A remark on Toeplitz and Laurent operators acting on ℓ^p spaces with power weights

To Albrecht Böttcher on the occasion of his 70th birthday

Oleksiy Karlovych[a], Eugene Shargorodsky[b]

[a] *Centro de Matemática e Aplicações, Departamento de Matematica,*
Faculdade de Ciencias e Tecnologia, Universidade Nova de Lisboa,
Quinta da Torre, Caparica, 2829–516, Portugal
[b] *Department of Mathematics, King's College London,*
Strand, London, WC2R 2LS, United Kingdom

Abstract

In the late 1980's, Böttcher and Silbermann asked whether the boundedness of the Toeplitz operator $T(a)$ on the space $\ell^p_\mu(\mathbb{Z}_+)$ implies boundedness of the Laurent operator $M(a)$ on the space $\ell^{p,p}_{0,\mu}(\mathbb{Z})$. We give the negative answer to this question.

Keywords: Toeplitz operator, Laurent operator, weighted ℓ^p space, power weight, discrete Hilbert transform, discrete Muckenhoupt condition.
2020 MSC: 47B35.

---

1. Introduction and the main result

The Banach algebra of all bounded linear operators on a Banach space X will be denoted by $\mathcal{B}(X)$. Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle on the complex plane. For a function $a \in L^1(\mathbb{T})$, the Fourier coefficients of a are defined by

$$a_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} a(e^{i\theta})e^{-in\theta}d\theta, \quad n \in \mathbb{Z}.$$

Email addresses: `oyk@fct.unl.pt` (Oleksiy Karlovych), `eugene.shargorodsky@kcl.ac.uk` (Eugene Shargorodsky)
URL: `https://docentes.fct.unl.pt/oyk` (Oleksiy Karlovych), `https://www.kcl.ac.uk/people/eugene-shargorodsky` (Eugene Shargorodsky)

Preprint submitted to Linear Algebra and its Applications *August 30, 2023*

Let α, β be any real numbers and p, r be real numbers satisfying $1 \le p, r < \infty$. Following [4, Section 1.49], we denote by $\ell^{r,p}_{\beta,\alpha}(\mathbb{Z})$ the Banach space of all sequences $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}}$ of complex numbers for which

$$\|\varphi\|_{r,p;\beta,\alpha} := \left(\sum_{n=1}^{\infty} |\varphi_{-n}|^r (n+1)^{r\beta}\right)^{1/r} + \left(\sum_{n=0}^{\infty} |\varphi_n|^p (n+1)^{p\alpha}\right)^{1/p} < \infty.$$

For $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ and α, p as above, let $\ell^p_\alpha(\mathbb{Z}_+)$ be the Banach space of all sequences $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}_+}$ satisfying

$$\|\varphi\|_{p,\alpha} := \left(\sum_{n=0}^{\infty} |\varphi_n|^p (n+1)^{p\alpha}\right)^{1/p} < \infty.$$

We will use the following abbreviations:

$$\ell^p_\alpha(\mathbb{Z}) := \ell^{p,p}_{\alpha,\alpha}(\mathbb{Z}), \quad \ell^p(\mathbb{Z}_+) := \ell^p_0(\mathbb{Z}_+), \quad \ell^p(\mathbb{Z}) := \ell^p_0(\mathbb{Z}).$$

The nonweighted space $\ell^p(\mathbb{Z})$ will be always equipped with the norm

$$\|\varphi\|_p := \left(\sum_{n=-\infty}^{\infty} |\varphi_n|^p\right)^{1/p},$$

which is equivalent to the norm $\|\varphi\|_{p,p;0,0}$ defined above. Finally, let $\ell^0(\mathbb{Z})$ and $\ell^0(\mathbb{Z}_+)$ denote the sets of sequences having finite support in $\mathbb{Z}$ and $\mathbb{Z}_+$, respectively.

Let $a \in L^1(\mathbb{T})$ have the Fourier coefficients sequence $\{a_n\}_{n\in\mathbb{Z}}$. Given $\varphi = \{\varphi_j\}_{j\in\mathbb{Z}} \in \ell^0(\mathbb{Z})$, define the sequence $a * \varphi$ by

$$(a * \varphi)_j := \sum_{k=-\infty}^{\infty} a_{j-k}\varphi_k, \quad j \in \mathbb{Z}.$$

For $1 \le p, r < \infty$ and $\alpha, \beta \in \mathbb{R}$, let $M^{r,p}_{\beta,\alpha}$ denote the collection of all $a \in L^1(\mathbb{T})$ for which $a * \varphi$ belongs to $\ell^{r,p}_{\beta,\alpha}(\mathbb{Z})$ whenever $\varphi \in \ell^0(\mathbb{Z})$ and

$$\sup\left\{\frac{\|a * \varphi\|_{r,p;\beta,\alpha}}{\|\varphi\|_{r,p;\beta,\alpha}} : \varphi \in \ell^0(\mathbb{Z}), \varphi \ne 0\right\} < \infty.$$

If $a \in M^{r,p}_{\beta,\alpha}$, then the operator $\ell^0(\mathbb{Z}) \to \ell^{r,p}_{\beta,\alpha}(\mathbb{Z})$, $\varphi \mapsto a * \varphi$ extends to a bounded operator

$$M(a) : \ell^{r,p}_{\beta,\alpha}(\mathbb{Z}) \to \ell^{r,p}_{\beta,\alpha}(\mathbb{Z}), \quad \varphi \mapsto a * \varphi,$$

which is referred to as the Laurent (or multiplication) operator on $\ell^{r,p}_{\beta,\alpha}(\mathbb{Z})$ generated by the function a.

Let P be the projection from $\ell^{r,p}_{\beta,\alpha}(\mathbb{Z})$ onto $\ell^p_\alpha(\mathbb{Z}_+)$ defined as follows:

$$(P\varphi)_n = 0 \quad \text{if} \quad n \in \mathbb{Z} \setminus \mathbb{Z}_+, \quad (P\varphi)_n = \varphi_n \quad \text{if} \quad n \in \mathbb{Z}_+.$$

For $\varphi \in \ell^p_\alpha(\mathbb{Z}_+)$, let $\widetilde{\varphi} \in \ell^{r,p}_{\beta,\alpha}(\mathbb{Z})$ be defined by

$$\widetilde{\varphi}_n = 0 \quad \text{if} \quad n \in \mathbb{Z} \setminus \mathbb{Z}_+, \quad \widetilde{\varphi}_n = \varphi_n \quad \text{if} \quad n \in \mathbb{Z}_+.$$

For $a \in L^1(\mathbb{T})$, the Toeplitz operator $T(a)$ is defined on $\ell^0(\mathbb{Z}_+)$ by

$$T(a)\varphi = P(a * \widetilde{\varphi}), \quad \varphi \in \ell^0(\mathbb{Z}_+).$$

Let $1 \le p < \infty$ and $\mu \in \mathbb{R}$. Following [4, Section 6.1], we write $T(a) \in \mathcal{B}(\ell^p_\mu(\mathbb{Z}_+))$ if $T(a)\varphi \in \ell^p_\mu(\mathbb{Z}_+)$ for all $\varphi \in \ell^0(\mathbb{Z}_+)$ and

$$\|T(a)\|_{\mathcal{B}(\ell^p_\mu(\mathbb{Z}_+))} := \sup\left\{ \frac{\|T(a)\varphi\|_{p,\mu}}{\|\varphi\|_{p,\mu}} : \varphi \in \ell^0(\mathbb{Z}_+), \varphi \neq 0 \right\} < \infty.$$

For $n \in \mathbb{Z}_+$, let $e_n \in \ell^0(\mathbb{Z}_+)$ be given by $(e_n)_k = \delta_{nk}$, where δ_{nk} is the Kronecker delta. For $\varphi \in \ell^p(\mathbb{Z}_+)$ and $\psi \in \ell^q(\mathbb{Z}_+)$, where $1/p + 1/q = 1$, let

$$(\varphi, \psi) = \sum_{n=0}^{\infty} \varphi_n \overline{\psi_n}.$$

It is easy to check that

$$(T(a)e_j, e_k) = a_{k-j} \quad \text{for all} \quad j, k \in \mathbb{Z}_+.$$

The following version of the Brown-Halmos theorem for Toeplitz operators on $\ell^p(\mathbb{Z}_+)$ is contained in [4, Theorem 2.7(b)].

Theorem 1.1 (à la Brown-Halmos)**.** *Let $1 \le p < \infty$ and $A \in \mathcal{B}(\ell^p(\mathbb{Z}_+))$. Suppose there exists a sequence of complex numbers $\{a_n\}_{n\in\mathbb{Z}}$ such that*

$$(Ae_j, e_k) = a_{k-j} \quad \textit{for all} \quad j, k \in \mathbb{Z}_+.$$

Then there exists $a \in M^{p,p}_{0,0}$ such that $A = T(a)$ and $\{a_n\}_{n\in\mathbb{Z}}$ is the Fourier coefficients sequence of a. Moreover,

$$\|T(a)\|_{\mathcal{B}(\ell^p(\mathbb{Z}_+))} = \|M(a)\|_{\mathcal{B}(\ell^p(\mathbb{Z}_+))}.$$

Let $W(\mathbb{T})$ be the Wiener algebra consisting of all continuous functions on $\mathbb{T}$ such that their sequences of Fourier coefficients belong to $\ell^1(\mathbb{Z})$.

In the late 1980's Böttcher and Silbermann observed that the above fact fails in the weighted case. More precisely, the following is true (see [3, Sections 6.2(b) and 6.3] and [4, Sections 6.2(b) and 6.3]).

Theorem 1.2 (Böttcher-Silbermann)**.** *Let* $1 < p < \infty$ *and* $\mu \in \mathbb{R}$*.*

(a) *If* $T(a) \in \mathcal{B}(\ell^p_\mu(\mathbb{Z}_+))$*, then* $M(a) \in \mathcal{B}(\ell^p(\mathbb{Z}))$ *and*

$$\|M(a)\|_{\mathcal{B}(\ell^p(\mathbb{Z}))} \le \|T(a)\|_{\mathcal{B}(\ell^p_\mu(\mathbb{Z}_+))}.$$

(b) *For every* $\mu > 0$*, there exists* $a \in W(\mathbb{T})$ *such that* $T(a) \in \mathcal{B}(\ell^p_\mu(\mathbb{Z}_+))$ *but* $M(a) \notin \mathcal{B}(\ell^p_\mu(\mathbb{Z}))$*.*

If $\mu \in \mathbb{R}$, then it is clear that

$$M(a) \in \mathcal{B}(\ell^{p,p}_{0,\mu}(\mathbb{Z})) \quad \Longrightarrow \quad T(a) \in \mathcal{B}(\ell^p_\mu(\mathbb{Z}_+)).$$

Böttcher and Silbermann asked in their famous monograph [3] (see Section 6.4) and, later, in its second edition ([4, Section 6.4]) whether

$$T(a) \in \mathcal{B}(\ell^p_\mu(\mathbb{Z}_+)) \quad \Longrightarrow \quad M(a) \in \mathcal{B}(\ell^{p,p}_{0,\mu}(\mathbb{Z})).$$

As far as we know, this problem was open until now. The following theorem gives the negative answer to this question.

Theorem 1.3 (Main result)**.** *Let* $1 < p < \infty$*,* $\mu \in (-1/p, 1 - 1/p) \setminus \{0\}$*, and* $a\left(e^{i\theta}\right) := \theta - \pi$ *for* $\theta \in (0, 2\pi)$*. Then* $T(a) \in \mathcal{B}(\ell^p_\mu(\mathbb{Z}_+))$ *but* $M(a) \notin \mathcal{B}(\ell^{p,p}_{0,\mu}(\mathbb{Z}))$*.*

The proof of this theorem is given in the next section. It is based on the observation that $M(a) = iH$, where H is the discrete Hilbert transform, which is bounded on the weighted ℓ^p space with the weight $\rho_n = (|n|+1)^\mu$, $n \in \mathbb{Z}$, and is unbounded on the weighted ℓ^p space with the weight

$$w_n = \begin{cases} 1, & n \in \mathbb{Z} \setminus \mathbb{Z}_+, \\ (n+1)^\mu, & n \in \mathbb{Z}_+. \end{cases}$$

Indeed, we show that the second weight fails to belong to the Muckenhoupt class $A_p(\mathbb{Z})$ under the assumptions of Theorem 1.3. On the other hand, as it is well known, under the same assumptions, the first weight belongs to $A_p(\mathbb{Z})$. A self-contained proof of this fact is given in the Appendix.

2. Proof of the main result

2.1. Discrete Muckenhoupt weights

A weight on $\mathbb{I} \in \{\mathbb{Z}, \mathbb{Z}_+\}$ is a sequence $\{w_n\}_{n\in\mathbb{I}}$ of positive numbers. Let $1 < p < \infty$ and $1/p + 1/q = 1$. A weight $w = \{w_n\}_{n\in\mathbb{I}}$ is said to belong to the Muckenhoupt class $A_p(\mathbb{I})$ if

$$c_{p,w}(\mathbb{I}) := \sup_{m,n\in\mathbb{I},\ m\le n} \frac{1}{n-m+1}\left(\sum_{k=m}^{n} w_k^p\right)^{1/p}\left(\sum_{k=m}^{n} w_k^{-q}\right)^{1/q} < \infty.$$

The following lemma is well known (it is implicit, for instance, in [4, Section 6.2(f)]). Since we are not able to provide an explicit reference, we give its proof in the Appendix.

Lemma 2.1. *Let $1 < p < \infty$ and $\mu \in \mathbb{R}$. Then the weight $v_n = (|n| + 1)^\mu$, $n \in \mathbb{Z}$, belongs to $A_p(\mathbb{Z})$ if and only if $\mu \in (-1/p, 1 - 1/p)$.*

The following result, analogous to [1, Example 2.6], is the first key ingredient of the proof of our main result.

Lemma 2.2. *Let $1 < p < \infty$ and $\mu \neq 0$. Then the weight*

$$w_n = \begin{cases} 1, & n \in \mathbb{Z}\setminus\mathbb{Z}_+, \\ (n+1)^\mu, & n \in \mathbb{Z}_+, \end{cases}$$

does not belong to $A_p(\mathbb{Z})$.

Proof. Let $n \in \mathbb{N}$. If $\mu < 0$, then $1 - \mu q > 1$,

$$\sum_{k=-n}^{n} w_k^p > n$$

and

$$\sum_{k=-n}^{n} w_k^{-q} > 1 + \sum_{k=1}^{n}(k+1)^{-\mu q} > 1 + \int_1^{n+1} x^{-\mu q}\,dx$$
$$= 1 + \frac{1}{1-\mu q}\left((n+1)^{1-\mu q} - 1\right) > \frac{n^{1-\mu q}}{1-\mu q},$$

whence

$$\frac{1}{2n+1}\left(\sum_{k=-n}^{n} w_k^p\right)^{1/p}\left(\sum_{k=-n}^{n} w_k^{-q}\right)^{1/q} > \frac{1}{3n}\cdot\frac{n^{\frac{1}{p}+\frac{1}{q}-\mu}}{(1-\mu q)^{1/q}} = \frac{n^{-\mu}}{3(1-\mu q)^{1/q}} \to \infty$$

as $n \to \infty$. Therefore, $w = \{w_n\}_{n\in\mathbb{Z}} \notin A_p(\mathbb{Z})$.

Similarly, if $\mu > 0$, then $1 + \mu p > 1$,

$$\sum_{k=-n}^{n} w_k^{-q} > n,$$

and

$$\begin{aligned}\sum_{k=-n}^{n} w_k^p > 1 + \sum_{k=1}^{n}(k+1)^{\mu p} > 1 + \int_1^{n+1} x^{\mu p}\,dx \\ = 1 + \frac{1}{\mu p+1}\left((n+1)^{\mu p+1} - 1\right) > \frac{n^{\mu p+1}}{\mu p+1}.\end{aligned}$$

Hence

$$\frac{1}{2n+1}\left(\sum_{k=-n}^{n} w_k^p\right)^{1/p}\left(\sum_{k=-n}^{n} w_k^{-q}\right)^{1/q} > \frac{1}{3n}\cdot\frac{n^{\mu+\frac{1}{p}+\frac{1}{q}}}{(\mu p+1)^{1/p}} = \frac{n^{\mu}}{3(1+\mu p)^{1/p}} \to \infty$$

as $n \to \infty$. Thus, $w = \{w_n\}_{n\in\mathbb{Z}} \notin A_p(\mathbb{Z})$. □

2.2. Discrete Hilbert transform on weighted ℓ^p spaces

Given a weight w on $\mathbb{Z}$, the weighted space $\ell^p(\mathbb{Z}, w)$ consists of all sequences $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}}$ such that

$$\|\varphi\|_{p,w} := \left(\sum_{n=-\infty}^{\infty} |\varphi_n w_n|^p\right)^{1/p} < \infty.$$

Given a sequence $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}}$, the discrete Hilbert transform of φ is defined by

$$(H\varphi)_n = \sum_{m\in\mathbb{Z}\setminus\{n\}} \frac{\varphi_m}{n-m}, \quad n \in \mathbb{Z}.$$

The following famous result is the second main ingredient in the proof of our main result (see [5, Theorem 10]).

Theorem 2.3 (Hunt-Muckenhoupt-Wheeden). *Let $1 < p < \infty$ and $w = \{w_n\}_{n\in\mathbb{Z}}$ be a weight. Then $H \in \mathcal{B}(\ell^p(\mathbb{Z}, w))$ if and only if $w \in A_p(\mathbb{Z})$.*

2.3. Proof of Theorem 1.3

It is easy to see that

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} a\left(e^{i\theta}\right) e^{-in\theta} d\theta = \begin{cases} \frac{i}{n}, & n \in \mathbb{Z}\setminus\{0\}, \\ 0, & n = 0. \end{cases}$$

Then $M(a) = iH$.

Let $\mu \in (-1/p, 1-1/p) \setminus \{0\}$. Consider two weights $\rho = \{\rho_n\}_{n\in\mathbb{Z}}$ and $w = \{w_n\}_{n\in\mathbb{Z}}$ defined by

$$\rho_n = (|n|+1)^\mu, \quad n \in \mathbb{Z}, \qquad w_n = \begin{cases} 1, & n \in \mathbb{Z}\setminus\mathbb{Z}_+, \\ (n+1)^\mu, & n \in \mathbb{Z}_+. \end{cases}$$

It is clear that $\ell^p(\mathbb{Z},\rho) = \ell^p_\mu(\mathbb{Z})$ and $\ell^p(\mathbb{Z},w) = \ell^{p,p}_{0,\mu}(\mathbb{Z})$ and the norms in the corresponding spaces are equivalent.

It follows from Lemma 2.1 that $\rho \in A_p(\mathbb{Z})$. Then Theorem 2.3 implies that $M(a) \in \mathcal{B}(\ell^p_\mu(\mathbb{Z}))$. Therefore $T(a) \in \mathcal{B}(\ell^p_\mu(\mathbb{Z}_+))$. On the other hand, Lemma 2.2 says that $w \notin A_p(\mathbb{Z})$. Then Theorem 2.3 yields $M(a) \notin \mathcal{B}(\ell^{p,p}_{0,\mu}(\mathbb{Z}))$. $\square$

3. Appendix. Symmetric power weights in the Muckenhoput class $A_p(\mathbb{Z})$

The aim of this Appendix is to provide a self-contained proof of Lemma 2.1.

3.1. Power weights in the Muckenhoupt class $A_p(\mathbb{Z}_+)$

For the sake of completeness, we give a direct elementary proof of the next well known lemma, which can also be extracted from a much more general result proved in [2, Theorem 2.2].

Lemma 3.1. *Let $1 < p < \infty$ and $\mu \in \mathbb{R}$. Then the weight $w_n := (n+1)^\mu$, $n \in \mathbb{Z}_+$, belongs to $A_p(\mathbb{Z}_+)$ if and only if $\mu \in (-1/p, 1-1/p)$.*

Proof. Clearly,

$$w \in A_p(\mathbb{Z}_+) \quad \Longleftrightarrow \quad w^{-1} \in A_q(\mathbb{Z}_+).$$

So, it is sufficient to show that for $\mu \ge 0$,

$$w \in A_p(\mathbb{Z}_+) \quad \Longleftrightarrow \quad \mu < 1 - \frac{1}{p}\,.$$

If $\mu = 1 - \frac{1}{p} = \frac{1}{q}$, then

$$\begin{aligned}
&\frac{1}{n+1}\left(\sum_{k=0}^{n} w_k^p\right)^{1/p}\left(\sum_{k=0}^{n} w_k^{-q}\right)^{1/q}\\
&\quad = \frac{1}{n+1}\left(\sum_{k=0}^{n}(k+1)^{p-1}\right)^{1/p}\left(\sum_{k=0}^{n}(k+1)^{-1}\right)^{1/q}\\
&\quad > \frac{1}{n+1}\left(\int_0^{n+1} x^{p-1}\,dx\right)^{1/p}\left(\int_1^{n+2} x^{-1}\,dx\right)^{1/q}\\
&\quad = \frac{(\log(n+2))^{1/q}}{p^{1/p}} \to \infty \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Hence $w \notin A_p(\mathbb{Z}_+)$ in this case.

If $\mu > 1 - \frac{1}{p}$, then $\mu + \frac{1}{p} - 1 > 0$, and

$$\begin{aligned}
&\frac{1}{n+1}\left(\sum_{k=0}^{n} w_k^p\right)^{1/p}\left(\sum_{k=0}^{n} w_k^{-q}\right)^{1/q}\\
&\quad = \frac{1}{n+1}\left(\sum_{k=0}^{n}(k+1)^{\mu p}\right)^{1/p}\left(\sum_{k=0}^{n}(k+1)^{-\mu q}\right)^{1/q}\\
&\quad > \frac{1}{n+1}\left(\int_0^{n+1} x^{\mu p}\,dx\right)^{1/p}\\
&\quad = \frac{(n+1)^{\mu+\frac{1}{p}-1}}{(\mu p+1)^{1/p}} \to \infty \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Hence $w \notin A_p(\mathbb{Z}_+)$.

Suppose now $0 \le \mu < 1 - \frac{1}{p} = \frac{1}{q}$. If $n/2 \le m < n$, then

$$\begin{aligned}
&\frac{1}{n-m+1}\left(\sum_{k=m}^{n} w_k^p\right)^{1/p}\left(\sum_{k=m}^{n} w_k^{-q}\right)^{1/q}\\
&\quad = \frac{1}{n-m+1}\left(\sum_{k=m}^{n}(k+1)^{\mu p}\right)^{1/p}\left(\sum_{k=m}^{n}(k+1)^{-\mu q}\right)^{1/q}\\
&\quad \le \frac{1}{n-m+1}\left((n-m+1)(n+1)^{\mu p}\right)^{1/p}\left((n-m+1)(n/2+1)^{-\mu q}\right)^{1/q}
\end{aligned}$$

$$= (n+1)^{\mu}(n/2+1)^{-\mu} < 2^{\mu}.$$

If $0 \le m < n/2$, then

$$\begin{aligned}
&\frac{1}{n-m+1}\left(\sum_{k=m}^{n} w_k^p\right)^{1/p}\left(\sum_{k=m}^{n} w_k^{-q}\right)^{1/q} \\
&\quad \le \frac{1}{n-m+1}\left(\sum_{k=0}^{n}(k+1)^{\mu p}\right)^{1/p}\left(\sum_{k=0}^{n}(k+1)^{-\mu q}\right)^{1/q} \\
&\quad < \frac{1}{n/2+1}\left(\int_0^{n+2} x^{\mu p}\,dx\right)^{1/p}\left(\int_0^{n+2} x^{-\mu q}\,dx\right)^{1/q} \\
&\quad = \frac{2}{n+2}\cdot\frac{(n+2)^{\mu+\frac{1}{p}}}{(\mu p+1)^{1/p}}\cdot\frac{(n+2)^{-\mu+\frac{1}{q}}}{(-\mu q+1)^{1/q}} = \frac{2}{(\mu p+1)^{1/p}(1-\mu q)^{1/q}}\,.
\end{aligned}$$

Hence, $w \in A_p(\mathbb{Z}_+)$ if $0 \le \mu < 1 - \frac{1}{p}$. □

3.2. Symmetric reproduction of Muckenhoupt weights

The following lemma might be known to experts, but we were not able to find it in the literature (cf. [1, Section 2.4], where similar results in the continuous case are considered).

Lemma 3.2. *Let* $1 < p < \infty$, $w : \mathbb{Z}_+ \to (0,\infty)$ *and* $v_n := w_{|n|}$, $n \in \mathbb{Z}$. *Then*

$$w \in A_p(\mathbb{Z}_+) \quad \Longleftrightarrow \quad v \in A_p(\mathbb{Z}). \tag{3.1}$$

Proof. The implication $\Longleftarrow$ is trivial, so one only needs to prove the implication $\Longrightarrow$ in (3.1). Suppose $w \in A_p(\mathbb{Z}_+)$. Take any $m, n \in \mathbb{Z}$, $m \le n$. If $m \ge 0$, then, clearly,

$$\begin{aligned}
&\frac{1}{n-m+1}\left(\sum_{k=m}^{n} v_k^p\right)^{1/p}\left(\sum_{k=m}^{n} v_k^{-q}\right)^{1/q} \\
&\quad = \frac{1}{n-m+1}\left(\sum_{k=m}^{n} w_k^p\right)^{1/p}\left(\sum_{k=m}^{n} w_k^{-q}\right)^{1/q} \le c_{p,w}(\mathbb{Z}_+).
\end{aligned}$$

Similarly, if $n \le 0$, then

$$\frac{1}{n-m+1}\left(\sum_{k=m}^{n} v_k^p\right)^{1/p}\left(\sum_{k=m}^{n} v_k^{-q}\right)^{1/q}$$

$$= \frac{1}{-m-(-n)+1}\left(\sum_{k=-n}^{-m} w_k^p\right)^{1/p}\left(\sum_{k=-n}^{-m} w_k^{-q}\right)^{1/q} \leq c_{p,w}(\mathbb{Z}_+).$$

Suppose now $m < 0 < n$, and let $\ell := \max\{-m, n\}$. Then

$$\begin{aligned}
&\frac{1}{n-m+1}\left(\sum_{k=m}^{n} v_k^p\right)^{1/p}\left(\sum_{k=m}^{n} v_k^{-q}\right)^{1/q} \\
&\quad\leq \frac{1}{n-m+1}\left(\sum_{k=0}^{n} w_k^p + \sum_{k=0}^{-m} w_k^p\right)^{1/p}\left(\sum_{k=0}^{n} w_k^{-q} + \sum_{k=0}^{-m} w_k^{-q}\right)^{1/q} \\
&\quad\leq \frac{1}{\ell+1}\left(2\sum_{k=0}^{\ell} w_k^p\right)^{1/p}\left(2\sum_{k=0}^{\ell} w_k^{-q}\right)^{1/q} \leq 2c_{p,w}(\mathbb{Z}_+).
\end{aligned}$$

Hence

$$\begin{aligned}
c_{p,v}(\mathbb{Z}) &= \sup_{m,n\in\mathbb{Z},\ m\leq n} \frac{1}{n-m+1}\left(\sum_{k=m}^{n} v_k^p\right)^{1/p}\left(\sum_{k=m}^{n} v_k^{-q}\right)^{1/q} \\
&\leq 2c_{p,w}(\mathbb{Z}_+) < \infty,
\end{aligned}$$

and $v \in A_p(\mathbb{Z})$. □

Lemma 2.1 follows immediately from the two previous results.

Acknowledgments

We would like to thank the anonymous referee for a useful suggestion concerning the current form of Lemma 2.2.

Funding

This work is funded by national funds through the FCT - Fundação para a Ciẽncia e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 and UIDP/00297/2020 (Center for Mathematics and Applications).

References

[1] A. Böttcher and Y. I. Karlovich. *Carleson curves, Muckenhoupt weights, and Toeplitz operators*, volume 154 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1997.

[2] A. Böttcher and M. Seybold. Discrete Wiener-Hopf operators on spaces with Muckenhoupt weight. *Studia Math.*, 143(2):121–144, 2000.

[3] A. Böttcher and B. Silbermann. *Analysis of Toeplitz operators*. Springer-Verlag, Berlin, first edition, 1990.

[4] A. Böttcher and B. Silbermann. *Analysis of Toeplitz operators*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2006.

[5] R. Hunt, B. Muckenhoupt, and R. Wheeden. Weighted norm inequalities for the conjugate function and Hilbert transform. *Trans. Amer. Math. Soc.*, 176:227–251, 1973.