Overlapping generations models with incomplete markets*

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Abstract

This paper addresses existence of equilibria for an overlapping generations exchange economy with incomplete markets, one-period real assets and bounded short sales, defined on an infinite event-tree where there is a continuum of branches at each node of the tree.

The demographic structure used in the paper is quite general and allows for infinite-lived as well as finite-lived agents.

Keywords: Overlapping generations, incomplete markets, continuum of states.

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1 Introduction

The existence of equilibria in models with finitely many agents, a continuum of states and incomplete markets was studied by Mas-Colell and Monteiro (1996), Hellwig (1996), Mas-Colell and Zame (1996), Monteiro (1996), Araujo, Monteiro and Páscoa (1998) for the finite horizon case, Araujo, Monteiro and Páscoa (1996) for the infinite horizon case. In the infinite horizon case, the stochastic and financial structure of the model is that of an infinite event tree with one-period real assets and bounded short-sales where there is a continuum of branches at each node of the tree. The existence of equilibrium is proved under the assumption that endowments are bounded from below and that discounting is sufficiently severe.

In this paper, we model overlapping generations of agents on the above stochastic and financial structure. The demographic structure we use is sufficiently general to encompass a finite number of infinite-lived agents as an extreme case as well as classical overlapping generations of finite-lived agents and economies where there are simultaneously finite-lived and infinite-lived agents.

To prove the existence of equilibrium, exactly as Araujo *et al.* (1996), we combine a finite truncation of the horizon with finite-dimensional approximations of utilities, endowments and asset returns. Each truncated economy has an equilibrium and each consumer's maximization problem satisfies Slater's condition. The main technical difficulty is to establish uniform bounds on the Kuhn-Tucker multipliers associated with these optimization problems. Then, applying Fatou's lemma sequentially allows to establish market clearing and optimality conditions in the limit economy.

The main required conditions are of two kinds. First, short-selling constraints are assumed to be stringent enough (or return coefficients to be small enough) in order to guarantee that ex-post incomes are positive and bounded away from zero. This hypothesis was used in all the above papers on existence of equilibrium with incomplete markets and a continuum of states, except Araujo *et al.* (1998) where a different equilibrium concept, allowing for default, was addressed. The second condition (used only for agents with infinite lifetime) states the same sufficiently severe discounting as in Araujo *et al.* (1996).

In addition, in the case of economies with infinite-lived agents, we also assume that aggregate endowments are uniformly bounded, over nodes: it is well known that, even in the deterministic overlapping generations models, the un-

boundedness of aggregate endowment may create serious difficulties for existence of equilibrium (see Geanakoplos and Polemarchakis (1991)). The assumption is satisfied under Wilson's (1981) condition that a finite set of agents owns a non-negligible fraction of aggregate endowment. It is also satisfied under an alternative condition bounding utilities from above and marginal utilities from below.

The paper is organized as follows. Section 2 describes the model. In Section 3, we state the results and discuss the specific assumptions. Section 4 is devoted to defining truncated economies and establishing bounds on multipliers. Section 5 develops the argument of the existence proof. An appendix states and proves a Radner-type theorem which asserts that each truncated economy has an equilibrium.

2 The model

We consider in this paper an overlapping generations model with countably many discrete time periods $n = 0, 1, \dots$. The stochastic structure of the model is given by an infinite event-tree with a unique initial node at date 0 and a continuum of branches at each node of the tree. As Araujo *et al.* (1996), we assume that the set of states branching at every node is $S = [0, 1]$ endowed with $\mathcal{B}$, the Borel σ -algebra of subsets. On S , the common probability measure at date n , given the history $(s_1, \dots, s_{n-1})$ of past realizations, is the Lebesgue measure μ . We will denote by μ_n the Lebesgue measure on S^n .

At each node, a finite number G of physical goods are traded on spot markets by alive consumers and a finite number J of one-period real assets are available for intertemporal transaction and insurance to any consumer who is currently alive and will be alive also in the following period. At date n , state dependent returns of asset j are given by $A_{n,s_1,\dots,s_n}^j \in \mathcal{R}_+^G$. Let $A_{n,s_1,\dots,s_n}$ be the $G \times J$ -matrix of returns whose j th column is $A_{n,s_1,\dots,s_n}^j$. The collection $A = (A_{n,s_1,\dots,s_n})_{n,s_1,\dots,s_n}$ describes the financial side of the economy.

The demographic structure of the economy is the one of a rather general overlapping generations model where at most countably many agents, finitely or infinitely lived, begin their lifetime at any discrete time period of the model. Commodity endowments and preferences are affected by uncertainty but not the birth and the death of agents who face in their individual optimization problem

the totality of events occurring during their lifetime.

More precisely, each agent $i \in I$ is described by a consumption set X^i , a portfolio set Z^i , a utility function U^i , an initial endowment ω^i , the collection of his lifetime periods T^i , and T'^i , the collection of periods where agent i can attend the financial market. Denote by X_n^i the set of bundles which may be consumed by i at date n and by Z_n^i the set of portfolios which may be traded by i at date n .

On these data together with A , we posit the following assumptions which fix the general framework of this paper:

Assumption A.1. For each time period n , $H(n) = \{i \in I \mid n \in T^i\}$ is finite.

Note that it follows from A.1 that I , the set of agents is at most countably infinite.

Assumption A.2. For each i , T^i is either a finite or an infinite nonempty interval of $\mathcal{N}$. If T^i is infinite, $T'^i = T^i$; if $T^i = [n, m]$, then $T'^i = [n, m - 1]$.

Let $\tilde{s}_n = (s_1, \dots, s_n)$ be a generic element of S^n . Occasionally for a given realization of $\tilde{s}_n$, we will use the notation $\tilde{s}_n^- = (s_1, \dots, s_{n-1})$ for the unique predecessor of $\tilde{s}_n$ at date $n - 1$ and $\tilde{s}_n^+ = (s_1, \dots, s_n, s_{n+1})$ for a generic successor at date $n + 1$ of $\tilde{s}_n$ (where the last component s_{n+1} is a random variable).

Assumption A.3. For each i ,

- If $n \in T^i$, X_n^i is the set of all $\mathcal{B}^n$ -measurable and essentially bounded functions $x_n^i : S^n \rightarrow \mathcal{R}_+^G$; if not, $X_n^i = \{0\}$. Finally, X^i is the subset of $\prod_n X_n^i$ consisting of functions which are uniformly bounded over n and $\omega^i \in X^i$.

- If $n \in T'^i$, Z_n^i is the set of all $\mathcal{B}^n$ -measurable and essentially bounded functions $\varphi_n^i : S^n \rightarrow \{v_n^i\} + \mathcal{R}_+^J$ with $v_n^i \ll 0$ in $\mathcal{R}^J$; if not, $Z_n^i = \{0\}$. Finally, $Z^i = \prod_n Z_n^i$.

- U^i is a time-separable discounted utility function given for every $x \in X^i$ by

$$U^i(x) = \sum_{n \in T^i} \beta_i^n \int_{S^n} u_{n, \tilde{s}_n}^i(x_{n, \tilde{s}_n}) d\mu_n(\tilde{s}_n)$$

where $\beta_i \in]0, 1[$, and for every $n \in T^i$, $u_{n, \tilde{s}_n}^i$ is a continuous, concave, strictly monotone real-valued function on $\mathcal{R}_+^G$ which is zero-valued outside of $\mathcal{R}_{++}^G$ and the function $\tilde{s}_n \rightarrow u_{n, \tilde{s}_n}^i(a)$ is essentially bounded, measurable for each $a \in \mathcal{R}_+^G$.

The vector v_n^i establishes a lower bound on short-sales of assets by agent i at

date n .

Assumption A.4. $A = (A_n)_n$ where $A_n : S^n \rightarrow \mathcal{R}_+^G$, defined by $A_n(\tilde{s}_n) = A_{n,\tilde{s}_n}$, is a $\mathcal{B}^n$ -measurable and essentially bounded function.

Remark 2.1 *In fact, the assumption that T^i is an interval is done merely for convenience. Indeed, since consumer i cannot transfer any wealth from time period n to a date outside the connected component of n in T^i , it is always possible to split the individual optimization problem of such an agent. We can replace this agent by a countable (or finite) collection of agents satisfying Assumption A.2 and keeping Assumption A.1 still satisfied.*

At node $\tilde{s}_n$, spot prices are denoted by $p_{n,\tilde{s}_n} \in \mathcal{R}^G$, asset prices by $\pi_{n,\tilde{s}_n} \in \mathcal{R}^J$. Consumer i 's budget constraint at this node is given by

$$p_{n,\tilde{s}_n}(x_{n,\tilde{s}_n} - \omega_{n,\tilde{s}_n}^i) + \pi_{n,\tilde{s}_n}\varphi_{n,\tilde{s}_n} \leq p_{n,\tilde{s}_n}A_{n,\tilde{s}_n}\varphi_{n-1,\tilde{s}_n^-}.$$

We can use this formula for the initial node ($n = 0$), by letting $A_0 = 0$ and $\varphi_{-1} = 0$.

Consumer i 's problem is to choose a consumption plan $x^i \in X^i$ and a portfolio plan $\varphi^i \in Z^i$ in order to maximize $U^i(x^i)$ subject to all above budget constraints.

Let us denote by Δ^ℓ the ℓ -dimensional simplex: $\Delta^\ell = \{x \in \mathcal{R}_+^{\ell+1} \mid \sum_{k=1}^{\ell+1} x_k = 1\}$.

Definition 2.1 *An equilibrium for the economy is a vector $(\bar{p}, \bar{\pi}, (\bar{x}^i, \bar{\varphi}^i)_{i \in I})$ such that:*

(a) $\bar{p} = (\bar{p}_n)$, $\bar{p}_n : S^n \rightarrow \mathcal{R}_+^G$ is measurable, $\bar{\pi} = (\bar{\pi}_n)$ and $\bar{\pi}_n : S^n \rightarrow \mathcal{R}_+^J$ is measurable with $(\bar{p}_{n,\tilde{s}_n}, \bar{\pi}_{n,\tilde{s}_n}) \in \Delta^{G+J-1}$ at every node of the event-tree;

(b) $(\bar{x}^i, \bar{\varphi}^i)$ solves consumer i 's problem at prices $(\bar{p}, \bar{\pi})$;

(c) For each n , $\sum_{i \in H(n)} (\bar{x}_n^i - \omega_n^i) = 0$ and $\sum_{i \in H(n)} \bar{\varphi}_n^i = 0$.

3 The results

3.1 Preliminary results

We will distinguish two cases depending whether there exists or not an "eternal" agent. The first case corresponds to the following assumption:

Assumption A.5. There exists $i_0 \in I$ such that $T^{i_0} = \mathcal{N}$.

The idea that an “eternal” agent performs a special role for getting equilibrium existence in dynamic economies with an infinity of agents was already noticed by Wilson (1981) in the different context of complete markets. In the second case, we need the following assumption:

Assumption A'.5. For every i , T^i is finite and for every $n \in \mathcal{N}$, there exists $i_n \in I$ such that $n \in T^{i_n}$ (or equivalently $\{n, n+1\} \subset T^{i_n}$).

In other words, with A'.5 we require at least one agent to be “active in asset markets” at each period. Assumption A'.5 is for example satisfied in classical overlapping generations models where agents live for at least two periods and where at least one agent is born at each period. Before additional comments on these assumptions on the demographic structure, we need to specify the assumptions on the endowments.

Assumption A.6. For all $i \in I$, there exists $\bar{\varepsilon}^i > 0$ such that for all $n \in T^i$, for all $\varphi \in Z^i$, $\omega_{n, \tilde{s}_n}^i + A_{n, \tilde{s}_n} \varphi_{n-1, \tilde{s}_n^-} \geq \bar{\varepsilon}^i 1$ a.e. $\tilde{s}_n \in S_n$.

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The Assumption (A.6) is by now classical in incomplete markets with a continuum of states where the agents are not allowed to default. We refer to Araujo *et al.* (1996) for a discussion of this assumption. It should be noticed that the lower bound on the ex-post endowment of each agent at each period of his lifetime is uniform over n . This lower bound is also uniform over i in the corresponding assumption (A'.6) for classical overlapping generations models. These assumptions implies in particular that ex-ante endowments are strictly positive.

Definition 3.1 *An economy is irreducible if at any feasible allocation, $(x^i)_{i \in I}$, and for any non-trivial partition $\{I_1, I_2\}$ of the set I , there exists an individual $i_2 \in I_2$ such that*

$$U^{i_2}(x^{i_2} + \sum_{i \in I_1} \omega_i) > U^{i_2}(x^{i_2}).$$

The previous notion of irreducibility is known to play an important role for existence results of equilibrium. We refer to Nikaido (1956), McKenzie (1959),

Debreu (1962) for the use of this condition in finite economies, and to Geanakoplos and Polemarchakis (1991) for the case of deterministic overlapping generations models. The following proposition proves that the assumptions stated here will guarantee this condition.

Proposition 3.1 *Under Assumption A.3, if we assume the assumptions A.5 and A.6, (respectively A'.5 and A'.6) then the economy is irreducible.*

Proof: Let us consider the sets of periods T_1 and T_2 associated to the partition $\{I_1, I_2\}$ of agents by: $T_j = \{n \in \mathcal{N} \mid H(n) \cap I_j \neq \emptyset\}$ ($j = 1$ or 2). By construction, T_1 and T_2 are nonempty. Since at each period, there is at least one agent alive, one deduces that $T_1 \cup T_2 = \mathcal{N}$. Recalling that ex-ante endowments are strictly positive, in view of strict monotonicity of preferences, it suffices to prove that $T_1 \cap T_2 \neq \emptyset$.

Under the assumption A.5, one of the sets T_1 or T_2 is equal to $\mathcal{N}$, and therefore the result is established.

Under the assumption A'.5, if $0 \in T_2$ (if $0 \in T_1$, the proof is symmetric) let us consider n in T_2 defined by $n = \max\{p \in \mathcal{N} \mid \{0, \dots, p\} \subset T_2\}$. By assumption A'.5, there exists some agent i_n alive at date n and $n+1$. By definition of n , one has $n+1 \notin T_2$, in particular $i_n \notin I_2$. It follows that $i_n \in I_1$, and consequently $\{n, n+1\} \subset T_1$. Therefore, $n \in T_1 \cap T_2$. $\square$

Remark 3.1 *In fact, the second part of Assumption A'.5 is done merely for convenience. Indeed, since any consumer cannot transfer any wealth from time period n to a date later, it is always possible to split the economy into two economies, one of them having a finite number of periods. By induction, we can replace the economy by a countable (or finite) collection of finite length economies and at most one of infinite length with all satisfying Assumption A'.5.*

Let us now add some notations: for all n , $\bar{\omega}_n = \text{ess sup}_{\bar{s}_n} (\sum_{i \in H(n)} \omega_{n, \bar{s}_n}^i)$ and $\bar{\bar{\omega}}_n = \max\{\bar{\omega}_{n, g} \mid g = 1, \dots, G\}$. For each $i \in H(n)$, $\bar{u}_n^i = \text{ess sup}_{\bar{s}_n} u_{n, \bar{s}_n}^i(\bar{\omega}_n)$. We will also define $\bar{u}^i = \sup_{n \in T^i} \bar{u}_n^i$.

Assumption A.7. The sequence $(\bar{\omega}_n)_n$ is bounded.

This assumption will only be used for the first case, it implies in particular that $\bar{u}^i$ will be finite. Let us now introduce two alternative conditions for Assumption A.7. Both of them imply Assumption A.7 under Assumption A.3.

Assumption A.8. A finite set of agents own a non-negligible fraction of the aggregate endowment of all but finitely many commodities.

The assumption A.8 is classical in overlapping generation models (cf., for example Wilson (1981)). More precisely, under the assumptions A.3, A.5 and A.6 the assumptions A.7 and A.8 are equivalent. The alternative condition implying Assumption A.7, is a boundedness condition on utilities and (non-smooth) marginal utilities stated only for the eternal agent.

Assumption A.9. $\bar{u}^{i_0}$ is finite and there exists some real number δ^{i_0} such that for a.e. $\tilde{s}_n$, there exists some vector $\mu_{n,\tilde{s}_n} \in \partial u^{i_0}(2\bar{\omega}_n)$ such that $\mu_{n,\tilde{s}_n} \geq \delta^{i_0} 1$, where ∂u^{i_0} denotes the supergradient.

3.2 Existence Results

For all i , let ε^i satisfy A.6 and $-\varepsilon^i 1 \geq v_n^i$, for all $n \in T^{v^i}$. We can now posit our equilibrium existence results.

Theorem 3.1 *Under the assumptions A.1–A.7, there exists an equilibrium if we assume that*

(i) *for all $n \in \mathcal{N}$, there exists $\gamma_n^{i_0} > 0$ satisfying $u_{n,\tilde{s}_n}^{i_0}(2\bar{\omega}_n) - u_{n,\tilde{s}_n}^{i_0}(\bar{\omega}_n) \geq \gamma_n^{i_0}$, a.e. $\tilde{s}_n \in S^n$.*

(ii) *for all $i \in I$ such that T^i is infinite, one has $H^i < +\infty$ where*

$$H^i = \sup_{n \in T^{v^i}} \left(\left(\text{ess sup}_{\tilde{s}_n} \|A_{n,\tilde{s}_n}\|_\infty \right) \left(\sum_{\{h|(n-1) \in T^{v^h}\}} \|v_{n-1}^h\|_1 \right) \right).$$

and if $H^i \neq 0$, $\beta_i < H^i/\varepsilon^i$.

In the setting of classical overlapping generations models, let ε satisfy A'.6 and $-\varepsilon 1 \geq v_n^i$ for all $n \in T^{v^i}$, and all i . The existence theorem is the following:

Theorem 3.2 *Under the assumptions A.1–A.4 and A'.5–A'.6, there exists an equilibrium if we assume that*

(iii) for all $n \in \mathcal{N}$, there exists $\gamma_n > 0$ and i_n with $n \in T'^{i_n}$ satisfying $u_{n,\bar{s}_n}^{i_n}(2\bar{\omega}_n) - u_{n,\bar{s}_n}^{i_n}(\bar{\omega}_n) \geq \gamma_n$, a.e. $\bar{s}_n \in S^n$.

(iv) there exists $\delta \in]0, 1[$, $\bar{u}$ such that for all $i \in I$, $\bar{u}^i \leq \bar{u}$ and if $H^i \neq 0$,

$$\beta_i \leq \delta \frac{\varepsilon^2}{2H^i \bar{u}}, \text{ where}$$

$$H^i = \sup_{n \in T'^i} ((\bar{\omega}_n \gamma_n^{-1}) (\text{ess sup}_{\bar{s}_n} \|A_{n,\bar{s}_n}\|_\infty) (\sum_{\{h|n-1 \in T'^h\}} \|v_{n-1}^h\|_1))$$

Remark 3.2 Assumption (i) in Theorem 3.1 is only made for one “eternal” agent. In view of strict monotonicity of utilities, it is automatically satisfied if the utility function $u_{n,\bar{s}_n}^{i_0}$ does only depend on n . It is also satisfied (in fact uniformly over n) under Assumption A.9. The corresponding assumption (Assumption (iii) in Theorem 3.2) for the classical overlapping generations model is made at each period for at least one “active” agent.

Remark 3.3 Assumption (i) in Theorem 3.1 is used in Lemma 4.2 in order to bound the marginal multipliers of finite lifetime agents. Hence, Assumption (i) is unnecessary if the economy only contains agents with infinite lifetime.

Remark 3.4 Theorem 3.1 contains as a particular case Theorem 4.1 in Araujo et al. (1996). Assumption (ii) in Theorem 3.1 corresponds to (iii) in Theorem 4.1 of Araujo et al. (1996). Made in Theorem 3.1 only for infinite lived agents, it imposes a strong form of discounting and guarantees, among other things, a transversality condition in the budget constraints of these agents. The corresponding assumption in Theorem 3.2 for classical overlapping generations models is also a strong form of discounting made for technical needs of the proof.

Let us consider the case of an economy where v_n^i is uniform over n and over i . If the number of agents who are financially active goes to infinity then $\text{ess sup}_{\bar{s}_n} \|A_{n,\bar{s}_n}\|_\infty$ needs to tend to zero if H^{i_0} is finite. In this sense, the finiteness of H^i in Theorem 3.1 means that either the financial market disappears with time or the number of agents who can attend the financial market cannot explode.

In the case of Theorem 3.2, the definition of H^i involves the aggregate endowment and Assumption A.7 might not be satisfied. With (iv) in Theorem 3.2,

increasing the population or the aggregate endowment implies higher discounting of the agents.

When agents cannot transfer revenue between lifetime periods and states of nature, one has the following result:

Proposition 3.2 *Under the assumptions A.1–A.4, when A is identically 0, there exists a pure spot market equilibrium if we assume:*

(i) *that for all $i \in I$, there exists $\varepsilon^i > 0$ such that for all $n \in T^i$, $\omega_{n, \tilde{s}_n}^i \geq \varepsilon^i 1$ a.e. $\tilde{s}_n \in S_n$.*

(ii) *either A.5 and A.7 or A'.5,*

It is worth noticing that in this case one has : H^i or H'^i are identically 0.

4 The truncated economy

4.1 Definition

Let us partition $S = [0, 1]$ in N subintervals of form $V_k^N = [\frac{k-1}{N}, \frac{k}{N}]$, where $k = 1, \dots, N$. We consider now a finite event-tree of length $(N+1)$ with a unique initial node at date $n = 0$ and, at each node, a constant branching number equal to N . Let us set $\tilde{N} = \{1, \dots, N\}$. At date $n \leq N$, the node $\tilde{m}_n = (m_1, \dots, m_n)$ will denote a generic element of $\tilde{N}^n$, $\tilde{m}_n^-$ its unique predecessor and, if $n < N$, $\tilde{m}_n^+$ a generic successor of $\tilde{m}_n$. On this event-tree, we consider an economy

$$\mathcal{E}^N = ((X^{iN}, U^{iN}, \omega^{iN}, Z^{iN})_{i \in I^N}, A^N)$$

defined as follows :

$$\begin{aligned} - I^N &= \{i \in I \mid n \in T^i, \text{ for some } n \leq N\} = \bigcup_{n \leq N} H(n) \\ - X_{n, \tilde{m}_n}^{iN} &= \begin{cases} \mathcal{R}_+^G & \text{if } n \in T^i \\ \{0\} & \text{if not} \end{cases} \quad \text{and } X^{iN} = \prod_{\substack{(n, \tilde{m}_n) \\ n \leq N}} X_{n, \tilde{m}_n}^{iN} \\ - Z_{n, \tilde{m}_n}^{iN} &= \begin{cases} \{v_n^i\} + \mathcal{R}_+^J & \text{if } n \in T^i \text{ and } n < N \\ \{0\} & \text{if not} \end{cases} \quad \text{and } Z^{iN} = \prod_{\substack{(n, \tilde{m}_n) \\ n \leq N}} Z_{n, \tilde{m}_n}^{iN}. \end{aligned}$$

$$- \omega^{iN} = (\omega_{n,\tilde{m}_n}^{iN})_{n,\tilde{m}_n} \text{ with } \omega_{n,\tilde{m}_n}^{iN} = N^n \int_{V_{\tilde{m}_1}^N \times \dots \times V_{\tilde{m}_n}^N} \omega_n^i(\tilde{s}_n) d\mu_n(\tilde{s}_n).$$

$$- U^{iN}(x) = \sum_{\substack{n=0 \\ n \in T^i}}^N \beta_i^n \cdot \sum_{\tilde{m} \in \tilde{N}^n} u_{n,\tilde{m}_n}^{iN}(x_{n,\tilde{m}_n}) \text{ with}$$

$$u_{n,\tilde{m}_n}^{iN}(x_{n,\tilde{m}_n}) = \int_{V_{\tilde{m}_1}^N \times \dots \times V_{\tilde{m}_n}^N} u_{n,\tilde{s}_n}^i(x_{n,\tilde{m}_n}) d\mu_n(\tilde{s}_n).$$

$$- A^N = \prod_{\substack{(n,\tilde{m}_n) \\ n \leq N}} A_{n,\tilde{m}_n} \text{ with } A_{n,\tilde{m}_n} = N^n \int_{V_{\tilde{m}_1}^N \times \dots \times V_{\tilde{m}_n}^N} A_n(\tilde{s}_n) d\mu_n(\tilde{s}_n).$$

Note that it follows from A.1 that I^N is finite.

It follows from the previous definitions that for all n , $\max_{\tilde{m}_n} \sum_{i \in H(n)} \omega_{n,\tilde{m}_n}^{iN} \leq \bar{\omega}_n$ and for each $i \in H(n)$, $N^n \max_{\tilde{m}_n} u_{n,\tilde{m}_n}^{iN}(\omega_{n,\tilde{m}_n}^{iN}) \leq \bar{u}_n^i$. Also for each infinite lived i , $\sup_{n \in T^i} (\max_{\tilde{m}_n} \|A_{n,\tilde{m}_n}^N\|_\infty (\sum_{\{h|(n-1) \in T^h\}} \|v_{n-1}^h\|_\infty)) \leq H^i$ in the case of Theorem 3.1 and in the case of Theorem 3.2, for every $i \in I$, $\sup_{n \in T^i} (\bar{\omega}_n(\gamma_n)^{-1} \max_{\tilde{m}_n} \|A_{n,\tilde{m}_n}^N\|_\infty (\sum_{\{h|(n-1) \in T^h\}} \|v_{n-1}^h\|_1)) \leq H^i$.

4.2 Truncated equilibrium

Under the assumptions of Theorem 3.1, one deduces immediately:

- (i) for all n and all $i \in H(n)$, for all $\varphi \in Z^{iN}$, $\omega_{n,\tilde{m}_n}^{iN} + A_{n,\tilde{m}_n}^N \varphi_{n-1,\tilde{m}_n} \geq \varepsilon^i 1$.
- (ii) for all $n \in \mathcal{N}$, $u_{n,\tilde{m}_n}^{i_0N}(2\bar{\omega}_n) - u_{n,\tilde{m}_n}^{i_0N}(\bar{\omega}_n) \geq \gamma_n^{i_0} N^{-n}$.

From the assumptions of Theorem 3.2, one deduces immediately:

- (iii) for all n and all $i \in H(n)$, for all $\varphi \in Z^{iN}$, $\omega_{n,\tilde{m}_n}^{iN} + A_{n,\tilde{m}_n}^N \varphi_{n-1,\tilde{m}_n} \geq \varepsilon 1$.
- (iv) for all $n \in \mathcal{N}$, for some i_n with $n \in T^{i_n}$, $u_{n,\tilde{m}_n}^{i_nN}(2\bar{\omega}_n) - u_{n,\tilde{m}_n}^{i_nN}(\bar{\omega}_n) \geq N^{-n} \gamma_n$.

As proven in the appendix, in view of Assumption A.6 or A'.6, the truncated economy $\mathcal{E}^N$ has an equilibrium, i.e. there exists $(p^N, \pi^N, (x^{iN}, \varphi^{iN})_{i \in I^N})$ such that

$$- p^N = (p_{n,\tilde{m}_n}^N)_{n,\tilde{m}_n}, \pi^N = (\pi_{n,\tilde{m}_n}^N)_{n,\tilde{m}_n} \text{ with } p_{n,\tilde{m}_n}^N \in \mathcal{R}^G, \pi_{n,\tilde{m}_n}^N \in \mathcal{R}^J \text{ and } (p_{n,\tilde{m}_n}^N, \pi_{n,\tilde{m}_n}^N) \in \Delta^{G+J-1} \text{ at every node } (n, \tilde{m}_n), n \leq N.$$

- For every $i \in I^N$, (x^{iN}, φ^{iN}) maximizes U^{iN} on $X^{iN} \times Z^{iN}$ subject to the budget constraints : $p_{n,\tilde{m}_n}^N (x_{n,\tilde{m}_n}^{iN} - \omega_{n,\tilde{m}_n}^{iN}) + \pi_{n,\tilde{m}_n}^N \varphi_{n,\tilde{m}_n}^{iN} \leq p_{n,\tilde{m}_n}^N A_{n,\tilde{m}_n}^N \varphi_{n-1,\tilde{m}_n}^{iN}$

$$-\sum_{i \in I^N} (x_{n,\tilde{m}_n}^{iN} - \omega_{n,\tilde{m}_n}^{iN}) = 0 \text{ and } \sum_{i \in I^N} \varphi_{n,\tilde{m}_n}^{iN} = 0.$$

Note that in view of the definition of Z^{iN} , $\varphi_{N,\tilde{m}_N}^{iN} = 0$ and one can assume that $\pi_{N,\tilde{m}_N}^N = 0$ and $\|p_{N,\tilde{m}_N}^N\|_1 = 1$.

4.3 Optimality Conditions

Lemma 4.1 *for each agent, there exist some real vector $((K_{n,\tilde{m}_n}^{iN})_{n \leq N})_{n \in T^i}$ such that for all x in X^{iN} and all φ in Z^{iN} , one has:*

$$\begin{aligned} \sum_{\substack{n \in T^i(n,\tilde{m}_n) \\ n \leq N}} \sum \beta_i^n (u_{n,\tilde{m}_n}^{iN}(x_{n,\tilde{m}_n}) - u_{n,\tilde{m}_n}^{iN}(x_{n,\tilde{m}_n}^{iN})) &\leq \sum_{\substack{n \in T^i(n,\tilde{m}_n) \\ n \leq N}} \sum K_{n,\tilde{m}_n}^{iN} (p_{n,\tilde{m}_n}^N (x_{n,\tilde{m}_n} - x_{n,\tilde{m}_n}^{iN})) \\ + \pi_{n,\tilde{m}_n}^N (\varphi_{n,\tilde{m}_n} - \varphi_{n,\tilde{m}_n}^{iN}) - p_{n,\tilde{m}_n}^N A_{n,\tilde{m}_n}^N (\varphi_{n-1,\tilde{m}_n^-} - \varphi_{n-1,\tilde{m}_n^-}^{iN}) \end{aligned} \quad (4.1)$$

$$\begin{aligned} \forall n \in T^i, n \leq N, \\ K_{n,\tilde{m}_n}^{iN} (p_{n,\tilde{m}_n}^N (x_{n,\tilde{m}_n}^{iN} - \omega_{n,\tilde{m}_n}^{iN}) + \pi_{n,\tilde{m}_n}^N \varphi_{n,\tilde{m}_n}^{iN} - p_{n,\tilde{m}_n}^N A_{n,\tilde{m}_n}^N \varphi_{n-1,\tilde{m}_n^-}^{iN}) = 0 \end{aligned} \quad (4.2)$$

$$\begin{aligned} \forall n \in T^i, n \leq N, \\ \beta_i^n (u_{n,\tilde{m}_n}^{iN}(x_{n,\tilde{m}_n}) - u_{n,\tilde{m}_n}^{iN}(x_{n,\tilde{m}_n}^{iN})) \leq K_{n,\tilde{m}_n}^{iN} p_{n,\tilde{m}_n}^N (x_{n,\tilde{m}_n} - x_{n,\tilde{m}_n}^{iN}) \end{aligned} \quad (4.3)$$

$$\begin{aligned} \forall n \in T^i, n < N, \\ 0 \leq (\varphi_{n,\tilde{m}_n} - \varphi_{n,\tilde{m}_n}^{iN}) (K_{n,\tilde{m}_n}^{iN} \pi_{n,\tilde{m}_n}^N - \sum_{\tilde{m}_n^+} K_{n+1,\tilde{m}_n^+}^{iN} p_{n+1,\tilde{m}_n^+}^N A_{n+1,\tilde{m}_n^+}^N) \end{aligned} \quad (4.4)$$

Proof: in view of the concavity assumptions made in Theorem 3.1 and Theorem 3.2, exactly as in Araujo *et al.* (1996), we can associate non-negative Lagrange multipliers $(K_{n,\tilde{m}_n}^{iN})$ to the maximization problem of any consumer $i \in I^N$, so as to get the successive assertions. $\square$

We now define $(k_{n,\tilde{m}_n}^{iN})_{n \in T^i}$ by $k_{n,\tilde{m}_n}^{iN} = N^n K_{n,\tilde{m}_n}^{iN}$. Using $x_{n,\tilde{m}_n} = 0$ in (4.3) and the monotonicity of $u_{n,\tilde{m}_n}^{iN}$, we have for any $n \leq N$, $i \in H(n)$ and $\tilde{m}_n \in \tilde{N}^n$, $\beta_i^n \bar{u}^i \geq N^n K_{n,\tilde{m}_n}^{iN} p_{n,\tilde{m}_n}^N x_{n,\tilde{m}_n}^{iN} = k_{n,\tilde{m}_n}^{iN} (-\pi_{n,\tilde{m}_n}^N \varphi_{n,\tilde{m}_n}^{iN} + p_{n,\tilde{m}_n}^N (\omega_{n,\tilde{m}_n}^{iN} + A_{n,\tilde{m}_n} \varphi_{n-1,\tilde{m}_n^-}^{iN})) \geq k_{n,\tilde{m}_n}^{iN} (-\pi_{n,\tilde{m}_n}^N \varphi_{n,\tilde{m}_n}^{iN} + \varepsilon^i p_{n,\tilde{m}_n}^N 1) = k_{n,\tilde{m}_n}^{iN} (\varepsilon^i - \pi_{n,\tilde{m}_n}^N (\varphi_{n,\tilde{m}_n}^{iN} + \varepsilon^i 1))$. This inequality can be rewritten in the following two equivalent ways:

$$(\varepsilon^i (\beta_i^n \bar{u}^i)^{-1}) k_{n,\tilde{m}_n}^{iN} (p_{n,\tilde{m}_n}^N 1) \leq 1 + (\beta_i^n \bar{u}^i)^{-1} k_{n,\tilde{m}_n}^{iN} \pi_{n,\tilde{m}_n}^N (\varphi_{n,\tilde{m}_n}^{iN}) \quad (4.5)$$

$$(\varepsilon^i (\beta_i^n \bar{u}^i)^{-1}) k_{n,\tilde{m}_n}^{iN} \leq 1 + (\beta_i^n \bar{u}^i)^{-1} k_{n,\tilde{m}_n}^{iN} \pi_{n,\tilde{m}_n}^N (\varphi_{n,\tilde{m}_n}^{iN} + \varepsilon^i 1) \quad (4.6)$$

4.4 Bounds on Lagrange multipliers

In order to get uniform bounds on $k_{n,\tilde{m}_n}^{iN}$, we will introduce the following notation, for all $n \leq N$, for all $i \in H(n)$,

$$\eta_n^{iN} = \frac{\varepsilon^i}{\beta_i^n \bar{u}^i} \sup_{\tilde{m}_n \in \tilde{N}^n} k_{n,\tilde{m}_n}^{iN} \quad \text{in the case of Theorem 3.1.}$$

$$\eta_n^{iN} = \frac{\varepsilon}{\beta_i^n \bar{u}} \sup_{\tilde{m}_n \in \tilde{N}^n} k_{n,\tilde{m}_n}^{iN} \quad \text{in the case of Theorem 3.2.}$$

The multipliers η_n^{iN} of the infinite-lived agents are uniformly bounded from above, by an inductive argument (as in Araujo *et al.* (1996)), using hypothesis (ii) and noticing that at the last period of the truncated economy the multiplier η_N^{iN} is bounded by 1.

However, it is not so immediate to bound the multipliers of agents that die before date N , since the multiplier at the death date is not known to be bounded. The proof of the following lemma shows that these multipliers can nevertheless be bounded from above since $\sum_{g=1}^G p_{n,\tilde{m}_n,g}^N$ is bounded from below as a consequence of the upper bound on the multipliers of the eternal agent.

Lemma 4.2 *Under assumptions A.1–A.6, (i) and (ii) of Theorem 3.1, for all agent i , and all $n \in T^i$, the sequence $(\eta_n^{iN})_N$ is bounded. More precisely, in the case where T^i is infinite, one has for all $n \in T^i$, $\eta_n^{iN} \leq \sum_{\ell \geq 0} ((\beta_i H^i)/\varepsilon^i)^\ell$.*

Proof. First by induction, we will compute explicit bounds for the agents with infinite lifetime, then we will deduce bounds for the other agents. Here, we will always suppose that the date N is large enough (after the birth of agent i).

Let us suppose that T^i is infinite. For $n = N$, we compute from (4.5) that $\eta_N^{iN} \leq 1$. If $n < N$, recall that equilibrium portfolio are in zero net supply, then using $\varphi_{n,\tilde{m}_n} = -\varepsilon^i 1$ in (4.4), one deduces that for all $i \in H(n)$ with infinite lifetime, one has $k_{n,\tilde{m}_n}^{iN} \pi_{n,\tilde{m}_n}^N (\varphi_{n,\tilde{m}_n}^{iN} + \varepsilon^i 1) \leq N^n \sum_{\tilde{m}_n^+} (k_{n+1,\tilde{m}_n^+}^{iN} p_{n+1,\tilde{m}_n^+}^N A_{n+1,\tilde{m}_n^+}^N (\varphi_{n,\tilde{m}_n}^{iN} + \varepsilon^i 1)) \leq H^i((\beta_i^{n+1} \bar{u}_i)/\varepsilon^i) \eta_{n+1}^{iN}$. This inequality, together with (4.6) leads to

$$\eta_n^{iN} \leq 1 + ((\beta_i H^i)/\varepsilon^i) \eta_{n+1}^{iN} \quad (4.7)$$

Recalling Assumption (ii) in Theorem 3.1, we get by decreasing induction the required inequality for infinite lived agents,

$$\eta_n^{iN} \leq \sum_{\ell=0}^{N-n} ((\beta_i H^i)/\varepsilon^i)^\ell \leq \sum_{\ell=0}^{\infty} ((\beta_i H^i)/\varepsilon^i)^\ell.$$

We will now use the bounds of agent i_0 to obtain a lower bound of $(p_{n,\tilde{m}_n}^N 1)$. Indeed, if we use $x_{n,\tilde{m}_n} = 2\bar{\omega}_n$ in (4.3), we have for any $n \leq N$ and $\tilde{m}_n \in \tilde{N}^n$, $N^{-n}\beta_{i_0}^n \gamma_{i_0}^{i_0} \leq K_{n,\tilde{m}_n}^{i_0 N} p_{n,\tilde{m}_n}^N (2\bar{\omega}_n - x_{n,\tilde{m}_n}^{i_0 N}) \leq 2N^{-n} k_{n,\tilde{m}_n}^{i_0 N} p_{n,\tilde{m}_n}^N \bar{\omega}_n \leq 2N^{-n} (\beta_{i_0}^n \bar{u}^{i_0} / \varepsilon^{i_0}) \eta_n^{i_0 N} \bar{\omega}_n (p_{n,\tilde{m}_n}^N 1)$. Hence, $p_{n,\tilde{m}_n}^N 1 \geq \gamma_{i_0}^{i_0} \varepsilon^{i_0} (2\bar{u}^{i_0} \bar{\omega}_n \eta_n^{i_0 N})^{-1}$.

For an agent i with a finite lifetime ($T^i = [\underline{n}, \bar{n}]$), to start the induction, we can remark that for all $N \geq \bar{n}$, the inequality (4.5) is now $\eta_n^{iN} \leq (p_{n,\tilde{m}_n}^N 1)^{-1}$, hence, it is bounded. If $\underline{n} \leq n < \bar{n}$, since the inequality (4.7) is still satisfied, this allows us to compute recursively an upper bound of η_n^{iN} from the one of η_{n+1}^{iN} . Since T^i is finite, one gets the conclusion. Note that, we can always define H^i for an agent i with a finite lifetime. $\square$

In the economy contemplated in Theorem 3.2 all agents have a finite life, but the multipliers η_n^{iN} can also be bounded from above, as shown in the next lemma, by a chain argument relating the multipliers at the death date to the multipliers of an agent still alive at the end of the truncated economy.

Lemma 4.3 *Under the assumptions of Theorem 3.2, for every agent i , and all $n \in T^i$, the sequence $(\eta_n^{iN})_N$ is bounded. More precisely, one has for all $n \in T^i$, $\eta_n^{iN} \leq (2\bar{\omega}_n \bar{u})(\gamma_n \varepsilon)^{-1} C$ with $C = \sum_{\ell \geq 0} \delta^\ell$; moreover if $n \in T^i$, $\eta_n^{iN} \leq C$.*

This lemma establishes a uniform upper bound on Lagrange multipliers and a more stringent uniform upper bound on multipliers of financially active agents. In fact, using assumptions A'.6, (iii) of Theorem 3.2 and the concavity of utility functions $u_{n,\tilde{s}_n}^{iN}$, it is easy to see that the quantity $(\bar{\omega}_n \bar{u})(\varepsilon \gamma_n)^{-1}$ is always greater than 1.

Proof. The proof follows an inductive argument as in the proof of Lemma 1. To start the induction, since the inequality (4.5), with ε in place of ε^i , is still satisfied, one has for the particular date N , $\eta_N^{iN} \leq 1 \leq C$, for all $i \in H(N)$.

If $n < N$, we will now prove that if the conclusion of Lemma 4.3 is satisfied for every agent alive at date $n+1$, then it is also satisfied for every agent alive at date n . Let $i \in H(n)$, we will have to distinguish whether the agent is allowed or not to trade assets.

If $n \in T^i$, using the portfolio $\varphi_{n,\tilde{m}_n} = -\varepsilon 1$ in (4.4), one deduces that one has $k_{n,\tilde{m}_n}^{iN} \pi_{n,\tilde{m}_n}^N (\varphi_{n,\tilde{m}_n}^{iN} + \varepsilon 1) \leq N^{-1} \sum_{\tilde{m}_n^+} (k_{n+1,\tilde{m}_n^+}^{iN} p_{n+1,\tilde{m}_n^+}^N A_{n+1,\tilde{m}_n^+}^N (\varphi_{n,\tilde{m}_n}^{iN} + \varepsilon 1)) \leq \gamma_{n+1} (\bar{\omega}_{n+1})^{-1} H^i ((\beta_i^{n+1} \bar{u}) \varepsilon^{-1}) \eta_{n+1}^{iN}$. This inequality, together with (4.6) leads to

$\eta_n^{iN} \leq 1 + ((\beta_i H^i \gamma_{n+1})(\varepsilon \bar{\omega}_{n+1})^{-1}) \eta_{n+1}^{iN}$. By induction hypothesis we deduce that $\eta_n^{iN} \leq 1 + \delta C = C$.

If $n \notin T^i$, then by (4.5), one obtains that $\eta_n^{iN} \leq (p_{n,\tilde{m}_n}^N 1)^{-1}$. If we use $x_{n,\tilde{m}_n} = 2\bar{\omega}_n$ in (4.3) for the particular agent i_n stated by assumption (iii) of Theorem 3.2, we have for any $\tilde{m}_n \in \tilde{N}^n$, $N^{-n} \beta_{i_n}^n \gamma_n \leq 2N^{-n} k_{n,\tilde{m}_n}^{iN} p_{n,\tilde{m}_n}^N \bar{\omega}_n \leq 2N^{-n} \beta_{i_n}^n \bar{u} \eta_n^{iN} \bar{\omega}_n (p_{n,\tilde{m}_n}^N 1) \varepsilon^{-1}$. Finally we deduce: $\eta_n^{iN} \leq (p_{n,\tilde{m}_n}^N 1)^{-1} \leq 2\bar{\omega}_n \bar{u} \eta_n^{iN} (\gamma_n \varepsilon)^{-1}$. Since $n \in T^{i_n}$, we have already proved $\eta_n^{i_n N} \leq C$, this inequality together with the last one gives the conclusion. $\square$

Under the assumptions of Proposition 3.2, since π^N can be set identically to 0, it follows that $p_{n,\tilde{s}_n}^N 1 = 1$ and consequently by (4.5) the multipliers η_n^{iN} are bounded from above by 1.

Finally, we will define simple functions $(\varphi_n^{iN}, k_n^{iN}, \varphi_n^{iN}, p_n^N, \pi_n^N)$. For example, we associate a simple function to the list of $(x_{n,\tilde{m}_n}^{iN})$. Formally, we define $x_n^{iN} = \sum_{\tilde{m}_n \in \tilde{N}_n} x_{n,\tilde{m}_n}^{iN} \chi_{V_{m_1}^N \times \dots \times V_{m_n}^N}$ where χ_A denotes the characteristic function of the set A .

5 Proofs

The proofs of Theorem 3.1 and 3.2 are very similar and will be done with the same arguments. The main steps of our proof are as follows. The first step is to take an appropriate limit of parameters of the truncated economy, the second step is to prove that budget constraints and market clearing conditions hold for the limit parameters, and the final step is to prove the optimality for the consumer. In the final step, the presence of agents with an infinite lifetime (only the case of Theorem 3.1) will require a specific argument.

STEP 1 Let us define $f^N = (f_n^N)_{n \leq N}$ where for every $\tilde{s}_n \in S^n$,

$$f_{n,\tilde{s}_n}^N = ((x_{n,\tilde{s}_n}^{iN}, k_{n,\tilde{s}_n}^{iN}, \varphi_{n,\tilde{s}_n}^{iN}, k_{n,\tilde{s}_n}^{iN} p_{n,\tilde{s}_n}^N A_{n,\tilde{s}_n}^N, p_{n,\tilde{s}_n}^N A_{n,\tilde{s}_n}^N \varphi_{n-1,\tilde{s}_{n-1}}^{iN})_{i \in H(n)}, p_{n,\tilde{s}_n}^N, \pi_{n,\tilde{s}_n}^N).$$

It follows from Assumption A.1 that f_n^N is a finite dimensional vector function. In view of Lemma 4.2 and 4.3, the function f_n^N is uniformly bounded. As in Araujo et al. (1996), using sequentially Fatou's lemma, it is then possible to find a measurable function $\bar{f} = (\bar{f}_n)_{n \geq 0}$ such that for all n , for almost every $\tilde{s}_n$,

- (I) $(\bar{f}_{1,s_1}, \dots, \bar{f}_{n-1,\bar{s}_n}, \bar{f}_{n,\bar{s}_n})$ is a cluster point of the sequence of vector $((f_{1,s_1}^N, \dots, f_{n-1,\bar{s}_n}^N, f_{n,\bar{s}_n}^N))_{N>0}$
- (II) $\int_S \bar{f}_{n,\bar{s}_n} d\mu(s_n)$ is a cluster point of the sequence $(\int_S f_{n,\bar{s}_n}^N d\mu(s_n))_{N>0}$.

We refer to Araujo et al. (1996) for the details of this result. It should be clear that the kind of convergence obtained here is not a pointwise convergence of the sequence $(f_n^N)_{N>0}$ to $\bar{f}_n$, even for a convenient subsequence. Nevertheless, we will see that some (in)equalities still hold at the limit.

More precisely, let us define the components of $\bar{f}_n = ((\bar{x}_n^i, \bar{k}_n^i, \bar{\varphi}_n^i, \bar{\lambda}_n^i, \bar{t}_n^i)_{i \in H(n)}, \bar{p}_n, \bar{\pi}_n)$ which allows to consider $\bar{x}^i = (\bar{x}_n^i)_{n \in T^i}$, $\bar{\varphi}^i = (\bar{\varphi}_n^i)_{n \in T^i}$, $\bar{p} = (\bar{p}_n)_{n \in \mathcal{N}}$ and $\bar{\pi} = (\bar{\pi}_n)_{n \in \mathcal{N}}$.

STEP 2 By (I) and by the construction of $(A_{n,\bar{m}_n}^N, \omega_{n,\bar{m}_n}^{iN})_{\bar{m}_n \in \bar{N}^n}$, there exists a full measure set $\Omega^n \subset S^n$ such that for every $\bar{s}_n \in \Omega^n$, there exists a subsequence defined by ψ (depending of $\bar{s}_n$) such that

$$(x_{n,\bar{s}_n}^{i\psi(N)}, \varphi_{n,\bar{s}_n}^{i\psi(N)}, A_{n,\bar{s}_n}^{\psi(N)}, \varphi_{n-1,\bar{s}_n}^{i\psi(N)}, k_{n,\bar{s}_n}^{i\psi(N)}, p_{n,\bar{s}_n}^{\psi(N)}, A_{n,\bar{s}_n}^{\psi(N)}, p_{n,\bar{s}_n}^{\psi(N)}, A_{n,\bar{s}_n}^{\psi(N)}, \varphi_{n-1,\bar{s}_n}^{i\psi(N)}, p_{n,\bar{s}_n}^{\psi(N)}, \pi_{n,\bar{s}_n}^{\psi(N)}) \\ \rightarrow (\bar{x}_{n,\bar{s}_n}^i, \bar{\varphi}_{n,\bar{s}_n}^i, A_{n,\bar{s}_n}, \bar{\varphi}_{n-1,\bar{s}_n}^i, \bar{\lambda}_{n,\bar{s}_n}^i, \bar{t}_{n,\bar{s}_n}^i, \bar{p}_{n,\bar{s}_n}, \bar{\pi}_{n,\bar{s}_n}) \text{ when } N \rightarrow \infty.$$

It follows from the equilibrium conditions of truncated economies that for every $\bar{s}_n \in \Omega^n$, $\bar{p}_{n,\bar{s}_n}(\bar{x}_{n,\bar{s}_n}^i - \omega_{n,\bar{s}_n}^i) + \bar{\pi}_{n,\bar{s}_n} \bar{\varphi}_{n,\bar{s}_n}^i \leq \bar{p}_{n,\bar{s}_n} A_{n,\bar{s}_n} \bar{\varphi}_{n-1,\bar{s}_n}^i$. The same argument, using now (II) instead, can be used to prove that almost everywhere markets clear, and that $(\bar{p}_{n,\bar{s}_n}, \bar{\pi}_{n,\bar{s}_n}) \in \Delta^{G+J-1}$ almost everywhere.

To end the proof, we must check the optimality condition b) for a consumer i .

STEP 3 Notice that $\bar{x}^i \in X^i$ (trivially in the case of Theorem 3.2 and, by market clearing and A.7, in the case of Theorem 3.1). Let (x, φ) be a point of $X^i \times Z^i$ satisfying budget conditions.

The inequalities (4.2) and (4.4) together with conditions (I) and (II) of the first step lead to the following properties (almost everywhere) :

$$\bar{k}_{n,\bar{s}_n}^i \bar{p}_{n,\bar{s}_n} (\bar{x}_{n,\bar{s}_n}^i - \bar{\omega}_{n,\bar{s}_n}) + \bar{k}_{n,\bar{s}_n}^i \bar{\pi}_{n,\bar{s}_n} \bar{\varphi}_{n,\bar{s}_n}^i \leq \bar{k}_{n,\bar{s}_n}^i \bar{p}_{n,\bar{s}_n} A_{n,\bar{s}_n} \bar{\varphi}_{n-1,\bar{s}_n}^i \quad (5.1)$$

$$0 \leq (\varphi_{n,\bar{s}_n} - \bar{\varphi}_{n,\bar{s}_n}^i) \left(\bar{k}_{n,\bar{s}_n}^i \bar{\pi}_{n,\bar{s}_n} - \int_S \bar{k}_{n+1,\bar{s}_n^+}^i \bar{p}_{n+1,\bar{s}_n^+} A_{n+1,\bar{s}_n^+} d\mu(s_{n+1}) \right) \quad (5.2)$$

The following inequality is more difficult to establish. It can be established from (4.3) by a direct pointwise argument in the case where $u_{n,\bar{s}_n}^i$ is independent of $\bar{s}_n$. The general case would require an adaptation of Lemma 8 in Monteiro (1996). One has in both cases

$$\beta_i^n(u_{n,\bar{s}_n}^i(x_{n,\bar{s}_n}) - u_{n,\bar{s}_n}^i(\bar{x}_{n,\bar{s}_n}^i)) \leq \bar{k}_{n,\bar{s}_n}^i \bar{p}_{n,\bar{s}_n}(x_{n,\bar{s}_n} - \bar{x}_{n,\bar{s}_n}^i) \quad (5.3)$$

Let us define $\alpha = U^i(x, \varphi) - U^i(\bar{x}^i, \bar{\varphi}^i)$. We will show that $\alpha \leq 0$, as desired. Using inequality (5.3), one has $\alpha = \sum_{n \geq 0} \beta_i^n \int_{S^n} (u_{n,\bar{s}_n}^i(x_{n,\bar{s}_n}) - u_{n,\bar{s}_n}^i(\bar{x}_{n,\bar{s}_n}^i)) d\mu(\bar{s}_n) \leq \limsup_{M \rightarrow \infty} \sum_{n=0}^M \int_{S^n} \bar{k}_{n,\bar{s}_n}^i \bar{p}_{n,\bar{s}_n}(x_{n,\bar{s}_n} - \bar{x}_{n,\bar{s}_n}^i) d\mu(\bar{s}_n)$.

Since (x, φ) satisfies all the budget conditions, it follows from property (5.1) that $\int_{S^n} \bar{k}_{n,\bar{s}_n}^i \bar{p}_{n,\bar{s}_n}(x_{n,\bar{s}_n} - \bar{x}_{n,\bar{s}_n}^i) d\mu(\bar{s}_n) \leq \int_{S^n} \bar{k}_{n,\bar{s}_n}^i \bar{p}_{n,\bar{s}_n} A_{n,\bar{s}_n}(\varphi_{n-1,\bar{s}_n} - \bar{\varphi}_{n-1,\bar{s}_n}^i) - \bar{k}_{n,\bar{s}_n}^i \bar{\pi}_{n,\bar{s}_n}(\varphi_{n,\bar{s}_n} - \bar{\varphi}_{n,\bar{s}_n}^i) d\mu(\bar{s}_n)$. We can apply Fubini's Theorem together with property (5.2) to get $\sum_{n=0}^M \int_{S^n} (\bar{k}_{n,\bar{s}_n}^i \bar{p}_{n,\bar{s}_n} A_{n,\bar{s}_n}(\varphi_{n-1,\bar{s}_n} - \bar{\varphi}_{n-1,\bar{s}_n}^i) - \bar{k}_{n,\bar{s}_n}^i \bar{\pi}_{n,\bar{s}_n}(\varphi_{n,\bar{s}_n} - \bar{\varphi}_{n,\bar{s}_n}^i)) d\mu(\bar{s}_n) \leq - \int_{S^M} \bar{k}_{M,\bar{s}_M}^i \bar{\pi}_{M,\bar{s}_M}(\varphi_{M,\bar{s}_M} - \bar{\varphi}_{M,\bar{s}_M}^i) d\mu(\bar{s}_M) \leq - \int_{S^{M+1}} \bar{k}_{M+1,\bar{s}_M^+}^i \bar{p}_{M+1,\bar{s}_M^+} A_{M+1,\bar{s}_M^+}(\varphi_{M,\bar{s}_M} - \bar{\varphi}_{M,\bar{s}_M}^i) d\mu(\bar{s}_M^+)$.

At this stage, we will have to distinguish whether lifetime of agent i is finite or infinite. If it is finite, then for M large enough, the right term in the previous inequality is equal to 0, hence we deduce $\alpha \leq 0$ which proves the optimality of $(\bar{x}^i, \bar{\varphi}^i)$.

If it is infinite, using Assumption (ii) of Theorem 3.1, we state the following claim that implies $\alpha \leq 0$

$$\liminf_{M \rightarrow \infty} \int_{S^{M+1}} \bar{k}_{M+1,\bar{s}_M^+}^i \bar{p}_{M+1,\bar{s}_M^+} A_{M+1,\bar{s}_M^+}(\varphi_{M,\bar{s}_M} - \bar{\varphi}_{M,\bar{s}_M}^i) d\mu(\bar{s}_M^+) \geq 0$$

Indeed, recalling that $\sum_{i \in H(M)} \bar{\varphi}_{M,\bar{s}_M}^i = 0$ and $\varphi_{M,\bar{s}_M}^i \geq v_n^i$, since $\bar{k}_{M+1,\bar{s}_M^+}^i$ is explicitly bounded by Lemma 4.2, one deduces

$$\begin{aligned} & \liminf_{M \rightarrow \infty} \int_{S^{M+1}} \bar{k}_{M+1,\bar{s}_M^+}^i \bar{p}_{M+1,\bar{s}_M^+} A_{M+1,\bar{s}_M^+}(\varphi_{M,\bar{s}_M} - \bar{\varphi}_{M,\bar{s}_M}^i) d\mu(\bar{s}_M^+) \\ & \geq \liminf_{M \rightarrow \infty} -(\beta_i^{M+1} \bar{u}^i H^i / \varepsilon^i) (\sum_{l=0}^{\infty} (\beta_i / \varepsilon^i)^l) = 0. \end{aligned}$$

6 Appendix

The model we study in this appendix is general enough to fit with each truncated economy considered in Section 4.

We consider an exchange model with $(T + 1)$ periods $t = 0, 1, \dots, T$ and a finite set I of agents. The stochastic structure of the model is described by an event-tree with a unique initial node $s = 0$ at date 0, and at each node of date $t < T$, a finite branching number (depending on the node) that determines the nodes of the following time-period. S denotes the (finite) set of nodes, S_t the set of nodes at date t ; if $k \leq T$, $S^k = \bigcup_{t=0}^k S_t$ and obviously, $S = S^T$. If $s \in S_t$, $t > 0$, s^- denotes the unique predecessor of s .

At each $s \in S$, there is a spot market for each of a finite number G of physical goods, and if $s \in S^{T-1}$, there is also a financial market for a finite number J of one-period real assets. Economy $\mathcal{E}$ is described by the following list:

$$\mathcal{E} = ((X^i, Z^i, u^i, \omega^i)_{i \in I}, A)$$

where for each i , $X^i, Z^i, u^i : X^i \rightarrow \mathcal{R}$ and ω^i denote classically the consumption set, portfolio set, utility function and initial endowment of agent i , while the financial side of the model is described by the collection A of $(G \times J)$ -matrices $(A_s)_{s \in S^T \setminus \{0\}}$, where the column-vector A_s^j represents the return at s of one unit of asset j bought at s^- , the unique predecessor of s .

We make on $\mathcal{E}$ the following assumptions:

For each i ,

- $X^i = \prod_{s \in S} X^i(s) \subset (\mathcal{R}_+^G)^{S^T}$
- $Z^i = \prod_{s \in S^{T-1}} Z^i(s) \subset (\mathcal{R}^J)^{S^{T-1}}$
- $u^i(x) = \sum_{s \in S^T} u_s^i(x_s)$, where each $u_s^i : X^i(s) \rightarrow \mathcal{R}$ is continuous and concave
- $u_s^i \equiv 0 \implies X^i(s) = \{0\}, \omega^i(s) = 0, Z^i(s) = \{0\}$
- $u_s^i \not\equiv 0 \implies X^i(s) = \mathcal{R}_+^G, Z^i(s)$ is a closed, convex and bounded below subset of $\mathcal{R}^J$ containing 0, $\omega^i(s) \gg 0$, and u_s^i is strictly monotone.

For every s , there exists at least one i (depending on s) such that $u_s^i \not\equiv 0$.

For every s , for every j , $A_s^j \in \mathcal{R}_+^G$.

It should be noticed that with these assumptions, rather than defining a life-time for each consumer, we distinguish for each i between nodes s where i is alive

$(u_s^i \neq 0)$ and nodes s where i is not alive ($u_s^i \equiv 0$). When alive, consumer i can consume and has a strictly positive initial endowment; the same i can participate in the financial market if $Z^i(s)$ is not reduced to $\{0\}$. When not alive, i cannot consume and cannot participate in the financial market.

Taking as given a market system of commodity and asset price processes $(p, \pi) = ((p(s))_{s \in S}, (\pi(s))_{s \in S^{T-1}})$, consumer i 's budget constraint at node s is

$$p(s)(x^i(s) - \omega^i(s)) + \pi(s)\varphi^i(s) \leq p(s)A(s)\varphi^i(s^-),$$

with the usual conventions $\varphi^i(0^-) = 0$, $A(0) = 0$ and $\varphi^i(s) = 0$ if $s \in S_T$. The budget set $B^i(p, \pi)$ is the set of all $(x^i, \varphi^i) \in X^i \times Z^i$ that satisfy all budget constraints and the problem of consumer i is to choose (x^i, φ^i) in $B^i(p, \pi)$ in order to maximize $u^i(x^i)$. Let Δ^{G+J-1} denote the unit-simplex of $\mathcal{R}^{G+J}$. An equilibrium for the above economy is a vector $(\bar{p}, \bar{\pi}, (\bar{x}^i, \bar{\varphi}^i)_{i \in I})$ such that

- (a) $(\bar{p}(s), \bar{\pi}(s)) \in \Delta^{G+J-1}$ at every node s ;
- (b) $(\bar{x}^i, \bar{\varphi}^i)$ solves consumer i 's problem at prices $(\bar{p}, \bar{\pi})$;
- (c) $\sum_{i \in I} \bar{\varphi}^i = 0$ and $\sum_{i \in I} (\bar{x}^i - \omega^i) = 0$.

Given the above assumptions on the economy, the just defined equilibrium is a Radner-type equilibrium. However, given our quite general assumptions on lifetimes of the agents, some classical assumptions of a Radner-type model are not satisfied, specially the ones concerning strong survival of agents and local nonsatiation for each agent, as assumed by Radner (1972), at every node and every component of an attainable consumption allocation. In what follows, we adapt the proof of Theorem 3.1 in Florenzano *et al.* (1998) and prove the following existence result:

Proposition 6.1 *Under the above assumptions, $\mathcal{E}$ has an equilibrium. Moreover, $\bar{p}(s) \gg 0 \forall s \in S$.*

Proof. Noticing that the set of attainable plans $\{(x, \varphi) \in X \times Z \mid \sum_{i \in I} (x^i - \omega^i) \leq 0, \sum_{i \in I} z^i = 0\}$ is nonempty and bounded, we can also replace the original economy $\mathcal{E}$ by an "equivalent" one. More precisely, let for each i , $\hat{X}^i(s)$ and $\hat{Z}^i(s)$ be respectively the sets of consumptions and portfolios at s attainable for i . We consider for each s closed balls C_s and D_s containing respectively in their interior all $\hat{X}^i(s)$ and all $\hat{Z}^i(s)$ and define $\tilde{X}^i(s) = X^i \cap \prod_{s \in S} C_s$, $\tilde{Z}^i(s) = Z^i \cap \prod_{s \in S} D_s$, $\tilde{X} = \prod_{i=1}^I \tilde{X}^i$, $\tilde{Z} = \prod_{i=1}^I \tilde{Z}^i$, and the truncated economy

$$\tilde{\mathcal{E}} = ((\tilde{X}^i, \tilde{Z}^i, u^i, \omega^i)_{i \in I}, A)$$

It is easily verified that an equilibrium of $\tilde{\mathcal{E}}$ is an equilibrium of $\mathcal{E}$.

We now claim that $\tilde{\mathcal{E}}$ has an equilibrium. Indeed, let $\tilde{\Delta} = (\Delta^{G+J-1})^S$. As in Gale and Mas-Colell (1975-1979), on $\tilde{\Delta} \times \prod_{i \in I} \tilde{X}^i \times \prod_{i \in I} \tilde{Z}^i$, we set the following reaction correspondences

$$\psi^0(p, \pi, x, \varphi) = \left\{ (p', \pi') \in \tilde{\Delta} \mid (p' - p) \left(\sum_{i \in I} (x^i - \omega^i) \right) + (\pi' - \pi) \left(\sum_{i \in I} \varphi^i \right) > 0 \right\}$$

and for each i ,

$$\psi^i(p, \pi, x, \varphi) = \begin{cases} B^i(p, \pi) & \text{if } (x^i, \varphi^i) \notin B^i(p, \pi) \\ B^i(p, \pi) \cap (P^i(x^i) \times \tilde{Z}^i) & \text{if } (x^i, \varphi^i) \in B^i(p, \pi) \end{cases}$$

where $P^i(x^i) = \{x'^i \in \tilde{X}^i \mid u^i(x'^i) > u^i(x^i)\}$ and $B^i(p, \pi)$ denotes now the budget set of i in $\tilde{\mathcal{E}}$.

Let us associate with each budget set the set $B'^i(p, \pi)$ deduced from $B^i(p, \pi)$ by replacing whenever $u_s^i \neq 0$ the inequality defining the budget constraint at s by a strict inequality. It is readily proved that for every i , $B'^i(p, \pi) \neq \emptyset \forall (p, \pi) \in \tilde{\Delta}$, hence that B^i and ψ^i are lower semicontinuous correspondences with convex values. Therefore, since the correspondences are irreflexive, by the lemma of Gale and Mas-Colell, there exists $(\bar{p}, \bar{\pi}, \bar{x}, \bar{\varphi})$ such that $\psi^i(\bar{p}, \bar{\pi}, \bar{x}, \bar{\varphi}) = \emptyset$, $i = 0$ and $i \in I$, i.e.

$$(\bar{x}^i, \bar{\varphi}^i) \in B^i(\bar{p}, \bar{\pi}) \text{ and } B^i(\bar{p}, \bar{\pi}) \cap (P^i(\bar{x}^i) \times \tilde{Z}^i) = \emptyset, \text{ for all } i \text{ in } I,$$

$$p \left(\sum_{i \in I} (\bar{x}^i - \omega^i) \right) + \pi \sum_{i \in I} \bar{\varphi}^i \leq \bar{p} \left(\sum_{i \in I} (\bar{x}^i - \omega^i) \right) + \bar{\pi} \sum_{i \in I} \bar{\varphi}^i, \forall (p, \pi) \in \tilde{\Delta}.$$

To complete the proof, we must now show that the allocation $(\bar{x}, \bar{\varphi})$ is attainable. If $s = 0$, since u_0^i is strictly monotone whenever $u_0^i \neq 0$, we get successively

$$\bar{p}(0) \left(\sum_{i \in I} (\bar{x}^i(0) - \omega^i(0)) \right) + \bar{\pi}(0) \sum_{i \in I} \bar{\varphi}^i(0) = 0,$$

$\sum_{i \in I} (\bar{x}^i(0) - \omega^i(0)) \leq 0$, and $\sum_{i \in I} \bar{\varphi}^i(0) \leq 0$. Since for some i , $u_0^i \neq 0$, we deduce $\bar{p}(0) \gg 0$ and $\sum_{i \in I} (\bar{x}^i(0) - \omega^i(0)) = 0$.

When $(A_s^j)_{s \in S_1}$ is not identically zero, if $\pi_j = 0$, then the optimal choice $\bar{\varphi}_j^i(0)$ would be positive, contradicting $\sum_{i \in I} \bar{\varphi}^i(0) \leq 0$. It then follows from $\bar{\pi}(0) \sum_{i \in I} \bar{\varphi}^i(0) = 0$ that $\sum_{i \in I} \bar{\varphi}^i(0) = 0$.

When $(A_s^j)_{s \in S_1}$ is identically zero, we can set $\bar{\varphi}_j^i(0)$ equal to 0. This portfolios allocation is actually a solution given by the lemma of Gale and Mas-Colell. Then, in both cases, we have $\sum_{i \in I} \bar{\varphi}^i(0) = 0$, as desired.

Using this market clearing condition for assets traded at the initial node, we can use the same argument to prove that for every immediate successor node, $s \in 0^+$, $\bar{p}(s) \gg 0$, $\sum_{i \in I} (\bar{x}^i(s) - \omega^i(s)) = 0$, and $\sum_{i \in I} \bar{\varphi}^i(s) = 0$. The same induction shows that for every node s , $\bar{p}(s) \gg 0$, $\sum_{i \in I} (\bar{x}^i(s) - \omega^i(s)) = 0$, and $\sum_{i \in I} \bar{\varphi}^i(s) = 0$. $\square$

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