above shows that if $\omega \in \text{int}(D_{[\bar{a}]})$, then $\omega \in S_{[\bar{a}]}$. Hence, defining

$$c_a = \ell(\{t \in [0,1] : \alpha(t) \in S_{[\bar{a}]} \text{ and } g(t) = a\}) \tag{41}$$

for all $a \in [\bar{a}]$, it follows that

$$\ell \circ \alpha^{-1}(\text{int}(D_{[\bar{a}]})) \leq \ell \circ \alpha^{-1}(S_{[\bar{a}]}) = \sum_{a\in[\bar{a}]} c_a \leq \ell \circ \alpha^{-1}(D_{[\bar{a}]}). \tag{42}$$

By Lemma 5, there exists a convex partition $\{C_a\}_{a\in[\bar{a}]}$ of $D_{[\bar{a}]}$ such that $\ell \circ \alpha^{-1}(\text{int}(C_a)) \leq c_a$. Define, again, $f(t) = a$ if $\alpha(t) \in \text{int}(C_a)$ and note that $\ell \circ \alpha^{-1}(\text{int}(C_a)) \leq \ell \circ g^{-1}(\{a\})$ since $c_a \leq \ell \circ g^{-1}(\{a\})$.

Since $\Omega = \left(\cup_{a\in A(g)\setminus A_i} C_a\right) \cup \left(\cup_{a\in A_i} D_{[a]}\right) = \cup_{a\in A(g)} C_a$, C_a is convex and $f(t) = a$ if $\alpha(t) \in \mathrm{int}(C_a)$ for all $a \in A(g)$, $\{C_a\}_{a\in A(g)}$ is our desired convex partition.

To complete the proof, we need to specify f on $\Omega \setminus \cup_{a\in A(g)}\mathrm{int}(C_a)$ so that $\ell \circ f^{-1} = \ell \circ g^{-1}$. This will follow from the Bollobás-Varopoulos Theorem (see Khan and Sun (1995, Theorem 4)). Indeed, define $Z = [0,1] \setminus \alpha^{-1}(\cup_a \mathrm{int}(C_a))$, $T_a = \cap_{a'\in A}\{t \in Z : U_{\alpha(t)}(a,\tau) \geq U_{\alpha(t)}(a',\tau)\}$ and $\theta_a = \ell \circ g^{-1}(\{a\}) - \ell \circ \alpha^{-1}(\mathrm{int}(C_a)) \geq 0$. Since we have that $\ell(\cup_{a\in B} T_a) \geq \sum_{a\in B} \theta_a$ for all (finite) subsets B of A, it follows that there exists a disjoint family of measurable sets $\{F_a\}_{a\in A}$ such that $F_a \subseteq T_a$ and $\ell(F_a) = \theta_a$ for all $a \in A$. Thus, define $f(t) = a$ if $t \in F_a$. Hence, f is a Nash equilibrium of G displaying behavioral conformity. ■

6 Behavioral Conformity in Large Finite Games

In this section we consider the case of games with a large, but finite, number of players. Since we can understand games with a continuum of players as the limit of large finite games (see, for example, Green (1984), Housman (1988) and Carmona (2004)), one expects that properties that hold for Nash equilibria in games with a continuum of players will also hold in approximate equilibria for large finite games. In fact, we use our previous results for dispersed induced anonymous games and for induced games to establish the existence of a Nash equilibrium displaying (strong) behavioral conformity in all sufficiently large finite games.

Theorem 4 states that all sufficiently large finite games with a one-dimensional attribute space and an injective attribute function have an approximate equilibrium displaying strong behavioral conformity.

Theorem 4 *Let $G = (\varphi, \Omega, U, A)$ be a pregame such that $\Omega \subseteq \mathbb{R}$ and $U(\Omega)$ is equicontinuous and quasi-intermediate. Then, for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ with the following* *property:*

For all $n \geq N$, every injective induced game $G_n = ((T_n, \nu_n), \Omega, U, A, \alpha)$ with n players has an ε – equilibrium that displays strong behavioral conformity.

The assumption that the attribute function is injective plays in Theorem 4 a role similar to the one that the dispersed assumption plays in Theorem 1. In fact, both guarantee that there is a small "number" of players with the same attribute.

The relationship between Theorems 1 and 4 is strengthened by noticing that the latter implies the former. This is illustrated by the following example. Let G_n be such that Ω, A and U are as defined in the first example of Section 5, and let $\alpha(t) = 2t/n$ for all $t = 1, \ldots, n$. To this induced game with n players, we associate a dispersed anonymous game defined as follows: for all $t \in T_n$, let Ω_t be an interval containing $\omega_t = 2t/n$. Then, let $g : \Omega \to \mathbb{R}$ be defined by $g(\omega) = 1/n\ell(\Omega_t)$ if $\omega \in \Omega_t$ and define a measure λ in Ω by $\lambda(E) = \int_E g\mathrm{d}\ell$ for all measurable $E \subseteq \Omega$. In this example, we can let $\Omega_1 = [0, 2/n]$ and $\Omega_t = (2(t-1)/n, 2t/n]$ for all $t > 1$. Hence, g is constantly equal to $1/2$ and $\lambda = \ell/2$.

Since $\tilde{G} = (\lambda, \Omega, U, A)$ is a dispersed anonymous game, then it has a Nash equilibrium $f : \Omega \to A$ satisfying strong behavioral conformity. In this example, we can let $C_1 = [0, 1]$, $C_2 = (1, 2]$ and f be such that $f(\omega) = b_1$ if and only if $\omega \in C_1$. Then, we can easily check that the strategy $f \circ \alpha : T_n \to A$ is a Nash equilibrium of G_n, and clearly displays strong behavioral conformity.

Proof of Theorem 4. Let $\varepsilon > 0$. Since $U(\Omega)$ is equicontinuous, let $\delta > 0$ be such that

$$|U_\omega(a, \tau) - U_\omega(a, \mu)| < \varepsilon/3 \tag{43}$$

whenever $||\tau - \mu|| < \delta$, $\tau, \mu \in \mathcal{M}$, $a \in A$ and $\omega \in \Omega$. Finally, let $N \in \mathbb{N}$ be such that $|A|/n < \delta$ whenever $n \geq N$.

Let $n \geq N$ and $G_n = ((T_n, \nu_n), \Omega, U, A, \alpha)$ be an injective induced game with n players. Let $\alpha(T_n) = \{\omega_1, \ldots, \omega_n\}$ and partition Ω into n intervals $\{\Omega_i\}_{i=1}^n$ such that $\omega_i \in \Omega_i$ and $\ell(\Omega_i) > 0$.

Define $g : \Omega \to \mathbb{R}$ by

$$g(\omega) = \frac{1}{n\ell(\Omega_i)} \tag{44}$$

if $\omega \in \Omega_i$ and a measure λ in Ω by

$$\lambda(E) = \int_E g \mathrm{d}\ell \tag{45}$$

for all measurable subsets E of Ω. Then, $\lambda(\Omega_i) = 1/n$ and λ is non-atomic.

By Theorem 1, it follows that there exists a Nash equilibrium f of $G = (\Omega, \lambda, U, A)$ in which all players best-reply and a partition $\{C_k\}_{k=1}^K$ of Ω into disjoint convex sets satisfying $f^{-1}(a_k) = C_k$ for all $a_k \in A$ such that $f^{-1}(a_k) \neq \emptyset$ (i.e., $A(f) = \{a_1, \ldots, a_K\}$).

An important step in the argument is provided by the following lemma. Let $X \subseteq \mathbb{R}$ be convex, $\mathcal{B} = \{B_i\}_{i=1}^N$ be a collection of pairwise disjoint convex subsets of X and $\mathcal{C} = \{C_j\}_{j=1}^K$ be a convex partition of X (i.e., C_j is convex for all $j = 1, \ldots, K$). Define

$$\rho(X, \mathcal{B}, \mathcal{C}) = |\{i \in \{1, \ldots, N\} : B_i \subseteq C_j \text{ for some } j \in \{1, \ldots, K\}\}| . \tag{46}$$

Lemma 6 *For all convex subsets X of $\mathbb{R}$ and all $N, K \in \mathbb{N}$, if $|\mathcal{B}| = N$ and $|\mathcal{C}| = K$, then $\rho(X, \mathcal{B}, \mathcal{C}) \geq N - K + 1$.*

It follows by Lemma 6 that for at least $n - K + 1$ sets in $\{\Omega_i\}_{i=1}^n$, there exists $k \in \{1, \ldots, K\}$ such that $\Omega_i \subseteq C_k$.

For all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, |A|\}$, let $\tau_{i,j} = \lambda(\{\omega \in \Omega_i : f(\omega) = a_j\})$ and $\tau_j = \sum_{i=1}^n \tau_{i,j}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Clearly, $\tau_j = \lambda \circ f^{-1}(\{a_j\})$. Note that if $\Omega_i \subseteq C_k$, then $f(\omega) = a_k$ for all $\omega \in \Omega_i$. So, in this case, $\tau_{i,k} = \lambda(\Omega_i) = 1/n$ and $\tau_{i,j} = \lambda(\emptyset) = 0$ if $j \neq k$.

Let $\tilde{\nu}_n = \nu_n \circ \alpha^{-1}$ and $\mu = \tilde{\nu}_n \circ f^{-1}$. Note that if $\Omega_i \subseteq C_k$, for some $1 \leq i \leq n$ and $1 \leq k \leq K$, then $f(\omega) = a_k$ for all $\omega \in \Omega_i$, implying that

$$\mu_{i,j} = \tilde{\nu}_n(\{\omega \in \omega_i : f(\omega) = a_j\}) = \begin{cases} 1/n \text{ if } j = k \\ 0 \quad\;\; \text{if } j \neq k. \end{cases} \tag{47}$$

Define

$$P_k = \{i \in \{1, \ldots, n\} : \Omega_i \subseteq C_k\}$$

and $P = \cup_{k=1}^{K} P_k$. We claim that $|\tau_j - \mu_j| \leq |A|/n$ for all $1 \leq j \leq |A|$: if $i \in P$, then $\mu_{i,j} = 1/n = \tau_{i,j}$ if $i \in P_j$ and $\mu_{i,j} = 0 = \tau_{i,j}$ if $j \in P_k$, $k \neq j$; therefore,

$$|\tau_j - \mu_j| \leq \sum_{i \notin P} |\tau_{i,j} - \mu_{i,j}| \leq \frac{1}{n}\,(n - (n - K + 1)) = \frac{K-1}{n} < \frac{|A|}{n} < \delta,$$

since $|\tau_{i,j} - \mu_{i,j}| \leq 1/n$ for all i and j. Thus, $||\tau - \mu|| < \delta$.

Since f is a Nash equilibrium in which all players best-reply, it follows that

$$U_{\omega_i}(f(\omega_i), \tau) \geq U_{\omega_i}(a, \tau) \tag{48}$$

for all $a \in A$ and $1 \leq i \leq n$. Hence, for all $i \in \{1, \ldots, n\}$ and $a \in A$, we obtain

$$\begin{aligned} U_{\omega_i}(f(\omega_i), \nu_n \circ f^{-1}) &= U_{\omega_i}(f(\omega_i), \mu) > U_{\omega_i}(f(\omega_i), \tau) - \frac{\varepsilon}{3} \\ &\geq U_{\omega_i}(a, \tau) - \frac{\varepsilon}{3} > U_{\omega_i}(a, \mu) - \frac{2\varepsilon}{3} > U_{\omega_i}(a, \nu_n \circ (f \setminus_{\omega_i} a)^{-1}) - \varepsilon, \end{aligned} \tag{49}$$

where the first and third inequalities follow from (50), the second from (48) and the last from Lemma 11. Therefore, $f \circ \alpha$ is an $\varepsilon -$ equilibrium of G_n. ■

Similarly to the case of induced games with a continuum of players, if the attribute function is not injective, then there is no hope of guaranteing the existence of approximate equilibria displaying strong behavioral conformity in sufficiently large games. For instance, if we modify the example in this section so that $\alpha(t) = 1$ for all $t \in T_n$, then the resulting game has no approximate equilibrium displaying strong behavioral conformity.

Nevertheless, we can use Theorem 3 to establish the existence of approximate equilibria displaying behavioral conformity in all sufficiently large finite games.

Theorem 5 *Let $G = (\Omega, U, A)$ be a pregame such that $U(\Omega)$ is equicontinuous and quasi-intermediate. Then, for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ with the following property:*

For all $n \geq N$, every induced game $G_n = (T_n, \nu_n, \Omega, U, A, \alpha)$ with n players has an ε – equilibrium that displays behavioral conformity.

Proof. Let $\varepsilon > 0$ and let $\eta > 0$ be such that $4\eta < \varepsilon$. Since $U(\Omega)$ is equicontinuous, let $\delta > 0$ be such that

$$||U_\omega(a, \tau) - U_\omega(a, \mu)|| < \eta/3 \tag{50}$$

whenever $||\tau - \mu|| < \delta$, $\tau, \mu \in \mathcal{M}$, $a \in A$ and $\omega \in \Omega$. Finally, let $N \in \mathbb{N}$ be such that $2|A|^2/n < \delta$ and $|A|/n < \eta$ whenever $n \geq N$.

Let $n \geq N$ and $G_n = ((T_n, \nu_n), \Omega, \alpha, U, A)$ be an induced game with n players. Define $I_1 = [0, 1/n]$, $I_i = ((i-1)/n, i/n]$ for all $i = 2, \ldots, n$ and consider the induced game $\tilde{G} = (([0,1], \ell), \Omega, \tilde{\alpha}, U, A)$ defined by $\tilde{\alpha}(t) = \alpha(i)$ if $t \in I_i$. That is, players $t \in I_1$ have payoff function $U_{\alpha(1)}$, players $t \in I_2$ have payoff function $U_{\alpha(2)}$, and so on. Then, $\tilde{G}$ has a Nash equilibrium f displaying behavioral conformity by Theorem 3. Let $\{a_1, \ldots, a_K\} = A(f)$ and $\{C_k\}_{k=1}^K$ be such that $\alpha(t) \in \mathrm{int}(C_k)$ implies that $f(t) = a_k$.

For all $1 \leq i \leq n$ and $1 \leq j \leq K$, let $\tau_{i,j} = \ell(\{t \in I_i : f(t) = a_j\})$ and $\tau_j = \sum_{i=1}^n \tau_{i,j}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. The equilibrium distribution can be represented in the following way:

$$\begin{array}{cccc}
\tau_{1,1} & \tau_{1,2} & \cdots & \tau_{1,K} \\
\tau_{2,1} & \tau_{2,2} & & \tau_{2,K} \\
\vdots & & & \\
\tau_{n,1} & \tau_{n,2} & & \tau_{n,K} \\
\hline
\tau_1 & \tau_2 & & \tau_K
\end{array}$$

Note that $\sum_{j=1}^{K} \tau_{i,j} = 1/n$ for all $1 \leq i \leq n$. Furthermore, since f is a Nash equilibrium, $\tau_{i,j} > 0$ implies that

$$U_{\alpha(i)}(a_j, \ell \circ f^{-1}) \geq U_{\alpha(i)}(a, \ell \circ f^{-1}) \tag{51}$$

for all $a \in A$.

Let $E = \{e_1, \ldots, e_K\}$ denote the standard basis of $\mathbb{R}^K$ and define

$$S_i = \left\{ \frac{1}{n} e_j : \tau_{i,j} > 0 \right\}.$$

Then, we have that $(\tau_{i,1}, \ldots, \tau_{i,K}) \in \text{co}(S_i)$ for every $1 \leq i \leq n$, since

$$(\tau_{i,1}, \ldots, \tau_{i,K}) = \sum_{j:\tau_{i,j}>0} \tau_{i,j} e_j = \sum_{j:\tau_{i,j}>0} n\tau_{i,j} \frac{e_j}{n}, \tag{52}$$

$n\tau_{i,j} \geq 0$ for all j and $\sum_{j:\tau_{i,j}>0} n\tau_{i,j} = 1$. This implies that

$$(\tau_1, \ldots, \tau_K) \in \text{co}\left(\sum_{i=1}^{n} S_i \right) = \sum_{i=1}^{n} \text{co}(S_i), \tag{53}$$

and so by the Shapley-Folkman Theorem (see Rashid (1983)), it follows that there are n points $(\alpha_{i,1}, \ldots, \alpha_{i,K}) \in \text{co}(S_i)$, $i = 1, \ldots, n$, such that

$$(\tau_1, \ldots, \tau_K) = \sum_{i=1}^{n} (\alpha_{i,1}, \ldots, \alpha_{i,K}) \tag{54}$$

and

$$|\{i : (\alpha_{i,1}, \ldots, \alpha_{i,K}) \notin S_i\}| \leq K = |A(f)|. \tag{55}$$

Let $P = \{i \in \{1, \ldots, n\} : n\alpha_i \in E\}$. Define a pure strategy g as follows: if $i \in P$, let e_j be such that $n\alpha_i = e_j$ and define $g(i) = a_j$; if $i \notin P$, choose $1 \leq j \leq K$ such that $\tau_{i,j} > 0$ and define $g(i) = a_j$. It then follows from inequality (51) that $U_{\alpha(i)}(g(i), \ell \circ f^{-1}) \geq U_{\alpha(i)}(a, \ell \circ f^{-1})$ for all $i \in \{1, \ldots, n\}$ and $a \in A$.

It follows by construction that g displays behavioral conformity. Indeed, if $i \in \{1, \ldots, n\}$ is such that $\alpha(i) \in \mathrm{int}(C_j)$ for some $1 \leq j \leq K$, then for all $t \in I_i$, $\tilde{\alpha}(t) = \alpha(i) \in \mathrm{int}(C_j)$. Therefore, $f(t) = a_j$ for all $t \in I_i$ and so $\tau_{i,j} = 1$. Hence, $S_i = \mathrm{co}(S_i) = \{e_j/n\}$, $n\alpha_i = e_j$ and $g(i) = a_j$. In conclusion, if $i \in \{1, \ldots, n\}$ is such that $\alpha(i) \in \mathrm{int}(C_j)$, then $g(i) = a_j$.

Let $1 \leq j \leq K$ and let $1_{\{g=a_j\}} : \{1, \ldots, n\} \to \{0, 1\}$ denote the function defined by $1_{\{g=a_j\}}(i) = 1$ if $g(i) = a_j$ and $1_{\{g=a_j\}} = 0$ if $g(i) \neq a_j$. Then we can write

$$\nu_n \circ g^{-1}(a_j) = \sum_{i \in P} \alpha_{i,j} + \frac{1}{n} \sum_{i \notin P} 1_{\{g=a_j\}}(i). \tag{56}$$

Hence,

$$|\ell \circ f^{-1}(a_j) - \nu_n \circ g^{-1}(a_j)| = \frac{1}{n} \left| \sum_{i \notin P} \left(n\alpha_{i,j} - 1_{\{g=a_j\}}(i) \right) \right| \leq \frac{K}{n} \tag{57}$$

and so $||\ell \circ f^{-1} - \nu_n \circ g^{-1}|| \leq K/n < \delta/2$.

By Lemma 11, it follows that

$$\begin{aligned} ||\ell \circ f^{-1} - \nu_n \circ (g^{-1} \setminus_i a)|| &\leq ||\ell \circ f^{-1} - \nu_n \circ g^{-1}|| + ||\nu_n \circ g^{-1} - \nu_n \circ (g^{-1} \setminus_i a)|| \\ &\leq \frac{\delta}{2} + \frac{1}{n} < \delta, \end{aligned} \tag{58}$$

for all $i \in \{1, \ldots, n\}$ and $a \in A$.

Hence, for all $i \in \{1, \ldots, n\}$ and $a \in A$, we obtain

$$\begin{aligned} &U_{\alpha(i)}(g(i), \nu_n \circ g^{-1}) > U_{\alpha(i)}(g(i), \ell \circ f^{-1}) - \frac{\varepsilon}{2} \\ &\geq U_{\alpha(i)}(a, \ell \circ f^{-1}) - \frac{\varepsilon}{2} > U_{\alpha(i)}(a, \nu_n \circ (g^{-1} \setminus_i a)) - \varepsilon. \end{aligned} \tag{59}$$

Therefore, g is a $\varepsilon -$ equilibrium of G_n. $\blacksquare$

7 Relation with Wooders et al.

Despite the fact that we have closely followed Wooders, Cartwright, and Selten (2006), we have presented the model and stated our assumptions in a different way. In this section, we show that we are considering the same model and that our assumptions are weaker.

In light of Subsection 2.4, we only discuss their model under the assumption that there is only one crowding type. In this case, let $\Omega = [0,1]^L$ for some $L \in \mathbb{N}$. Let W be the space of all non-negative, real-valued functions in $\Omega \times A$ with finite support. A *WCS – pregame* is a triple (Ω, h, A), where h is a non-negative, real-valued function in $\Omega \times A \times W$. We shall prove that every WCS – pregame satisfying the large game and within type anonymity of Wooders, Cartwright, and Selten (2006) defines a uniformly continuous pregame that induces the same n-player games.

Let $\tilde{G} = (\Omega, h, A)$ be a WCS – pregame satisfying the large game property. Recall that we can identify $\mathcal{M}$ with the standard unit Δ in $\mathbb{R}^{|A|}$. Define $\Delta_q = \Delta \cap \mathbb{Q}$ and, for all $a \in A$, $\Delta_a = \{x \in \Delta \cap \mathbb{Q} : x_a > 0\}$. For all $\omega \in \Omega$ and $\mu \in \Delta \cap \mathbb{Q}$, define $w_{\omega,\mu} \in W$ by

$$w_{\omega,\mu}(\bar{\omega}, a) = \begin{cases} \mu_a & \text{if } \bar{\omega} = \omega, \\ 0 & \text{otherwise.} \end{cases} \tag{60}$$

Also, define $\tilde{U}^h_\omega : A \times \Delta \cap \mathbb{Q} \to \mathbb{R}$ by $\tilde{U}^h_\omega(a, \mu) = h(\omega, a, w_{\omega,\mu})$.

Lemma 7 *For all $\omega \in \Omega$ and $a \in A$, the function $\mu \mapsto \tilde{U}_\omega(a, \mu)$ is uniformly continuous in Δ_a.*

Proof. Let $\omega \in \Omega$, $a \in A$ and $\varepsilon > 0$. Let $\delta > 0$ be as in the definition of the global interaction property and let $\mu, \mu' \in \Delta_a$ be such that $||\mu - \mu'||_1 = \sum_{\bar{a} \in A} |\mu_{\bar{a}} - \mu'_{\bar{a}}| < \delta$. If $\mu = (p_{\bar{a}}/q)_{\bar{a} \in A}$ and $\mu' = (p'_{\bar{a}}/q')_{\bar{a} \in A}$ for some $p_{\bar{a}}$, $p'_{\bar{a}}$, q, $q' \in \mathbb{N}$ for all $\bar{a} \in A$, then $\mu = (q' p_{\bar{a}}/qq')_{\bar{a} \in A}$ and $\mu' = (q p'_{\bar{a}}/qq')_{\bar{a} \in A}$.

Consider a game with $n = qq'$ players and $\alpha(t) = \omega$ for all $t \in T_n$. Let f and g be strategies such that $\nu_n \circ f^{-1} = \mu$, $\nu_n \circ g^{-1} = \mu'$ and $f(1) = g(1) = a$. Since α is constant, then $\rho_\alpha(f, g) = ||\nu_n \circ f^{-1} - \nu_n \circ g^{-1}|| = ||\mu - \mu'|| < \delta$. Furthermore, $w_{\omega,\mu} = w_{\alpha,f}$ and $w_{\omega,\mu'} = w_{\alpha,g}$. Hence, it follows by the global interaction property that

$$|\tilde{U}^h_\omega(a, \mu) - \tilde{U}^h_\omega(a, \mu')| = |h(\omega, a, w_{\alpha,f}) - h(\omega, a, w_{\alpha,g})| < \varepsilon$$

and so $\mu \mapsto \tilde{U}_\omega(a, \mu)$ is uniformly continuous in Δ_a. ■

Since the set $\cup_{a\in A}(\{a\} \times \Delta_a)$ is dense in $A \times \Delta$, it follows that $\tilde{U}^h_\omega$ can be extended to a uniformly continuous function U^h_ω in $A \times \Delta$ for all $\omega \in \Omega$ (see DePree and Swartz (1988, Lemma 14, p. 279)). Hence, $U^h_\omega \in \mathcal{U}$ for all $\omega \in \Omega$ and so indeed we have a function $U^h : \Omega \to \mathcal{U}$.

Lemma 8 *The function U^h is uniformly continuous.*

Proof. Let $\varepsilon > 0$ and let $\delta > 0$ be as in the definition of the continuity in attributes property and corresponding to $\varepsilon/2$. Let $\omega, \omega' \in \Omega$ be such that $||\omega - \omega'|| < \delta$.

Let $a \in A$ and $\mu \in \Delta_a$. Then, $\mu = (p_{\bar{a}}/n)_{\bar{a}\in A}$ for some n and $p_{\bar{a}} \in \mathbb{N}$ for all $\bar{a} \in A$. Consider two n-player games, one with $\alpha(t) = \omega$ and the other with $\alpha'(t) = \omega'$ for all $t \in T_n$. Let f be a strategy such that $\nu_n \circ f^{-1} = \mu$ and $f(1) = a$. Then, by the continuity in attributes property, it follows that

$$|U^h_\omega(a, \mu) - U^h_{\omega'}(a, \mu)| = |\tilde{U}^h_\omega(a, \mu) - \tilde{U}^h_{\omega'}(a, \mu)| = |h(\omega, a, w_{\alpha,f}) - h(\omega', a, w_{\alpha,f})| < \varepsilon/2.$$

Since $\cup_{a\in A}\{a\} \times \Delta_a$ is dense in $A \times \Delta$ and both U_ω and $U_{\omega'}$ are continuous, it follows that

$$||U_\omega - U_{\omega'}|| = \sup_{(a,\mu)\in A\times\Delta} |U^h_\omega(a, \mu) - U^h_{\omega'}(a, \mu)| \leq \varepsilon/2 < \varepsilon.$$

Hence, U is uniformly continuous. ■

It follows from Lemmas 7 and 8 that every WCS – pregame with a large game property induces a uniformly continuous pregame.

Theorem 6 *If $\tilde{G} = (\Omega, h, A)$ is a WCS – pregame satisfying the large game property, then $G = (\Omega, U^h, A)$ is a uniformly continuous pregame. Hence, $U^h(\Omega)$ is a compact subset of $\mathcal{U}$.*

We shall prove next that if $\tilde{G} = (\Omega, h, A)$ satisfies the large game property and within-type anonymity, then $G = (\Omega, U^h, A)$ induces the same n-player games as $\tilde{G}$. In order to establish this claim, note first that for every attribute function α and strategy f in a game with n players, the weight function $w_{\alpha,f}$ relative to f is equal to n times the distribution $\nu_n \circ (\alpha, f)^{-1}$ in $\Omega \times A$ induced by α and f, i.e., $w_{\alpha,f}(\omega, a) = n\nu_n \circ (\alpha, f)^{-1}(\omega, a)$ for all $\omega \in \Omega$ and $a \in A$. Therefore, slightly abusing the notation, we can write $h(\omega, a, w_{\alpha,f}) = h(\omega, a, \nu_n \circ (\alpha, f)^{-1})$ for all $\omega \in \Omega$ and $a \in A$.

Furthermore, if $\tilde{G}$ satisfies within-type anonymity, then h depends only on the distribution $\nu_n \circ f^{-1}$ on A. Hence, there exists $\tilde{h} : \Omega \times A \times \Delta \to \mathbb{R}$ such that $h(\omega, a, w_{\alpha,f}) = \tilde{h}(\omega, a, \nu_n \circ f^{-1})$ for all $\omega \in \Omega$ and $a \in A$. Letting $\bar{\alpha}(t) = \omega$ for all $t \in T_n$, it follows that $w_{\omega,\nu_n \circ f^{-1}} = n\nu_n \circ (\bar{\alpha}, f)^{-1}$, and so

$$U^h_\omega(a, \nu_n \circ f^{-1}) = h(\omega, a, \nu_n \circ (\bar{\alpha}, f)^{-1}) = \tilde{h}(\omega, a, \nu_n \circ f^{-1}).$$

Hence, $U^h_{\alpha(t)}(f(t), \nu_n \circ f^{-1}) = h(\alpha(t), f(t), w_{\alpha,f})$ for all players $t \in T_n$, strategies $f : T_n \to A$ and attribute functions $\alpha : T_n \to \Omega$, establishing Theorem 7

Theorem 7 *If $\tilde{G} = (\Omega, h, A)$ is a WCS – pregame satisfying both the large game property and within-type anonymity, then $\tilde{G}$ and its associated pregame $G = (\Omega, U^h, A)$ induce the same games and the same anonymous games.*

Since Theorem 3 in Wooders, Cartwright, and Selten (2006) concerns only games induced by WCS – pregames satisfying both the within-type anonymity and the large game property, it follows from Theorem 7 that they can be obtained simply by using pregames. Therefore, their Theorem 3 follows from Theorem 5. In fact, if $\tilde{G} = (\Omega, h, A)$ is a WCS – pregame satisfying the large game property, within-type anonymity and linearity in taste types, then $G = (\Omega, U^h, A)$ is a pregame with U^h uniformly continuous and intermediate and $U^h(\Omega)$ compact, hence equicontinuous. Therefore, we can conclude from Theorem 5 for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that every induced game G_n (by $\tilde{G}$ or, equivalently, by G) with $n \geq N$ players has an ε – equilibrium that displays behavioral conformity. In conclusion, Theorem 5 extends Theorem 3 of Wooders, Cartwright, and Selten (2006) by allowing more general attribute spaces (Ω can be any convex, bounded, measurable subset of $\mathbb{R}^L$) and more general preference functions (U needs only to be measurable, quasi-intermediate and such that $U(\Omega)$ is equicontinuous).

8 Concluding Remarks

In this paper, we have approached the behavioral conformity problem of Wooders, Cartwright, and Selten (2006) using the framework of games with a continuum of players. This has led to some improvements of their Theorem 3, not only in terms of weakening some of its assumptions, but also in terms of a simpler proof, able to accommodate easily the case of multi-dimensional attributes and arbitrary finite action spaces.

Although artificial, the case of a continuum of players is interesting even for someone who is only interested in games with a large but finite number of players. In fact, as we have shown in Carmona (2004), the limit points of the approximate equilibria of large finite games correspond to the Nash equilibria of games with a continuum of players. In this paper, this correspondence between Nash equilibria of games with a continuum of players and approximate equilibria of games with a large, finite number of players was precisely the tool we used to derive our generalization of Wooders et al.'s Theorem 3 from our Theorem 3 pertaining to games with a continuum of players. Furthermore, it was also used to derive our strong behavioral conformity result for large, finite games (Theorem 4) from Theorem 1 pertaining to dispersed anonymous games.

We conclude with a remark on the definition of behavioral conformity: Theorems 1 and 2 (and their proofs) imply that if G is an induced anonymous game satisfying the assumptions of any of them, then G has a continuous λ – almost everywhere equilibrium. Such equilibrium also reflects the intuitive notion of behavioral conformity, since it implies that individuals that have similar attributes behave similarly, with possibly the exception of a small fraction of them. With this weaker notion of behavioral conformity one loses the notion of a society, but one may gain a more general behavioral conformity result. Also, it allows us to fit some interesting examples. One of those is the inner vs outer city behavior where cities are represented by circles: the inner city forms a society, but not the outer city.[8]

A Appendix

Proof of Lemma 1. Note that if $\lambda(C) = 0$, then we can let $C_1 = C$ and $C_i = \emptyset$ for all $i = 2, \ldots, k$. Also, if $c_i = 0$ for some $1 \leq i \leq k$, then we can let $C_i = \emptyset$. Therefore, we may assume that $c_i > 0$ for all $i = 1, \ldots, k$.

Let $x \notin C$. By the supporting hyperplane theorem, let $p \in \mathbb{R}^L$ be such that $p \neq 0$

[8] I thank João Rosal for suggesting this example to me.

and $p \cdot y \leq p \cdot x$ for all $y \in C$. Let $d = p \cdot x$.

Consider the function $m : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$m(\eta) = \lambda(C \cap \{z \in \mathbb{R}^L : d - \eta \leq p \cdot z \leq d\}).$$

One easily sees that m is continuous, $m(0) = 0$ and $\lim_{\eta \to \infty} m(\eta) = \lambda(C)$. Since C is bounded, in fact, $m(\bar{\eta}) = \lambda(C)$ for some $\bar{\eta} > 0$. Hence, there is $\eta_1 > 0$ such that $m(\eta_1) = c_1$, and so define $C_1 = C \cap \{z \in \mathbb{R}^L : d - \eta_1 \leq p \cdot z \leq d\}$, which is a convex subset of C.

Furthermore, there is $\eta_2 > \eta_1$ such that $m(\eta_2) = c_1 + c_2$. Then, defining $C_2 = C \cap \{z \in \mathbb{R}^L : d - \eta_2 \leq p \cdot z < d - \eta_1\}$, we see that $\lambda(C_1) + \lambda(C_2) = \lambda(C_1 \cup C_2) = c_1 + c_2$, which implies that $\lambda(C_2) = c_2$.

Let $i \geq 3$ and assume that we have found $\{\eta_j\}_{j=1}^{i-1}$ such that $\eta_j > \eta_{j-1}$ and $c_j = \lambda(C \cap \{z \in \mathbb{R}^L : d - \eta_j \leq p \cdot z < d - \eta_{j-1}\})$ for all $j = 2, \ldots, i-1$. Then there is $\eta_i > \eta_{i-1}$ such that $m(\eta_i) = \sum_{j=1}^{i} c_j$. Then, defining $C_i = C \cap \{z \in \mathbb{R}^L : d - \eta_i \leq p \cdot z < d - \eta_{i-1}\}$, we see that $\sum_{j=1}^{i} \lambda(C_i) = \lambda(\cup_{j=1}^{i} C_j) = \sum_{j=1}^{i} c_i$, which implies that $\lambda(C_i) = c_i$. Clearly, C_i is a convex subset of C.

Finally, let $C_k = C \cap \{z \in \mathbb{R}^L : p \cdot z < d - \eta_{k-1}\}$, a convex subset of C. Clearly, $\cup_{i=1}^{k} C_i = C \cap \{z \in \mathbb{R}^L : p \cdot z \leq d\} = C$. Reasoning as above, one sees that $\lambda(C_k) = c_k$. ■

Proof of Lemma 2. We may assume that $B := \{\omega \in \Omega : U_\omega(a, \mu) = U_\omega(\bar{a}, \mu)\}$ is non-empty, since otherwise this result is trivial. For concreteness assume that $C := \{\omega \in \Omega : U_\omega(a, \mu) > U_\omega(\bar{a}, \mu)\}$ is non-empty and let $\omega' \in C$

By the separating hyperplane theorem, there exist $q \in \mathbb{R}^L$ and $c \in \mathbb{R}$ such that $q \neq 0$ and $q \cdot \tilde{\omega} \geq c \geq q \cdot \omega$ for all $\tilde{\omega} \in C$ and all $\omega \in B$.

Let $\omega \in B$. Then, $q \cdot \omega \leq c$. Let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence in $(0, 1)$ such that $\alpha_k \to 1$ and consider $\omega_k = \alpha_k \omega + (1 - \alpha_k)\omega'$. Then, $\omega_k \in C$ and so $q \cdot \omega_k \geq c$. Since $\omega_k \to \omega$, it follows that $q \cdot \omega \geq c$. Hence, $q \cdot \omega = c$. So, let $q_{a,\bar{a}} = q$ and $c_{a,\bar{a}} = c$. ■

Lemma 9 *Let $n \geq 2$ and $H_i \subseteq \mathbb{R}^n$, $i = 1, 2$, be two hyperplanes. If H_1 is not parallel to H_2, then they intersect (i.e., $H_1 \cap H_2 \neq \emptyset$).*

Proof. Let L_i be the subspace of $\mathbb{R}^n$ parallel to H_i, $i = 1, 2$ and recall that $\dim(L_i) = n - 1$ (see Rockafellar (1970, Theorem 1.2, p. 4)). Then, $L_1 \neq L_2$, since otherwise H_1 and H_2 would be parallel. Hence, there exists $w \in L_2 \setminus L_1$, since otherwise $L_2 \subseteq L_1$, which together with $\dim(L_1) = \dim(L_2)$, would imply $L_1 = L_2$ (see Friedberg, Insel, and Spence (1997, Theorem 1.11, p. 49)).

It then follows that $L_1 + L_2 = \mathbb{R}^n$: if $w \in L_2 \setminus L_1$ and $\{w_1, \ldots, w_{n-1}\}$ is a basis for L_1, then $\operatorname{span}(\{w_1, \ldots, w_{n-1}, w\}) = \mathbb{R}^n$ since $\{w_1, \ldots, w_{n-1}, w\}$ is linearly independent by Friedberg, Insel, and Spence (1997, Theorem 1.8, p. 41). Furthermore, $\{w_1, \ldots, w_{n-1}, w\} \subseteq L_1 \cup L_2$, which implies that $\operatorname{span}(\{w_1, \ldots, w_{n-1}, w\}) \subseteq \operatorname{span}(L_1 \cup L_2) = L_1 + L_2$. So, $\mathbb{R}^n \subseteq L_1 + L_2$, and, of course, $L_1 + L_2 \subseteq \mathbb{R}^n$.

Since $L_2 = -L_2$, then $L_1 - L_2 = \mathbb{R}^L$. Let $a_i \in \mathbb{R}^L$, $i = 1, 2$, be such that $H_i = L_i + a_i$. Let $x_1 \in L_1$ and $x_2 \in L_2$ be such $x_1 - x_2 = a_2 - a_1$. So, defining $x = x_1 + a_1 = x_2 + a_2$, it follows that $x \in H_1 \cap H_2$. ■

Proof of Lemma 3. Let $\{\eta_k\}$ be such that $\eta_k \searrow \eta$. We claim that $\lambda(C_{\eta_k}) \searrow \lambda(C_\eta)$. We start by establishing that $\{C_{\eta_k}\}$ decreases: since $\eta_k \searrow \eta$, it follows that $\eta_k > 0$ for all k. Since $q \cdot w = d \leq q \cdot z$ and $(1 - \eta_{k+1})/\eta_{k+1} \geq (1 - \eta_k)/\eta_k$, then if $z \in C_{\eta_{k+1}}$, it follows that

$$\begin{aligned} p \cdot z &\leq p \cdot w + \frac{1 - \eta_{k+1}}{\eta_{k+1}}(q \cdot w - q \cdot z) \\ &\leq p \cdot w + \frac{1 - \eta_k}{\eta_k}(q \cdot w - q \cdot z). \end{aligned} \tag{61}$$

Hence, $\eta_k p \cdot z + (1 - \eta_k) q \cdot z \leq \eta_k p \cdot w + (1 - \eta_k) q w$ and $z \in C_{\eta_k}$.

We next show that $\cap_k C_{\eta_k} \subseteq C_\eta$ and that $\cap_k C_{\eta_k} = C_\eta$ if $\eta > 0$. If $z \in C_\eta$, and $\eta > 0$, then a similar argument to the one used to establish that $C_{\eta_{k+1}} \subseteq C_{\eta_k}$ shows that $z \in C_{\eta_k}$ for all $k \in \mathbb{N}$. Conversely, if $z \in \cap_k C_{\eta_k}$, then $\eta_k p \cdot z + (1-\eta_k) q \cdot z \geq \eta_k p \cdot w + (1-\eta_k) q w$ for all k, and so $\eta p \cdot z + (1-\eta) q \cdot z \geq \eta p \cdot w + (1-\eta) q w$. Hence, $z \in C_\eta$.

Finally, we show that if $\eta = 0$, then $\lim_k \lambda(C_{\eta_k}) = \lambda(C_\eta) = 0$: We have that $C_0 = \Omega \cap \{z : q \cdot z = d\}$ and so $\lambda(C_0) = 0$. Since $\cap_k C_{\eta_k} \subseteq C_0$ and $C_{\eta_k} \searrow \cap_k C_{\eta_k}$ then, $\lim_k \lambda(C_{\eta_k}) = \lambda(\cap_k C_{\eta_k}) \leq \lambda(C_0) = 0$, and so $\lim_k \lambda(C_{\eta_k}) = 0$.

Similarly, we can show that if $\eta_k \nearrow \eta$ then $C_{\eta_k} \nearrow \cup_k C_{\eta_k}$ and $\cup_k C_{\eta_k} \subseteq C_\eta$ (note that $\eta > 0$). Hence, $\lambda(C_{\eta_k}) \nearrow \lambda(\cup_k C_{\eta_k}) \leq \lambda(C_\eta)$. We claim that $\lambda(\cup_k C_{\eta_k}) = \lambda(C_\eta)$. Note that it is enough to show that $\lambda(C_\eta \setminus \cup_k C_{\eta_k}) = 0$. If $z \in C_\eta \setminus \cup_k C_{\eta_k}$, then $q_\eta \cdot z \leq d_\eta$ and $q_{\eta_k} \cdot z > d_{\eta_k}$ for all k. This implies that $q_\eta \cdot z = d_\eta$, and so $C_\eta \setminus \cup_k C_{\eta_k} \subseteq \{z \in \mathbb{R}^L : q_\eta \cdot z = d_\eta\}$. Thus, $\lambda(C_\eta \setminus \cup_k C_{\eta_k}) = 0$.

We have shown that both $\eta_k \nearrow \eta$ and $\eta_k \searrow \eta$ imply that $\lambda(C_{\eta_k}) \to \lambda(C_\eta)$. This shows that the function $\eta \mapsto \lambda(C_\eta)$ is continuous. ■

Proof of Lemma 5. Note that if $\lambda(C) = 0$, then we can let $C_1 = C$ and $C_i = \emptyset$ for all $i = 2, \ldots, k$. Also, if $c_i = 0$ for some $1 \leq i \leq k$, then we can let $C_i = \emptyset$. Therefore, we may assume that $c_i > 0$ for all $i = 1, \ldots, k$.

Let $x \notin C$. By the supporting hyperplane theorem, let $p \in \mathbb{R}^L$ be such that $p \neq 0$ and $p \cdot y \leq p \cdot x$ for all $y \in C$. Let $d = p \cdot x + 1$.

For all $\eta \in \mathbb{R}_+$, define

$$C_\eta = C \cap \{z \in \mathbb{R}^L : d - \eta \leq p \cdot z \leq d\} \tag{62}$$

and define a function $m : \mathbb{R}_+ \to \mathbb{R}_+$ by $m(\eta) = \lambda(C_\eta)$.

One easily establishes the following properties of m: (1) m is increasing, (2) $\lim_{\eta \downarrow \eta^*} m(\eta) = m(\eta^*)$ for all η^*, (3) $\lim_{\eta \uparrow \eta^*} m(\eta) = \lambda(C \cap \{z \in \mathbb{R}^L : d - \eta^* <$

$p \cdot z \leq d\}$) for all η^* and (4) $m(0) = 0$. Note that property 2 means that m is upper semicontinuous. Since C is bounded let $\bar{\eta} > 0$ be such that $C_{\bar{\eta}} = C$. Therefore, $m(\eta) = \lambda(C)$ for all $\eta \geq \bar{\eta}$.

We will define $C_1 = C \cap \{z \in \mathbb{R}^L : d - \eta_1 \leq p \cdot z \leq d\}$ and

$$C_i = C \cap \{z \in \mathbb{R}^L : d - \eta_i \leq p \cdot z < d - \eta_{i-1}\}$$

for all $i = 2, \ldots, k$, where $\eta_1, \ldots, \eta_k$ are chosen so that $0 < \eta_1 \leq \ldots \leq \eta_k$, $C = \cup_{i=1}^k C_i$, $\lambda(\mathrm{int}(C_i)) \leq c_i$ for all $i = 1, \ldots, k$ and $\lambda(\cup_{i=1}^n C_i) \geq \sum_{i=1}^n c_i$ for all $1 \leq n \leq k$. Note that if $z \in C$, then $p \cdot z \leq p \cdot x < p \cdot x + 1 = d$; hence, letting $\eta_0 = 0$, we can write $C_1 = C \cap \{z \in \mathbb{R}^L : d - \eta_1 \leq p \cdot z < d - \eta_0\}$. Our task is to show the existence of such $\eta_1, \ldots, \eta_k$.

For all $j = 1, \ldots, k-1$, define $\eta_j = \min\{\eta : m(\eta) \geq \sum_{i=1}^j c_j\}$. Note that η_j is well defined since $\{\eta : m(\eta) \geq \sum_{i=1}^j c_j\}$ is closed and nonempty; in fact, the second property follows because m is upper semicontinuous, while the first from the fact that $\bar{\eta} \in \{\eta : m(\eta) \geq \sum_{i=1}^j c_j\}$. Finally, define $\eta_k = \bar{\eta} \geq \min\{\eta : m(\eta) \geq \sum_{i=1}^k c_j\}$. For convenience, let $\hat{\eta}_k = \min\{\eta : m(\eta) \geq \sum_{i=1}^k c_j\}$.

Since $c_1 > 0$ and $\{\eta : m(\eta) \geq \sum_{i=1}^{j+1} c_j\} \subseteq \{\eta : m(\eta) \geq \sum_{i=1}^j c_j\}$), it follows that $\eta_1 > 0$ and $\eta_j \leq \eta_{j+1}$ for all $j = 1, \ldots, k-2$. Also, $\eta_{k-1} \leq \hat{\eta}_k \leq \eta_k$, and so the inequality $\eta_j \leq \eta_{j+1}$ holds for all $j = 1, \ldots, k-1$.

Since $\cup_i^n C_i = C_{\eta_n}$ it follows that $\lambda(\cup_{i=1}^n C_i) = \lambda(C_{\eta_n}) = m(\eta_n) \geq \sum_{i=1}^n c_i$ for all $1 \leq n \leq k$.

We next show that $\lambda(\mathrm{int}(C_i)) \leq c_i$ for all $i = 1, \ldots, k$. Let $1 \leq i \leq k$. If $\eta_i = \eta_{i-1}$, then $C_i = \emptyset$ and so the conclusion follows. Therefore, we may assume that $\eta_i > \eta_{i-1}$.

In this case, it follows that

$$\begin{aligned}\operatorname{int}(C_i) &= \operatorname{int}(C) \cap \operatorname{int}(\{z \in \mathbb{R}^L : d - \eta_i \leq p \cdot z < d - \eta_{i-1}\}) \\ &= \operatorname{int}(C) \cap \{z \in \mathbb{R}^L : d - \eta_i < p \cdot z < d - \eta_{i-1}\} \\ &\subseteq \operatorname{int}(C) \cap \left(C \cap \{z \in \mathbb{R}^L : d - \eta_i < p \cdot z \leq d\} \setminus \cup_{j=1}^{i-1} C_j\right).\end{aligned} \tag{63}$$

Therefore, if $i \leq k-1$, then $\eta_i = \min\{\eta : m(\eta) \geq \sum_{j=1}^{i} c_j\}$ and so $m(\eta) < \sum_{j=1}^{i} c_j$ for all $\eta < \eta_i$. Thus, $\lim_{\eta \uparrow \eta_i} m(\eta) \leq \sum_{j=1}^{n} c_i$ and so

$$\lambda(\operatorname{int}(C_i)) \leq \lim_{\eta \uparrow \eta_i} m(\eta) - \lambda(\cup_{j=1}^{i-1} C_j) \leq \sum_{j=1}^{n} c_j - \sum_{j=1}^{i-1} c_j = c_i. \tag{64}$$

If $i = k$, then

$$\lambda(\operatorname{int}(C_k)) \leq \lambda(\operatorname{int}(C)) - \lambda(\cup_{j=1}^{k-1} C_j) \leq \sum_{j=1}^{k} c_j - \sum_{j=1}^{k-1} c_j = c_k.$$

In conclusion, $\lambda(\operatorname{int}(C_i)) \leq c_i$ for all $i = 1, \ldots, k$. ■

As we have already pointed out, a measure with a finite support can be thought of as a vector in some Euclidean space. Roughly, Lemma 10 says that the Prohorov distance between two measures whose support is contained in some finite set is proportional to their Euclidean distance.

Lemma 10 *Let $\tau, \mu \in \mathcal{M}(X)$ be such that* $\operatorname{supp}(\tau) \cup \operatorname{supp}(\mu) \subseteq \Psi$*, where Ψ is a finite set. If there exists $\varepsilon > 0$ such that $||\tau_l - \mu_l|| \leq \varepsilon$ for all $1 \leq l \leq |\Psi|$, then $\rho(\tau, \mu) \leq |\Psi|\varepsilon$.*

Proof. Let $\varepsilon > 0$ and $B \subseteq X$ be Borel measurable. Then,

$$\tau(B) = \sum_{l \in \Psi \cap B} \tau(\{l\}) \leq \sum_{l \in \Psi \cap B} (\mu(\{l\}) + \varepsilon) \leq \sum_{l \in \Psi \cap B} \mu(\{l\}) + |\Psi|\varepsilon \leq \mu(\overline{B}_{|\Psi|\varepsilon}(B)) + |\Psi|\varepsilon. \tag{65}$$

Similarly, we can show that $\mu(B) \leq \tau(\overline{B}_{|\Psi|\varepsilon}(B)) + |\Psi|\varepsilon$. This implies that $\rho(\tau, \mu) \leq |\Psi|\varepsilon$. ■

Lemma 11 shows that in large games a player deviation has a small impact on the distribution of actions.

Lemma 11 *Let $G_n = ((T, \nu_n), U, A)$ be a game and let f be a strategy. Then,*

$$||\nu_n \circ f^{-1} - \nu_n \circ (f \setminus_\omega a)^{-1}|| \leq \frac{1}{n}, \tag{66}$$

for all $\omega \in T$ and $a \in A$.

Proof. Let $\omega \in \Omega$ and $a \in A$. Let $\tau = \nu_n \circ f^{-1}$, $\mu = \nu_n \circ (f \setminus_\omega a)^{-1}$ and $\bar{a} = f(\omega)$. If $\bar{a} = a$, then $\tau = \mu$. If $\bar{a} \neq a$, then $|\tau(\{\tilde{a}\}) - \mu(\{\tilde{a}\})| = 0$ if $\tilde{a} \neq a$ and $\tilde{a} \neq \bar{a}$, while $|\tau(\{\tilde{a}\}) - \mu(\{\tilde{a}\})| = 1/n$ otherwise, and so $||\tau - \mu|| = 1/n$. ■

Proof of Lemma 6. The proof is by induction on K. If $K = 1$, then $C_1 = X$ and so $\rho = N = N - K + 1$ for all X and N.

Then, suppose that the conclusion holds for all $K = 1, \ldots, k$, $N \in \mathbb{N}$ and all convex subsets X of $\mathbb{R}$. Consider $K = k + 1$ and order the sets in $\mathcal{B}$ (resp. $\mathcal{C}$) so that if $x \in B_i$, $y \in B_j$ (resp. $x \in C_i$, $y \in C_j$) and $i < j$, then $x < y$. These orderings are possible since for all $i = 1, \ldots, k+1$ and $j = 1, \ldots, N$ are convex subsets of $\mathbb{R}$, and so intervals. It then follows that $\cup_{j=1}^{k} C_j$ is convex. Let $\tilde{X} = \cup_{j=1}^{k} C_j$ and $c = \sup \tilde{X}$.

We consider two cases: the first is when there exists i^* such that $c \in B_{i^*}$. In the case, $\cup_{i<i^*} B_i \subseteq \tilde{X}$ and $\{C_j\}_{j=1}^{k}$ is a partition of $\tilde{X}$, which implies that at least $i^* - 1 - k + 1 = i^* - k$ sets in $\{B_i\}_{i=1}^{i^*-1}$ are contained in some C_j, $j \in \{1, \ldots, m\}$. Since $\cup_{i>i^*} B_i \subseteq C_{k+1}$ and $|\{B_i\}_{i=i^*+1}^{N}| = N - i^*$, then $\rho \geq i^* - k + N - i^* = N - k = N - (k+1) + 1$, as required.

Finally, if there is no i such that $c \in B_i$, let $i^* = \max\{i : \sup B_i \leq c\}$ and the conclusion follows from the same argument used above. ■

References

Carmona, G. (2004): "Nash Equilibria of Games with a Continuum of Players," Universidade Nova de Lisboa.

DePree, J., and C. Swartz (1988): *Introduction to Real Analysis*. Wiley, New York.

Friedberg, S., A. Insel, and L. Spence (1997): *Linear Algebra*. Prentice-Hall, Upper Saddle River, NJ, third edn.

Grandmont, J. (1978): "Intermediate Preference and the Majority Rule," *Econometrica*, 46, 317–330.

Green, E. (1984): "Continuum and Finite-Player Noncooperative Models of Competition," *Econometrica*, 52, 975–993.

Housman, D. (1988): "Infinite Player Noncooperative Games and the Continuity of the Nash Equilibrium Correspondence," *Mathematics of Operations Research*, 13, 488–496.

Khan, M., and Y. Sun (1995): "Pure Strategies in Games with Private Information," *Journal of Mathematical Economics*, 24, 633–653.

——— (2002): "Non-Cooperative Games with Many Players," in *Handbook of Game Theory, Volume 3*, ed. by R. Aumann, and S. Hart. Elsevier, Holland.

Mas-Colell, A. (1984): "On a Theorem by Schmeidler," *Journal of Mathematical Economics*, 13, 201–206.

Rashid, S. (1983): "Equilibrium Points of Non-atomic Games: Asymptotic Results," *Economics Letters*, 12, 7–10.

Rath, K. (1995): "Representation of Finite Action Large Games," *International Journal of Game Theory*, 24, 23–35.

Rockafellar, R. T. (1970): *Convex Analysis*. Princeton University Press, Princeton.

Schmeidler, D. (1973): "Equilibrium Points of Nonatomic Games," *Journal of Statistical Physics*, 4, 295–300.

Wooders, M., E. Cartwright, and R. Selten (2006): "Behavioral Conformity in Games with Many Players," *Games and Economic Behavior*, 57, 347–360.