A Work Project, presented as part of the requirements for the Award of a Master Degree in Economics from the NOVA – School of Business and Economics.

A Yield Curve Model with Macroeconomic and Financial Variables.

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Abstract

This thesis has the goal to fit the Portuguese yield curve constructing a model with not only latent factors, but also macroeconomic factors, such as inflation, marginal lending rate and industrial production.

The model will incorporate the Nelson and Siegel (1987) decomposition and it will be made using a state-space framework, where the estimation results gave me a good fitting of the yield curve. Afterwards, I analyze the correlation between the yield curve components and the macroeconomic variables by impulse response functions and variance decompositions, where I find a causality relationship between macro variables and the latent factors.

Keywords: Yield Curve, Portugal, Macroeconomy, State-space model

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1. Introduction

This work project has the goal to construct a model that fits the Portuguese yield curve with not only latent factors, but also macroeconomic factors. The various possibilities to model the yield curve will be on Section 2, where I will describe the shortcomings and advantages of many yield curve models on my literature review.

Thus, my thesis is an attempt to construct a model that fits better the Portuguese yield curve, using the same framework as Diebold, Rudebusch and Aruoba (2006), but adding some new specifications, where I will link macroeconomics and finance conclusions. The model uses the common Nelson and Siegel (1987) term structure model with a state-space representation that integrates macroeconomic variables. Nonetheless, I will also need to use some inputs defined in Diebold and Li (2002), as the Diebold, Rudebusch and Arouba (2006) model that I'm working is an improvement over this model. The methodology used will be described on section 3.

The reason to choose this model is the possibility to incorporate exogenous variables as my known factors, which will give me a better fit of the term structure, as the macroeconomic variables will improve the fitting of the yield curve. Furthermore, a factor model was suggested by the literature as the ideal to build a yield curve model.

I obtained a good fit of the yield curve using two types of dynamic factor models: a yields-only model and a yields-macro model, where I incorporate exogenous variables. Subsequently, I tested the possibility of a causality connection between macroeconomic variables and the yield curve components, but there's only significant results when I have a shock on the macroeconomic variables and expect a response of the yield curve components. However, the opposite does not happen, which means that the macroeconomic variables may explain the variation on the yield curve, but the vice-versa does not seem to happen in the Portuguese case.

The process of gathering and adjusting the data will be described on section 4. The analysis of the dynamics of the latent factors and the incorporation of the macroeconomic variables will be made on section 5. Finally, section 6 corresponds to the conclusions and the final remarks.

2. Literature Review

There are two different strands of the literature that diverge on their modelling of the yield curve: Nelson-Siegel yield curve decomposition and affine no-arbitrage yield curve models. Nelson and Siegel (1987) proposed a three-latent factor term structure model as the best way to catch the yield curve dynamics over time, improving a limitation on the affine no-arbitrage term structure models, as they seem to function worst at catching this dynamics, although they seem to fit well the yield curve at a specific time period (Brousseau (2002)).

Using the decomposition proposed by Nelson and Siegel (1987), Litterman and Scheinkman (1991) interpreted it as a three latent factors model, which was then clarified by Diebold and Li (2006). Diebold, Li and Li (2004) proved that the Nelson-Siegel decomposition was a good forecasting instrument for the yield curve.

Alternatively, the affine no-arbitrage yield curve models continued to be implemented, as some authors believed that is the best way to compute a term structure model, since this kind of models have a much better numerical tractability (Duffie and Kan (1996). Heath, Jarrow and Morton (1991) developed a framework essential to this strand of the literature. Also, Bikbov and Chernov (2004), Ang and Piazzesi (2003), Hördahl, Tristani and Vestin (2002) and Dai and Singleton (2000) studied the yield curve using an affine no arbitrage term structure model.

Nevertheless, there are some improvements regarding the latent factor yield curve model in the last few years that allowed some simplification in the tractability of the data. Diebold, Rudebusch and Aruoba (2006) proposed a dynamic three-factor model with the incorporation of macroeconomic factors in the estimation, as they wanted to understand better the bidirectional relations among the known factors and the latent factors. They stated that the two-steps approach of the Nelson-Siegel decomposition was not optimal and proposed a one-step approach of the state-space representation of the Nelson-Siegel decomposition and the estimation was made with the Kalman Filter with the goal of obtaining the maximum likelihood of the estimators.

The two-way relationships between the known factors and the latent factors were previously analyzed by Wu (2001), where he studied the relationship between the slope factor and monetary policy shocks. Kozicki and Tinsley (2001) and Dewachter and Lyrio (2003) also investigated a relationship between two factors (inflation and yield curve components), but in an affine term structure model. Rudebusch and Williams (2008) analysed the bidirectional relation between inflation and Level. The incorporation of macroeconomic factors also was not unprecedented, as Carriero, Favero and Kaminska (2004) and Stock and Watson (2003) incorporated a set of macroeconomic factors in a dynamic factor model, proving the importance of this kind of variables. After Diebold, Rudebusch and Aruoba (2006) presented their study, other authors followed their representation and tried to incorporate some new specifications like Afonso and Martins (2010), Hoffmaister, Roldós and Tuladhar (2010) and Favero and Giglio (2006). The first and the last ones tried to implement the effect of fiscal policy in the yield curve model. Hoffmaister, Roldós and Tuladhar (2010) tried to understand better the two-way relationship amid the macroeconomic variables and the yield curve.

3. Methodology

As I said before, the framework used to analyze the dynamics of the Portuguese yield curve is the one made by Diebold, Rudebusch and Aruoba (2006). Also, this model follows some of the assumptions previously made by Diebold and Li (2002), where they interpret the Nelson-Siegel equation as a “latent factor model in which β_1, β_2 and β_3 are time-varying level, slope and curvature factors and the terms that multiply these factors are factors loadings”. The model derived by them is:

$$y(\tau) = L_t + S_t\left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau}\right) + C_t\left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) \tag{1}$$

Where the time-varying level, slope and curvature parameters are represented by L_t, S_t and C_t, respectively, and λ is a decay parameter defined by Nelson and Siegel (1987). In order to estimate and forecast Level, Slope and Curvature, I will use a state-space model, where the dynamics will be estimated using a VAR(1) to be parsimonious on my analysis.

$$\begin{bmatrix} L_t \\ S_t \\ C_t \end{bmatrix} = \begin{bmatrix} \mu_L \\ \mu_S \\ \mu_C \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} L_{t-1} - \mu_L \\ S_{t-1} - \mu_S \\ C_{t-1} - \mu_C \end{bmatrix} + \begin{bmatrix} \eta_t(L) \\ \eta_t(S) \\ \eta_t(\mathrm{C}) \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{bmatrix} = \begin{bmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1} \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} - e^{-\lambda\tau_N} \end{bmatrix} \begin{bmatrix} L_t \\ S_t \\ C_t \end{bmatrix} + \begin{bmatrix} \varepsilon_t(\tau_1) \\ \varepsilon_t(\tau_2) \\ \vdots \\ \varepsilon_t(\tau_N) \end{bmatrix} \tag{3}$$

, maturity of the bonds is represented by τ.

Equation 2 corresponds to the transition equation, which will govern the dynamics of the state-space model. We can see that the latent factors follow an AR(1) process, as we maintain the same assumption as Diebold and Li (2002) and Diebold, Rudebusch and Aruoba (2006). This equation has the following vector notation:

$$f_t = \mu_t + A_t f_{t-1} + \eta_t \tag{4}$$

Equation 3 corresponds to the measurement equation. This equation follows the common Nelson-Siegel representation as it relates a set of N yields along the sample period to the latent factors. The following vector notation corresponds to this equation:

$$y_t = \Lambda f_t + \varepsilon_t \tag{5}$$

These two equation form a state-space model, but before making any estimations, we need to make some assumptions:

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim WN \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix} \right] \tag{6}$$

$$E(f_0 \eta_t') = 0 \tag{7}$$

$$E(f_0 \varepsilon_t') = 0 \tag{8}$$

As it's possible to see in equation 6, 7 and 8, it is essential that the white noise transition and measurement disturbances are orthogonal to each other and to the initial state. Like Diebold, Rudebusch and Aruoba (2006), I assume that there is no correlation between the errors of the different maturities, so matrix H is diagonal. Nonetheless, I assume that the Q matrix will be non-diagonal, allowing correlation among the shocks of the yield curve factors. This will allow me to estimate all parameters simultaneously using the Kalman Filter, so I can get "maximum-likelihood estimates and optimal filtered and smoothed estimates of the underlying factors" Diebold, Rudebusch and Aruoba (2006). With this procedure, I can get the estimated factors for all the time periods in just one-step.

Anyway, the analysis will begin with the two-step approach developed by Diebold and Li (2002), so I can evaluate the dynamics of the latent factors and compare with the real values. The one-step procedure will only be used after that, when I estimate the yield curve using the macroeconomic variables.

4. Data

This state-space approach will require historical data regarding the Portuguese yield curve. The data was obtained, through Bloomberg, using Portuguese zero-coupon yields of the following maturities: 6 Months, 1 Year, 2 Years, 3 Years, 5 Years, 7 Years and 10 Years. These zero-coupon yields provide us a direct observation of the spot rates, but there's also an inconvenient as some of them may have liquidity problems due to limited supply. These spot interest rates range from January 2000 to January 2016 and were obtained on a monthly basis, being collected at the last business day of the month. These yields are bid-ask averages and some of their descriptive statistics can be seen on the following table:

	6M	12M	24M	36M	60M	84M	120M
Mean	2.744	3.176	4.009	4.354	4.782	5.112	5.225
Median	2.469	2.714	3.506	3.691	3.999	4.265	4.612
Maximum	8.956	16.888	22.783	24.134	22.537	19.070	13.687
Minimum	0.010	0.010	0.130	0.435	0.869	1.232	1.672
Std. Dev.	1.813	2.507	3.485	3.535	3.215	2.976	2.119
Skewness	0.698	2.271	2.819	2.803	2.627	2.323	1.664
Kurtosis	3.663	10.748	12.381	12.081	11.124	8.781	5.882
Observations	193	193	193	193	193	193	193

Table 1 – Historical yield data descriptive statistics

The macroeconomic variables were collected from several places and are the Inflation, the Industrial Production Index and the Marginal Lending Rate.

The Portuguese Inflation rate was computed using the Harmonized Consumers Price Index, collected on Eurostat on a monthly basis and it will be called "INFL". The Industrial Production Index is also a global index and it will be used as a proxy for the level of real economic activity. It will be called "IP" and it was collected on Eurostat. The Marginal Lending Rate it is used as a proxy for the monetary policy instrument. This rate was collected on European Central Bank and it is basically the overnight credit rate offered to banks, on a monthly basis and it will be called "MLR".

	Inflation	IP	MLR
Mean	2.147	109.653	0.029
Median	2.500	115.400	0.030
Maximum	5.100	138.000	0.058
Minimum	-1.800	81.500	0.003
Std. Dev.	1.543	18.124	0.017
Skewness	-0.539	-0.136	0.037
Kurtosis	2.424	1.580	1.868

Table 2 – INFL, IP and MLR descriptive statistics.

5. Estimation Analysis

Firstly, it is important to understand how the loadings of the model affect the evolution of the yields over time. I have to fix the λ, so I can get the loadings. As I can see on the graphic below, the loading on β_1 is always one, the loading on β_2 starts near 1 and converges to zero and the loading on β_3 starts around zero, increases until around 24 months and after that it begins to converge to zero again. The similarity between the behavior of this factor loadings and the one presented on Diebold and Li (2006), where λ is fixed at 0.0609, is evident.

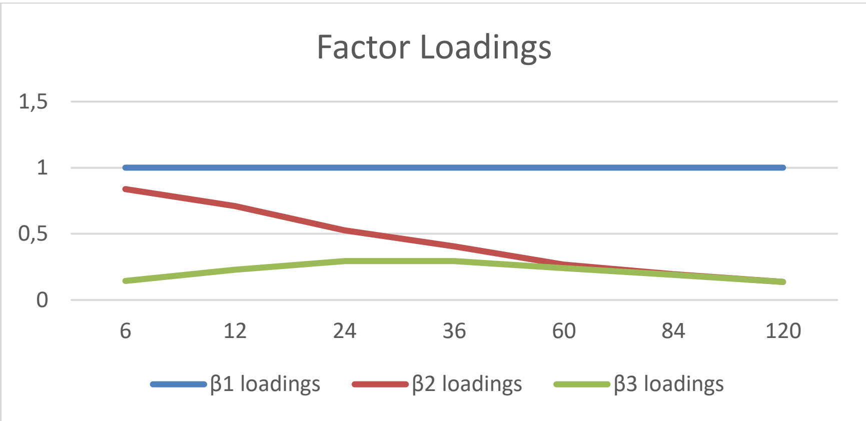

Figure 1 – Factor loadings with constant λ=0.0609

It is possible to analyse the historical level, slope and curvature, using the definitions given by Diebold and Li (2006) for those factors. Historical Level is defined as y_t(120M), Historical Slope is defined as y_t(120M)- y_t(3M), but since the zero-coupon yield with the shortest maturity that I use is a 6-month zero-coupon yield, I have defined it as y_t(120M)- y_t(6M). Lastly, Historical Curvature is defined as $2 \times y_t(24\text{M}) - y_t(120\text{M}) - y_t(6\text{M})$. The descriptive statistics of these variables can be seen in table 3.

	Hist. Level	Hist. Slope	Hist. Curvature
Mean	5.225	2.481	0.050
Median	4.612	1.955	-0.955
Maximum	13.687	8.836	26.125
Minimum	1.672	0.227	-2.636
Std. Dev.	2.119	1.873	3.925
Skewness	1.664	1.207	4.105
Kurtosis	5.882	4.202	21.331
Observations	193	193	193

Table 3 – Level, Slope and Curvature as defined by Diebold and Li (2006).

Diebold and Li (2006) also presented another method to estimate the latent factors, where I can use in all the data periods an OLS to obtain series of time-varying β_1, β_2 and β_3, using an equation, like equation (9) that we can see below:

$$y(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + \beta_{3t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) \tag{9}$$

The descriptive statistics of the series $\{\beta_{1t}, \beta_{2t}\beta_{3t}\}$ are disposed on Table 4.

	β_1	β_2	β_3
Mean	5.641	-3.595	0.638
Median	5.348	-2.637	-2.605
Maximum	12.554	1.289	86.674
Minimum	2.333	-16.627	-7.409
Std. Dev.	1.707	3.209	12.402
Skewness	1.572	-1.715	4.051
Kurtosis	6.481	6.305	21.238
Observations	193	193	193

Table 4 – β_1, β_2 and β_3 computed using OLS.

I can compare the evolution of the betas and the historical latent factors, by looking at their plots, where they have very similar behaviors, apart from the comparison between the Historical Level and β_1, who still have a high correlation between them; corr(β_1, Hist.Level)=0.752, corr(-β_2, Hist.Slope)=0.974, corr(β_3, Hist.Curvature)=0.995.

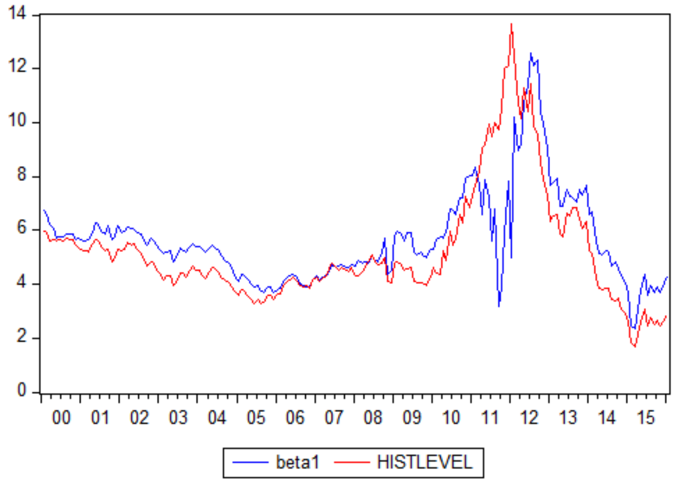

Figure 2 – Estimated β1 versus Historical Level

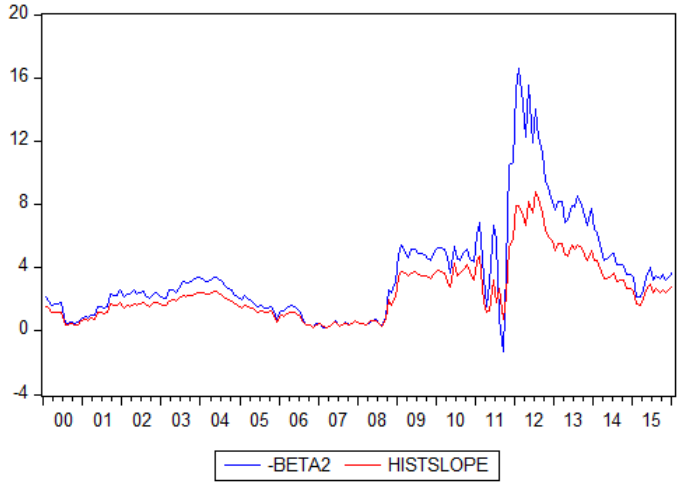

Figure 3 – Estimated -β2 versus Historical Level

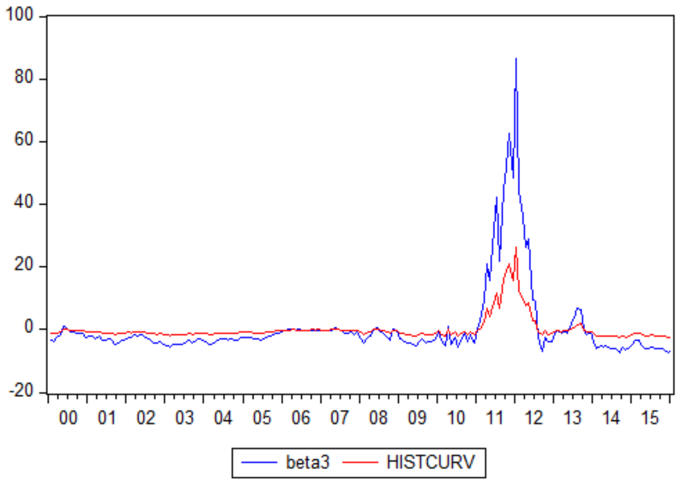

Figure 4 – Estimated β3 versus Historical Level

Additionally, I can also examine the average estimated yield curve and the average actual yield curve to evaluate the quality of this fit, which is clearly perceivable in figure 5.

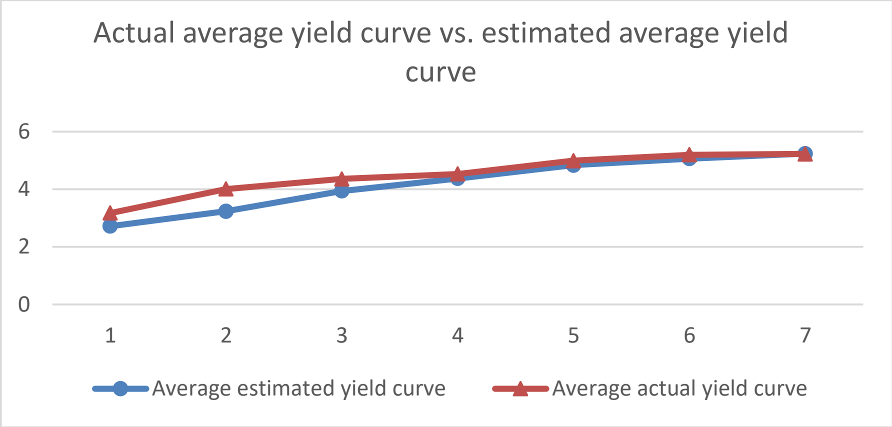

Figure 5 – Average actual yield curve vs. average estimated yield curve

The average estimated yield curve resembles quite well the pattern of the average actual yield curve, which makes me conclude that the estimated betas are a good instrument to our state-space model VAR.

5.1. Yields-only model

The first model I have to estimate is the yields-only model, where the dependent variables of our transition equation correspond only to the latent factors, as I can see using equations (2) and (3).

$$\begin{bmatrix} L_t \\ S_t \\ C_t \end{bmatrix} = \begin{bmatrix} \mu_L \\ \mu_S \\ \mu_C \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} L_{t-1} - \mu_L \\ S_{t-1} - \mu_S \\ C_{t-1} - \mu_C \end{bmatrix} + \begin{bmatrix} \eta_t(L) \\ \eta_t(S) \\ \eta_t(C) \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} y_t(6) \\ y_t(12) \\ \vdots \\ y_t(120) \end{bmatrix} = \begin{bmatrix} 1 & \frac{1-e^{-\lambda\times 6}}{\lambda\times 6} & \frac{1-e^{-\lambda\times 6}}{\lambda\times 6} - e^{-\lambda\times 6} \\ 1 & \frac{1-e^{-\lambda\times 12}}{\lambda\times 12} & \frac{1-e^{-0.0609\times 12}}{0.0609\times 12} - e^{-\lambda\times 12} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\times 120}}{\lambda\times 120} & \frac{1-e^{-\lambda\times 120}}{\lambda\times 120} - e^{-\lambda\times 120} \end{bmatrix} \begin{bmatrix} L_t \\ S_t \\ C_t \end{bmatrix} + \begin{bmatrix} \varepsilon_t(6) \\ \varepsilon_t(12) \\ \vdots \\ \varepsilon_t(120) \end{bmatrix} \tag{11}$$

Equation (9) corresponds to the transition equation and the measurement equation is equation (10), where Y_t is the dependent variable, where I have all the collected historical yields with maturities from 6 months to 120 months.

The error distribution will be defined as:

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim WN\left[\begin{pmatrix} 0_{(3\times 1)} \\ 0_{(7\times 1)} \end{pmatrix}, \begin{pmatrix} Q_{(3\times 3)} & 0 \\ 0 & H_{(7\times 7)} \end{pmatrix}\right] \tag{12}$$

Matrix Q corresponds to the disturbance matrix of the transition equation and won't be diagonal, but the matrix H will be diagonal and is the disturbance matrix of the measurement equation.

Altogether, I have to estimate a total of 25 parameters: matrix Q has 6 parameters to be estimated (3 variances and 3 covariances), vector μ_t has 3 parameters to be estimated, matrix H has 7 parameters to be estimated, the transition matrix A_t has 9 parameters to be estimated and, finally, I also have to estimate λ.

Nevertheless, before I start estimating, I have to define the initial values. Matrix A_t is initialized with the coefficients obtained with an unrestricted VAR(1) between $\{\beta_{1t}, \beta_{2t}\beta_{3t}\}$, vector μ_t is initialized with the means of the same factors, matrix H is initialized with the variances of the historical yields and matrix Q is also initialized using the factors computed in section 5. This estimation of the optimal yields and the state variables was made through a multivariate Kalman Filter, using the Marquardt and Berndt-Hall-Hall-Hausman algorithms.

The coefficients obtained can be seen on the table below:

	L_{t-1}	S_{t-1}	C_{t-1}	μ
L_t	**0,760**	-0,109	0,011	**5,406**
	(0,104)	(0,088)	(0,008)	(0,738)
S_t	**0,141**	**1,015**	**-0,014**	**-3,125**
	(0,074)	(0,057)	(0,008)	(1,228)
C_t	**1,245**	0,879	**0,951**	-0,080
	(0,726)	(0,577)	(0,059)	(8,248)
Q Matrix				
L_t	0,840	**-0,423**	**-4,661**	
	(1,273)	(0,158)	(0,735)	
S_t		0,556	**1,550**	
		(1,243)	(0,803)	
C_t			**31,998**	
			(1,087)	

Bold coefficients represent parameter estimates significant at the 10 percent level and standard errors are in parentheses.

Table 5 – Coefficients of matrix A_t and matrix Q.

In matrix A it's possible to denote that the own dynamics of L_t, S_t and C_t are very persistent. I can see also that the cross-factor dynamics are sometimes statistically significant, with the lags of Level and Curvature having a significant effect on Slope and the lag of Level also having a significant effect on Curvature. The average level, slope (don't forget that the slope is defined as short-term minus long-term, so a negative average slope means that yields increase as maturity rises) and curvature are approximately 5,4 -3,125% and insignificantly different from zero, respectively. There are three significant covariance terms, as I can see on the Q matrix. The estimated λ is 0.0456, which means that the loading on the curvature factor ix maximized at a maturity of 39 months.

5.2. Yields-Macro model

The second model to be estimated is the yields-macro model, where I also add the macroeconomic and financial variables (inflation, spread, industrial production and marginal lending rate) to our dependent variables in the transition equation (13).

$$\begin{bmatrix} L_t \\ S_t \\ C_t \\ IP_t \\ INFL_t \\ MLR_t \end{bmatrix} = \begin{bmatrix} \mu_L \\ \mu_S \\ \mu_C \\ \mu_{IP} \\ \mu_{INFL} \\ \mu_{MLR} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \times \begin{bmatrix} L_{t-1} - \mu_L \\ S_{t-1} - \mu_S \\ C_{t-1} - \mu_C \\ INFL_{t-1} - \mu_{IP} \\ IP_{t-1} - \mu_{INFL} \\ MLR_{t-1} - \mu_{MLR} \end{bmatrix} + \begin{bmatrix} \eta_t(L) \\ \eta_t(S) \\ \eta_t(\mathrm{C}) \\ \eta_t(\mathrm{IP}) \\ \eta_t(\mathrm{INFL}) \\ \eta_t(\mathrm{MLR}) \end{bmatrix} \quad (13)$$

$$\begin{bmatrix} y_t(6) \\ y_t(12) \\ \vdots \\ y_t(120) \end{bmatrix} = \begin{bmatrix} 1 & \frac{1-e^{-\lambda\times6}}{\lambda\times6} & \frac{1-e^{-\lambda\times6}}{\lambda\times6} - e^{-\lambda\times6} & 0 & 0 & 0 \\ 1 & \frac{1-e^{-\lambda\times12}}{\lambda\times12} & \frac{1-e^{-0.0609\times12}}{0.0609\times12} - e^{-\lambda\times12} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\times120}}{\lambda\times120} & \frac{1-e^{-\lambda\times120}}{\lambda\times120} - e^{-\lambda\times120} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} L_t \\ S_t \\ C_t \\ IP_t \\ INFL_t \\ MLR_t \end{bmatrix} + \begin{bmatrix} \varepsilon_t(6) \\ \varepsilon_t(12) \\ \vdots \\ \varepsilon_t(120) \end{bmatrix} \quad (14)$$

In the extended model, the structure remains almost unchanged, but the dimensions of some matrices have increased. The loading matrix on equation 14 will be 6x6, where the yields will load only on the yield curve factors, so the three columns on the right hold only zeros, as Diebold, Rudebusch and Aruoba (2006).

The error distribution will be similar as the one defined in section 5.1, but the dimensions of Matrix Q will grow to 6x6, as I can see below:

$$\begin{pmatrix}\eta_t\\ \varepsilon_t\end{pmatrix} \sim WN\left[\begin{pmatrix}0_{(6\times1)}\\ 0_{(7\times1)}\end{pmatrix}, \begin{pmatrix}Q_{(6\times6)} & 0\\ 0 & H_{(7\times7)}\end{pmatrix}\right] \tag{15}$$

Now, in this extended model, I will have to estimate 71 parameters: matrix Q has 21 parameters to be estimated (6 variances and 15 covariances), the transition matrix A_t has 36 parameters to be estimated, vector μ_t has 6 parameters to be estimated, matrix H has 7 variances to be estimated and, finally, I also have to estimate λ.

The initialization and estimation procedures will be similar to the one used on section 5.1., but now the coefficient matrix A_t will be initialized using the coefficients estimated on the unrestricted VAR(1) of the previously estimated $\{\beta_{1t}, \beta_{2t}\beta_{3t}\}$ and also the inflation, the industrial production, the marginal lending rate and the spread between the German and the Portuguese yields. This will be made with recursive causal ordering of the variables. First, I have the term structure factors and, then, I have the macroeconomic variables.

The coefficients obtained can be seen on the table below:

	L_{t-1}	S_{t-1}	C_{t-1}	IP_{t-1}	MLR_{t-1}	$INFL_{t-1}$	μ
L_t	**0,499**	**-0,397**	0,006	0,001	37,182	0,007	**5,288**
	(0,212)	(0,178)	(0,021)	(0,038)	(40,150)	(0,232)	(1,265)
S_t	0,238	**1,128**	**-0,013**	0,003	-19,707	0,020	**-3,601**
	(0,172)	(0,122)	(0,017)	(0,020)	(20,556)	(0,180)	(3,912)
C_t	**3,111**	**2,910**	**0,987**	-0,0003	-265,904	-0,012	**1,224**
	(1,478)	(1,026)	(0,152)	(0,245)	(241,583)	(1,522)	(19,243)
IP_t	-0,074	-0,006	-0,016	**0,981**	13,007	0,008	**107,537**
	(0,650)	(0,619)	(0,076)	(0,027)	(60,799)	(0,242)	(45,258)
MLR_t	0,0004	0,0004	0,000	0,000	**0,934**	-0,0001	0,024
	(0,001)	(0,001)	(0,000)	(0,000)	(0,053)	(0,0002)	(0,043)
$INFL_t$	0,067	0,070	0,005	0,005	-5,019	**0,903**	1,794
	(0,078)	(0,062)	(0,012)	(0,005)	(7,913)	(0,046)	(2,796)

Q Matrix	L_t	S_t	C_t	IP_t	MLR_t	$INFL_t$
L_t	-0,388	-0,335	**3,700**	0,145	0.000	0,044
	(0,049)	(0,294)	(1,229)	(0,704)	(0,001)	(0,089)
S_t		**-0,696**	1,074	-0,051	0,0002	-0,002
		(0,362)	(1,766)	(0,364)	(0,0003)	(0,061)
C_t			**3,250**	-0,670	0,001	-0,170
			(0,175)	(4,405)	(0,004)	(0,573)
IP_t				**1,625**	0,0004	0,093
				(0,129)	(0,001)	(0,098)
MLR_t					**-12,867**	0,0001
					(0,157)	(0,0000)
$INFL_t$						**-1,928**
						(0,129)

Bold entries denote parameter estimates significant at the 10 percent level and standard errors are in parentheses.

Table 6 – Coefficients of matrix A_t and matrix Q.

The first part of table 6 contains the estimates of the coefficients of matrix A_t of the yields-macro model, which correspond to the vital macro and term structure connections. Nonetheless, some of the off-diagonal elements appear insignificant, but they are jointly significant. On the second part of table 6, the matrix Q is presented.

5.3. Evaluating the Fit of the model

Both of the previous models were obtained with the goal of getting an optimal term structure prediction from January 2000 to January 2016. The Yields-only model only estimates the time series of β_1, β_2 and β_3, but the Yields-macro model also estimates the time series of Industrial Production index, marginal lending rate and inflation.

In an attempt to evaluate the fit of these two models in comparison with the historical yield curves, I have computed the measurement errors, for both models, and also some statistics were obtained in table 7.

		6M	12M	24M	36M	60M	84M	120M
MSE	Yields-Only	0.204	1.006	1.257	1.074	0.757	0.518	0.186
	Yields-Macro	0.182	0.828	1.003	0.868	0.690	0.439	0.164
Correlation	Yields-Only	0.968	0.917	0.949	0.956	0.963	0.974	0.979
	Yields-Macro	0.972	0.932	0.960	0.965	0.966	0.978	0.982

Table 7 – Measures of forecasting accuracy for the 2 models.

Table 7 gives a good overview of the quality of the fit, which is pretty alike in both models. It's possible to see that both of the models are a really good fit for the historical values of the yield curves, because we can see that the correlation is really high and the computed mean square errors are low.

Subsequently, I plotted the actual Level, Slope and Curvature, like Diebold and Li (2006), with the estimated latent factors using both models.

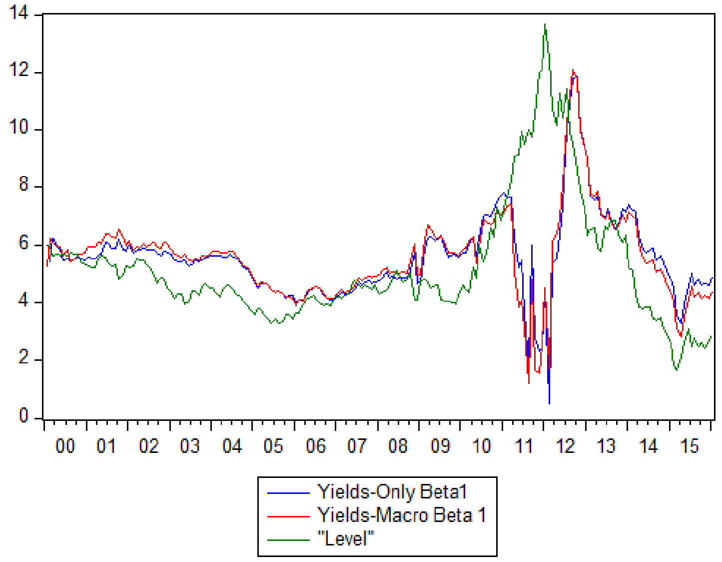

Figure 6 – Historical Level versus Estimated Betas 1

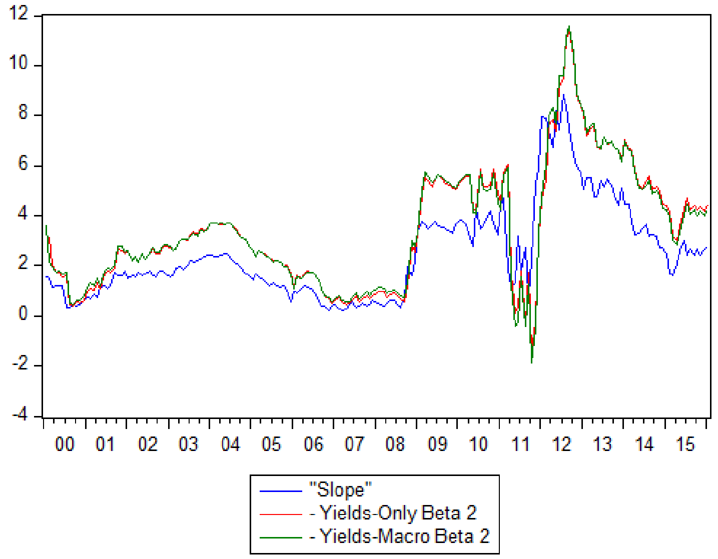

Figure 7 – Historical Slope versus Estimated Betas 2

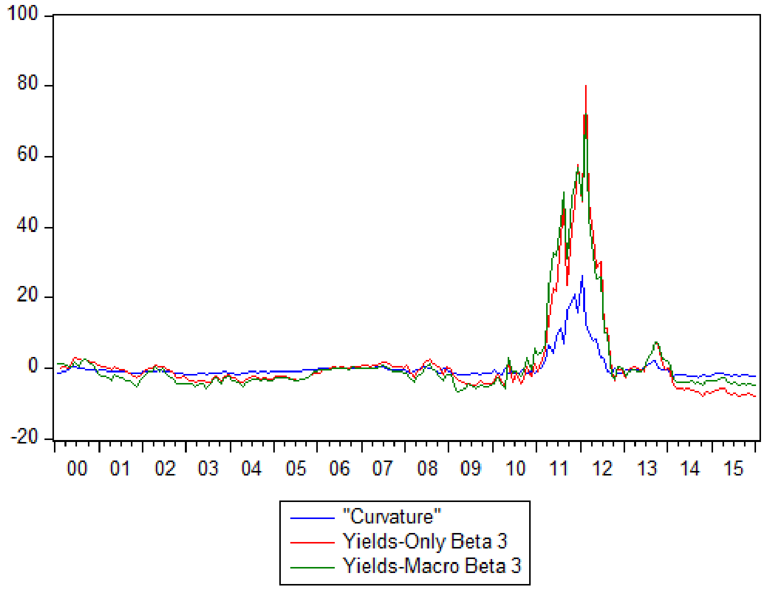

Figure 8 – Historical Curvature versus Estimated Betas 3

Regardless of some differences, especially in crisis years, the behavior is similar, principally in the “normal” years of the yields, so both models match really well the dynamics that I wanted to achieve. The Level of the estimated yield curve models doesn’t follow the same path as the variable estimated using Diebold and Li (2006) method in the 2008-2009, where it decreases to the its lowest values, but aside from a greater impact in the state-space model, that matches quite well the behavior of the long-term yields in those years. The Level in those models also reaches its peak between 2011-2012, which also resembles quite well the behavior on the longer-term zero-coupon yields. The (symmetric of the) Slope of the estimated yield curve models matches fits well the expected from the estimation. In the crisis years there is a lot of turbulence on the yield curve behavior, which leads to very different yield curve patterns from 2008 to 2012, just like what I observed from the yields collected. Finally, the Curvature of the estimated yield curve models also has a very similar behavior along the years without a financial crisis, but on those years, although I verify that the curvature rises through 2010 to 2012, the impact on this component is way higher on the estimated curvature through our yields-only and yields-macro models than using the Diebold and Li (2006) method.

5.4. Macroeconomic and yield curve impulse response functions

To get a better view of the dynamics of the yields-macro model, we can see the impulse response functions, which are represented on the figure below:

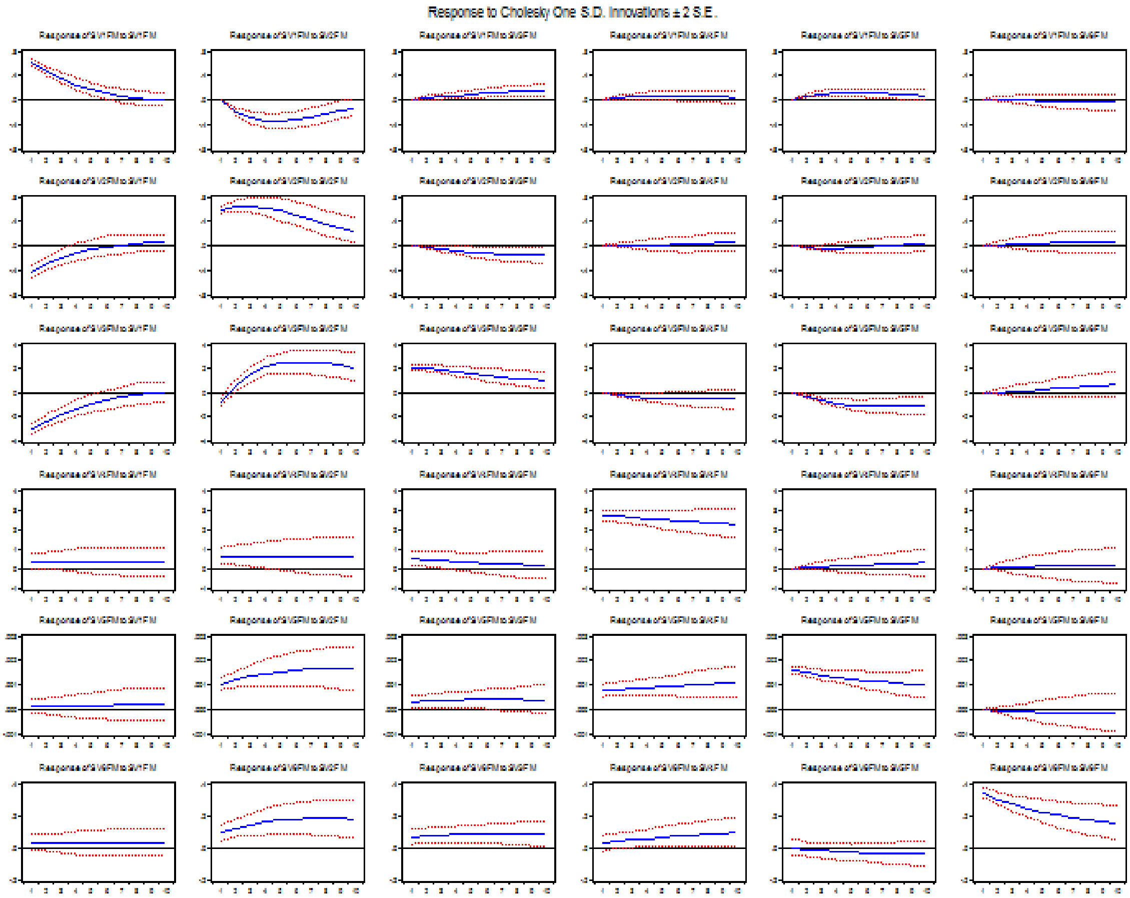

Figure 9 – Impulse response functions of the yield-macro model

The responses of macro variables to macro shocks (on the bottom right quarter) show the typical persistence of macro variables in these kind of models, but it's possible to see that there's no significant connection between them, unlike the expected in these kind of models. The only difference is the shocks on Industrial Production that lead also to an increase on the marginal lending rate and inflation.

On the other hand, the responses of macro variables to yield curve components shocks (on the bottom left quarter) are a bit more interesting. An increase in the slope factor leads to an increase in the marginal lending rate and in inflation, which is an interesting effect of this yield curve component. If yields increase (slope increases), it's very likely that the marginal lending rate also increases in a reaction to this shock. In contrast, an increase on the other two components (level and curvature) does not seem to cause any big effects on macro variables. This may be due to the specificities of this time period with the crisis and the quantitative easing period, where the yields does not seem to behave as normally as in the past.

The response of the yield curve components to the macro variables shocks (on the top right quarter) are not significant, but I can see that at a lesser extent a shock on the marginal lending rate has an unexpected effect on the curvature of the yield curve, decreasing it.

Finally, the own-dynamics of the yield curve factors (on the top left quarter) exhibit significant persistence and there are also some significant off-diagonal responses, specially a shock in the level factor leads to an immediate effect on the other term structure components that slowly decreases along time and a shock in the slope factor also leads to an immediate effect on the other two term structure components, where the level factor decreases immediately and the curvature factor has the opposite reaction.

5.5. Variance decomposition of macro variables and yield curve components

If I want to improve our analysis on the macro and yield curve interactions, then the variance decomposition is a popular metric to do it. I extract it from the VAR of the yield-only model and the yields-macro model. This analysis will show us how is the change of an interest rate explicated (by the latent factors, the macro variables and itself). I will do two variance decompositions analysis: one with the yields-only model and one with the yields-macro model and I will use three different yields (short-term, medium-term and long-term).

Horizon	6M Yield	L	S	C
10	93,584	5,338	1,042	0,036
		83,199%	16,236%	0,566%
60	81,275	16,521	2,168	0,037
		88,228%	11,576%	0,196%
120	80,431	16,685	2,845	0,039
		85,262%	14,539%	0,200%

Table 8 – Variance decomposition of a 6-Month yield, using yields-only model

Horizon	6M Yield	L	S	C	IP	MLR	INFL
10	93,852	2,205	1,463	0,470	0,051	0,019	1,939
		35,861%	23,800%	7,652%	0,836%	0,3158%	31,536%
60	80,771	3,447	2,838	1,055	4,878	0,858	6,153
		17,926%	14,758%	5,488%	25,369%	4,461%	31,997%
120	73,417	3,334	2,896	2,765	10,415	1,443	5,731
		12,542%	10,894%	10,401%	39,179%	5,428%	21,558%

Table 9 – Variance decomposition of a 6-Month yield, using yields-macro model

Under the yields-only model, I can see Table 8, where the variation of the 6-month yield is mainly explained by itself at a horizon of 10 months, while the remaining variance is explained mainly by the level factor (83%). Nevertheless, at a longer horizon, the remaining variance is mainly explained by level factor (85%) and slope factor (14%).

Under the yields-macro model, I can see Table 9, where the 6-month yield is mostly explained by itself (73%), but with a greater impact of the IP (10%). This means that the GDP growth proxy explains a bit of the variance of the short-term end of the yield curve.

Horizon	60M Yield	L	S	C
10	83,233	0,054	16,065	0,649
		0,320%	95,812%	3,868%
60	31,274	11,120	55,978	1,628
		16,180%	81,451%	2,369%
120	25,503	15,325	57,249	1,923
		20,572%	76,847%	2,582%

Table 10 – Variance decomposition of a 60-Month yield, using yields-only model

Horizon	60M Yield	L	S	C	IP	MLR	INFL
10	45,102	0,517	44,125	5,193	0,093	4,553	0,417
		0,941%	80,376%	9,459%	0,170%	8,295%	0,759%
60	27,738	1,159	43,546	7,512	1,749	12,146	6,149
		1,604%	60,262%	10,395%	2,420%	16,809%	8,510%
120	26,331	1,520	40,006	7,382	6,073	12,974	5,713
		2,064%	54,305%	10,020%	8,244%	17,611%	7,756%

Table 11 – Variance decomposition of a 60-Month yield, using yields-macro model

As I can see on both tables, it is evident that on a 60-month yield, the bigger the horizon, the smaller will be the percentage of the yield variance due to itself (around 25% on yields-only model, around 26% on yields-macro model). The most relevant factor is the slope, unlike what is said on the literature, but on yields-macro model the marginal lending rate also accounts for a good portion of the variation in the yield (17%).

Horizon	120M Yield	L	S	C
10	89,977	0,046	9,944	0,033
		0,463%	99,208%	0,329%
60	42,320	10,399	47,176	0,105
		18,029%	81,790%	0,182%
120	37,404	12,936	49,548	0,112
		20,666%	79,156%	0,178%

Table 12 – Variance decomposition of a 120-Month yield, using yields-only model

Horizon	120M Yield	L	S	C	IP	MLR	INFL
10	74,159	0,655	21,016	1,532	0,004	2,586	0,048
		2,534%	81,329%	5,929%	0,015%	10,007%	0,186%
60	53,563	0,359	25,449	6,266	3,095	8,855	2,413
		0,773%	54,803%	13,494%	6,664%	19,070%	5,196%
120	48,742	0,383	21,779	6,433	10,987	9,463	2,210
		0,747%	42,490%	12,552%	21,436%	18,463%	4,312%

Table 13 – Variance decomposition of a 60-Month yield, using yields-macro model

At the longest horizon, the slope factor is what mostly explains the variation of the yield (49%) on the yields-only model, but on the yields-macro model, it is the yield itself that mainly explains its variation. The marginal lending rate also accounts for a good portion of the variation (18%) at the largest horizon, suggesting that the monetary policy instrument also accounts for an important portion of the variation of a yield.

6. <u>Conclusion</u>

I have succeeded in the specification and estimation of the yield curve with both yield curve models suggested by Diebold, Rudebusch and Arouba (2006). I started using only latent factors (level, slope and curvature) and, then, I added macroeconomic variables (inflation, marginal lending rate and industrial production). So, I have managed to find a good fit of the yield curve using a yields-only model and a yields-macro model.

The estimated Level, Slope and Curvature follow a reasonable path for the yield curve components. The Level maintains a similar value between 4% and 6% in the pre-crisis years. Along those years, it first starts to decrease to a value around zero and then it makes a high jump to values above 10%, which perfectly describes the instability felt on those years. After

2015, the stability arrives again. The (symmetric of the) Slope starts to have a value slightly above zero, but in 2008 it has a high jump to a value around 6 from some time. When I reach 2011, the slope goes to values below zero and almost immediately rises to values above 10, which describes again the instability felt on those years. After 2012, the slope is slowly decreasing and it reached normal values again. The curvature has a residual value between 2000 and 2010. When I reach 2011, the curvature rises exponentially to values around 60 and 75, depending on the estimated model (yields-only or yields-macro), obtaining higher values on the yields-only model. After 2012, it goes back again to residual values. So, what I may conclude is that the yield curve assumes standard values on the pre-crisis years in Portugal, as it seems that they follow a similar path as the one I have on the whole EU. When the crisis arrives, the yield curve assumes all kinds of forms, with a lot of instability on the yields, and a greater systematic risk affecting the bond market.

I found some signs of macroeconomic effects on the future yield curve, but there doesn't seem to exist a yield curve effect on the macroeconomic variables. So, there is no evidence of a bidirectional relationship between the yield curve components and the macroeconomic variables.

From the variance decomposition analysis, the level accounted for most of the remaining variance of the yields on the short-term end of the yield curve, but the industrial production (GDP growth proxy) explained most of the remaining variance on longer horizons.

Also from this analysis it was possible to perceive that on the medium and long-term yields the slope and the marginal lending rate accounted for most of the variance on the term structure, which is not what was expected from Nelson and Siegel (1987), as the theory predicted that the level would explain most of the variation of the yields with a longer term.

For future research, my main goal is to implement a new exogenous variable which could catch the inherent risk on the yield curve and to add more maturities to my analysis, without having a collinearity problem.

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Appendix

Matrix H

	$(6M)_{t-1}$	$(12M)_{t-1}$	$(24M)_{t-1}$	$(36M)_{t-1}$	$(60M)_{t-1}$	$(84M)_{t-1}$	$(120M)_{t-1}$
$(6M)_t$	0,0001	0	0	0	0	0	0
$(12M)_t$		0,501	0	0	0	0	0
$(24M)_t$			0,170	0	0	0	0
$(36M)_t$				0.000	0	0	0
$(60M)_t$					0,051	0	0
$(84M)_t$						0,095	0
$(120M)_t$							0,014

Matrix Q

	$\beta_{1,\,t-1}$	$\beta_{2,\,t-1}$	$\beta_{3,\,t-1}$
$\beta_{1,\,t}$	0,870	-0,450	-4,710
$\beta_{2,\,t}$	-0,450	0,581	1,593
$\beta_{3,\,t}$	-4,710	1,593	32,077

Final State Vector

L_t	5,054
S_t	-4,587
C_t	-8,067

Appendix 1 – Yields-only model final values of Matrix H, Q and the state vector

Matrix H

	$(6M)_{t-1}$	$(12M)_{t-1}$	$(24M)_{t-1}$	$(36M)_{t-1}$	$(60M)_{t-1}$	$(84M)_{t-1}$	$(120M)_{t-1}$
$(6M)_t$	0,001	0	0	0	0	0	0
$(12M)_t$		0,498	0	0	0	0	0
$(24M)_t$			0,168	0	0	0	0
$(36M)_t$				0.000	0	0	0
$(60M)_t$					0,050	0	0
$(84M)_t$						0,094	0
$(120M)_t$							0,018

Matrix Q

	$\beta_{1,\,t-1}$	$\beta_{2,\,t-1}$	$\beta_{3,\,t-1}$	IP_{t-1}	MLR_{t-1}	$INFL_{t-1}$
$\beta_{1,\,t}$	0,717	-0,368	-3,761	0,143	0.000	0,044
$\beta_{2,\,t}$	-0,368	0,529	1,129	-0,049	0,0002	-0,002
$\beta_{3,\,t}$	-3,761	1,129	25,894	-0,667	-0,001	-0,170
IP_t	0,143	-0,049	-0,667	5,077	0,0004	0,093
MLR_t	0.000	0,0002	-0,001	0,0004	0.000	0,0001
$INFL_t$	0,044	-0,002	-0,170	0,093	0,0001	0,145

Final State Vector

L_t	4,596
S_t	-4,383
C_t	-5,070
IP_t	84,038
MLR_t	0,003
$INFL_t$	0,650

Appendix 2 – Yields-macro model final values of Matrix H, Q and the state vector