Fixed Point Theorems for Correspondences with Values in a Partially Ordered Set and Extended Supermodular Games

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Abstract

We present fixed point theorems for correspondences with values in a partially ordered set, generalizing Knaster-Tarski's fixed point theorem. We prove the existence of a Nash equilibrium for extended supermodular games, non-cooperative games where the strategy space is a quasi-lattice and the best reply satisfies certain monotonicity requirements.

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1 Introduction

In this paper we present fixed point theorems for correspondences (set valued functions) with values in partially ordered sets and prove the existence of a Nash equilibrium for extended supermodular games (symmetric games in which the strategy space of each player is a quasi-lattice).

Knaster (1928) and Tarski (1955) obtained fixed point theorems for functions with values in partially ordered sets.¹ We present here an extension of those results, proving fixed point theorems for correspondences with values in partially ordered sets.

Fixed point theorems for functions with values in partially ordered sets are now playing an important role in economics. The proof of the existence of a Nash equilibrium in supermodular games, developed by Topkis (1979) and initially applied to economic problems by Vives (1990) and Milgrom and Roberts (1990), relies on Tarski's fixed point theorem. The recent work of Milgrom and Shannon (1994) and Milgrom and Roberts (1994) on comparative statics also provide examples of economic applications of fixed point theorems for functions with values in partially ordered sets.

We generalize previous results on supermodular games by proving the existence of a Nash equilibrium for extended supermodular games. These are symmetric games where the strategy space of each player is no longer required to be a lattice. The strategy space of each player will be a quasi-lattice, a partially ordered set with somewhat more structure than the one imposed by the partial order. We use the results derived in this paper about the existence of a fixed point for correspondences with values in a partially

¹See Aliprantis and Border (1994), p. 15, for a recent textbook with a clear and concise presentation of these results.

ordered set to prove the existence of a Nash equilibrium for the games just defined.

The remainder of the paper is organized as follows. In the next section we present our main results. In the third section we define quasi-lattices, supermodularity for functions defined over a partially ordered set, extended supermodular games and prove that these games have a Nash equilibrium. In the final section we sum up.

2 Main Results

Let X be a nonempty set and $\leq$ be a reflexive, antisymmetric and transitive binary relation defined in X , i.e., a partial order defined in X . We shall say that $(X, \leq)$ is a partially ordered set, with $\leq$ being the partial order. As in Davey and Priestley (1990), p. 15, we shall say that $x \in B \subseteq X$ is a maximal element of B if $\forall y \in B : x \leq y$ we have $x = y$.

Let

$$\tau : X \rightarrow \mathcal{P}(X)$$

be a correspondence, a set valued function with domain in X and range in the set of the parts of X , $\mathcal{P}(X)$. We assume in the sequel that τ is a nonempty valued correspondence, meaning that $\forall x \in X, \tau(x) \neq \emptyset$.

Results obtained previously on fixed points of functions with domain and range in partially ordered sets required the functions to obey a nondecreasing condition. A similar requirement is imposed on correspondences:

Definition 1. Let $(X, \leq)$ be a partially ordered set and $\tau : X \rightarrow \mathcal{P}(X)$, a

nonempty correspondence. τ is said to be nondecreasing if

$$\forall a, b \in X : a \leq b, a \neq b \Rightarrow (\forall x \in \tau(a), \forall y \in \tau(b), x \leq y)$$

Note that this definition extends in an obvious way the nondecreasing definition usually imposed on functions with domains and ranges in partially ordered sets.

In the context of fixed point theorems, when the algebraic structure involved is a partial order, one usually imposes a requisite about the behavior of the function at some point in the set. More specifically, one usually requires not only the comparability of a certain point and its image, a strong requirement in a partially ordered set, but also that the image of the point is, at least, equal or higher, in the partial order sense, than the point. Geometrically the function has to stay on or jump over the diagonal at a particular point. One can extend the previous condition to correspondences, leading to the following

Definition 2 Let $(X, \leq)$ be a partially ordered set and $\tau : X \rightarrow \mathcal{P}(X)$, a nonempty correspondence. We say that τ jumps the diagonal if

$$\exists a \in X : (\forall x \in \tau(a), a \leq x)$$

Note that a correspondence jumps the diagonal if there exists a certain point, a , whose image is a subset of that point's up-set, $a \uparrow$, where $a \uparrow = \{x \in X : a \leq x\}$. We can weaken the previous definition by requiring only that the image of a certain point intersects that point's up-set. This will be done in the following definition.

Definition 3 Let $(X, \leq)$ be a partially ordered set and $\tau : X \rightarrow \mathcal{P}(X)$, a correspondence. We say that τ weakly jumps the diagonal if

$$\exists a \in X : (\exists x \in \tau(a), a \leq x)$$

From any nonempty correspondence that weakly jumps the diagonal we can derive a correspondence that jumps the diagonal. Let $\tau : X \rightarrow \mathcal{P}(X)$ be a nonempty correspondence that weakly jumps the diagonal and $H = \{x \in X : \exists y \in \tau(x) : x \leq y\}$ be the set of points where τ weakly jumps the diagonal. Let $\tau^d : X \rightarrow \mathcal{P}(X)$ be the derived correspondence, defined in the following way:

$$\tau^d(x) = \tau(x), \forall x \in X - H$$

and

$$\tau^d(x) = \{y \in \tau(x) : x \leq y\}, \forall x \in H.$$

Given that τ weakly jumps the diagonal, τ^d is a nonempty correspondence that, by construction, jumps the diagonal. It is immediate to see that the set of points where τ^d jumps the diagonal is equal, by construction, to the set of points where τ weakly jumps the diagonal. We also have that $\forall x \in X, \tau^d(x) \subseteq \tau(x)$.

Given the previous definitions we are now able to state the following lemma.

Lemma 1 *Let $(X, \leq)$ be a partially ordered set. Let $\tau : X \rightarrow \mathcal{P}(X)$ be a nonempty correspondence that jumps the diagonal. Let*

$$H = \{x \in X : \forall y \in \tau(x), x \leq y\},$$

be the set of points where τ jumps the diagonal. Suppose that $\emptyset \neq A \subseteq H$ is closed under τ , $\tau(A) \subseteq A$, and that A has maximal points. Then τ has a fixed point.

Proof Let $a \in A$ be a maximal point of A . As $a \in A \subseteq H$, where H is the subset of X where τ jumps the diagonal, we know that $\forall b \in \tau(a), a \leq b$. As $\tau(a) \subseteq \tau(A) \subseteq A$, maximality of a in A ensures that $\forall b \in \tau(a), a = b$.

proving that $\{a\} = \tau(a)$, ending the proof. ■

The previous lemma showed that any maximal element of a nonempty and τ -closed subset of H , will be a fixed point of τ . We will now prove that, when τ is a nondecreasing correspondence, H is closed under τ .

Lemma 2 *Let $(X, \leq)$ be a partially ordered set. Let $\tau : X \rightarrow \mathcal{P}(X)$ be a nonempty and nondecreasing correspondence that jumps the diagonal. Let*

$$H = \{x \in X : \forall y \in \tau(x), x \leq y\},$$

be the set of points where τ jumps the diagonal. Then $\tau(H) \subseteq H$.

Proof Let $y \in \tau(H)$. Then $\exists x \in H : y \in \tau(x)$. As $x \in H$ we have that $\forall z \in \tau(x), x \leq z$. In particular, it is true that $x \leq y$. If $x = y$ the result follows. If $x \neq y$ then, as τ is nondecreasing, we have that $\forall z_1 \in \tau(x), z_2 \in \tau(y), z_1 \leq z_2$. As $y \in \tau(x)$ we can conclude that $\forall z_2 \in \tau(y), y \leq z_2$, thus proving that $y \in H$. ■

We will now find a sufficient condition on X that will ensure that any nondecreasing correspondence defined in X that jumps the diagonal, has a fixed point. Given the previous lemmas, if we prove that the nonempty set H , where τ jumps the diagonal, has a maximal element, then τ has a fixed point. This will be done in the following theorem.

Theorem 1 *Let $(X, \leq)$ be a partially ordered set in which every nonempty chain has a supremum. Let $\tau : X \rightarrow \mathcal{P}(X)$, be a nonempty nondecreasing correspondence that jumps the diagonal. Then the set*

$$\mathcal{F}_\tau = \{x \in X : x \in \tau(x)\}$$

is nonempty and contains maximal elements of the set of elements where τ jumps the diagonal. Moreover, there exists some $x \in X$ such that $\tau(x) = x$.

Proof Consider the set

$$H = \{x \in X : \forall y \in \tau(x), x \leq y\}.$$

Since τ jumps the diagonal, H is a nonempty set. Take a nonempty chain C in H . By hypothesis, C has a supremum, b , in X , meaning that

$$c \leq b, \forall c \in C.$$

If $b = c$, for some $c \in C$, then $b \in P$ is trivially true. Suppose now that $b \neq c, \forall c \in C$. Then we have that $c < b, \forall c \in C$. Given that τ is nondecreasing, we can conclude that

$$\forall y \in \tau(c), \forall z \in \tau(b), y \leq z.$$

As $c \in C \subseteq H$ we know that

$$c \leq y, \forall y \in \tau(c)$$

and, by transitivity, we can infer that

$$c \leq z, \forall z \in \tau(b).$$

As c was a general element of C , we have that any z belonging to $\tau(b)$ is a majorant of C . As, by definition, b is the smallest of all majorants,

$$b \leq z, \forall z \in \tau(b),$$

meaning that b is in H . By Zorn's lemma we know that H has a maximal element, x_0 . As we proved in Lemma 2 that H is closed under τ and Lemma 1 allows us to conclude that any maximal element of a τ -closed nonempty set is a fixed point for τ , we can conclude that x_0 is a fixed point of τ such

that $\{x_0\} = \tau(x_0)$, proving the theorem. ■

The previous theorem can be extended for a correspondence that is nonempty, nondecreasing, and that weakly jumps the diagonal. We will no longer be able to prove that there are fixed points whose image is the singleton formed by that point, but will be able to prove that $\mathcal{F}_\tau$, the set of fixed points of τ , is nonempty. This will be done in the next theorem.

Theorem 2 *Let $(X, \leq)$ be a partially ordered set in which every nonempty chain has a supremum. Let $\tau : X \rightarrow \mathcal{P}(X)$, be a nonempty nondecreasing correspondence that weakly jumps the diagonal. Then the set*

$$\mathcal{F}_\tau = \{x \in X : x \in \tau(x)\}$$

is nonempty and contains maximal elements of the set of elements where τ weakly jumps the diagonal.

Proof Let τ^d be the derived correspondence from τ . As τ^d jumps the diagonal and is nondecreasing, from Theorem 1 we know that τ^d has fixed points and that there exists a point $x \in X$ such that $\tau^d(x) = \{x\}$, where x is a maximal element of the set where τ^d jumps the diagonal. As $\{x\} = \tau^d(x) \subseteq \tau(x)$, the theorem is proved. ■

An interesting application of the previous Theorem arises if X is a partially ordered set that has a bottom element, $\perp$, an element of X such that $\forall x \in X, \perp \leq x$ and in which every nonempty chain has a supremum. In such a partially ordered set any nonempty and nondecreasing correspondence has a fixed point. This is now proved.

Theorem 3 *Let $(X, \leq)$ be a partially ordered set in which every nonempty chain has a supremum. Assume that X has a bottom element, $\perp$. Let $\tau : X \rightarrow \mathcal{P}(X)$, be a nonempty nondecreasing correspondence. Then the set*

$$\mathcal{F}_\tau = \{x \in X : x \in \tau(x)\}$$

is nonempty and contains maximal elements of the set of elements where τ weakly jumps the diagonal.

Proof As X has a bottom element and τ is nonempty, $\tau(\perp) \neq \emptyset$. Then $\exists x \in \tau(\perp) : \perp \leq x$. This means that τ weakly jumps the diagonal. Then as all the hypotheses of Theorem 2 are fulfilled, the thesis follows. ■

In this section we proved several theorems ensuring that a nonempty, nondecreasing correspondence, defined over a partially ordered set, has a fixed point. In the next section we will use Theorem 3 to prove the existence of an equilibrium for a certain class of games, that generalize, in a particular way, supermodular games initially considered by Vives (1990).

3 The existence of a Nash equilibrium for general supermodular games

Supermodular games, considered by Vives (1990), require the strategy space of each player to be a lattice. Lattices are very special partially ordered sets, as any two elements have a supremum and an infimum. In a general partially ordered set this is not necessarily true, and any two elements may not even be linked, either directly or indirectly, by means of the *meet* and *join* operations. Nevertheless, the structure over a set provided by the partial

order is a rather weak one. To obtain some interesting results one has to enrich it and the lattice environment gives us some hints about possible directions for enrichment. In order to do so, thus generalizing the lattice structure over a partially ordered set, we put forward the following definition.

Definition 4 *Let X be a partially ordered set and $x, y \in X$. We shall say that x meets y if $x \downarrow \cap y \downarrow \neq \emptyset$ and that x joins y if $x \uparrow \cap y \uparrow \neq \emptyset$. $x, y \in X$ are in a regular position, $xRPy$, if x and y meet and join. If X is such that all pairs of points are in a regular position we shall say that X is a quasi-lattice.*

By definition any partially ordered set in which the up-sets and down-sets of any two points have nonempty intersections is a quasi-lattice. For any subset of $\mathbb{R}^2$, if the *inf* of any two points' up-sets intersection and the *sup* of any two points' down-sets intersection exists, in the subset considered, then the set is not only a quasi-lattice but also a lattice. Then, obviously, every lattice is a quasi-lattice, as in a lattice every two elements are in a regular position. This is ensured by the existence of their meet and join. But notice that not all quasi-lattices are lattices. For instance, take the set $\{(x, y) : x, y \in [0, 1]\} - \{(x, y) : x = y, 0 < x < 1\} \subseteq \mathbb{R}^2$, endowed with the usual partial order defined in $\mathbb{R}^2$. It is obviously not a lattice, although it is a quasi-lattice.

The definition of supermodularity is well known for real functions defined over a lattice. We extend this concept for real functions defined over a partially ordered set, in which at least points are in a regular position.

Definition 5 *Let X be a partially ordered set and $f : X \rightarrow \mathbb{R}$ be a real function defined over X . We shall say that f is supermodular if $\forall x, y \in X : xRPy$.*

$$\exists a \in x \downarrow \cap y \downarrow, \exists b \in x \uparrow \cap y \uparrow : f(a) + f(b) \geq f(x) + f(y).$$

f is strictly supermodular if the inequality in the previous definition is strict when x and y are in a regular position and unrelated by the $\leq$ relation defined over X .

A concept related to supermodularity is increasing differences. In the sequel we will be using an extension of the previous definition of increasing differences, as we are considering real functions defined over a product of partially ordered sets, adapting the definition of Vives (1990).

Definition 6 *Let X and T be two partially ordered sets and $f : X \times T \rightarrow \mathbb{R}$ be a real function defined over $X \times T$. Then f has (strictly) increasing differences in (x, t) if $f(x, t) - f(x, t')$ is (strictly) increasing in x for all $t' \leq t$ ($t' \leq t, t \neq t'$).*

As Topkis (1978) proved supermodularity implies increasing differences for real functions defined over a lattice. The reverse implication is also true when, under the notation of the previous definition, X and T are chains.

Consider now a real function, f , defined over the product of two partially ordered sets, X and T , $f : X \times T \rightarrow \mathbb{R}$. Given f we can form the nonempty correspondence $\tau : T \rightarrow X$ defined as

$$\tau(t) = \operatorname{argmax}_{x \in X} f(x, t),$$

that assigns to each $t \in T$ the subset of X consisting of the maximizing elements of the function $f(\cdot, t)$, defined over X . We are interested in the properties of the correspondence τ , and some relevant facts about τ are established in the following theorem.

Theorem 4 *Let X be a quasi-lattice and T a partially ordered set and let $f : X \times T \rightarrow \mathbb{R}$ be a real function defined over $X \times T$. Assume that, $\forall t \in T$, $f(\cdot, t)$ is strictly supermodular on X . Then*

i) $\forall t \in T, \tau(t)$ is a chain.

ii) If f has strictly increasing differences in (x, t) , the correspondence τ is nondecreasing.

Proof i) We shall prove the desired result by contradiction. Consider $x, y \in \mathcal{O}(t)$ and assume that they are not comparable under the $\leq$ relation defined over X . As X is a quasi-lattice, we know that $\exists a \in x \downarrow \cap y \downarrow$ and that $\exists b \in x \uparrow \cap y \uparrow$. As f is strictly supermodular,

$$0 \geq f(b, t) - f(x, t) > f(y, t) - f(a, t) \geq 0.$$

reaching the desired contradiction. Therefore either $x \leq y$ or $y \leq x$, proving the desired result.

ii) We will show that $\forall b, t \in T : b \leq t, b \neq t$ we have that $\forall x \in \tau(b), \forall y \in \tau(t), x \leq y$. If it is not true that $x \leq y$, then either x and y are unrelated or $y \leq x, y \neq x$. If x and y are unrelated, then as $xRPy$,

$$\exists z_1 \in x \downarrow \cap y \downarrow$$

and

$$\exists z_2 \in x \uparrow \cap y \uparrow,$$

with $z_1 \neq x \neq z_2$ and $z_1 \neq y \neq z_2$. As f has strictly increasing differences and is strictly supermodular, the following string of inequalities are obeyed:

$$0 \geq f(z_2, t) - f(y, t) \geq f(x, t) - f(z_1, t) > f(x, b) - f(z_1, b) \geq 0.$$

reaching a contradiction.

Another case must now be dealt with, namely when $y \leq x, y \neq x$. In this case the following string of inequalities is true:

$$0 \geq f(x, t) - f(y, t) > f(x, b) - f(y, b) \geq 0.$$

thus also reaching a contradiction. ■

We now define an extended supermodular game. It is a symmetric game with two players, where X , the strategy space common to both players, is assumed to be a quasi-lattice with a bottom element, $\perp$, and in which every nonempty chain has a supremum. Each player's payoff or utility function gives rise to a best reply function, f , defined over $X \times X$, that is strictly supermodular and has strictly increasing differences. Given the properties of X and f , this game can be considered as an extension of the supermodular games studied by Vives (1990), thus justifying the name of extended supermodular game. A Nash equilibrium for this game is an element x_0 of X such that $x_0 \in \operatorname{argmax}_{x \in X} f(x, x_0)$. Using the previous results, we will prove the existence of a Nash equilibrium for the extended supermodular game.

Consider $\tau : X \rightarrow \mathcal{P}(X)$, a nonempty correspondence, defined in the following way:

$$\forall x' \in X \quad \tau(x') = \operatorname{argmax}_{x \in X} f(x, x').$$

τ is obviously nonempty. Given the properties of f , Theorem 4 ensures that τ is nondecreasing. Then, as all hypotheses of Theorem 3 are fulfilled, we know that τ has a fixed point, x_0 . But x_0 being a fixed point for τ means that

$$x_0 \in \operatorname{argmax}_{x \in X} f(x, x_0),$$

thus proving the existence of a Nash equilibrium for the game considered above.

4 Conclusions

In this paper we showed that, when considering a partially ordered set in which every nonempty chain has a supremum, and imposing certain condi-

tions about correspondences with values in that set, fixed point theorems can be obtained. A weak fixed point theorem is obtained when the correspondence is nondecreasing and jumps the diagonal. In this case we are able to prove that at certain fixed points the correspondence is a function. A stronger fixed point theorem is also obtained when the correspondence is still nondecreasing and weakly jumps the diagonal but in this case we are no longer able to prove that the correspondence is a function at certain fixed point. We also obtain the result that, if the partially ordered set has a bottom element, any nondecreasing correspondence has a fixed point. We provide a simple application of the strong fixed point theorem for a correspondence defined over a partially ordered set that has a bottom element, showing that an extended supermodular game has a Nash equilibrium.

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