Equilibrium with Default and Endogenous Collateral

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Abstract

We study a two-period general equilibrium model with incomplete asset markets and default. We make collateral endogenous by allowing each seller of assets to fix the level of collateral. Sellers are required to provide collateral whose first-period value, per unit of asset, exceeds the asset price, by an arbitrarily small amount. Moreover, borrowers are also required to be fully covered by the purchase, in the first-period, of state by state default insurance. These insurance contracts are offered by lenders. The insurance cost or revenue is a linear charge and plays the role of a spread penalizing borrowers who will incur in default and benefiting lenders who will suffer default. Under these assumptions, equilibrium always exists.

Keywords: Incomplete Markets; Collateral; Default Insurance; Spread.

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1 Introduction

In spite of the enormous current importance of loans secured by collateral, general equilibrium theory has devoted little attention to this phenomenon. An exception is the recent work by Dubey, Geanakoplos and Zame(1995) on default, collateral and derivatives. These authors model a two period economy with uncertainty in the second period and incomplete markets, where debtors are required to buy durable goods, according to some fixed collateral coefficients, whenever assets are sold in the first period. Then, in each state of nature of the second period, a debtor will pay his debt only when it does not exceed the value of the collateral. Similarly, each creditor should expect to receive only the minimum between the claim and the value of the collateral.

The purpose of the present paper is to explain how the coefficients of collateral may be endogenously determined, in the context of a general equilibrium model with incomplete markets. We set up a model where debtors may choose the collateral to offer when selling assets. In order to discourage borrowers from excess reliance on default and encourage them to provide an adequate level of collateral, we require sellers of assets to buy default insurance in the first period. More precisely, borrowers are required to be fully covered by the purchase of state by state default insurance. There are S insurance contracts and each has an explicit price per unit of state-contingent default. These insurance contracts are offered by the lenders(the buyers of assets).

The cost of the insurance contracts is a linear charge, state by state, which is added to the usual portfolio cost. One may interpret this charge as a spread that makes the actual asset prices become personalized, due to the insurance premium charged according to the collateral offered. It is as if sales prices for assets became personalized to take into account expected repayment according to the collateral chosen. The spread takes the form of a discounted expectation of future default, with respect to some measure on states to be endogenously determined(which is given by the prices of the state-contingent default insurance).

Anonymity in market transactions is still preserved. In our model, lenders do not trade directly with individual borrowers, but rather buy collateralized obligations backed by a pool of individual promises, each of which is secured by a specific collateral. That is, the assets bought by lenders are secured

by a collateral which is a weighted average of the collateral chosen by individual borrowers, with the individual sales serving as weights. This model may be interpreted as an attempt to illustrate well known situations where agencies mix together many mortgages into composite securities or even as a step towards modeling the recent development of collateralized mortgage obligation (CMO) markets. We allow actually for some form of tranching. In CMO markets, mortgages pools are split into different tranches sold to individual investors, in the form of derivatives securities. In our model, in each asset market, the pool of promises backed by individual collaterals, is tranced across buyers. To simplify, we mix promises by different sellers of a same asset, but we do not mix different assets into collateralized derivatives. But this simplification should not hurt the interpretation of our model as an attempt towards modeling collateralized derivatives, since different sellers of a same asset choose different collateral and, therefore, sell personalized assets.

All lenders face the same risk when buying an asset, which should now be perceived as a composite asset, as a derivative on the personalized assets. In this way, lenders avoid the problem of ending up buying assets only from borrowers with high propensity to default. All lenders face a same purchase price which is equal to the no-default sale price net of the insurance revenue, per unit of asset(which is a discounted expectation of the default suffered in the future, per unit of asset, and is the same for all lenders).

The present paper is motivated by the pioneering study of collateral by Dubey-Geanakoplos-Zame(1995), where results on existence of equilibria are provided, for exogenously given collateral coefficients. We are also indebted to a enlightening talk by Geanakoplos (1996) where the importance of collateralized derivatives is extensively discussed. But we should trace back the origins of the modern general equilibrium literature on default to the model of unsecured claims of Dubey - Geanakoplos - Shubik (1989), where defaulters are subject to utility penalties and equilibria are shown to exist in the case of nominal or numeraire assets(or for more general real assets under bounded short sales), assuming that lenders expect to face mean risk and, therefore, discount returns using the mean repayment rate. Araújo - Monteiro - Páscoa(1996, 1998) extended the model to the continuum of states (and an infinite- tree), replacing the usual assumption on non negativity of ex post endowments by the default-penalty structure.

Penalties play nowadays a small deterrence role in the default world.

Many legal penalties are seldomly used or difficult to enforce. Moreover in the presence of utility penalties, adverse selection problems may arise, since lenders do not know the default likelihood of borrowers, which depends on the comparison between marginal penalties and marginal utilities of income, in each state of nature. This asymmetric information context is richer but also harder to deal with than the anonymous model of collateralized promises without utility penalties. In this paper we suppose that there are no utility penalties in case of default.

Mathematically, the choice of collateral levels by agents creates a non convexity in the individual decision problem, since the second period budget constraint, for each state of nature, does not define anymore a convex set. To overcome this difficulty we model the agents set as a continuum and appeal to the convexifying effect of large numbers given by Liapunov's theorem. More precisely, we show that, for each truncated economy where portfolios are bounded a priori, a non-atomic generalized game played by consumers and fictitious auctioneers has a pure strategies strategic equilibrium, which is an equilibrium for the truncated economy. Then, we look at the asymptotic properties of the equilibria of truncate economies, as portfolios bounds tend to infinity. We show that truncated equilibrium allocations are uniformly bounded and use a multi- dimensional version Fatou's lemma to infer the existence of an equilibrium for the untruncated economy. The paper is organized as follows, In the next section we describe the model, in section 3 we state our result on existence of equilibria, in section 4 we address the equilibria of truncated economies and in section 5 we complete the existence proof by studying the asymptotics of truncated equilibria. Finally we provide a short section discussing the constrained efficiency properties and an appendix with some mathematical definitions and results used throughout the paper.

2 The Model

Let L be the number of physical commodities being traded in the market. We consider an economy over two periods. In this economy there is a continuum of traders which is modeled by the Lebesgue probability space $(H = [0, 1], \mathcal{B}, \lambda)$. There are J assets that are traded in the initial period and whose returns are paid in the second period. Each promise (sale of asset) j

should be backed by collateral, which will consist of goods of some durability, this durability being measured by a linear transformation which depends on the state of nature that occurs in the second period. There is a finite number of states, namely S states of nature.

An allocation of commodity bundles to trades, on for each state of nature is an integrable map $x : H \rightarrow \mathbb{R}_+^L \times \mathbb{R}_+^{LS}$. A portfolios allocation consists of two integrable maps, $\theta : H \rightarrow \mathbb{R}_+^J$, and $\varphi : H \rightarrow \mathbb{R}_+^J$ representing the allocations of purchase and sale of assets. Each agent is characterized by an utility function, $U^h : \mathbb{R}_+^L \times \mathbb{R}_+^{LS} \rightarrow \mathbb{R}$ and an initial endowment, $\omega^h \in \mathbb{R}_+^L \times \mathbb{R}_+^{LS}$. The goal of this paper is to make the collateral endogenous, and we do it as follows:

2.1 Endogenizing the Collateral

We allow for each seller of assets to choose the collateral coefficient of each asset sold and we suppose that there exists an anonymous collateral which will be taken as given by each buyer of assets. A function $M : H \mapsto \mathbb{R}_+^{LJ}$ representing the assignments of collateral coefficients for each portfolio sold is called a collateral allocation.

The anonymous collateral is denoted by $C = (C_1, \dots, C_J) \in \mathbb{R}_+^{LJ}$ which will be determined endogenously and is taken as given when agents purchase assets and will be, in equilibrium, a weighted average of the collateral allocation. The budget constraint in the second period, of agent h when the state of the nature $s \in S$ occurs is :

$$p_s x_s^h + \sum_{j \in J} D_{sj}^h \varphi_j^h \leq p_s w_s^h + \sum_{j \in J} R_s^j \theta_j^h + \sum_{j \in J} p_s \cdot Y_s M_j^h \varphi_j^h \quad (1)$$

where $D_{sj}^h = \min\{p_s A_s^j, p_s \cdot Y_s M_j^h\}$ and $R_s^j = \min\{p_s A_s^j, p_s \cdot Y_s C_j\}$ are what he delivers and receives by the sale and purchase of one unit of asset j respectively. $Y_s \in \mathbb{R}^{L \times L}$ is a non singular matrix that represents the durability of goods, (for example $Y_s M_j^h \in \mathbb{R}_+^L$ is the collateral depreciated in the second period if state s occurs). Notice that consumer h defaults on asset j in state s when D_{sj}^h is $p_s \cdot Y_s M_j^h$ and its default becomes $(p_s A_s^j - p_s \cdot Y_s M_j^h) \varphi_j^h$. Consumers suffer default on asset j in state s when $R_s^j = p_s \cdot Y_s C_j$ and this default becomes $(p_s A_s^j - p_s \cdot Y_s C_j) \theta_j^h$.

A seller of assets (a borrower) would like his default to be as high as possible and this would be done by decreasing M^h and setting it equal to the

zero vector, unless we introduce incentives to offer collateral. Before writing the first period budget constraints for this model, we recall the first period budget constraint of the model where the collateral is fixed exogenously (see Geanakoplos 1996 or Orrillo 1997)

$$p_o x_o^h + \pi \theta^h + p_o M \varphi^h \leq p_o w_o^h + \pi \varphi^h$$

where the left hand side represents the amount spent in consumption and purchase of assets and collateral, all this being financed by the sale of assets and initial endowment. To give borrowers an incentive to offer collateral, we required them to purchase, in the first period, default insurance. There are S state-contingent default insurance contracts, each having a price β_s , per unit of state s default. Clearly, future default in state s , $(p_s A_s - p_s Y_s M^h)^+ \varphi^h$, depends linearly on portfolios and decreases with the collateral provided. The insurance contracts are offered by lenders: together with a purchase θ^h of assets, consumer h offers a state by state default insurance which, for state s , is equal to $(p_s A_s - p_s Y_s C)^+ \theta^h$ and is sold at price β_s .

Then, we add to the linear portfolio cost ($\pi(\theta^h - \varphi^h)$) the following overall insurance net premium:

$$\sum_{s \in S} \beta_s [(p_s A_s - p_s Y_s M^h)^+ \varphi^h - (p_s A_s - p_s Y_s C)^+ \theta^h]$$

where $\beta \in \mathbb{R}_+^S$ is being determined endogenously. It is as if a linear spread were being added to the usual portfolio cost. Sellers of assets who offer higher collateral will sell at higher actual prices (i.e., will pay lower interest in the future) since they are required to buy lower insurance. Buyers of assets face an effective purchase price which is equal to the apparent price reduced by the, per unit of asset, insurance revenue. Let $k = \sum_{s=1}^S \beta_s$ and $\eta = \frac{\beta}{k}$, where the former is interpreted as a discount factor and the latter as a probability measure on states. The first-period budget constraint of agent h can be rewritten as

$$\begin{aligned} p_o x_o^h + (\pi - k \sum_{s \in S} \eta_s (p_s A_s - p_s Y_s C)^+) \theta^h + p_o M^h \varphi^h \leq \\ p_o w_o^h + (\pi - k \sum_{s \in S} \eta_s (p_s A_s - p_s Y_s M^h)^+) \varphi^h \end{aligned} \quad (2)$$

That is, asset purchase prices receive a discount which is a present value expectation of future losses due to default by other agents. Asset sale prices

are reduced by an amount which is a present value expectation of the future default given by the agents. Clearly, the reduction in sale prices due to the spread (i.e., the aggravation in interests paid by a borrower) is larger for sellers offering lower collateral coefficients and, therefore, prone to defaulting easier). In order to simplify the notation, we define for each $s \in S$

$$g_s(\theta^h, \varphi^h, M^h) := (p_s A_s - p_s Y_s M^h)^+ \varphi^h - (p_s A_s - p_s Y_s C)^+ \theta^h,$$

which is the net default in state s , by the agent h over all assets. Then the budget constraint in the first period for the agent $h \in H$ is the following:

$$p_o x_o^h + \pi \theta^h + (p_o M^h - \pi) \varphi^h + \sum_{s \in S} \beta_s g_s(\theta^h, \varphi^h, M^h) \leq p_o w_o^h \quad (3)$$

In the model with exogenous collateral we had $p_o M_j > \pi_j, \forall j$ as a non arbitrage necessary condition. However with endogenous collateral the earlier condition is no longer a non arbitrage condition since such a condition would depend on each agent h . Nevertheless, we introduce an additional budget constraint which is in spirit of above condition. we require

$$p_o M_j^h \geq \pi_j + \epsilon, \text{ with } \epsilon > 0 \quad (4)$$

This condition says that for asset j , the value of the collateral, per unit of asset, must exceed the asset price, by a certain, arbitrarily small, fixed amount. There are many examples of loans satisfying this kind of collateral constraint. In residential mortgages, the loan finances only up to a certain fraction of the value of the house. In automobile and consumer durables in general, the same rule applies. However, in some business loans this is not the case.

The Budget set

For each consumer h , the budget set is defined by

$$B^h(p, \pi, C, \beta) := \{(x^h, \theta^h, \varphi^h, M^h) \in \mathbb{R}_+^{L(S+1)+(2+L)J} : (1), (3) \text{ and } (4) \text{ hold} \}$$

Hence each agent $h \in H$ faces the following problem:

$$\max_{(x^h, \theta^h, \varphi^h, M^h) \in B^h(p, \pi, C, \beta)} U^h(x_o^h + M^h \varphi^h, x_{-o}^h)$$

where $x_{-o}^h \in \mathbb{R}_+^{LS}$ denotes the vector which results from $x^h \in \mathbb{R}_+^{L(S+1)}$ by deleting the o -th component.

Since the net price of purchase of assets $q := \pi - k \sum_{s \in S} (p_s A_s - p_s Y_s C)^+$ and prices of commodities p are the same for all agents, and since the returns of the purchase are positives, a necessary condition of non arbitrage is $(p, q) \in \mathbb{R}_{++}^{L(S+1)+J}$. Otherwise, agents could buy infinitely many units increasing unboundedly the consumption in the second period and therefore, the consumer's problem would not have a solution. Let us define $v^h(x^h, \theta^h, \varphi^h, M^h)$ as $U^h(x_o^h + M^h \varphi^h, x_{-o}^h)$

Definition 1 An allocation $(x, \theta, \varphi, M) \in L^1(H, \mathbb{R}_+^{S(L+1)}) \times L^1(H, \mathbb{R}_+^J) \times L^1(H, \mathbb{R}_+^J) \times \mathcal{M}(H, \mathbb{R}_+^{LJ})$ is an equilibrium allocation for $\mathcal{E}$ if there exists (p, π, β, C) such that the following conditions hold

1. Markets clear, i.e.,

$$\begin{aligned} \int_H (x_o^h + M^h \varphi^h - \omega_o^h) dh &= 0 \\ \int_H (x_s^h - \omega_s^h dh - Y_s M^h \varphi^h) dh &= 0; s \in S \\ \int_H \theta^h dh - \int_H \varphi^h dh &= 0 \end{aligned}$$

2. The choice is optimal, i.e.;

$z^h := (x^h, \theta^h, \varphi^h, M^h)$ maximizes v^h on $B^h(p, \pi, C, \beta)$

3. $C_j \int_H \theta_j^h dh = \int_H M_j^h \varphi_j^h dh \forall j \in J$

4. $\int_{S_s^j} (p_s A_s^j - p_s Y_s C_j)^+ \theta_j^h = \int_{\mathcal{D}_s^j} (p_s A_s^j - p_s Y_s M_j^h)^+ \varphi_j^h, \forall s \in S, \forall j \in J$

Where $S_s^j = \{h : p_s A_s^j > p_s Y_s C_j\}$ is the set of agents that suffered default in the state of nature s on asset j . Similarly, $\mathcal{D}_s^j = \{h : p_s A_s^j > p_s Y_s M_j^h\}$ is the set of agents that give default in the state of nature s on asset j . It should be noted that S_s^j either is H or ϕ , since the comparison between $p_s A_s^j$ and $p_s Y_s C_j$ does not depend on h .

Remarks

- (i) Condition 3 says that in equilibrium the anonymous collateral C_j treated as given by consumers is perfectly anticipated as the weighted average of the collateral allocation M_j .
- (ii) The last condition says that in equilibrium the default suffered should be equal to the default given (for each state and for each asset). In equilibrium (more precisely when the markets of assets clear) this condition implies that

$$\int_H R_s^j \theta_j^h = \int_H D_{js}^h \varphi_j^h$$

i.e., aggregate actual yields equal aggregate actual payments.

In fact, say

$$\int_H R_s^j \theta_j^h > \int_H D_{js}^h \varphi_j^h \quad (\text{the opposite case is proven similarly}).$$

It is known that S_s^j either is H or ϕ . Let us assume the first case that is the more interesting. Hence the last inequality is

$$\int_H p_s Y_s C_j \theta_j^h > \int_{\mathcal{D}_s^j} p_s Y_s M_j^h \varphi_j^h dh + \int_{(\mathcal{D}_s^j)^c} p_s A_s^j \varphi_j^h dh$$

After multiplying by a minus sign and after adding and subtracting the term $\int_{\mathcal{D}_s^j} p_s A_s^j \varphi_j^h dh$ in the right hand side we obtain

$$- \int_H p_s Y_s C_j \theta_j^h < \int_{\mathcal{D}_s^j} p_s A_s^j \varphi_j^h dh - \int_{\mathcal{D}_s^j} p_s Y_s M_j^h \varphi_j^h dh - \int_H p_s A_s^j \varphi_j^h dh$$

Passing the last term of the right hand side to the left hand side and using $\int_H \theta_j^h dh = \int_H \varphi_j^h dh, \forall j \in J$ we obtain

$$\int_H (p_s A_s^j - p_s Y_s C_j)^+ \theta_j^h < \int_{\mathcal{D}_s^j} (p_s A_s^j - p_s Y_s M_j^h)^+ \varphi_j^h$$

which is a contradiction with 4. above. ■

- (iii) Condition 3. together with absence of excess demand in asset markets, imply already one of the inequalities in the condition 4., more precisely, the following

$$\int_{S_s^j} (p_s A_s^j - p_s Y_s C_j)^+ \theta_j^h \leq \int_{\mathcal{D}_s^j} (p_s A_s^j - p_s Y_s M_j^h)^+ \varphi_j^h, \forall j \in J.$$

That is, for each state s and each asset j aggregate default given agents is greater or equal to aggregate default suffered by agents (i.e., aggregate net default is nonnegative). In fact, we assume as in the item above that $S_s^j = H$, then for asset j one has

$$\begin{aligned}
\int_H (p_s A_s^j - p_s Y_s C_j)^+ \theta_j^h &= \int_H p_s A_s^j \theta_j^h - \int_H p_s Y_s C_j \theta_j^h dh \\
&\leq \int_H p_s A_s^j \varphi_j^h - \int_H p_s Y_s M_j^h \varphi_j^h dh \\
&= \int_{\mathcal{D}_s^j} (p_s A_s^j - p_s Y_s M_j^h) \varphi_j^h dh + \int_{(\mathcal{D}_s^j)^c} (p_s A_s^j - p_s Y_s M_j^h) \varphi_j^h dh \\
&\leq \int_{\mathcal{D}_s^j} (p_s A_s^j - p_s Y_s M_j^h)^+ \varphi_j^h dh
\end{aligned}$$

the last step is true since the second term of the last inequality is negative.

- (iv) Notice that 4. implies that the S default insurance markets are in equilibrium, that is,

$$\sum_{j \in J} \int_{S_s^j} (p_s A_s^j - p_s Y_s C_j)^+ \theta_j^h dh = \sum_{j \in J} \int_{\mathcal{D}_s^j} (p_s A_s^j - p_s Y_s M_j^h)^+ \varphi_j^h dh, \text{ for each } s \in S$$

There is a partial converse to this implication. In the absence of excess demand in asset and insurance markets(i.e., when both $\int_H (\theta^h - \varphi^h) dh \leq 0$ and $\int_H g_s^h dh \leq 0, \forall s \in S$) it follows that 4. holds, using item (iii) above.

3 Existence of Equilibrium

Our main result is the following

Theorem 1 *Suppose that*

1. $U^h : \mathbb{R}_+^L \times \mathbb{R}_+^{LS} \mapsto \mathbb{R}$ is concave, continuous and strictly increasing, for each h , and

2. $\omega \in L^\infty(H, \mathbb{R}_{++}^{L(S+1)}),$

then the economy $\mathcal{E}$ has an equilibrium.

We prove this theorem by looking at the asymptotics of truncated equilibria, as we let a priori portfolio and consumption bounds tends to infinity.

4 Truncated Economy

In this section we are going to consider a sequence of economies $\mathcal{E}^n$ such that the budget sets of each agent $h \in H$ are

$$B_n^h(p, \pi, C, \beta) := \{(x_n^h, \theta_n^h, \varphi_n^h, M_n^h) \in [0, n]^{L(S+1)+(2+L)J} : (1), (3) \text{ and } (4) \text{ hold} \}$$

Due to homogeneity of the budget set we always assume that

$$(p_o, \pi, \beta) \in \Delta^{L+J+S-1} \text{ and } p_s \in \Delta^{L-1}, s \in S$$

We also assume that $C \in [0, n]^{LJ}$.

For each $n \in \mathbb{N}$ we define the following generalized game played by the continuum of consumers and $S + 1 + JL$ additional players; where $S + 1$ are auctioneers and the other JL players are also fictitious agents. Denote this game by $\mathcal{J}_n$ which is described as follows:

- Each consumer $h \in H$ maximizes U^h in the constrained strategy set $B_n^h(p, \pi, C, \beta)$.

- The auctioneer of the first period chooses $(p_o, \pi, \beta) \in \Delta^{L+J+S-1}$ in order to maximize

$$p_o \int_H (x_o^h + \sum_j M_j^h \varphi_j^h - \omega_o^h) dh + \pi \int_H (\theta^h - \varphi^h) dh + \beta \int_H g(\theta^h, \varphi^h, M^h) dh$$

- The auctioneer of state s of the second period chooses $p_s \in \Delta^{L-1}$ in order to maximize $p_s \int_H (x_s^h + \sum_j Y_s M_j^h \varphi_j^h - \omega_s^h) dh$.
- Each of the remaining JL fictitious agents chooses $C_{jl} \in [0, n]$ in order to minimize $(C_{jl} \int_H \theta^h dh - \int_H M_{jl}^h \varphi_j^h dh)^2$.

Our next objective is to show that $\mathcal{J}_n$ has an equilibrium in pure strategies, since this equilibrium will correspond to an equilibrium of the truncated economy.

4.1 Extended Game

We are going to extend the game defined before by allowing for mixed strategies both in portfolios and collateral. Remember that, for each player a mixed strategy is a probability distribution on his set of pure strategies. In this case the set of measures on the Borelians of $K_n = [0, n]^J \times [0, n]^J \times [0, n]^{LJ}$. We denote by $\mathcal{M}$ the set of mixed strategies of each consumer.

Since we are not interested in a mixed strategies equilibrium, per se, we will extend the previous game to a game $\bar{\mathcal{J}}_n$ over mixed strategies (that we call extended game) whose equilibria: 1) exist 2) can be purified and 3) a pure version is an equilibrium for the original game.

First, before extending the game to mixed strategies, let us rewrite the payoffs of the fictitious agents replacing consumption bundles by the following function of portfolios and collateral:

$$d^h(\theta^h, \varphi^h, M^h) = \arg \max \{v^h : x^h \in [0, n]^{L(S+1)} \text{ satisfies (1), (3) and (4)}\}$$

That is, function d^h solves the utility maximization problem for a given portfolio (θ^h, φ^h) and a given collateral bundle M^h . By the maximum theorem, d^h is continuous.

Secondly, we extend the payoffs to mixed strategies.

- (i) Each consumer $h \in H$ chooses $(x^h, \mu^h) \in [0, n]^{L(S+1)} \times \mathcal{M}$ in order to maximize $\int_{K_n} U^h(x_o^h + M^h \varphi^h, x_{-o}^h) d\mu^h$ subject to the constraints:

$$p_o(x_o^h - \omega_o^h) + \int_{K_n} [\pi \theta^h + (p_o M^h - \pi) \varphi^h + \sum_s \beta_s g_s^h(\theta^h, \varphi^h, M^h)] d\mu^h \leq 0$$

$$p_s(x_s^h - \omega_s^h) \leq \int_{K_n} \sum_j (R_s^j \theta_j^h - D_{js}^h \varphi_j^h + p_s Y_s M_j^h \varphi_j^h) d\mu^h \text{ for } s \in S$$

and

$$p_o \int_{K_n} M_j d\mu^h \geq \pi_j + \epsilon \forall j \in J$$

- (ii) The auctioneer of the first period chooses $(p_o, \pi, \beta) \in \Delta^{L+J+S-1}$ in order to maximize

$$\begin{aligned} & p_o \int_H \int_{K_n} [d_o^h(\theta^h, \varphi^h, M^h) + \sum_j M_j^h \varphi_j^h - \omega_o^h] d\mu^h dh + \\ & \pi \int_H \int_{K_n} (\theta^h - \varphi^h) d\mu^h dh + \sum_s \beta_s \int_H \int_{K_n} g_s(\theta^h, \varphi^h, M^h) d\mu^h dh \end{aligned}$$

- (iii) The auctioneer of state s in the second period chooses $p_s \in \Delta^{L-1}$ in order to maximize

$$p_s \int_H \int_{K_n} [d_s^h - \sum_j Y_s M_j^h \varphi_j^h - \omega_s^h] d\mu^h dh$$

and

- (iv) each of the remaining JL fictitious agents chooses $C_{jl} \in [0, n]$ in order to minimize

$$\left(\int_H \int_{K_n} [C_{jl} \theta_j^h - M_{jl}^h \varphi_j^h] d\mu^h dh \right)^2$$

Lemma 1 $\mathcal{J}_n$ has an equilibrium, possibly in mixed strategies over portfolio and collateral together.

Proof.

The existence argument in Ali Khan (1986) can be modified to allow for some atomic players. Consumer's best response correspondences ν^h are convex-valued and upper semicontinuous on the strategies of fictitious agents.

Now, define the correspondence

$\alpha(p, \pi, \beta, C) = \{f \equiv (x, \mu) \in ([0, n]^{L(S+1)} \times \mathcal{M})^H : f(h) \in \nu^h(p, \pi, \beta, C)\}$, which is also convex value and upper semicontinuous. The best response correspondences $\mathcal{R}^i$ of the $r = S + 1 + JL$ fictitious agents are convex valued and upper semicontinuous on the profile of consumers' probability measures on $K_n = [0, n]^J \times [0, n]^J \times [0, n]^{LJ}$ (with respect to the weak * topology on the dual of $L^1(H, C(K_n))$). The profiles set is compact for the same topology and Fan - Glicksberg fixed point theorem applies to $\alpha \times \prod_{i=1}^r \mathcal{R}^i$. ■

Purification of the equilibrium

In this part Liapunov's theorem will be fundamental. First, notice that the payoffs of the atomic players in $\overline{\mathcal{J}}_n$ depend on the profile of mixed strategies $(\mu^h)_h$ only through finitely many m indicators of the form ($m = L + J + S + SL + JL$).

$$\int_H \int_{K_n} Z_m^h(\theta^h, \varphi^h, M^h) d\mu^h dh \text{ where } Z_m \in L(H, C(K_n))$$

Secondly, let $E^h(p, \pi, C, \beta) = \Pi_2 \nu^h(p, \pi, C, \beta)$ and $Z = (Z_1, \dots, Z_m)$
Now,

$$\int_{K_n} Z^h(\theta^h, \varphi^h, M^h) dE^h(p, \pi, C, \beta) = \text{conv} \int_{K_n} Z^h(\theta^h, \varphi^h, M^h) d(\text{ext} E^h(p, \pi, C, \beta))$$

where the integral on the left hand side is the set in $\mathbb{R}^m$ of the all integrals of the form $\int_{K_n} Z^h(\theta^h, \varphi^h, M^h) d\mu^h$, for $\mu^h \in E^h(p, \pi, \beta, C)$. The integral on the right hand side is defined endogenously. The equality above follows by linearity of the map

$$\mu^h \mapsto \int_{K_n} Z^h(\theta^h, \varphi^h, M^h) d\mu^h$$

Then, theorem I.D.4 in Hildenbrand(1974) implies

$$\int_H \int_{K_n} Z^h(.) dE^h(p, \pi, C, \beta) dh = \int_H \int_{K_n} Z^h(.) d(\text{ext} E^h(p, \pi, C, \beta)) dh$$

Then, given a mixed strategies equilibrium profile $(\mu^h)_h$, there exists $(\theta^h, \varphi^h, M^h)$ such that the Dirac measure at $(\theta^h, \varphi^h, M^h)$ is an extreme point of E^h (evaluated at the equilibrium levels of the variables chosen by the atomic players) and $(\theta^h, \varphi^h, M^h)_h$ can replace $(\mu^h)_h$ and keep all equilibrium conditions satisfied, without changing the equilibrium levels of the variables chosen by the atomic players but replacing the former equilibrium bundles by $d^h(\theta^h, \varphi^h, M^h)$.

Lemma 2 *An equilibrium in pure strategies of the generalized game $\mathcal{J}_n$ is an equilibrium for the truncated economy $\mathcal{E}^n$ for n large enough.*

Proof:

Let $z = (x, \theta, \varphi, M) : H \mapsto [0, n]^{L(S+1)+2J+LJ}$, (p_o, π, β) , p_{-o} and C be an equilibrium in pure strategies for $\mathcal{J}_n$.

Then by definition of equilibrium one has the following:

$$p_o(x_o^h - \omega_o^h + M^h \varphi^h) + \pi(\theta^h - \varphi^h) + \beta g^h \leq 0,$$

$$p_s(x_s^h - \omega_s^h - Y_s M^h \varphi^h) \leq R_s \theta^h - D_s^h \varphi^h, \forall s$$

$$p_o M_j^h \geq \pi_j + \epsilon, \forall j \in J$$

and

$$v^h(z^h) \geq v^h(z'), \forall z' \in B_n^h(p, \pi, C, \beta)$$

Integrating the budget constraint of the first period we have

$$p_o \int_H (x_o^h - \omega_o^h + M^h \varphi^h) dh + \pi \int_H (\theta^h - \varphi^h) dh + \beta \int_H g^h dh \leq 0,$$

Now the optimality conditions of the auctioneers' problems imply that

$$\int_H (x_o^h - \omega_o^h + M^h \varphi^h) dh \leq 0 \quad (5)$$

$$\int_H (\theta^h - \varphi^h) dh \leq 0 \quad (6)$$

$$\int_H g^h dh \leq 0 \quad (7)$$

$$\int_H (x_s^h - \omega_s^h - Y_s M^h \varphi^h) dh \leq 0 \quad (8)$$

Moreover,

$$C_j \int_H \theta_j^h dh = \int_H M_j^h \varphi_j^h dh \quad \forall j \in J$$

Claim 1

$$\int_{\mathcal{D}_j^i} (p_s A_s^j - Y_s M_j^h) \varphi_j^h dh = \int_{S_i^j} (p_s A_s^j - Y_s C_j) \theta^h dh, \quad (9)$$

Proof

One of the inequalities follows from (6) (see item (iii) of the remark of definition 1) and the other follows from the nonpositivity of aggregate excess default insurance state by state (i.e., from (7)). See item (iv) of the remark following definition 1. ■

Using item (ii) of the remark of the definition 1 and (9) implies that

$$\int_H (R_s \theta^h - D_s^h \varphi^h) dh = 0, \forall s \quad (10)$$

Then, after integrating the budget constraint of the second period and using (10) we obtain

$$p_s \int_H (x_s^h - \omega_s^h - Y_s M^h \varphi^h) dh = 0, \forall s \in S \quad (11)$$

since the utility function is strictly increasing.

For n larger enough, we must have $p_{ol} > 0, \forall l \in L$. Otherwise, every consumer would choose $x_{ol}^h = n$ and we would have contradicted (5). But when $p_{ol} > 0$ we must have

$$\int_H (x_{ol}^h - \omega_{ol}^h + (M^h \varphi^h)_l) dh = 0 \quad \forall l \in L \quad (12)$$

since the aggregate budget constraint of the first period is a null sum of non positive terms. Therefore a sum of null terms.

From (12) follows that $\int_H (M^h \varphi^h)_l dh$ is bounded by the aggregate endowment in period 0 and hence $\int_H (Y_s M^h \varphi^h)_l dh < \infty$ therefore, For n larger enough, we must have $p_{sl} > 0, \forall (s, l) \in S \times L$. Otherwise, every consumer would choose $x_{sl}^h = n$ and we would have contradicted (8). Therefore equality holds in (8).

In similar way, $\pi_j > 0, \forall j \in J$, otherwise, each consumer would choose $\theta_j^h - \varphi_j^h = n$ implying contradicting (6). But when $\pi_j > 0, \forall j \in J$ we must have $\int_H (\theta_j^h - \varphi_j^h) dh = 0$ ■

5 Asymptotics of truncated equilibria

Let $\{(x_n, \theta_n, \varphi_n, M_n) \in ([0, n]^{S(L+1)+2J+LJ})^H, p_n, \pi_n, C_n, \beta_n\}$ be the sequence of equilibria corresponding to $\mathcal{E}^n$. Then by definition, for any n ,

1. Markets clear: $\int_H (x_{on}^h + M_n^h \varphi_n^h - \omega_o^h) dh = 0$,
 $\int_H (x_{sn}^h - \omega_s^h - Y_s M_n^h \varphi_n^h) dh = 0, s \in S$ and
 $\int_H \theta_n^h dh - \int_H \varphi_n^h dh = 0$
2. $(x_n^h, \theta_n^h, \varphi_n^h, M_n^h)$ maximizes U^h on $B^h(p_n, \pi_n, C_n, \beta_n)$
3. $C_{jn} \int_H \theta_{jn}^h dh = \int_H M_{jn}^h \varphi_{jn}^h dh \in \mathbb{R}_+^L, \forall j \in J$
4. $\int_{S_s^j} (p_{ns} A_s^j - p_s Y_s C_{jn})^+ \theta_{jn}^h = \int_{D_s^j} (p_{ns} A_s^j - p_s Y_s M_{nj}^h)^+ \varphi_{jn}^h \quad \forall j \in J$

Claim 2 $\{\int_H (x_n^h, \theta_n^h, \varphi_n^h, M_n^h \varphi_n^h) dh\}$ is bounded sequence.

Proof: By 1., $\int_H x_{no}^h dh \leq \int_H \omega_o^h dh$ and $\int_H M_n^h \varphi_n^h dh \leq \int_H \omega_o^h dh$.

So

$$\int_H x_{ns}^h dh < \int_H (\omega_s^h + Y_s \omega_o^h) dh, \forall s \in S. \quad (13)$$

For each $l \in L$ the following holds

$$\int_H M_{lnj}^h \varphi_n^h dh = C_{lnj} \int_H \theta_{jn}^h dh \quad (14)$$

Now, using item 1 and the last equality we obtain

$$C_{jnl} \int_H \theta_{nj}^h dh \leq \int_H \omega_{\alpha}^h dh, \forall l \in L \quad (15)$$

Therefore, $\int_H \theta_{nj}^h dh = \int_H \varphi_{nj}^h dh$ is bounded as long as C_{nj} does not go to zero, which follows from $p_o^n M_{jn}^h \geq \pi_j^n + \epsilon$ and from (3). More precisely, we have

$$\begin{aligned} p_o^n M_{nj}^h \varphi_{nj}^h &\geq \pi_j^n \varphi_{nj}^h + \epsilon \varphi_{nj}^h \\ \int_H p_o^n M_{nj}^h \varphi_{nj}^h dh &\geq \int_H \pi_j^n \varphi_{nj}^h dh + \int_H \epsilon \varphi_{nj}^h dh \\ \sum_l p_{lo}^n \int_H M_{njl}^h \varphi_{nj}^h dh &\geq \pi_j^n \int_H \varphi_{nj}^h dh + \epsilon \int_H \varphi_{nj}^h dh \\ p_{on} C_{jn} \int_H \theta_{jn}^h dh &\geq \pi_j^n \int_H \theta_{nj}^h dh + \epsilon \int_H \theta_{nj}^h dh \\ p_{on} C_{jn} &\geq \pi_j^n + \epsilon > \epsilon \end{aligned}$$

This last inequality implies that for some $l \in L$, $\lim_{n \rightarrow \infty} C_{jnl} > 0$, therefore we can divide in (15) by $\lim_{n \rightarrow \infty} C_{jnl} > 0$ in both sides obtaining, thus, a bound for

$$\int_H \varphi_{nj}^h dh = \int_H \theta_{nj}^h dh \leq \frac{\int_H \omega_{\alpha}^h dh}{\lim_{n \rightarrow \infty} C_{jnl}} \text{ for some } l(j) \quad \blacksquare$$

Let us define $\rho_{slj}^n := p_{sl}^n C_{jl}^n$.

Claim 3: $\{\rho^n\}$ is a bounded sequence.

Proof:

If $\rho_{slj}^n > \bar{a} \equiv \max_{s,l,j} A_{sl}^j$, the demand correspondence becomes unchangable since the value of collateral is greater than the debt, in the second period; that is, in each state of nature. In fact,

$$\rho_{slj}^n > \bar{a} \Rightarrow \sum_l \rho_{slj}^n > \bar{a} \Rightarrow p_s^n C_j^n > p_s^n A_s^j$$

Therefore we can always suppose that $p_s^n C_j^n \leq \bar{a}$. Then $\{\rho^n \in \mathbb{R}_+^{LSJ}\}$ is a bounded sequence of vectors since $p_{sl}^n C_{jl}^n \leq \sum_l p_{sl}^n C_{jl}^n \leq \bar{a}$. By (6) and 1., the claim is established. $\blacksquare$

Now we are going to modify the problem of each consumer. The key consists of reformulating the consumer's problem in terms of total collateral instead of the collateral backing the sale of each unit of asset.

Let $m^h \in \mathbb{R}_+^{LJ}$ be the total collateral that he gets by the sale of assets φ^h and which is defined as $m_j^h := M_j^h \varphi_j^h \in \mathbb{R}_+^L, \forall j \in J$.

Claim 4 The sequence of allocations $\{x_n, \theta_n, \varphi_n, m_n, (p_{sn} A_s^j \varphi_{jn} - p_{sn} m_n)\}$ is uniformly bounded.

Proof:

We rewrite the consumer's problem in the following way:
Given (p, π, β, ρ) , then each agent h chooses $(x^h, \theta^h, \varphi^h, m^h)$ in order to maximize $U^h(x_o^h + m^h, x_{-o}^h)$ subject to

$$p_o(x_o^h - \omega_o^h) + \pi(\theta^h - \varphi^h) + p_o m^h + \sum_{s,j} \beta_s [p_s A_s^j \varphi_j^h - p_s m_j^h]^+ - \sum_{s,j} \beta_s [p_s A_s^j - \rho_s]^+ \theta_j^h \leq 0 \quad (A),$$

$$p_s x_s^h + \sum_j \min\{p_s A_s^j \varphi_j^h, p_s m_j^h\} \leq p_s \omega_s^h + \sum_j \min\{p_s A_s^j, \sum_l \rho_{lsj}\} \theta_j^h + p_s m^h \quad (B)$$

and

$$p_o m_j^h \geq \pi_j \varphi_j^h + \epsilon \varphi_j^h, \forall j \in J \quad (C)$$

We also rewrite other equilibrium conditions replacing $M_j^h \varphi_j^h$ by m_j^h . Now the equilibrium sequence $((x_n, \theta_n, \varphi_n, M_n); p^n, \pi^n, \beta^n, C^n)$ induces the equilibrium sequence $(z_n = (x_n, \theta_n, \varphi_n, m_n); p^n, \pi^n, \beta^n, \rho^n)$ since $(x_n^h, \theta_n^h, \varphi_n^h, m_n^h)$ solves the above redefined consumer h 's problem given $(p^n, \pi^n, \beta^n, \rho^n)$ if and only if $(x_n^h, \theta_n^h, \varphi_n^h, M_n^h)$ solves the formerly problem, for the same parameter. Now, by claims 2 and 3, the sequence z_n satisfies the hypothesis of the weak version of Fatou's Lemma. Therefore $\exists z$ integrable such that

$$z^h \in cl\{z_n(h)\} \text{ for a.e } h$$

This imply that z^h is budget feasible at $(p, \pi, \beta, \rho) = \lim_{n \rightarrow \infty} (p^n, \pi^n, \beta^n, \rho^n)$ (passing to a subsequence if necessary). By applying the lower hemi-continuity

of the budget set of the above redefined problem, parametrized on ρ now (see, lemma 3 in appendix), we show that $(x^h, \theta^h, \varphi^h, m^h)$ maximizes U^h at the cluster point of $(p^n, \pi^n, \rho^n, \beta^n)$. In fact, suppose that it is not optimal, i.e; $\exists \bar{z}^h \in B^h(p, \pi, \beta, \rho)$ such that $v^h(\bar{z}) > v^h(z)$. Then by the lemma 1, $\exists \bar{z}_n^h \in B^h(p^n, \pi^n, \beta^n, \rho^n)$ and $\bar{z}_n^h \rightarrow \bar{z}^h$. Now, for $n \geq n_o$ one has $\bar{z}_n^h \in B_n^h(p^n, \pi^n, \rho^n, \beta^n)$. Since v^h is continuous, the following holds

$$v^h(\bar{z}_n^h) > v^h(z_n^h), \forall n \geq n_1$$

Therefore for $n \geq \max\{n_o, n_1\}$, z_n^h is not optimal in the truncated economy $\mathcal{E}^n$, a contradiction. Hence non arbitrage necessary conditions are satisfied at the cluster point, implying $\lim_{n \rightarrow \infty} (p^n, q^n) >> 0$.

Define q^n and q_n^h as

$$q_j^n = \pi - \sum_s \beta_s \sum_j [p_{sn} A_s^j - p_{sn} C_j^n]^+$$

and

$$q_{jn}^h = \sum_j p_o M_j^h - \pi_j + \sum_s \beta_s \sum_j [p_{sn} A_s^j - p_{sn} M_{jn}^h]^+ \geq \epsilon > 0$$

First period budget constraint, in terms of parameters (p_o^n, q^n, q_n^h) , is written as $p_o^n x_o^h + q^n \theta_n^h + q_n^h \varphi_n^h \leq p_o \omega_o^h$. Then

$$x_{on}^h, \theta_{jn}^h, \varphi_{jn}^h \leq \frac{\text{ess sup}_{h,l} \omega_{ol}^h}{\delta}$$

where $\delta = \min\{(\lim_{n \rightarrow \infty} p_{ol}^n)_{l=1}^L, (\lim_{n \rightarrow \infty} q_j^n)_{j=1}^J, \epsilon\}$. Therefore, by (2), $\{\sum_j p_o^n M_{jln}^h \varphi_{jn}^h\}$ becomes also a uniformly bounded sequence, implying that $\{M_{jln}^h \varphi_{jn}^h\}$ is uniformly bounded since p_{ol}^n is bounded away from zero, for any l . Hence, from (1), $\{x_{sln}^h\}$ is also uniformly bounded. All this facts imply that the sequence $(x_n, \theta_n, \varphi_n, m_n, (p_s^n A_s^j \varphi_{jn} - p_s^n m_n)^+)$ is uniformly bounded.

Since the allocations sequence in the above claim is uniformly bounded Fatou's Lemma (stronger version) can be applied: $\int_H x^h dh = \lim_{n \rightarrow \infty} \int_H x_n^h dh$, $\int_H \theta^h dh = \lim_{n \rightarrow \infty} \int_H \theta_n^h dh$, $\int_H \varphi^h dh = \lim_{n \rightarrow \infty} \int_H \varphi_n^h dh$ and

$$\int_H m_j^h dh = \lim_{n \rightarrow \infty} \int_H m_{jn}^h dh = \lim_{n \rightarrow \infty} \int_H M_{jn}^h \varphi_{jn}^h dh \quad (16)$$

Defining

$$M_j^h = \begin{cases} \frac{1}{\varphi_j^h} m_j^h \in \mathbb{R}_+^L, & \varphi_j^h \neq 0 \\ \text{anything}, & \varphi_j^h = 0 \end{cases}$$

we have that

$$\int_H M_j^h \varphi_j^h dh = \int_H \frac{m_j^h}{\varphi_j^h} \varphi_j^h dh = \int_H m_j^h dh$$

Therefore by using (16) one has

$$\int_H M_j^h \varphi_j^h dh = \lim_{n \rightarrow \infty} \int_H M_{jn}^h \varphi_{jn}^h dh$$

And so

$$\int_H M^h \varphi^h dh = \lim_{n \rightarrow \infty} \int_H M_n^h \varphi_n^h dh$$

Thus all markets clear in the $\mathcal{E}$. It remains to find a vector $C \in \mathbb{R}_+^{JL}$ such that

$$(a) \quad C_j \int_H \theta_j^h dh = \int_H M_j^h \varphi_j^h dh \in \mathbb{R}_+^L, \forall j \in J$$

$$(b) \quad \int_H (p_s A_s^j - p_s C_j)^+ \theta_j^h = \int_H (p_s A_s^j - p_s M^h j)^+ \varphi_j^h, \forall j \in J$$

Define $\bar{C}_{lj} = \frac{\rho_{slj}}{p_{sl}}$, where $\rho_{slj} = \lim_{n \rightarrow \infty} \rho_{slj}^n = \lim_{n \rightarrow \infty} p_{sl}^n C_{lj}^n$. Then

$$\begin{aligned} \bar{C}_{jl} \int_H \theta_j^h dh &= \frac{\rho_{slj}}{p_{sl}} \int_H \theta_j^h dh \\ &= \lim_{n \rightarrow \infty} \frac{\rho_{slj}^n}{p_{sl}^n} \lim_{n \rightarrow \infty} \int_H \theta_{jn}^h dh \\ &= \lim_{n \rightarrow \infty} C_{lj}^n \int_H \theta_{jn}^h dh \\ &= \lim_{n \rightarrow \infty} \int_H M_{jn}^h \varphi_{jn}^h dh \\ &= \int_H M_j^h \varphi_j^h dh \end{aligned}$$

Now, $\int_H (p_s A_s^j - p_s M_j^h)^+ \varphi_j^h dh = \lim_{n \rightarrow \infty} \int_H (p_s^n A_s^j \varphi_{jn}^h - p_s^n m_{jn}^h)^+ dh$. By definition of $\bar{C}$ one has

$$p_s \bar{C}_j = \lim_{n \rightarrow \infty} \sum_l p_{sl} C_{lj}^n = \lim_{n \rightarrow \infty} p_{sn} C_j^n$$

Then we have

$$\int_{S_s^j} (p_s A_s^j - p_s \bar{C}_j)^+ \theta_j^h = \int_{D_s^j} (p_s A_s^j - p_s M_j^h)^+ \varphi_j^h, \forall j \in J,$$

as desired.

6 A remark on efficiency

We conclude the paper with a final remark on the efficiency properties of equilibria. We show that an equilibrium allocation is constrained efficient among all feasible allocations that provide income across states through the same spot prices (i.e., the given equilibrium prices). However, we argue that it might be dominated when spot prices are free to vary and only the structure of asset returns is taken as given.

As in previous work on efficiency properties of incomplete markets (see Magill-Shafer (1991)), we compare the equilibrium allocation with other feasible allocations whose portfolios do not necessarily result from trading competitively in asset markets. That is, in alternative allocations agents pay participation fees which may differ from the market portfolio cost. Equivalently, we allow for transfers across agents which are being added to the usual market portfolio cost and these transfers do not necessarily coincide with the net default insurance premium considered in our model.

Proposition 1 *Let $((\bar{x}, \bar{\theta}, \bar{\varphi}, \bar{M}), \bar{p}, \bar{\pi}, \bar{C}, \bar{\beta})$ be an equilibrium. The allocation $(\bar{x}, \bar{\theta}, \bar{\varphi}, \bar{M})$ is efficient among all allocations (x, θ, φ, M) for which there are transfers $T^h \in \mathbb{R}$ across agents and a vector $C \in \mathbb{R}_+^{JL}$, such that*

- (i) $\int_H (x_o^h + M^h \varphi^h) dh = \int_H \omega_o^h dh, \int_H x_s^h = \int_H (\omega_s^h + Y_s M^h \varphi^h) dh,$
 $\int_H \theta^h dh = \int_H \varphi^h dh$
- (ii) $\bar{p}_s (x_s^h - \omega_s^h) + \sum_{j \in J} \min\{\bar{p}_s A_s^j, \bar{p}_s Y_s M_j^h\} \varphi_j^h = \sum_{j \in J} \min\{\bar{p}_s A_s^j, \bar{p}_s Y_s C_j\} \theta_j^h + \sum_{j \in J} \bar{p}_s Y_s M_j^h \varphi_j^h, \forall s, \text{ a.e. } h$
- (iii) $\bar{p}_o (x_o^h + M^h \varphi^h - \omega_o^h) + \bar{\pi} (\theta^h - \varphi^h) + T^h = 0$
- (iv) $\bar{p}_o M^h \geq \bar{\pi} + \epsilon$ with $\epsilon > 0$ for a.e. h
- (v) $\int_H T^h dh = 0$

$$(vi) \ C_j \int_H \theta^h dh = \int_H M_j^h \varphi_j^h, \quad \forall j$$

Proof: Suppose not, say $(x, \theta, \varphi, M, C)$ together with some transfer fraction T satisfies (i) through (vi); $u^h(x_o^h + M^h \varphi^h, x_{-o}^h) \geq u^h(\bar{x}_o^h + \bar{M}^h \bar{\varphi}^h, \bar{x}_{-o}^h)$ for a.e h and $u^h(x_o^h + M^h \varphi^h, x_{-o}^h) > u^h(\bar{x}_o^h + \bar{M}^h \bar{\varphi}^h, \bar{x}_{-o}^h)$ for h in some positive measure set G of agents. Then, for $h \in G$, the first period constraint must be violated, that is,

$$\bar{p}_o(x_o^h + M^h \varphi^h - \omega_o^h) + \bar{\pi}(\theta^h - \varphi^h) + \sum_{s \in S} \bar{\beta}_s g_s^h > 0 \quad (17)$$

where

$$\begin{aligned} g_s^h &= (\bar{p}_s A_s - \bar{p}_s Y_s M^h)^+ \varphi^h - (\bar{p}_s A_s - \bar{p}_s Y_s C)^+ \theta^h \\ &= (\bar{p}_s A_s - D_s^h) \varphi^h - (\bar{p}_s A_s - R_s) \theta^h \end{aligned}$$

where

$$\begin{aligned} D_s^h &= \min\{\bar{p}_s A_s, \bar{p}_s Y_s M^h\} \\ R_s &= \min\{\bar{p}_s A_s, \bar{p}_s Y_s C\} \end{aligned}$$

By continuity of preferences and monotonicity we can take $G = H$, without loss of generality. Then $\int_H g_s^h dh > 0$ for some s , by (17) and (i), implying $\int_H R_s \theta^h dh > \int_H D_s^h \varphi^h dh$.

Now, by (ii),

$$\bar{p}_s \cdot \int_H (x_s^h - \omega_s^h) dh = \int_H R_s \theta^h dh - \int_H D_s^h \varphi^h dh + \int_H \bar{p}_s Y_s M^h \varphi^h dh$$

where the right hand side is strictly positive, contradicting $\int_H (x_s^h - \omega_s^h) dh = 0$

■ The above weak constrained efficiency property extends similar the properties found in the incomplete markets model without default (see Magill - Shafer (1991)) and also in the exogenous collateral model (without utility penalties) of Geanakoplos - Zame - Dubey (1995). As in these models it does not seem to be possible to show that equilibrium allocations are undominated when prices are no longer assumed to be constant at the equilibrium levels.

The weakly constrained efficient transfer among agents is given by the equilibrium net insurance premium . Besides the incompleteness of markets, there is another potential source of inefficiency in our model which is default. This second source of inefficiency can be dealt with by transferring among

agents default insurances and requiring borrowers to be fully covered. But the inefficiency caused by incompleteness might make the equilibrium allocation become dominated when spot prices are free to differ from the equilibrium levels.

7 Appendix

- Let $C(K)$ the Banach space of continuous functions on the compact metric space K . Let $L^1(H, C(K))$ be the Banach space of Bochner integrable functions whose values belong to $C(K)$. For $z \in L^1(H, C(K))$,

$$\|z\|_1 := \int_H \sup_K |Z^h| dh < \infty$$

Let $\mathcal{M}(K)$ denotes the set of regular measures on the Borelians of K . The dual space of $L^1(H, C(K))$ is $L^\infty_\omega(H, \mathcal{M}(K))$, the Banach space of essentially strong bounded weak * measurable functions from H into $\mathcal{M}(K)$. We say that $\{\mu_n\} \subset L^\infty(H, \mathcal{M}(K))$ converges to $\mu \in L^\infty_\omega(H, \mathcal{M}(K))$ with respect to the weak * topology on the dual $L^1(H, C(K))$, if

$$\int_H \int_K z^h d\mu_n^h dh \rightarrow \int_H \int_K z^h d\mu^h dh, \forall f \in L^1(H, C(K))$$

- We will use in this work the following Lemmas (in m-dimension).

Fatou's Lemma (Weak Version)

Let $\{f_n\}$ be a sequence of integrable functions of a measure space $(\Omega, \mathcal{A}, \nu)$ into $\mathbb{R}_+^m$. Suppose that $\lim_{n \rightarrow \infty} \int_\Omega f_n d\nu$ exists. Then there exists an integrable function $f : \Omega \mapsto \mathbb{R}_+^m$ such that:

1. $f(w) \in cl\{f_n(w)\}$ for a.e w , and
2. $\int_\Omega f d\nu \leq \lim_{n \rightarrow \infty} \int_\Omega f_n d\nu$

Fatou's Lemma (Stronger version)

If in addition the sequence $\{f_n\}$ above is uniformly integrable, then the inequality in 2 holds as an equality.

- **Lemma 3** *The budget correspondence defined by*

$$B^h(p, \pi, \beta, \rho) = \{(x^h, \theta^h, \varphi^h, m^h) : A, B \text{ and } C \text{ are satisfied}\}.$$

is lower hemi-continuous at any (p, π, β, ρ) strictly positive, provided that $\omega^h \gg 0$.

(to be used in claim 3, where A, B y C are defined)

Proof

We define $B_o^h(p, \pi, \beta, \rho)$ to be the interior of $B^h(p, \pi, \beta, \rho)$, that is; the following holds:

$$p_o x_o^h + \pi \theta^h - \pi \varphi^h + p_o m^h + \sum_s \beta_s [p_s A_s \varphi^h - p_s m^h]^+ - \sum_s \beta_s [p_s A_s - \rho_s]^+ \theta^h < p_o \omega_o^h,$$

$$p_s x_s^h + \sum_j \min\{p_s A_s^j \varphi_j^h, p_s m_j^h\} < p_s \omega_s^h + r_s \theta^h + p_s m^h$$

and

$$p_o m_j^h > \pi_j \varphi_j^h + \epsilon \varphi_j^h, \forall j \in J$$

Let $x^h = 0, \theta^h = 0, \varphi^h = 0$ and m_j^h such that $\epsilon < p_o m_j^h \leq p_o \omega_o^h$. It is easy to verify that these variables thus chosen satisfy the budget constraint of agent h with strict inequality. So, $B_o^h(p, \pi, \beta, \rho) \neq \emptyset$. Let $\lim_{n \rightarrow \infty} (p^n, \pi^n, \beta^n, \rho^n) = (p, \pi, \beta, \rho)$ and $(x^h, \theta^h, \varphi^h, m^h) \in B_o^h(p, \pi, \beta, \rho)$. Then for every $\{(x_n^h, \theta_n^h, \varphi_n^h, m_n^h)\}$ such that $\lim_{n \rightarrow \infty} (x_n^h, \theta_n^h, \varphi_n^h, m_n^h) = (x^h, \theta^h, \varphi^h, m^h)$ and for n large enough, the following holds

$$p_o^n x_{no}^h + \pi^n \theta_n^h - \pi^n \varphi_n^h + p_o^n m_n^h + \sum_s \beta_s^n [p_s^n A_s \varphi_n^h - p_s^n m_n^h]^+ - \sum_s \beta_s^n [p_s^n A_s - \rho_s^n]^+ \theta_n^h < p_o^n \omega_o^h,$$

$$p_s^n x_{ns}^h + \sum_j \min\{p_s^n A_s^j \varphi_{nj}^h, p_s^n m_{nj}^h\} < p_s^n \omega_s^h + r_s^n \theta_n^h + p_s^n m_n^h$$

and

$$p_o^n m_{nj}^h > \pi_j^n \varphi_{nj}^h + \epsilon \varphi_{nj}^h, \forall j \in J$$

Thus $(x^n, \theta^n, \varphi^n, m_n^h) \in B_o^h(p^n, \pi^n, \beta^n, \rho^n)$ for n large enough, which implies that B_o^h is lower hemi-continuous. Then the result follows from Hildenbrant (1974), pag. 26, fact 4. ■

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