A Work Project, presented as part of the requirements for the Award of a Masters Degree in Finance from Faculdade de Economia da Universidade Nova de Lisboa.

Portfolio Optimization with Options

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A Project carried out on the Finance course, with the supervision of:

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Abstract

In order to address the options returns non normality problem in the investment portfolio theory, this work project aims to discuss and present alternatives to the classic Markowitz risk/return paradigm. The following pages will exploit the Portfolio Selection Theory developed over the last decade, maximazing a standard CRRA utility function, and simulating (MonteCarlo) or deriving from the past data (Bootstrap) the path taken by the S&P 500 stock Index. To conclude, a 5 year back test is developed to evidence the practical implications of the several models exposed.

1.Introduction

1.1 General Overview

Given the non-linear payout structure of an option, it is difficult, if not impossible, to assume a normal return distribution for this kind of securities, even when the underlying asset returns follows such distribution. Generally an option ends up returning -100% (the underlying finishes out of the money), losing the entire premium paid initially by the investor. When the underlying finishes well above the strike price, the return on these securities could be well above 100% or 200%, given this instrument extreme leverage. On the other hand, writing options (the equivalent of being short) has the exact opposite payout, having a maximum gain of 100%, if the option expires worthless. In this scenario, the option writer keeps the entire premium. Another possible situation is the writer's exposure to a large payout, if the option ends up in the money.

This concentrated distribution over both extremes make the normality assumption on these returns impossible.

Parallel to the classic set up of Normal returns, many researchers have implemented a utility maximization as an alternative, which in this case has the advantage of not assuming any distribution on the security.

In order to form a 5 securities portfolio, I've used one month to maturity options on the S&P 500: an at the money call, an at the money put, an out of the money call, an out of the money put and a time deposit (or loan) at a risk free rate.

The underlying asset path is then simulated (MonteCarlo) and resampled from past data (Bootstraped) in order to find the optimal weight for each option, maximizing a CRRA utility function.

On section 2, I analysed both methods and imposed some restrictions. Thought section 3, comments on the results of a 5 years backtest on both strategies are offered and finally, on section 4, I've presented conclusions and some comments on further research.

1.2 Literature Review

After the classic Markowitz (1952) set up, which lead to significant development in portfolio theory, with the risk/return paradigm[1], many challenges appeared defending dynamic portfolio choices. This line of thinking was pioneered by Merton (1969, 1971) and Samuelson (1969) in continuous time and Fama (1970) in discrete time. More recently, a renewed interest has surface about this issue. For instance Brandt and Santa-Clara (2006) present a novel approach which mimics dynamic portfolio selection by creating a larger set of assets to include managed portfolios, using the static framework to determine the optimum portfolio weights. In Brandt Santa-Clara and Valkanov (2007) the portfolio weight are modelled

[1] This paradigm assumes a Normal Distribution on asset return what leads to important closed formula results in portfolio selection.

directly from the asset's characteristics, optimizing the investor's average utility. More related to this Work Project is Brandt, Goyal, Santa-Clara and Stroud (2005), where a simulation-based method is used to determine the optimal portfolio weight for a multi period problem.

In the field of options, since the formula for pricing these securities has been put forward by Black and Scholes (1973) much work have been done over their mathematical elegant solution, mainly in risk management. For instance, see Leland (1980) on portfolio insurance, a direct result from the Black and Scholes formula. One connection between these two worlds could be found in Kostakis, Panigirtzoglou and Skiadopolos (2008), where the market prices for option are used to determine the typical agent risk aversion.

2.Methodology

2.1 Formal Problem

Using a similar approach as Brandt, Goyal, Santa-Clara and Stroud (2005), consider an investor with a universe of N risky assets and a risk free, who maximizes a CRRA utility function u(.) for 1 period from t to t+1.

$$V_t(W_{t+1}) = \max E_t\left[u\left(W_{t+1}\right)\right] \qquad (1)$$

where,

$$W_{t+1} = x_t^{'} R_{t+1}^{e} + R^{f} \qquad \forall 1 \le t \le T \qquad (2)$$

being x_t a vector of portfolio weights on the risky assets, chosen at time t, R_{t+1}^{e} is the vector of excess returns on the N risky assets from time t to t+1, and R^f is the gross return on the risk-free asset.

The function u(.) measures the investor's utility at time t, given its wealth W_{t+1}. The function $V_t(W_{t+1})$ represents the expectation at time t of the utility from the subsequent optimal portfolio weights x_t.

2.2 Underlying Evolution Simulation

In order to maximize the portfolio, the evolution of the underlying must be simulated numerically or extracted (bootstrapped) from the data. To achive this purpose two models were used, a MonteCarlo simulation and a Bootstraped sample.

The MonteCarlo simulation is used abundantly on the pricing of exotic options, which could not be priced with closed formulas. This model simulates the underlying market variable path, in a risk-neutral world, through a process which can be described as the following:

$$dS = \hat{\mu} S dt + \sigma S dz \tag{3}$$

where dz is a Wiener process, $\hat{\mu}$ is the expected return in a risk-neutral world, and σ is the volatility. In practice, it is more accurate to simulate ln(S) rather than S. From the Itô's lemma the process followed by ln(S) is

$$d\ln(S) = \left(\hat{\mu} - \frac{\sigma^2}{2}\right) dt + \sigma dz \tag{4}$$

In order to simulate the S path, the derivative's life is divided in short intervals Δt, approximating equation (4) as

$$\ln S(t + \Delta t) - \ln S(t) = \left(\hat{\mu} - \frac{\sigma^2}{2}\right) \Delta t + \sigma \varepsilon \sqrt{\Delta t} \tag{5}$$

and rearranging to

$$S(t + \Delta t) = S(t) \exp\left[\left(\hat{\mu} - \frac{\sigma^2}{2}\right) \Delta t + \sigma \varepsilon \sqrt{\Delta t}\right] \tag{6}$$

As the aim of this simulation is not to find the market variable path of the in a risk-neutral world, a risk premium must be added to $\hat{\mu}$. Substituting on equation (6) we have

$$S(t+\Delta t) = S(t)\exp\left[\left(\hat{\mu} + R^{prem} - \frac{\sigma^2}{2}\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}\right] \qquad (7)$$

This is the equation used to simulate a path for S. The advantage of working with ln(S), instead of S, is that it follows a generalized Wiener process.

Bootstrap is a derivation of Monte Carlo technique introduced by Efron in 1979. It uses the resampling with replacement method (unlike the resampling with no replacement method that we used in a Lotto Game for example). It is a convenient tool not only to extract estimates that do not have a closed form (cannot be expressed in an equation), but also to estimates from a non-parametric data set (where no underling distribution is assumed). Bootstrap can also be used to increase sample population when the original sample size is small (despite Bootstrap usually works best with large sample size as all other statistical methods). The main difference from the MonteCarlo simulation is that no distribution has to be assumed.

In order to approximate the monthly returns from the S&P 500 to an iid Normal Distribution a similar method to the one presented by Andersen, Bollerslev, Diebold and Labys (1999) was used, showing $X_{t+1} = \frac{r_{t+1}}{\sigma_t} \sim \text{iid Normal Distribution}$ in high frequency return observations.[2]

[2] Andersen, Bollerslev, Diebold and Labys (1999) work is based on a unique high-frequency dataset consisting of ten years of continuously recorded 5-minute returns on DM/$ and Yen/$ spot exchange rates.

For an S&P 500 volatility approximation, I used the VIX index, which was only calculated since 1990. As the sample of monthly returns on the S&P 500 is small and maybe bias since 1990, I used another transformed variable for the Bootstrap, removing the average up to date t

$$X_{t+1} = \frac{r_{t+1} - \bar{r}_t}{VIX_t} \tag{8}$$

The later variable statistics can be found on the Appendix A (Mean, Standard Deviation, Skew, Kurtosis and AR(1) coefficient).

This modification has another advantage as results comparable to the MonteCarlo simulation can be achieved, adding the risk premium when the current VIX is taken into account:

$$\hat{r} = (X_i \times VIX) + R^{prem} \tag{9}$$

Four options compose the investment universe (1 call option at the money (ATM), 1 call out of the money (OTM), 1 put option ATM, 1 put option OTM) and a 1 month deposit (or loan) made at Libor rate flat. Both OTM options will be 5% OTM from the spot level.

In order to estimate the portfolio weights I used the Solver module provided by Microsoft Excel. This module uses an algorithm called Generalized Reduced Gradient (GRG2) developed by Leon Lasdon (University of Austin, Texas) and Allan Waren (Cleveland State University), which attempts to minimizing or maximizing the value of a given computed cell by varying systematically the values of some input cells, given a set of restrictions.

In order to cap the leverage in some simulations and to resolve some non converging optimizations, a restriction was imposed on the sum of the squared weights. They have to be smaller than or equal to 0,25.

2.3 Data

For the Time Deposit Rate, I used the 1 month Libor Rate on the US Dollar, assuming not only a flat funding cost, but also that deposits and loans could be done at the same rate. This rate could be viewed as the risk free rate, despite the spread charged above US Government T-Bills for the same maturity. The reason for this spread lies in a higher perceivable risk between the lending rate among the largest banks and the US Government. Despite this difference, the market standard is to assume Libor as a risk free rate.

The monthly returns from S&P 500 and VIX were taken from Bloomberg at the end of each month since 1990 up to April 2009, which constitutes a 232 months nonoverlapping sample. The new VIX started being calculated in 2003 by the Chicago Board of Option Exchange, substituting the old VIX which used prices from S&P 100 to calculate the general market implied volatility. History up to 1990 is given to us in this new VIX.

The option prices as well as the implied volatility and dividend yield needed to parameterize the MonteCarlo simulation were also taken from Bloomberg, with one month to maturity, where the historical values go as far as October 2004. It is assumed that options could be bought and sold at the same price. Although this later scenario is not verified at the market, this was made for simplify, as the historical prices Bid/Offer of most options could not be retrieved.

3. Results

The results from an out of sample backtest can be found on the table below.

Table 1

	MonteCarlo	Bootstrap
Mean	8.4028%	22.3223%
Standard Error	1.6729%	7.5280%
Median	7.0849%	25.7885%
Standard Deviation	12.4069%	55.8295%
Sample Variance	1.5393%	31.1693%
Kurtosis	6.717334	33.848775
Skewness	-1.419545	-5.178938
Range	81.0256%	427.9888%
Minimum	-47.8617%	-340.3458%
Maximum	33.1639%	87.6430%
Sum	462.1515%	1227.7281%
Count	55	55

Legend: Descriptive statistics from both strategies employed.

For the Cumulative Returns see Graphs 1 and 2 (for MonteCarlo and Bootstrap respectively) in Appendix B. The average return and standard deviation from both methods are quite different. The MonteCarlo method achieves an average monthly return of 8,40% with a standard deviation of 12,41%, comparing with 22,32% average monthly return and 55,83% standard deviation using the Bootstraped sample. The worst month for both models was October 2008, after the Lehman bankruptcy and the consequent downfall in the market.

In both cases, the strategy suggested after the optimization is similar to a straddle, with the difference of a highly leverage short of the OTM options, both call and put. This makes the investor exposed to large changes in the underlying asset, in this case the S&P 500 Index. See Graph 3 and 4 in Appendix C where two portfolio examples are presented for each model, showing the possible outcomes for the underlying asset percentage change, over the following month. This leverage can reach over 1000 times the amount invested, in scenarios where the options are very cheap and the optimization point to a large investment (long or short) on that option. On average, the leverage on the Bootstap strategy is around 20 times and 3 times on the MonteCarlo.

Finally, an important aspect is the capital consumption of both strategies. As Appendix D shows, the deposit weight on the portfolio is very close to 100%, which means almost no capital is needed to implement the MonteCarlo strategy (the premiums received from writing the short option almost cover premiums paid for the long options). In the Bootstrap method this effect is even more relevant, as the deposit weight is almost always above 100%, reaching more than 190%. This can be viewed as a funding opportunity for market participants, but a very risky one.

4.Conclusions

This Work project pretends to show how the theoretical framework on Portfolio Selection can be applied in the context of option portfolios. The two methods used seem fit to banks proprietary desks or hedge funds, as a secondary strategy given its high volatility and also very significant returns. The main problems are risk asymmetry (large downside with very limited upside) and high leverage. Given this asymmetry, capitalizing gains can be risky as the increased leverage can lead to superior losses in the most extreme outcomes. My advice is to perform this strategy always with the same initial Wealth, removing potential gains from this strategy and adding the amount lost in any given month.

This strategy works better in a low realized volatility context, where the underlying does not deviate from strike more than 5%. The performed backtest showed that the most difficult period was on the late 2008, mainly in October, after the bankruptcy of Lehman Brothers. The later observation exemplifies that this strategy does not deal well with fat tails, or extreme events, which can lead to very large swings on the equity index (mainly on the downside). This fact relates to the book Black Swan, by the former trader Nassim Taleb, which defends that the bell curve used in many models is the "great intellectual fraud", given its failure to accommodate extreme events ("Black Swans"), something that can be pointed to this work project.

Usually an option portfolio is constructed taken into account the Greek letters, representing the portfolio sensitivity to changes on the underlying asset (Delta), underlying volatility (Vega), time (Theta), Delta itself (Gamma) or the interest rates (Rho), among other. These measures are helpful to manage risk dynamically, avoiding large movement on the portfolio value. The portfolios proposed in this Work Project could use some risk management, adding a dynamically hedged investment on the underlying asset, in order to reduce the exposure to large movements on the underlying asset. Another important issue not dealt with was the mark to market and the portfolio value evolution during the 1-month period. With today restrictions on trading and with Value at Risk calculations determining the liquidation of highly leverage positions, this strategy may be difficult to implement in many banks or insurance companies, following the Basel II policies, which are very restrictive on these matters.

For further research, a few aspects could be deepened. First of all, other distributions for the MonteCarlo simulation can be used to account for fat tails (e.g. t distribution). Another aspect is the use of a larger Database for option prices with data previous to 2004. Also important is the use of Bid and Offer prices for options, depending on the weight on the portfolio as well as different rate for borrowing and lending. This later aspect is regardless of the fact that the results suggest a very low net investment in premiums, resulting in available cash. However, this can be a result from the restriction on the maximum leverage on the portfolio. Finally an additional enhancement could come from using other degrees of risk aversion and moneyness further than 5%, probably resulting on the higher leverage.

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Appendix A

Table 2

	r	x	r^2	x^2	abs [r]	abs [x]
Mean	0.422%	-0.018%	0.193%	0.000%	3.348%	0.158%
std dev	4.378%	0.198%	0.360%	0.001%	2.844%	0.120%
Skew	-0.8642	-0.5241	5.0126	3.2172	1.7102	1.0229
Kurtosis	1.9094	-0.0866	35.7558	15.1744	4.4564	1.3483

Legend: Main descriptive statistics for the r and x variables, as well as some other derived from them.

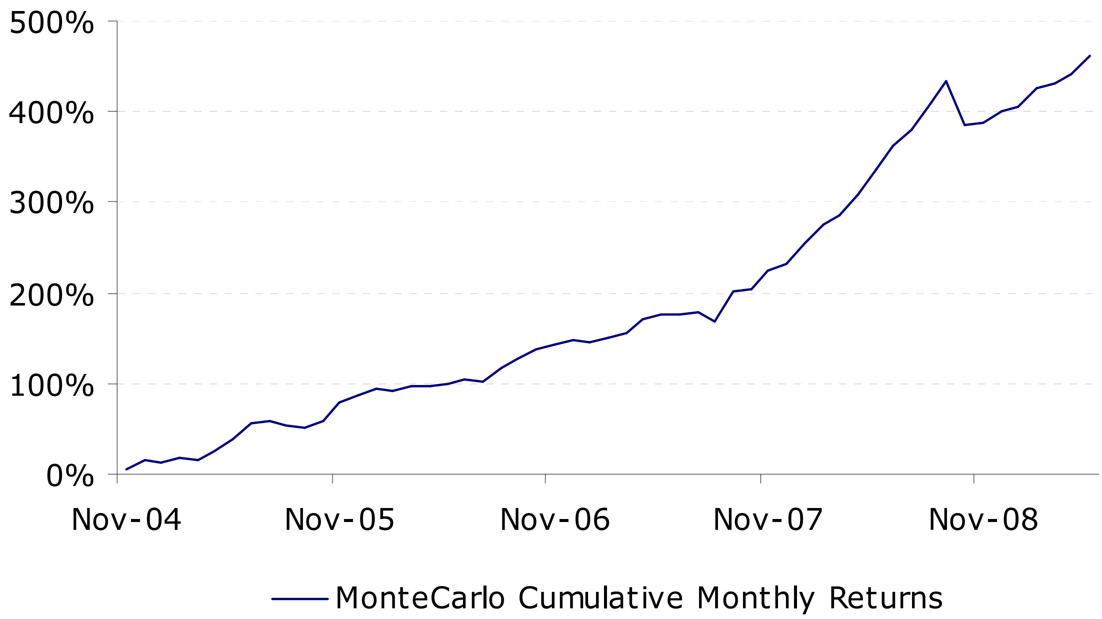
Table 3

Dependent Variable	Variable	Coefficient	Std. Error	t-Statistic	Prob.
R	C	0.004237	0.003160	1.340934	0.181276
	AR(1)	0.085819	0.066542	1.289693	0.198464
X	C	-0.000180	0.000130	-1.384930	0.167427
	AR(1)	-0.008882	0.066388	-0.133786	0.893690
r^2	C	0.001944	0.000298	6.518928	0.000000
	AR(1)	0.219707	0.065003	3.379969	0.000853
x^2	C	0.000004	0.000000	10.745689	0.000000
	AR(1)	-0.022115	0.066141	-0.334363	0.738413
Abs [r]	C	0.033670	0.002293	14.685391	0.000000
	AR(1)	0.195398	0.065420	2.986805	0.003127
Abs [x]	C	0.001591	0.000076	20.991499	0.000000
	AR(1)	-0.045585	0.065918	-0.691539	0.489931

Legend: Autoregressive coefficient for the r and x variables, as well as some other derived from them.

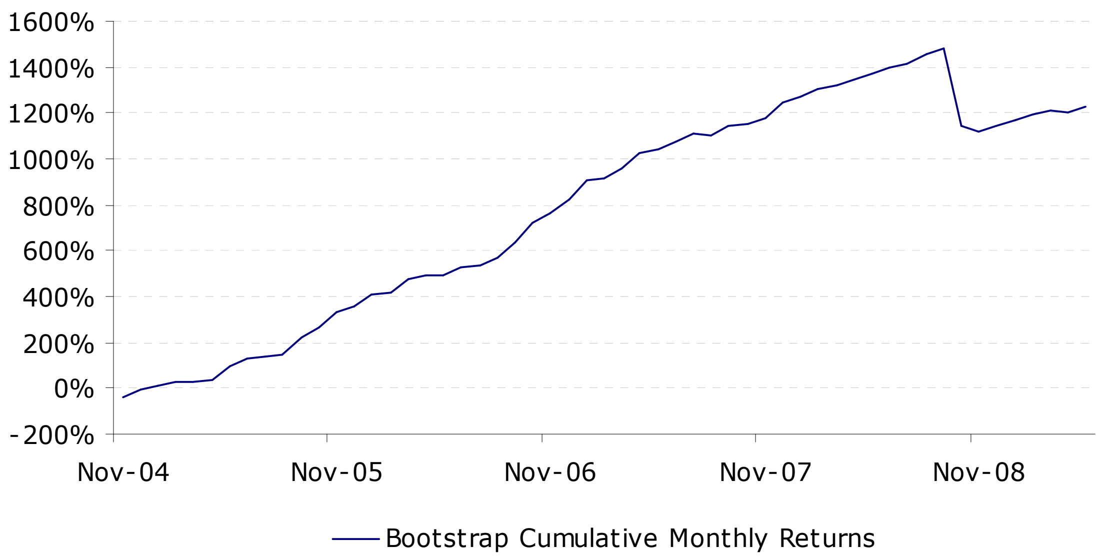
Appendix B

Graph 1 – MonteCarlo Cumulative Monthly Returns



Legend: Cumulative Monthly Returns from the MonteCarlo strategy with constant investment (no return capitalization). Any positive return is taken out of the strategy and every negative month, the loss is added for the following month.

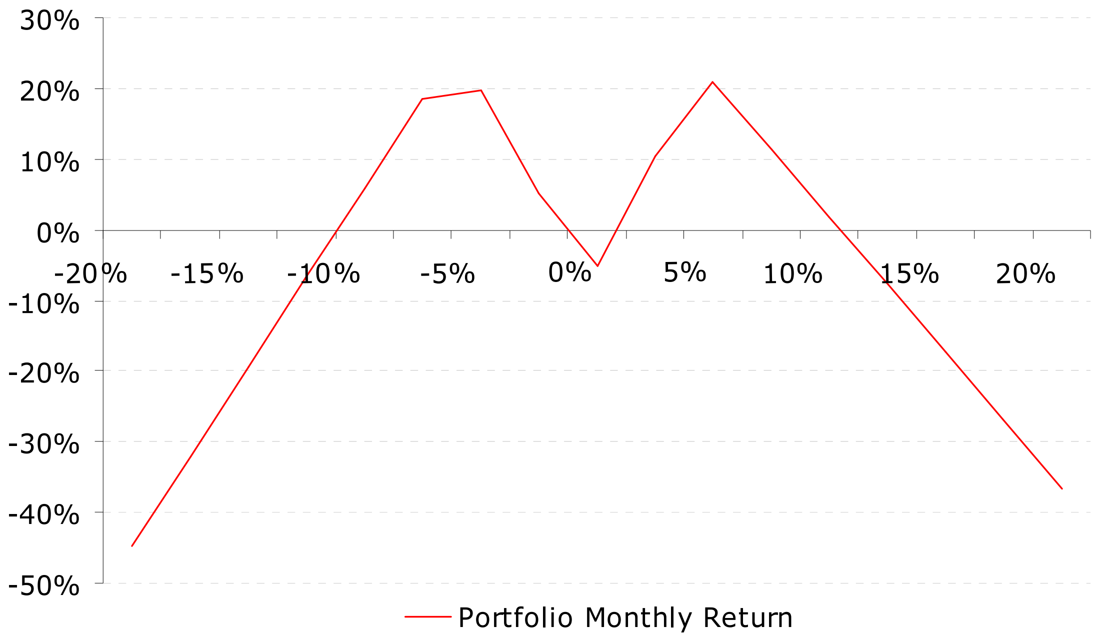
Graph 2 – Bootstrap Cumulative Monthly Returns



Legend: Cumulative Monthly Returns from the Bootstrap strategy with constant investment (no return capitalization). Any positive return is taken out of the strategy and every negative month, the loss is added for the following month.

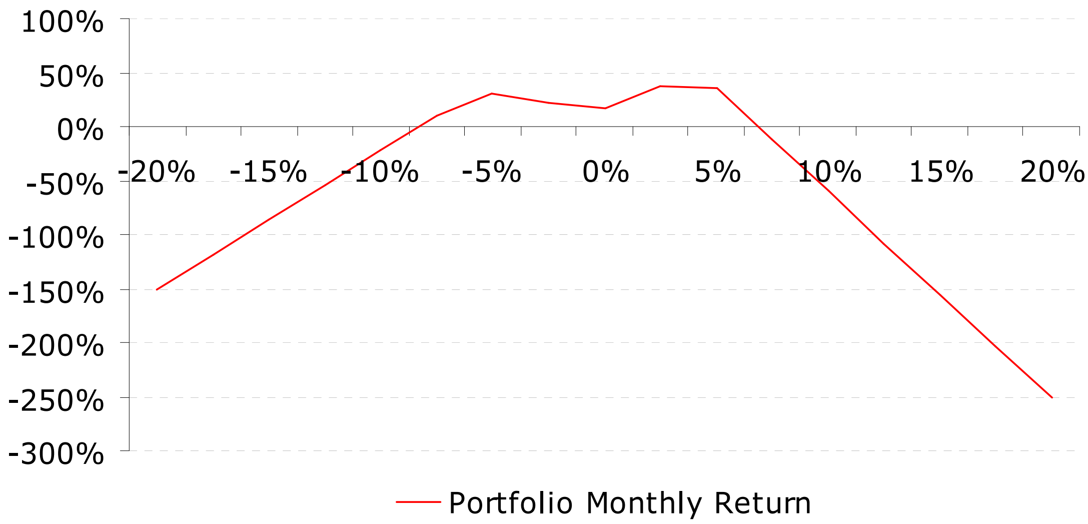
Appendix C

Graph 3 – MonteCarlo Generic Portfolio Behaviour



Legend: Given the average portfolio weights in the MonteCarlo strategy this graphs shows how the strategy behaves in different underlying performances.

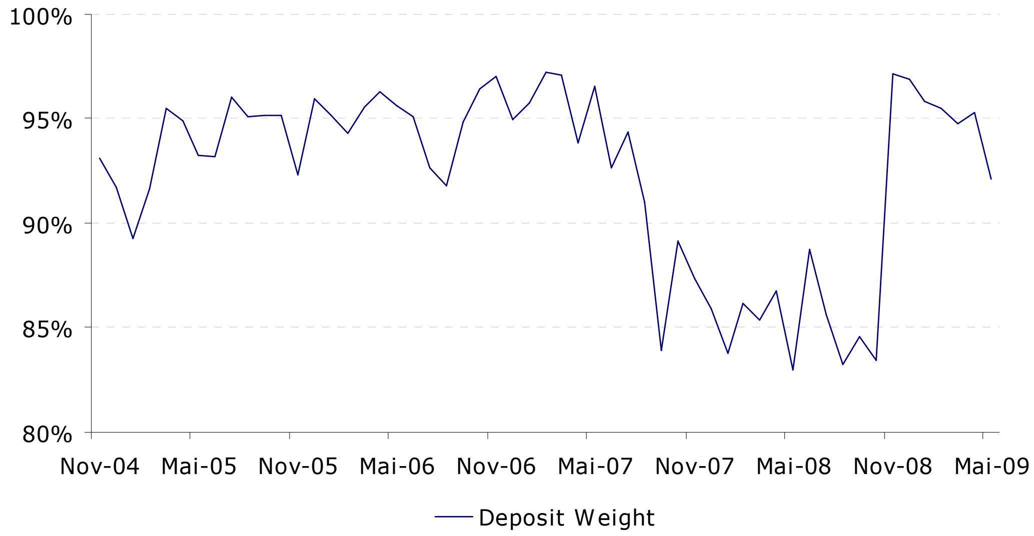
Graph 4 – Bootstrap Generic Portfolio Behaviour



Legend: Given the average portfolio weights in the Bootstrap strategy this graphs shows how the strategy behaves in different underlying performances.

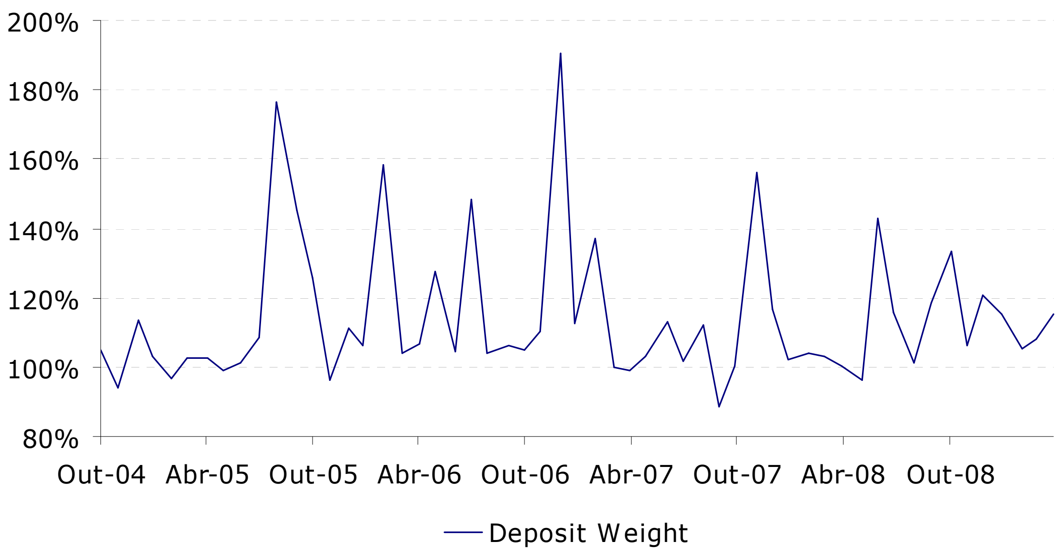
Appendix D

Graph 5 – MonteCarlo Deposit Weight



Legend: The invested amount added to the result from the premiums received from writing option minus the premiums paid to buy option in the MonteCarlo strategy. This represents the value invested in time deposit (if positive) or borrowed (if negative).

Graph 6 – Bootstrap Deposit Weight



Legend: The invested amount added to the result from the premiums received from writing option minus the premiums paid to buy option in the Bootstrap strategy. This represents the value invested in time deposit (if positive) or borrowed (if negative).