BANKRUPTCY IN A MODEL OF UNSECURED CLAIMS

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Summary

We study a two periods model of incomplete markets with nominal assets unsecured by collateral, where agents can go bankrupt but there are no bankruptcy penalties entering directly in the utility function. We address two cases: first, a proportional reimbursement rule under bounded short sales and limited liability and, secondly, a nonproportional reimbursement rule, favoring smaller claims, without bounds on short-sales, but assuming that liability approaches total garnishment as debt goes to infinity.

INTRODUCTION

In the past ten years, a few papers have studied default in the context of the general equilibrium incomplete markets model. The pioneering work was done by Dubey-Geanakoplos-Shubik (1989) in a two-period model where defaulters suffer utility penalties and anticipate that asset returns are discounted according to the average repayment rate. In this model, default occurs when the marginal utility of income, in a particular state of nature, exceeds the marginal penalty of defaulting in a specific asset. Zame (1993) discussed efficiency properties of a single-good, countable states, version of this model and Araujo-Monteiro-Pascoa (1996, 1998) extended the model to an infinite-tree with a continuum of states at each node. However, the current decreasing deterrence role of penalties tends to question the approach. More recently, collateral was brought into the default model, by Dubey-Geanakoplos-Zame (1995). When assets are backed by collateral and utility penalties are absent, adverse selection problems are immediately ruled out, since all lenders know what to expect to receive in each state of nature (and this is the minimum between the value of the claim and the value of the exogenously fixed collateral). There is even no need to suppose that lenders deal indirectly with borrowers, through a pool of mortgages, in order to rationalize the hypothesis that returns are discounted using the average repayment rate. However, when collateral becomes endogenously chosen by borrowers this pooling step may be required again (see Araujo-Orrillo-Pascoa (1998)).

In the present paper we addresss bankruptcy, instead of default, in the context also of the general equilibrium incomplete markets model. By bankruptcy we mean a situation where an agent has no means to pay back his debt, or more precisely, that his garnishable wealth and income do not cover his debt. Bankruptcy is a very important institutional arrangement that offers protection to agents on the basis of the concept of limited liability. When agents are liable only up to some fraction of their wealth and income, it is in their interest to go bankrupt when the debt exceeds that fraction of their estates. Creditors will then be reimbursed using the garnished estates. This procedure should not be confused with default, which is plainly a situation where debtors fail to honor their commitments, even when they could afford to do it.

If all claims were secured by collateral, bankruptcy would become uninteresting, since it could never offer the debtor a protection putting at risk the reposession of the collateral by the lender. There are, however, many promisses not backed by collateral. To simplify the analysis we will concentrate actually on a model where all claims are unsecured. A more complex version could be worked out, taking into account the priority of secured claims in the partition of the garnished estate of a bankrupt agent (or, possibly, only after tax authorities and employees were reimbursed). We assume also that bankrupt agents do not suffer any penalties entering directly in the utility function. Legal punishments are no longer contemplated in the bankruptcy laws of many countries, although reputation punishments, related to future access to credit, are clearly relevant and could still be represented in the form of subjective utility penalties, which should be taken into account when filing for bankruptcy. However, we do not attempt to model these asymmetric information aspects of the problem.

We develop a two-periods model where uncertainty may affect endowments and preferences in the second period, through the realization of S states of nature. Assets have nominal payoffs and are not secured by collateral. For each state of nature, an agent goes bankrupt when his financial surplus is negative and its absolute value exceeds the value of the garnishable endowments. We assume that some liability rule has already determined the fraction of the endowments that creditors can garnish in case of bankruptcy. This structure for second-period budget constraints introduces a nonconvexity in the consumer's budget set of bundles and portfolios (actually, in the latter). This difficulty is overcome by considering a continuum of consumers and appealing to Liapunov's theorem.

Our portrait of bankruptcy was contemplated in Araujo-Pascoa (1994) and has some similarities with the one used by Modica-Rustichini-Tallon (1995) in the context of an interesting temporary equilibrium model, but it differs from the latter since we assume limited liability and do not restrict bankruptcy only to unforeseen states (to which the consumer attaches zero subjective probability). We model bankruptcy as a predictable and less dramatic contingency: it may be in the consumer's interest to pick a portfolio that provides interesting transfers of wealth across periods and states, in spite of the known risk of leading to bankruptcy and the resulting partial confiscation of estates in some states of nature.

In the first part of the paper (section 2) we assume proportional reimbursement to creditors. That is, the garnished estates of bankrupt agents are partitioned among lenders proportionally to the claims involved. Since we model transactions anonymously, assuming that lenders deal with a pool of borrowers, the above proportional rule implies that, in a rational expectations equilibrium, lenders should discount future returns using the average repayment rate (the ratio of effective payments to promissed payments). We show that, under bounded short-sales, a bankruptcy equilibrium will exist.

Bounded short-sales is a convenient but arguable assumption, avoidable when assets are nominal, both in the no-default model and in the default model with penalties or collateral. However, in our model, as in the real assets no-default model, the rank of the returns matrix depends on endogenous variables, which are the repayment rates, taking into account bankruptcy by other agents. Besides, the nonconvexity of the budget set prevents us from establishing that consumer demand is differentiable (since the Jacobian of first order conditons can not be shown to be nonsingular). This difficulty seems to rule out any attempt to use differential topology approaches in the existence argument.

In the second part of the paper (section 3) we discuss implications of deviations from proportional reimbursement to creditors. There are frequent real life deviations from the proportional rule and some are actually contemplated by law in the form of seniority criteria (giving priority to tax authorities, employees, secured creditors and some among the unsecured creditors). Theoretical work has also questioned the interest of proportional rules. The bargaining approach by a Aumann-Maschler (1985) suggested equal division up to the minimum of the smallest claim and the estate (due to equal bargaining positions when the estate is smaller than any claim) and a more complex rule, according to equal-division of contested additional claims, in general (coinciding with the nucleolus of the cooperative game played by the creditors).

We examine a reimursement rule that tends to favor smaller claims, by making the reimbursement rate decrease with the size of the claim. The resulting strict concavity of positive returns, contrasted with the linearity of negative returns, forces the financial surplus to go to $-\infty$, as budget-feasible portfolios become unbounded along many of the possible directions (avoiding linear positive returns of non-defaulting assets). On the other hand, we assume, in this part of the paper, that agents are required to provide increasing liability as the debt goes to infinity. Such a liability rule should be seen as a mechanism intended to discourage irresponsible borrowers. This assumption, together with Inada's condition an utility, makes financial surplus bounded from below by some negative amount. The existence argument is carried out by taking a sequence of equilibria of truncated economies and appealing to a multi-dimensional version of Fatou's lemma which requires the sequence of allocations to be uniformly bounded (or at least uniformly integrable). The above reimbursement rule, together with the liability assumption, imply non-arbitrage conditions that allow us to uniformly bound portfolios of assets subject to default.

In this second part of the paper, the nonlinearity of returns requires nonlinear pricing for assets. The portfolio cost includes, in addition to the usual linear component, a spread which is a discounted expectation of the future net default (i.e., the difference between default given and default suffered), for some endogenously determined, in equilibrium, measure on states. That is, agents more prone to go bankrupt end up selling assets at lower prices (i.e., borrow at higher interest rates). Also, lenders receive discounts when buying assets due to future expected default (and these discounts are nonlinear, due to the above nonlinear returns scheme). This structure for the spread plays a crucial role in guaranteeing that, when markets clear, default is also correctly anticipated (i.e., net aggregate default is zero).

We close the paper with a short section discussing the constrained efficiency properties of equilibria.

2. PROPORTIONAL REIMBURSEMENT

2.1 The model

We consider a two period general equilibrium model. In the first period A nominal assets and L physical goods are traded. In the second period, there are S of nature which may affect asset returns and the consumers' endowments and preferences over L physical goods.

The continuum of consumers is modelled as the Lebesgue measure space $(I, \mathfrak{I}, \lambda)$ of the unit interval $I \equiv [0, 1]$. Preferences are time and state separable and given by a utility function $u^h\colon \mathbb{R}_+^{L(S+1)} \to \mathbb{R}$ of the form $u^h(x^h) = \sum_{s=0}^{S} u_s^h(x^h)$ for $h \in I$. Endowments are vectors $\omega^h \in \mathbb{R}_{++}^{L(S+1)}$, for $h \in I$. We allow for bankruptcy: agents will not pay by their entire debt if this exceeds the value of the alienable endowment. That is, consumers have limited liability in the sense that, in case of bankruptcy, only same share $\gamma_{s\ell} \in [0,1]$ of the endowment of good ℓ in state s may be confiscated by the creditors. If the total nominal return is lower then $-\sum_{\ell=1}^{L} p_{s\ell}\,\gamma_{s\ell}\,\omega_{s\ell}^h$, at some spot price vector $p_s \in \mathbb{R}_+^L$, then consumer h will decide to pay the latter instead.

Default by others is anticipated according to a proportional reimbursement rule. Denote consumer h's portfolio by $y^h \in \mathbb{R}$ and assume that the returns matrix has nonnegative elements r_{sa}. Let θ_{sa} be the mean reimbursement rate, to be formally defined below. Discounted returns $G_{sa}(r_{sa}\,y_a)$ are given by

$$G_{sa}(r_{sa}\,y_a) = \begin{cases} r_{sa}\,y_a\,, & \text{if} \quad y_a < 0 \\ \theta_{sa} r_{sa} y_a\,, & \text{otherwise} \end{cases}$$

We can now write down the budget constraints of consumer h, given asset price vector $q \in \mathbb{R}_+^A$ and spot price vector $p_s \in \mathbb{R}_+^L$. Consumption plan $x^h \in \mathbb{R}_+^{SL}$ and portfolio $y^h \in \mathbb{R}^A$ must satisfy the following first period budget constraint

$$p_0 \cdot (x_0^h - w_0^h) + q \cdot y^h = 0. \tag{1}$$

In each state s of the second period the budget constraint is

$$p_s \cdot (x_s^h - \omega_s^h) = \max\left(\sum_{a=1} G_{sa}(r_{sa}\,y_a), -\sum_{\ell=1}^{L} p_{s\ell}\,\gamma_{s\ell}\,\omega_{s\ell}^h\right) \equiv f_s^h \tag{2}$$

INSERT FIGURE 1

Fig. 1

Figure 1 illustrates the shape of the function f_s.

When consumer h goes bankrupt, the total amount available to partially reimburse creditors will be $\sum_\ell p_{s\ell}\,\gamma_{s\ell}\,\omega^h_{s\ell} + \sum_a (G^h_{sa})^+$. This amount will be allocated across assets sold by agent h according to the weights $\tau^h_{sa} = r_{sa}(y^h_a)^- \,/\, \sum_b r_{sa}(y^h_b)^-$.

Default is correctly anticipated in each asset market provided that

$$\text{(3)} \qquad \int_{B_s} \Big(\sum_\ell p_{s\ell}\gamma_{s\ell}\omega^h_{s\ell} + \sum_b (G^h_{sb})^+ \Big)\tau^h_{sa}\, d\lambda(h)$$

$$- r_{sa} \int_{B'_s} (y^h_a)^-\, d\lambda(h) = \int_I (G^h_{sa})^+\, d\lambda(h)$$

where $B_s = \{h\colon f^h_s = -\sum_\ell p_{s\ell}\,\gamma_{s\ell}\,\omega^h_{s\ell}\}$.

The above equation states that the aggregate payments, by defaulters and non-defaulters, related to asset a, should match aggregate revenues due to this asset, in any state s.

Notice that the left hand side of equation (3) can be rewritten as $r_{sa} \int_I (y^h_a)^+\, d(\lambda(h) - \int_I (f^h_s - \sum_b G^h_{sb})\tau^h_{sa}\, d\lambda(h)$. The mean reimbursement ratio is defined by (3) above, since the right hand side is $\theta_{sa} \int_I (y_a^h)^+ \, d\lambda(h)$, or equivalenty by:

$$\theta_{sa} = 1 - \frac{\int_I (f_s^h - \sum_b G_{sb}^h) r_{sa}^h \, d\lambda(h)}{r_{sa} \int_I (y_a^h)^+ \, d\lambda(h)} . \tag{3'}$$

There is no need to contemplate the possibility of selling short and buying a same asset, since this action would be dominated ($G_{sa}(0) > -r_{sa}y_a + \theta_{sa}r_{sa}y_a$), for $\theta_{sa} < 1$).

An equilibrium is a vector $(x, y, p, q, \theta) \in L^1(I, \mathbb{R}_+^{(S+1)L} \times \mathbb{R}^A \times \mathbb{R}_+^{(S+1)L+A} \times [0,1]^{SA}$ such that

- $(\bar{x}^h, \bar{y}^h)$ maximizes u^h subject to (1) and (2), for a.e. $h \in I$
- markets clear, i.e. $\int_I (x^h - \omega^h) d\lambda(h) = 0$ and $\int_I y^h \, d\lambda(h) = 0$
- (3) holds, for any (s, a).

2.2 The result

Theorem 1. Suppose the endowment allocaiton w is uniformly bounded, from above and from below, and asset short-sales are required to be bounded from below, by $t \in \mathbb{R}_{--}^A$, say. Then, an equilibrium exists.

Proof: Consider a sequence of economies E^n with truncated consumption sets $L_n = [0, n]^{L(S+1)}$ and truncated portfolio sets $K_n = [-d, n]^A$. Define, for each n, a generalized game played by the continuum of consumers and $S + 1 + SA$ additional players. Each consumer maximizes his utility on $L_n \times K_n$ subject to the budget constraints (1) and (2), given (q, p, θ). The additional players are as follows

(i) S auctioneers choose $p_s \in \Delta^{L-1}$ in order to maximize $p_s \cdot \int_I (x_s^k - \omega_s^h) d\lambda(h)$

(ii) one auctioneer chooses $(p_0, q) \in \Delta^{L+A-1}$ in order to maximize

$$p_0 \cdot \int_I (x_0^h - \omega_0^h) d\lambda(h) + q \cdot \int_I y^h d\lambda(h)$$

(iii) SA additional agents choose $\theta_{sa} \in [0,1]$ according to (3') above (which is equivalent to (3)).

Lemma 1. A pure strategies noncooperative equilibrium of the generalized game of a truncated economy is an equilibrium for this economy, for n sufficiently large.

Proof: At an equilibrium of the game, the optimality conditions of consumers' problems and of problems (i) and (ii) imply that

$$\int_I (x_0^h - \omega_0^h) d\lambda \leq 0, \quad \int_I y^h \, d\lambda \leq 0,$$

and, therefore, by (iii), aggregating (3') over assets, we have

$$\int_I f^h \, d\lambda(h) = \int_I \sum_a G_{sa}^h \, d\lambda(h) + \sum_a (1-\theta_{sa}) r_{sa} \int_I (y_a^h)^+ \, d\lambda(h) = R_s \int_I y^h \, d\lambda(h)$$

then,

$$p_s \int_I (x_0^h - \omega_0^h) d\lambda(h) = \int_I f^h \, d\lambda(h) \leq R_s \int_I y^h \, d\lambda \leq 0.$$

Now, $p_{s\ell} > 0$, for any $s = 0, 1, \ldots, s$ and any $\ell = 1, \ldots, h$, for n sufficiently large. Otherwise, every consumer would choose $x_{s\ell}^h = n$, implying that $\int_I x_{s\ell}^h \, d\lambda = n > \int_I w_{s\ell}^h \, d\lambda$, for $n >$ ess sup $w_{s\ell}$. But when $p_{o\ell} > 0$ we must have $\int_I (x_{o\ell}^h - w_{o\ell}^h) d\lambda(h) = 0$, since $\sum_{\ell=1}^{L} p_{o\ell}(x_{o\ell} - w_{o\ell}^h) d\lambda(h) + \sum_{j=1} q_j \int_I y_j^h \, d\lambda(h)$ is a null sum of non positive terms, hence a sum of null terms.

Moreover, $q_j > 0$, for any $j = 1, \ldots, A$. Otherwise, every consumer would choose $y_j = n$, implying $\int_I y_j^h \, d\lambda(h) = n > 0$. But when $q_j > 0$ we must have $\int_I y_j^h \, d\lambda(h) = 0$, by the argument above.

Finally, since $\int_I f_s^h \, d\lambda(h) = R_s \int_I y_s^h \, d\lambda(h)$ and the latter is zero, it follows that $p_s \cdot \int_I (x_s^h - w_s^h) d\lambda(h) = \int_I f_s^h \, d\lambda(h) = 0$ and therefore $\int_I (x_s^h - w_s^h) d\lambda(h) = 0$, for n

sufficiently large. In fact, we showed that $p_s \gg 0$ and therefore excess supply is ruled out in any spot market, since $\sum_{\ell=1}^{L} p_{s\ell} \int_I (x^h_{s\ell} - w^h_{s\ell}) d\lambda(h)$ is a null sum of h nonpositive terms, hence a sum of L null terms. $\square$

To show that the game has an equilibrium we have to solve the difficulties created by the nonconvexity of the budget set.

Lemma 2. The generalized game of a truncated economy has a noncooperative equilibrium, in pure strategies.

Proof: First, let us extend the game to allow for consumers' mixed strategies in portfolios. Let ρ^h be a regular probability measure on the Borel sets of K_n. We replace budget constraints (1) and (2) by the extended budget constraints

$$\int_{K_n} q \cdot y^h d\rho^h(y^h) = p_0 \cdot (w^h_0 - x^h_0)$$

$$p_s \cdot (x^h_s - \omega^h_s) = \int_{K_n} f^h_s \, d\rho^h(y^h)$$

We also replace, in problem (ii), $\int_I y^h \, d\lambda(h)$ by $\int_I \int_{K_n} (id) d\rho^h(y^h) d\lambda(h)$ and $\int_I f^h_s \, d\lambda(h)$ by $\int_I \int_{K_n} f^h_s \, d\rho^h(y^h) d\lambda(h)$.

Let us replace, in problem (iii) θ_{sa} by

$$1 - \int_I \int_{K_n} \left(f^h_s - \sum_b G^h_{sb} \right) \tau^h_{sa} \, d\rho^h(y^h) d\lambda \Big/ r_{sa} \int_I \int_{[-n,n]} (id)^+ \, d\rho^h_a(y^h_a) d\lambda(h)$$

Moreover, let us replace, in problems (i) and (ii), $\int_I (x^h_s - w^h_s) d\lambda(h)$ $(s = 0, 1, \dots, s)$ by $\int_I \int_{K_n} (d^h_s(y^h) - w^h) d\rho^h(y^h) d\lambda(h)$, where $d^h(y^h) =$ argmax$\{u^h : x^h$ satisfies the constraints (1) and (2), given the portfolio $y^h\}$. In this framework, consumers choose pairs

(x^h, ρ^h) of bundles and mixed strategies over portfolios and other agents choose (p, q, θ) given the consumers' profile of mixed strategies over portfolios. Even though one might argue that an equilibrium of this extended game is not a mixed strategies equilibrium, all that matters for our purpose is the fact that a degenerate equilibrium of the extended game is an equilibrium of the original game.

The extended game has an equilibrium, possibly in mixed strategies over portfolios. In fact, the best response correspondence of agents (i), (ii) and (iii) is upper semicontinuous on the profile of consumers' probability measures on K_n (with respect to the weak* topology on the dual of $L^1(I, C(K_n))$) and the profiles set is compact (for the same topology).

Now, it is possible to show the existence of a pure strategies equilibrium. First, notice that the functions in problems (i), (ii) and (iii) depend on the profile of mixed strategies $(\rho^h)_h$ only through finitely many m indicators of the form $\int_I \int_{K_n} z^h(y^h) d\rho^h(y^h) d\lambda(h)$, where $z_j \in L^1(I, C(K_n))$, for $j = 1, \ldots, m$.

Secondly, let B^h be the mixed strategies (in portfolios) best response correspondence of consumer h, given by $\rho \in B^h(p, q, \theta)$ if there exists $x \in [0, n]^{L(S+1)}$ such that (x, ρ) maximizes u^h subject to the extended budget constraints, given (p, q, θ). Now

$$\int_I \int_{K_n} z^h(y^h) dB^h(y^h) d\lambda(h) = \int_I \int_{K_n} z^h(y^h) d \text{ ext } B^h(y^h) d\lambda(h),$$

for $z \equiv (z_1, \ldots, z_m) \in L^1(I, (C(K_n))^m)$, since

$$\text{con} \int_{K_n} z^h(y^h) d \text{ ext } B^h(y^h) = \int_{K_n} z^h(y^h) dB^h(y^h) \subseteq \mathbb{R}^m$$

and theorem I.D.4 in Hildenbrand (1974) applies.

Then, given a mixed strategies (in portfolios) equilibrium profile $(\rho^h)_h$, there exists a portfolio profile $(y^h)_h$ such that the Dirac measure at y^h is an extreme point of

B^h (evaluated at the equilibrium levels of the variables chosen by the atomic players) and $(y^h)_h$ can replace $(\rho^h)_h$ and keep all equilibrium conditions satisfied (without changing the equilibrium levels of the variables chosen by the atomic players but replacing the former equilibrium bundles by $d^h(y^h)$). $\square$

Therefore, each truncated economy has an equilibrium (x, y, p, q, θ).

Let $Z_n = (x_n, y_n, p_n, q_n, \theta_n)$ be an equilibrium sequence for E^n and let $y_{jn}^{+h} = \max(0, y_{jn}^h)$, $y_{jn}^{-h} = \max(0, -y_{jn}^h)$.

Fatou's lemma, part I (on functions mapping into the positive orthant of a finite-dimensional space, see appendix), allows us to infer from the convergence of $\int_I (x_n^h, y_n^{+h}, y_n^{-h}) d\lambda$ (passing to a subsequence if necessary, since the sequence is bounded by $(\int_I w^h \, d\lambda, t, t)$) that there exists an integrable function (x, y^+, y^-) mapping into $\mathbb{R}_+^{L(S+1)+2A}$ such that $\int_I (x^h, y^{+h}, y^{-h}) d\lambda = \lim \int_I (x_n^h, y_n^{+h}, y_n^{-h}) d\lambda$ and, for a.e. (x^h, y^{+h}, y^{-h}) is a cluster point of $(x_n^h, y_n^{+h}, y_n^{-h})$.

Claim: $(x^h, y^h) \equiv (x^h, y^{+h} - y^{-h})$ is an optimal solution of consumer h's problem at the cluster point (p, q, θ).

Suppose $(\tilde{x}^h, \tilde{y}^h)$ satisfies the budget constraints at (p, q, θ) and $u^h(\tilde{x}^h) > u^h(x^h)$. Lower hemi-continuity of the budget correspondence (by lower hemi-continuity of its interior and result 4 in Hildenbrand (1974), page 26) implies that there is a sequence $(\tilde{x}^{hn}, \tilde{y}^{hn})$ converging to $(\tilde{x}^h, \tilde{y}^h)$ such that $(\tilde{x}^{hn}, \tilde{y}^{hn})$ satisfies the budget constraints at (p^n, q^n, θ^n). Then $u^h(\tilde{x}^{hn}) > u^h(x^{hn})$, for n sufficiently large. Moreover, the quantity constraints of the truncated economies are satisfied, since $(\tilde{x}^{hn}, \tilde{y}^{hn})$ converges. This would contradict optimality of (x^{hn}, y^{hn}). $\square$

Then the cluster point (p, q, θ) must satisfy $(p, q) \gg 0$, which is a necessary optimality condition for the consumers' problems, in the untruncated economy (where consumption and asset purchases are not bounded). Hence, $\int_I (x^h - w^h) d\lambda(h) = 0$ and

$\int_I y^h = 0$, since the sequences of allocations $(x_n^h, (y_n^h)^+)$ became uniformly bounded (and part II of Fatou's lemma applies): $x_{no\ell}^h \le (\operatorname*{esssup}_{h,\ell} w_{o\ell}^h / p_{o\ell}) + t \sum_j q_j$ and $(y_{na}^h)^+ \le (\operatorname*{esssup}_{h,\ell} w_{0\ell}^h / q_a) + t \sum_j q_j$ . . $\square$

3. NONPROPORTIONAL REIMBURSEMENT

3.1 The model

We assume now that reimbursement ratios decrease with the size of the claim. This nonproportional reimbursement scheme will be crucial for our result on existence of equilibria which dispenses with bounded short-sales. Discounted returns are given by $G_{sa}(r_{sa}\, y_a)$. The function G_{sa} coincides with the identity map on $(-\infty, 0]$ and is strictly concave on $[0, +\infty)$ when other agents default on asset a in state s. More precisely, let θ_{sa} be the mean reimbursement rate. For $\theta_{sa} = 1$, G_{sa} is still the identity map on $[0, +\infty)$, but, for $\theta_{sa} < 1$, the derivative of G_{sa} is equal to 1 at the origin and tends to zero as $y_a \to +\infty$. Actually, we assume the precise form of the function G_{sa} is endogenously determined within the family of functions of the form $G_{sa}(z) = g_{sa}(z) z$ where the reimbursement ratio is given by $g_{sa}(z) = 1 - \exp(-\delta z^{-\gamma})$. The parameters δ and γ are related to the inflection point (c, d) of the curve g by $\gamma = [1 - \ell n(1 - d)]^{-1}$ and $\delta = c^\gamma (1 - \gamma)/\gamma$. Now $d = \theta_{sa}$ and c is to be determined endogenously. Notice that $\theta_{sa} \to 1$ implies $g_{sa} \to 1$ and that $dG/dc > 0$, $d^2G/dc^2 < 0$, for $\theta_{sa} < 1$ (see appendix).

INSERT FIGURES 2 AND 3

Fig. 2

Fig. 3

We assume also that the liability coefficients $\gamma_{s\ell}$ tend to one as the debt goes to $-\infty$. More precisely, suppose there are minimal liability coefficients $\bar{\gamma}_{s\ell} \in [0,1]$ and that the actual liability ratios are given by $\gamma_{s\ell} = \bar{\gamma}_{s\ell} + (1 - \bar{\gamma}_{s\ell})\varphi(\sum_a G_{sa})$, where $\varphi: \mathbb{R} \to [0,1]$ is such that $\varphi(z) = 0$ for $z \geq -\sum_\ell p_{s\ell}\,\bar{\gamma}_{s\ell}\,\omega_{s\ell}^h$ and $\varphi(z) \to 1$ as $z \to -\infty$. This specification for the liability structure together with Inada's condition on utility will imply the existence a lower bound on the debt. Figure 4 illustrates the shape of the function f_s.

INSERT FIGURE 4

Fig. 4

The second-period budget constraints are given by (2) as before, where G_{sa} has this new specification.

In the first period, we add to the linear cost of the portfolio $q \cdot y^h$ a spread which is a discounted expected value of the difference between the default given by agent h and the default suffered by this agent. Formally, the first period constraint is written as

$$(1') \qquad q \cdot y^h + t\sum_{s=1}^{S} \eta_s (f_s^h - R_s y^h) + p_0 \cdot (x_0^h - \omega_0^h) = 0.$$

Here $f_s^h - R_s y^h = f_s^h - \sum_a G_{sa} - \left(R_s y^h - \sum_a G_{sa}\right)$ is the difference between default given by agent h in state s and default suffered by this agent in this state. The discount factor $t \in \mathbb{R}_+$ and the probability measure $\eta \in \Delta^{S-1}$ are to be determined endogenously.

In equilibrium, (t, η) will be such that aggregate net default is nonpositive, that is, the mean of the map $h \mapsto (f_s^h - R_s y)$ is nonpositive, for each state s. To ensure that, for each asset default is correctly anticipated, requires, in addition, the endogeneous adjustment of the abcisses c_{sa} of the inflection point of the reimbrusement schedule g_{sa}.

Equation (3), requiring default to be correctly anticipated, still holds. Now, the mean reimbursement ratio θ_{sa} is defined by

$$\theta_{sa} = \frac{\int_I (G_{sa}^h)^+ \, d\lambda(h)}{r_{sa} \int_I (y_a^h)^+ \, d\lambda(h)} \cdot$$

and, therefore, (3') still holds also.

We did not contemplate the possibility of buying and setting short in a same asset, since this action would be dominated in terms of second period returns (due) to the strict concavity of G_{sj}) and would lead to no gains in the first period either, assuming that the negative part of the spread (that is, the compensation for default suffered, given by $t \sum_s \eta_s \; (\sum_j G_{sj} - R_s y))$ is calculated on the basis of the net holdings of each asset.

We can now define an equilibrium. Given an economy $(\omega, R, \bar{\gamma}) \in L^1(I, \mathbb{R}_+^{(S+1)L}) \times \mathbb{R}_+^{SA} \times [0,1]^{SL}$, a vector $(\bar{x}, \bar{y}, p, q, t, \eta, c, \theta) \in L^1(I, \mathbb{R}_+^{(S+1)L}) \times L^1(I, \mathbb{R}^A) \times \mathbb{R}_+^{(S+1)L} \times \mathbb{R}^A \times \mathbb{R}_+ \times \Delta^{S-1} \times \mathbb{R}_+^{2SA}$ is an equilibrium if

- $(\bar{x}^h, \bar{y}^h)$ maximizes u^h subject to (1') and (2), for a.e. $h \in I$
- markets clear, i.e. $\int_I (x^h - \omega^h) d\lambda(h) = 0$ and $\int_I y^h \, d\lambda(h) = 0$

- (3) holds, for any (s, a).

3.2 The result

Theorem 2. *Suppose*

(A1) u_s^h *is* C^1 *on* $\mathbb{R}_{++}^L$ *and* $||\nabla u_s^h(x_s^h)|| \to \infty$ *as* $x_s^h \to 0$

(A2) $\{u^h\}_h$ *and* $\{Du^h\}_h$ *are equicontinuous*

(A3) the endowment allocation $\omega \in L(I, \mathbb{R}^{L(S+1)})$ *is continuous and uniformly bounded from above and from below*

(A4) for $({}^n x^h, {}^n y^h)$ *satisfying the budget constraints, at* $(p^n, q^n, \theta^n, t^n, \eta^n, c^n)$*, we have* $||\nabla u_s^h({}^n x_s^h)|| \quad (f_s^h)'(\sum_j G_{sj}(r_{sj}{}^n y_j^h)) \to +\infty$ *when* $\sum_j G_{sj}(r_{sj}{}^n y^h) \to -\infty$ *(and therefore* ${}^n x_s^h \to 0$*), for any* h

Then, an equilibrium exists.

Assumptions (A2) and (A3) introduce compactness on the set of consumers' characteristics. Assumption (A4) says that, as the debt becomes unbounded, in some state, and therefore, consumer's income and consumption tend to zero (since the liability coefficients approach one), the marginal utility of income will go to $+\infty$ faster than the derivative of the alienable endowment (with respect to the debt) tends to zero. We knew already that marginal utility of income would go to $+\infty$ as income approaches zero, as the endowment of a defaulter is eventually totally confiscated. We knew also that this endowment is being confiscated at a decreasing rate, as the debt increases. We now assume the former dominates the latter, so that the indirect marginal utility, with respect to portfolio returns, explodes as the debt becomes unbounded.

The following four sections present the proof of the theorem.

3.3 Equilibria of truncated economies

As in the proof of Theorem 1 above, consider a sequence of turncated economies D^n. Consumption sets are $L_n = [0,n]^{L(S+1)}$, as before, but portfolio sets are now $K_n = [-n,n]$. For each n, the generalized game is now played by the continuum of consumers, the $S+1+SA$ fictitions agents contemplated earlier (choosing prices and mean reimbursement rates) and, now, also SA additional agents choosing $c_{sa} = [0, r_{sa}n]$ in order to adjust the reimbursement schedules. These last SA players maximize $\min(0, \int_I (G^h_{sa}(r_{sa}y_a, c_{sa}))^+ \, d\lambda(h) - \theta_{sa}r_{sa}\int_I (y^h_a)^+ \, d\lambda(h))$. These last SA players are refered to as the schedulers.

Lemma 1 can be redone to show that a pure strategies equilibrium of the generalized game is an equilibrium for the truncated economy, for n sufficiently large. Notice that, at a strategic equilibrium, $\int_I (x^h_0 - w^h_0) d\lambda \leq 0$, $\int_I y^h \, d\lambda \leq 0$ and $\int_I (f^h_0 - R_s y^h) d\lambda \leq 0$ (by the optimality of the strategy of the first period auctioneer and Walras' law). Then, $p_s \cdot \int_I (x^h_0 - w^h_0) d\lambda = \int_I f^h \, d\lambda \leq R_s \int_I y^h \, d\lambda \leq 0$.

Notice that at, an optimum of the problem of the SA schedulers, the objective function always takes value zero and $\int_I (G^h_{sa})^+ \, d\lambda - \theta_{sa} r_{sa} \int_I (y^h_a)^+ \, d\lambda \geq 0$. Since we knew already that $\int_I (f^h_s - R_s y^h) d\lambda \leq 0$ it follows that equation (3) holds, for any (s,a), and $\int_I f^h_s \, d\lambda(h) = R_s \int_I y^h \, d\lambda(h)$.

In fact, $\int_I (f^h_s - \sum\limits_j G^h_{sj}) \tau^h_{sa} \, d\lambda = (1-\theta_{sa}) r_{sa} \int_I (y^h_a)^+ \, d\lambda$, by definition of θ_{sa}. Now $\int_I (G^h_{sa})^+ \, d\lambda \geq \theta_{sa} r_{sa} \int_I (y^h_a)^+ \, d\lambda$ implies (I) $\int_I (f^h_s - \sum\limits_j G^h_{sj}) \tau^h_{sa} \, d\lambda \geq r_{sa} \int_I (y^h_a)^+ \, d\lambda - \int_I (G^h_{sa})^+ \, d\lambda$, for any (s,a). On the other hand, $\int_I (f^h_s - R_s y^h) d\lambda \leq 0$ is equivalent to (II) $\int_I (f^h_s - \sum\limits_j G^h_{sj}) d\lambda \leq \int_I (R_s y^h - \sum\limits_j G^h_{sj}) d\lambda$. Now (II) implies that (I) must hold as an equality, for any a, since $\sum_a \tau^h_{sa} = 1$.

As in the proof of Lemma 1, $p \gg 0$ and $q \gg 0$ for n large enough, implying that commodity and asset markets will not be in excess supply either.

Lemma 2 can also be reduce to show the existence of a price strategia equilibrium. The extended game, over mixed strategies, is as in Lemma 2, except for the extended first period budget constraint and the extended problems of the SA schedulers. The former is now $\int_{K_n}[q \cdot y^h + \sum_s \beta_s(f^h_s - R_s y^h)]d\rho^h(y^h) = p_0 \cdot (w^h_0 - x^h_0)$, where ρ^h is, as before, a probability measure on Borelians of K_n. In the latter, $\int_I (y^h_a)^+ \, d\lambda$ should be replaced by $\int_I \int_{-n,n]}(\mathrm{id})^+ \, d\rho^h_a \, d\lambda$ and $\int_I (G^h_{sa})^+ \, d\lambda$ by $\int_I \int_{-n,n]}(G^h_{sa})^+ \, d\rho^h_a \, d\lambda$, where ρ^h_a is the marginal on coordinate a. The argument follows as in Lemma 2, and therefore, each truncated economy has an equilibrium $(x_n, y_n, p_n, q_n, t_n, \eta_n, \theta_n)$, where $t_n = \sum_s \beta^n_s$ and $\eta_n = \beta^n / t_n$.

3.4 A default barrier

Before discussing the properties of the sequence of truncated equilibria, as $n \to \infty$, we need to show that, at each vector of parameters of a consumer's problem, the debt will be bounded, and then use this fact to derive a set of nonarbitrage conditions, which will be satisfied by the equilibrium sequence, beyond a certain order.

Lemma 3. For each vector $(p, q, \theta, t, \eta, c)$ and each state s, $\operatorname*{ess\,inf}_h \sum_j G^h_{sj} > -\infty$ on the set of portfolios which are admissible and undominated in consumers' problem.

Proof: Suppose that for some consumer there is a sequence y_n of portfolios which satisfies the budget constraints and $||y_n|| \to +\infty$. Looking first at the subsets J^+ and J^- of assets bought and sold which diverge at the fastest rate, we assume that the set $S_1 = \{s : \sum_{j\in J^+} r_{sj}\delta_{sj} - \sum_{j\in J^-} r_{sj} < 0\}$ is nonempty, for $\delta_{sj} = 1$ when $\theta_{sj} = 1$ and $\delta_{sj} = 0$ otherwise. If S_1 were empty, for any pair (J^+, J^-), we were done. Consider also $S_2 = \{s : \sum_{j\in J^+} r_{sj}\delta_{sj} - \sum_{j\in J^-} r_{sj} = 0\}$ and $S_3 = \{s : \sum_{j\in J^+} r_{sj}\delta_{sj} - \sum_{j\in J^-} r_{sj} > 0\}$. Notice that for $s \in S_1$, $\sum_j G^h_{sj} \to -\infty$ and $f'_s \to 0$, whereas for $s \in S_3$, $\sum_j G_{sj} \to +\infty$ and $f'_s \to 1$. For $s \in S_2$, if there exists $j \in J^+$ such that $\theta_{sj} < 1$ for some $s \in S_2$, then $\sum_j G_{sj} \to +\infty$ and $f'_s \to 1$ in this state; otherwise, $\sum_s G_{sj}$ stays bounded in S_2.

Consider now the problem of utility maximization over bundles, at the financial income derived from a given portfolio. That is, $\max\{u^h : x^h$ satisfies (1) and (2), given $y^h\}$. The indirect utility function v^h is differentiable on portfolios. We claim that $\sum_{j\in J^+} \frac{\partial v}{\partial y_j} = \sum_{j\in J^-} \frac{\partial v}{\partial y_j}$ goes to $-\infty$, when the pair (J^+, J^-) is such that $S_1 \neq \emptyset$. This would imply that such a sequence y_n would never be chosen. Let us evaluate this directional derivative, denoting by λ_s $(s = 0, 1, \ldots, s)$ the multipliers of the constraints and by r_s the difference $\sum_{j\in J^+} r_{sj}\delta_{sj} - \sum_{j\in J^-} r_{sj}$.

$$\sum_{j\in J^+} \frac{\partial v}{\partial y_j} - \sum_{j\in J^-} \frac{\partial v}{\partial y_j} = \sum_s \lambda_s f'_s\Big(r_s + \sum_{\substack{j\in J^+\\ \theta_{sj}<1}} G'_{sj}\Big) -$$

$$\lambda_0\Big[\sum_{j\in J^+} q_j - \sum_{j\in J^-} q_j + t\sum_s \eta_s f'_s\Big(r_s + \sum_{\substack{j\in J^+\\ \theta_{sj}<1}} G'_{sj}\Big) - t\sum_s \eta_s\Big(\sum_{j\in J^+} r_{sj} - \sum_{j\in J^-} r_{sj}\Big)\Big].$$

Now the expression inside squared brackets tends to

$$\sum_{j\in J^+} q_j - \sum_{j\in J^-} q_j - t\sum_{s\in S_2\cup S_3} \eta_s \sum_{j\in J^+:\theta_{sj}<1} r_{sj} - t\sum_{s\in S_1} \eta_s\Big(\sum_{j\in J^+} r_{sj} - \sum_{j\in J^-} r_{sj}\Big) \equiv N.$$

If $N < 0$, then $qy + t\sum_s \eta_s(f_s - R_s y)$ would go to $-\infty$ and x_0 to $+\infty$, implying that

$\lambda_0 = ||\nabla u_0||/||p_0||$ would stay bounded (possible converge to zero). That is, N negative is incompatible with λ_0 becoming unbounded (as a result of x_0 tending to zero).

Now, let us look at the second period terms. For $s \in S_3$, $r_s + \sum_{\substack{j \in J^+ \\ \theta_{sj} < 1}} G'_{sj}$ tends to $r_s > 0$, but $\lambda_s f'_s$ stays bounded (since $\sum_j G_{sj} \to +\infty$ and $x_s \to +\infty$). For $s \in S_1$, $r_s + \sum_{\substack{j \in J^+ \\ \theta_{sj} < 1}} G'_{sj}$ tends to $r_s < 0$ and $\lambda_s f'_s \to +\infty$, by assumption (A4), as $\sum_j G_{sj} \to -\infty$, $f_s \to -p_s w_s$ and $x_s \to 0$.

For $s \in S_2$, $r_s + \sum_{\substack{j \in J^+ \\ \theta_{sj} < 1}} G'_{sj} \to r_s = 0$. Now, if (a) slower diverging assets determine that $\sum_j G_{sj} \to +\infty$ (or if there is $j \in J^+$ such that $\theta_{sj} < 1$), then $\lambda_s f'_s$ stays bounded (as in S_3). When (b) slower diverging assets determine that $\sum_j G_{sj} \to -\infty$, we have $\lambda_s \to \infty$ and $f'_s \to 0$. If in these states there are no fastest bought assets for which $\theta_{sj} < 1$, the term $\lambda_s f'_s [r_s + \sum_{\substack{j \in J^+ \\ \theta_{sj} < 1}} G'_{sj}]$ will be zero throughout the sequence. Otherwise, $\sum_j G_{sj}$ might tend to $-\infty$ in these states (when the slower assets follow a sequence n and the faster ones follow n^α, with $\alpha < 1/(1-\gamma)$), but at lower rates than in states belonging to S_1. Hence, even if $\lambda_s f'_s \sum_{\substack{j \in J^+ \\ \theta_{sj} < 1}} G'_{sj}$ would tend to $+\infty$ for some states in S_2, we could still claim that these terms would be dominated by $\sum_{s \in S_1} \lambda_s f'_s r_s$, which tends to $-\infty$. Finally, when (c) $\sum_j G_{sj}$ stays bounded, $\lambda_s f'_s$ will stay bounded also and $\lambda_s f'_s \sum_{\substack{j \in J^+ \\ \theta_{sj} < 1}} G'_{sj} \to 0$.

Then $\sum_{j\in J^+} \frac{\partial v}{\partial y_j} - \sum_{j\in J^-} \frac{\partial v}{\partial y_j} \to -\infty$ and this implies that the sequence y_n leads to unbounded utility losses, even if there were other assets, outside of $J^+ \cup J^-$, diverging at lower rates.

The equicontinuity assumption on utility and marginal utility, together with the continuity assumption on endowments, imply that the lower bound on $\sum_j G^h_{sj}$ is uniform on consumers. In fact, let $L^h = \inf \sum_j G^h_{sj}$ and pick $\widetilde{L}^h < L^h$, for each h. Now, for each h, there exists $\varepsilon^h > 0$, such that for $h' \in (h - \varepsilon^h, h + \varepsilon^h)$ we still have $\sum_j G^{h'}_{sj} > \widetilde{L}^h$. Passing now to a finite subcover $\{(h - \varepsilon^h, h + \varepsilon^h)\}_{h=1}^H$ we find a uniform lower bound on $\sum_j G^h_{sj}$. $\square$

Remark 1: Suppose (θ^n) is such that $r_s(\theta^n) \equiv \sum_{J+} r_{sj}\delta^n_{sj} - \sum_{J^-} r_{sj} < 0$, for some state s and some pair (J^+, J^-) of subsets of assets, but at $\theta = \lim_n \theta^n$ we get $r_s(\theta) = 0$ (that is, r_s suffers a positive jump at the limit point of θ^n). Let y^{hn} be consumer h's choice at $(p^n, q^n, \theta^n, t^n, \eta^n, c^n)$. Then, we can still claim that the sequence $\sum_j G_{sj}(y^{hn}_j)$ is uniformly bounded from below. If it were not bounded, then $\sum_{J+} \frac{\partial v^h}{\partial y_j} - \sum_{J^-} \frac{\partial v^h}{\partial y_j}$ would tend to $-\infty$. The uniformity of the bound follows from equicontinuity of utility and marginal utility, together with continuity of endowments, as in the proof of the lemma above.

3.5 Nonarbitrage conditions

Let us examine now the conditions that asset prices q_a, mean reinbursement rates

θ_{sa} and spread coefficients (t, η) must satisfy to rule out the possibility of arbitrage. Given $y \in \mathbb{R}^A$, let $f_s(y) = t \sum_{s=1}^{s} \eta_s(R_s y - f_s(y)) - q \cdot y$. In the context of our model, by arbitrage we mean the existence of a sequence $(y_n) \subseteq \mathbb{R}^A$ of portfolios, satisfying the budget constraints, such that $f_s(y_n) \to +\infty$, for at least one s $(s = 0, 1, \ldots, s)$, and $f_s(y_n)$ stays bounded for other states. Let δ be the $S \cdot A$ matrix whose elements are given by $\delta_{sj} = 1$, when $\theta_{sj} = 1$, and $\delta sj = 0$, otherwise.

Lemma 4: $q \in \mathbb{R}^A$, $\theta \in [0,1]^{SA}$, $t \in \mathbb{R}_+$ and $\eta \in \Delta^{S-1}$ do not allow for arbitrage only if

(i) $q_j \geq t \sum_{s:\theta_{sj}<1} \eta_s r_{sj}$, when $\theta_j \neq 1$.

(ii) $q_j > 0$, when $\theta_j = 1$

(iii) for each pair (J^+, J^-) of subsets of assets, such that $\theta_j \neq 1$ for some $j \in J^+$, when $\left(\sum_{j\in J^+} r_{sj}\delta_{sj} - \sum_{j \in J^-} r_{sj}\right)_{s=1}^{S} \geq 0$, we have $\sum_{j \in J^+} \left(q_j - t \sum_{s:\theta_{sj}<1} \eta_s r_{sj}\right) - \sum_{j\in J^-} q_j \geq 0$.

(iv) for each pair (J^+, J^-) of subseets of assets, such that $\theta_j = 1$, $\forall j \in J^+$, when $\left(\sum_{j\in J^+} r_{sj}\delta_{sj} - \sum_{j \in J^-} r_{sj}\right)_{s=1}^{S} \geq 0$, we have $\sum_{j\in J^+} q_j - \sum_{j \in J^-} q_j > 0$.

Proof: The first period budget constraint can be written as $q \cdot y + t \sum_s \eta_s \left(f_s - \sum_j G_{sj}\right) - t \sum_{j:y_j>0} \sum_{s:\theta_{sj}>0} (r_{sj}y_j - G_{sj}) = p_0 \cdot (w_0 - x_0)$. By lemma 3, $f_s - \sum_j G_{sj}$ is bounded, $\forall s$. Now, suppose (i) does not hold and let $y_j \to +\infty$. Then, $f_s \to +\infty$ for some s, $\left(q_j - t \sum_{s:\theta_{sj}<1} \eta_s r_{sj}\right) y_j \to -\infty$ and $G_{sj} \to +\infty$ $(s = 1, \ldots, S)$. By L'Hopital's rule,

$$\lim f_0(y) = \left[\lim \left(q_j - t \sum_{s:\theta_{sj}<1} \eta_s r_{sj}\right) y_j\right] \left\{1 + \lim \left[t \sum_{s:\theta_{sj}<1} \eta_s G'_{sj} \Big/ \left(q_j - t \sum_{s:\theta_{sj}<1} \eta_s r_{sj}\right)\right]\right\}$$

$= +\infty$, since $G'_{sj} \to 0$ in states where $\theta_{sj} < 1$. Then, (q, θ, t, η) would allow for arbitrage. We have shown (ii) earlier (in the proof of lemma 1).

Suppose now that (iii) does not hold and let $y_j^n = n$ for $j \in J^+$ and $y_j^n = -n$ for $j \in J^-$. When $n \to +\infty$, $\lim_n f_s(y^n) = +\infty$, for same $s \in \{1, \ldots, s\}$ where $\sum_{j \in J^+} r_{sj}\delta_{sj} - \sum_{j \in J^-} r_{sj} > 0$ or $\theta_{sj} \neq 1$ for some $j \in J^+$ (since $\sum_j G_{sj}$ would go to $+\infty$, even if the linear terms would cancel out). Now, $\lim_n f_0(y^n) = -\left[\lim_n \left(\sum_{j \in J^+} \left(q_j - t \sum_{s:\theta_{sj}<1} \eta_s r_{sj} \right) n - \sum_{j \in J^-} q_j n \right)\right] \left\{ 1 + \lim_n \left[t \sum_{j \in J^+} \sum_{s:\theta_{sj}<1} \eta_s G'_{sj} \Big/ \left(\sum_{j \in J^+} \left(q_j - t \sum_{s:\theta_{sj}<1} \eta_s r_{sj} \right) - \sum_{j \in J^-} q_j \right) \right] \right\} = +\infty$ and arbitrage is allowed.

If (iii) does not hold, it must be the case that $\sum_{j \in J^+} r_{sj} - \sum_{j \in J^-} r_{sj} > 0$, for some state $s = \bar{s}$. Otherwise, the rank of R could not be A. Now, let $y_j^n = n$, for $j \in J^+$, and $y_j = -n$, for $j \in J^-$. When $n \to \infty$, $f_{\bar{s}} \to +\infty$. If $\sum_{j \in J^+} q_j - \sum_{j \in J^-} q_j < 0$, then $\lim_n f_0(y^n) = -\lim_n \left(\sum_{j \in J^+} q_j - \sum_{j \in J^-} q_j \right) n = +\infty$ and if $\sum_{j \in J^+} q_j = \sum_{j \in J^-} q_j$, $f_0(y^n)$ would stay bounded. In either case, arbitrage is allowed.

Notice that, by lemma 3, the pairs (J^+, J^-) for which $\sum_{j \in J^-} r_{sj}\delta_{sj} - \sum_{j \in J^-} r_{sj} < 0$ in some state are of no interest. $\square$

3.6 Asymptotic properties of truncated equilibria

If n is sufficiently large, the equilibrium values of (q, θ, t, η) in a truncated economy E^n are nonarbitrage vectors. Otherwise, every consumer would choose $x_{s\ell}^h = n$ (for some $s = 0, 1, \ldots, s$ and some $\ell = 1, \ldots, L$) and feasibility would fail for n sufficiently large.

Now, $(p_0^n, q^n, \beta^n) \subseteq \Delta^{L+A+S-1}$, where $\beta^n = t^n\eta^n$, $(p_s^n) \subseteq \Delta^{L-1}$ $(s = 1, \ldots, s)$ and $\theta^n \in [0,1]^{SA}$. Then these sequences have cluster points $(p_0, q, \beta), (p_s)_{s=1}^s$ and θ. We would like to infer that (q, θ, t, η) is of nonarbitrage (for $t = \sum\limits_s \beta_s, \eta = \beta/t$).

Lemma 5: A cluster point (q, θ, t, η) of the equilibrium sequence $(q^n, \theta^n, t^n, \eta^n)$ of the truncated economies is still a nonarbitrage vector.

Proof: Suppose not. Then either (a) there is a pair (J^+, J^-) such that $\sum\limits_{J^+} r_{sj}\delta_{sj}^n - \sum\limits_{J^-} r_{sj}$ is negative for some state along the sequence but becomes zero at the cluster point, which might not satisfy $\sum\limits_{J^+}\left(q_j - t \sum\limits_{s:\theta_{sj}<1} \eta_s r_{sj}\right) \geq \sum\limits_{J^-} q_j$, since this inequality was not required along the sequence, or (b) there is a pair (J^+, J^-) such that $\theta_j^n \to 1, \forall j \in J^+$, and $\left(\sum\limits_{J^+} r_{sj} - \sum\limits_{J^-} r_{sj}\right) > 0$ and $\left(\sum\limits_{J^-} r_{sj}\delta_{sj}^n - \sum\limits_{J^-} r_{sj}\right)_s \geq 0$ along the sequence. In case (b), by (iii) in lemma 4, we know that $\sum\limits_{J^+} q_j^n - \sum\limits_{J^-} q_j^n > 0$, but, at the cluster point we know only that the weak inequality holds, which, by (iv) in lemma 4, is not enough to prevent arbitrage. All the other cases, lead to the fullfilment of the desired weak inequalities at the cluster point.

Now, in case (a), (y_n^h) is bounded, for any consumer h, by the remark following lemma 3, and let y^h be its cluster point. In case (b), we can try to find a cluster point for (y_n^h) by inverting a Cramer subsystem of budget constraints (since $\theta_j = 1$ for $j \in J^+$), provided that we already know that (x_n^h) has itself a cluster point.

Fatou's lemma, part 1, establishes the existence of an integrable function x mapping into $\mathbb{R}_+^{L(S+1)}$ such that $\int_I x^h \, d\lambda \leq \lim\limits_n \displaystyle\int_I x_n^h \, d\lambda$ (passing to a subsequence if necessary) and x^h is a cluster point of (x_n^h) for a.e. h. Then, in case (b), we can pick $\# J^+ + \# J^-$ states and solve the respective Cramer system of budget constraints: $p_s \cdot (x_s^h - w_s^h) = \sum_{j \in J^+ \cup J^-} r_{sj} y_j^h - \sum_{j \notin J^+ \cup J^-} G_{sj}^h$, noticing that the last term is bounded, by absence of arbitrage opportunities involving these assets. The solution $(y_j^h)_{j \in J^+ \cup J^-}$ is a cluster point of the sequence $(y_j^{hn})_{j \in J^+ \cup J^-}$.

In fact, apply the implicit function theorem to the system of $\# J^+ \cup \# J^-$ equations $f_s(\sum_{j \in J^+ \cup J^-} G_{sj}^h + \sum_{j \notin J^+ \cup J^-} G_{sj}^h) - p_s \cdot (x_s - w_s) = 0$. We know that at the cluster point $(p, q, \theta, t, \eta, c)$ this system has a solution at which the Jacobian with respect to y_j, $j \in J^+ \cup J^-$, is nonsingular. Hence, there is a continuous function φ mapping a neighborhood V of $\left(p, q, \theta, t, \eta, c, \sum_{j \notin J^+ \cup J^-} G_{sj}^h, x\right)$ into a neighborhood U of $(y_j)_{j \in J^+ \cup J^-}$ and this function describes the unique solution of the system in these neighborhoods. Now, for n sufficiently large, $\left(p^n, q^n, \theta^n, t^n, \eta^n, c^n, \sum_{j \notin J^+ \cup J^-} G_{sj}^{hn}, x^n\right) \in V$ and we must have $(y_j^{hn})_{j \in J^+ \cup J^-}$ given by φ. The continuity of φ implies that $(y_j^h)_{j \in J^+ \cup J^-}$ is a cluster point of $(y_j^{hn})_{j \in J^+ \cup J^-}$.

We want to show that (x^h, y^h) is an optimal solution of consumer h's problem and that, therefore, the cluster point (q, θ, t, η) does not allow for arbitrage. Budget feasibility follows by closed graph of the budget correspondence and optimality uses the lower hemi-continuity of this correspondence (as in the proof of Theorem 1). $\square$

We are now ready to infer that the sequence of equilibria allocations of the truncated economies is uniformly bounded, which will allow us to apply Fatou's lemma (part II, see appendix).

Lemma 6. If $(p^n, q^n, \theta^n, t^n, \eta^n, c^n)$ converges to $(p, q, \theta, t, \eta, c)$ and both the sequence

$(q^n, \theta^n, t^n; \eta^n)$ and its limit point are contained in the set of nonarbitrage vectors, then the sequence of demanded bundles and portfolios (x^{hn}, y^{hn}) is uniformly bounded.

Proof: Let $f_0(y; q, \theta, t, \eta, c) = -q \cdot y - t \sum_s \eta_s (f_s - R_s y)$. Suppose (y^{hn}) is not uniformly bounded. Then $\operatorname{ess\,sup}_h (y^{hn}) \to \infty$, for some $j \in \{1, \dots, A\}$. By lemma 3 we can decompose the nonuniformly bounded assets into a pair (J^+, J^-) of subsets of assets such that $\sum_{J^+} r_{sj} \delta_{sj} - \sum_{J^-} r_{sj} \geq 0, \ \forall s$. Now, $\operatorname{ess\,sup}_h f_0(y^{hn}) \to -\infty$, either because condition (iii) in lemma 4 holds in the limit and $\operatorname{ess\,sup}_h t \sum_{J^+} \sum_{s:\theta_{sj}<1} \eta_s G_{sj} \to +\infty$ or because condition (iv) in lemma 4 holds in the limit. Then, $-f_0(y^{hn}; q, \theta, t\eta, c) > \sup_\ell \operatorname{ess\,sup}_h w^h_{o\ell}$, for h belonging to a set H of consumers of positive measure and $n > N$. Then, $-f_0(y^{hn}; q^n, \theta^n, t^n, \eta^n, c^n) > \sup_\ell \operatorname{ess\,sup}_h w^h_{o\ell}$, for $h \in H$ and n sufficiently large, contradicting budget feasibility along the sequence. $\square$

Lemma 7. There exists an integrable function (x, y) mapping into $\mathbb{R}_+^{L(S+1)} \times \mathbb{R}^A$ such that $\int_I x^h \, d\lambda(h) = \int_I w^h \, d\lambda(h)$, $\int_I y^h \, d\lambda(h) = 0$, (x^h, y^h) is a cluster point of (x^{hn}, y^{hn}), for a.e. h, and (3) holds.

Proof: By lemma 6, we can apply Fatou's lemma (part II, see appendix) to the sequence $(z^n) \equiv (x^n, y^n, \tilde{z}^n)$ where $\tilde{z}^n = ((y^+)^n, \tau^n, G^n, (G^+)^n, f^n, (f_s^n - \sum_j G_{sj}^n)\tau_{sj}^n - (1 - \theta_{sj}^n) r_{sj} (y_j^+)^n, (G_{sj}^+)^n - \theta_{sj}^n r_{sj} (y_j^+)^n)$. Then, there is an integrable function z such that $\int_I z(h) d\lambda(h) = \lim_n \int z^n(h) d\lambda(h)$ and, for almost every h, $z(h)$ is a cluster point of $(z^n(h))$. Now, $z(h) = (x(h), y(h), \tilde{z}(h))$ and, by continuity of the functions involved, $\tilde{z}(h)$ must be equal to $(y^+, \tau, G, G^+, f, (f_s - \Sigma G_{sj})\tau_{sj} - (1 - \theta_{sj}) r_{sj} y_j^+, G_{sj}^+ - \theta_{sj} r_{sj} y_j^+)$ evaluated at $y(h)$. This implies that markets clear and equation (3) holds, for any (s, a).

Moreover $(x(h), y(h))$ is an optimal solution, as in the end of the proof of lemma 6. $\square$

This completes the proof of the theorem. We close this section with two remarks on the role of some assumptions.

Remark 2. Without assumption (A4), the default barrier established in lemma 3 would not exist and one of the nonarbitrage conditions would be as follows. For any pair (J^+, J^-) of subsets of assets, such that (i) $\sum_{j\in J^+} r_{sj}\delta_{sj} - \sum_{j\in J^-} r_{sj} \equiv r_s$ is negative for $s \in S_1 \neq \emptyset$, positive for $s \in S_3 \neq \emptyset$ and possibly zero for $s \in S_2$, but (ii) $\theta_{sj} = 1$, $\forall j \in J^+$, $s \in S_2 \cup S_3$, we must have $\sum_{j\in J^+} q_j - \sum_{j\in J^-} q_j > t\left(\sum_{s\in S_2\cup S_2} \eta_s \sum_{j\in J^+} r_{sj} - \sum_s \eta_s \sum_{j\in J^-} r_{sj} \right)$. This strict inequality condition would create a problem when we would try to redo lemma 5, since the condition accommodates a case where bought assets do not reimburse completely (i.e., $\theta_{sj} < 1$), for some $s \in S_1$. In fact at a cluster point, only the weak inequality is guaranteeed to hold, but we might not find a Cramer subsystem of budget constraints to solve for (y_{J^+}, y_{J^-}).

Remark 3. If the returns function G_{sj} were piecewise linear (that is, if G_{sj} were given by $\theta_{sj} r_{sj} y_j$, for $y_j > 0$, as in the proportional reimbursement scheme adopted by Dubey-Geanakoplos-Shubik and Araujo-Monteiro-Pascoa), the nonarbitrage conditions (under assumption (A4) and a default barrier established as in lemma 3) would be as follows. For any pair (J^+, J^-) of subsets of assets, such that $\sum_{j\in J^+} r_{sj}\theta_{sj} - \sum_{j\in J^-} r_{sj} \equiv r_s \geq 0$, for any s, and $r_s > 0$, for some s, we must have $\sum_{j\in J^+} \left(q_j - t \sum_{s:\theta_{sj}<1} \eta_s (1 - \theta_{sj}) \right) > \sum_{j\in J^-} q_j$. This strict inequality would create a problem when we would try to redo lemma 5. In fact, at a cluster point only the weak inequality is guaranteed to hold, but we might not find a Cramer subsystem of budget constraints to solve for (y_{J^+}, y_{J^-}), if θ_{sj} were less than one for some $j \in J^+$. This is a case of ex-post redundancy of assets, due to default.

4. CONSTRAINED EFFICIENCY

We will discuss now the efficiency properties of equilibria. We will show in Proposition 1 that an equilibrium allocation is always weakly constrained efficient and in Proposition 2 that, generically, it will not be strongly constrained efficient. Since proofs are a bit harder for the nonproportional reimbrusement model, we deal only with this case in the proofs.

Proposition 1. Let $(\bar{x}, \bar{y}, \bar{p}, \bar{q}, \bar{\theta})$ be an equilibrium (together with a vector $(\bar{t}, \bar{\eta}, \bar{c})$ in de case of section 3 above). The allocation $(\bar{x}, \bar{y})$ is efficient among all allocations (x, y) such that

(i) $\int_I x^h \, d\lambda(h) = \int_I \omega^h \, d\lambda(h)$ and $\int_I y^h \, d\lambda(h)$

(ii) $\bar{p}_s \cdot (x_s^h - \omega_s^h) = \bar{f}_s^h(y^h) \equiv \max\left(\sum_{j=1}^{A} G_{sj}(r_{sj} y_j), - \sum_{\ell=1}^{L} \bar{p}_{s\ell} \gamma_{s\ell} \omega_{s\ell}^h \right)$, for any $s = 1, \ldots, S$ and a.e. h

(iii) $\bar{p}_0 \cdot (x_0^h - \omega_0^h) + \bar{q} \cdot y^h + T^h = 0$, for a.e. h, where

(iv) $T \in L^1(I, \mathbb{R})$ is such that $\int_I T^h \, d\lambda(h) = 0$.

Proof: Suppose there exists an allocation (x, y) which, together with a transfers-function T, satisfies conditions (i) through (iv) and $u^h(x^h) > u^h(\bar{x}^h)$ for a.e. a. Then the first period budget constraint must be violated, that is,

$$\bar{p}_0 \cdot (x_0^h - \omega_0^h) + \bar{q} \cdot y^h + \bar{t} \sum_s \bar{\eta}_s (\bar{f}_s^h - R_s y^h) > 0, \quad h \in I.$$

Now, aggregating,

$$\bar{p}_0 \cdot \int_I (x_0^h - \omega_0^h) d\lambda(h) + \bar{q} \cdot \int_I y^h \, d\lambda(h) + \bar{t} \sum_s \bar{\eta}_s \int (\bar{f}_s^h - R_s y^h) d\lambda(h) > 0.$$

But we know that $\int_I (x_0^h - \omega_0^h) d\lambda(h) = 0$, $\int_I y^h \, d\lambda(h) = 0$ and $R_s \int_I y^h \, d\lambda(h) = 0$. Then, $\sum_s \eta_s \int_I \bar{f}_s^h \, d\lambda(h) > 0$ and, therefore, there is a state s for which $\int_I \bar{f}_s^h \, d\lambda(h) > 0$. Now, by (ii) we must have $\int_I (x_{s\ell}^h - \omega_{s\ell}^h) d\lambda(h) > 0$ for some $\ell = 1, \ldots, L$, contradicting (i). □

Remark 4. It is interesting to notice the role played by the spread in Proposition 1. The weakly constrained efficient transfer among agents is given by the equilibrium spread $\bar{t} \sum_s \bar{\eta}_s (\bar{f}_s^h - R_s \bar{y}^h)$, which is a discounted expected value of the difference between the consumer's default and the default suffered by the consumer.

Our default model can be seen as a model of externalities, where default by others decreases returns according to the function G_{sj}. The inefficiency introduced by this externality can be corrected when a market for the externality is created, assuming that legal rights stipulate that agents are entitled to full reimbursement of claims. In our model, there is a market for the externality in each state and the respective price is given by $t\eta_s$. As suggested by Coase (1960), the resulting market equilibrium is free from the inefficiency caused by the externality.

There is, however, another source of inefficiency, which is the incompleteness of markets. We will examine now the performance of the equilibrium allocations in the light of a stronger efficiency criterion.

The above weak constrained efficiency property of equilibria is analogous to the properties found in the incomplete markets model without default and also in the collateral model (without utility penalties) of Geanakoplos-Zame-Dubey. As in these two models, the equilibria of our model will, generically, be dominated when prices are no longer assumed to be kept constant at the equilibrium levels.

Proposition 2. Generically (in endowments and utility functions), equilibrium allocations are inefficient among the allocations (x, y) for which there exist a transfer function $T \in L^1(I, \mathbb{R})$ and prices $(p, q) \in \mathbb{R}_+^{L(S+1)} \times \mathbb{R}_+^A$ satisfying

(i) $\int_I x^h \, d\lambda(h) = \int_I \omega^h \, d\lambda(h)$ and $\int_I y^h \, d\lambda(h) = 0$

(ii') $p_s \cdot (x_s^h - \omega_s^h) = f_s^h(y^h) \equiv \max\left(\sum_j G_{sj}(r_{sj}y_j), -\sum_\ell p_{s\ell}\gamma_{s\ell}\omega_{s\ell}^h\right)$, for any $s = 1, \ldots, S$ and a.e. h

(iii') $p_0 \cdot (x_0^h - \omega_0^h) + q \cdot y^h + T^h = 0$, for a.e. h,

(iv') $\int_I T^h \, d\lambda(h) = 0$.

Proof: It is enough to show that an equilibrium allocation does not satisfy the necessary first order condtions of the following welfare maximization problem, given $a \in L^1(I, \mathbb{R}_+)$,

$$\max_{(x,y,T,p,q)} \int_I a^h u^h(x^h) d\lambda(h)$$

subject to (i), (ii'), (iii') and (iv').

Let ρ_s^h be the Lagrange multiplier of constraint $(\text{ii'})_{h,s}$ and let ρ_0^h be the Lagrange multiplier of constraint $(\text{iii'})_h$. Then, setting the derivative of the Lagrangean with respect to $p_{s\ell}$ equal to zero we obtain the following equation:

$$\int_I \rho_s^h [(x_{s\ell}^h - \omega_{s\ell}^h) + \gamma_{s\ell}\omega_{s\ell}^h \chi_{B_s}^h] d\lambda(h) = 0 \tag{4}$$

where $B_s = \{h : f_s^h = -\sum_\ell p_{s\ell}\gamma_{s\ell}\omega_{s\ell}^h\}$.

Let us check if this necessary condition for efficiency can be satisfied in equilibrium. Notice that the first order conditions with respect to (x, y) are satisfied at an equilibrium $(\bar{x}, \bar{y}, \bar{p}, \bar{q}, \bar{\theta}, \bar{t}, \bar{\eta}, \bar{c})$ (answer already know, by Proposition 1), when we set the multipliers of the commodity-feasibility constraints equal to $\bar{t}\bar{\eta}_s\bar{p}_s$ and $\rho_s^h = \frac{\lambda_s^h}{\lambda_0^h} - \bar{t}\bar{\eta}_s$ $(s = 1, \ldots, S)$, where $\lambda_s^h = ||\nabla u_s^h(x_s^h)||/||\bar{p}_s||$ $(s = 0, 1, \ldots, S)$. Replacing in equation (4) we obtain:

$$\int_I \frac{\lambda_s^h}{\lambda_0^h}[(\bar{x}_{s\ell}^h - \omega_{s\ell}^h) + \gamma_{s\ell}\omega_{s\ell}^h \chi_{B_s}^h] d\lambda(s) =$$

$$\bar{t}\bar{\eta}_s \int_I (\bar{x}_{s\ell}^h - \omega_{s\ell}^h) d\lambda(h) + \bar{t}\bar{\eta}_s\gamma_{s\ell} \int_{B_s} \omega_{s\ell}^h \, d\lambda(h) \tag{5}$$

Notice that the first integral on the right hand side is zero and that when B_s is of measure zero the equation becames $\int_I \frac{\lambda_s^h}{\lambda_0^h}(\bar{x}_{s\ell}^h - \omega_{s\ell}^h) d\lambda(h)$, which was the necessary condition for efficiency in the incomplete markets model without default (see Geanakoplos-Polemarchakis (1986) and Magill-Shaffer (1990)). Just like this condtion was generically (in endowments and utility functions) violated in equilibria of the model without default, now equation (5) will not hold, generically, in equilibria. (since the LS equations (5), together with the equilibrium equations, constitute an overdetermined system). $\square$

APPENDIX

Fatou's lemma (in m-dimension)

I) Let (f_n) be a sequence of integrable functions of a measure space $(\Omega, \mathcal{A}, \nu)$ into $\mathbb{R}_+^m$. Suppose that $\lim_n \int_\Omega f_n \, d\nu$ exists. Then there is an integrable function f of Ω into $\mathbb{R}_+^m$ such that $f(w)$ is a cluster point of $(f_n(w))$, for a.e. w, and

$\int_\Omega f\, d\nu \leq \lim_n \int_\Omega f_n\, d\nu$.

II) Let (f_n) be a uniformly integrable sequence of integrable functions of a measure $(\Omega, \mathcal{A}, \nu)$ into $\mathbb{R}^m$. Suppose also that $\lim_n \int f_n\, d\nu$ exists. Then there is an integrable function f of Ω into $\mathbb{R}^m_+$ such that $f(w)$ is a cluster point of $(f_n(w))$, for a.e. w, and $\int_\Omega f\, d\Omega = \lim_n \int_\Omega f_n\, d\nu$ (see Arstein (1979) or Hildenbrand-Mertens (1971)).

Lemma: Let $G\colon \mathbb{R}_+ \to \mathbb{R}$ given by $G(z) = z(1 - \exp(-\delta z^{-\gamma}))$, where $\delta = c^\gamma(1-\gamma)/\gamma$, $\gamma \in (0,1)$ and $c > 0$. Then $dG/dc > 0$ and $d^2G/dc^2 < 0$.

Proof:

$$\frac{dG}{dc} = z\exp(-\delta z^{-\gamma})z^{-\gamma}\,\gamma\, c^{\gamma-1}(1-\gamma)/\gamma > 0$$

$$\frac{d^2G}{dc^2} = z\left[z^{-\gamma}\left(\frac{d\delta}{dc}\right)^2 \exp(-\delta z^{-\gamma})(-z^{-\gamma}) + \exp(-\delta z^{-\gamma})z^{-\gamma}\frac{d^2G}{dc^2}\right]$$

now $\dfrac{d^2\delta}{dc^2} = \gamma(\gamma-1)c^{\gamma-2}(1-\gamma)/\gamma < 0$ and therefore

$$\frac{d^2G}{dc^2} = z\left[-z^{-2\gamma}\left(\frac{d\delta}{dc}\right)^2 \exp(-\delta z^{-\gamma}) + \exp(-\delta z^{-\gamma})z^{-\gamma}\frac{d^2\delta}{dc^2}\right]$$

is negative.

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