Nash Equilibrium and the Law of Large Numbers

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Summary

Pascoa (1993a) showed that the failure of the law of large numbers for a continuum of independent realizations implies that Schmeidler's (1973) concept of a measure-valued profile function in equilibrium might not coincide with the true concept of mixed strategies equilibrium of a nonatomic game. The latter should be defined as a probability measure on pure strategies profiles which is induced by the product measure of players' mixed strategies. This paper addresses existence and approximate purification of the latter and presents an assumption on continuity of payoffs that guarantees the equivalence between the two equilibrium concepts.

1 Introduction

In games with a continuum of players there is a convexifying effect of large numbers that tends to obviate the use of mixed strategies. In an influential paper, Schmeidler (1973) addressed the existence and purification of equilibria in these games. However, the failure of the strong law of large numbers, for a continuum of independent random variables, casted some doubt on the true significance of Schmeidler's (1973) concept of equilibrium for a nonatomic game. Feldman-Gilles (1985) showed that sample averages of realizations of a continuum of independent random variables are not, almost surely, equal to the corresponding theoretical means and in Pascoa (1993a) there is an example of a nonatomic game where Feldman-Gilles argument was used to point out an equilibrium in Schmeidler's (1973) sense that fails to be a mixed-strategies equilibrium.

Schmeidler (1973) considered a nonatomic game where payoffs are defined on measure - valued profiles and established the existence of an equilibrium for this abstract game. However, if we look at the underlying concrete payoffs defined on pure-strategy profiles (by taking the degenerate-measure-valued profiles) and integrate them with respect to the distribution on pure-strategy profiles that is induced by the product measure of players' mixed strategies, we may not recover the original abstract payoffs, as the example in Pascoa (1993a) illustrates. This observation seemed to assert that Schmeidler's (1973) approach to nonatomic games (like the behavioral strategies approach to uncertainty games) could not be used as a substitute for modelling mixed strategy plays as distributions on the space of pure strategy realizations.

In the context of games under uncertainty, Aumann (1964) had pointed out long before that expected payoffs should be computed by integrating payoffs with respect to distributions on pure strategy profiles, but he suggested also that the mathematical difficulties involved should be avoided by using the behavioral strategies approach, where a mixed strategy play is described by a measure-valued mapping defined on the continuum of states (or types).

In this paper we prove the existence of an equilibrium distribution on pure-strategy profiles; this distribution is induced by the product measure of

players' mixed strategies. Arbitrarily close to the equilibrium mixed strategies profile is a degenerate profile which is an approximate equilibrium in pure strategies. The existence argument uses a continuity assumption on payoffs that allows us to deal separately with countably many realization subspaces and we can make the law of large numbers hold in each by extending appropriately the distributions induced by the mixed strategies profile. This proof is inspired in Judd's (1985) approach to the problem of the law of large numbers.

By using the law of large numbers we make the expected payoff functionals inherit from the concrete payoff a continuity property which is precisely the condition assumed in the existence theorems for nonatomic games with measure-valued profiles (see Schmeidler (1973) for the case of finitely many strategies, and Ali-Khan (1986) and Pascoa (1993a), for the case of a compact strategy set). Moreover, under this continuity property, the expected payoffs of an abstract game, over measure-valued profiles, will pass the recoverability test which Pascoa's (1993a) example failed.

Alternative approaches to alleviate the problem with the law of large numbers in models with a continuum of random variables, such as E. Green's (1988) or Keisler's (1977), have the disadvantage of using nonseparable atomless measure spaces of types, which are not isomorphic to the Lebesgue measure space of the unit interval. For these larger atomless spaces of indices of the random variables, the law of large numbers will hold, but there are not yet any existence results. The known result on existence of equilibria for nonatomic games uses the metrizable topology on the set of mixed strategies profiles, which relies on the separability of the pre-dual implied by the separability of the measure space.

The nonatomic games addressed in this paper differ also from the anonymous games studied by E. Green (1984), Mas-Colell (1985) and Pascoa (1993a, section 2). In the current paper, as in Schmeidler (1973), the games are specified by a priori assignments of players to payoff functions, whereas, in the other papers, that assignment can only be found a posteriori, given a play of the game. In E. Green's (1984) anonymous game, payoffs depend on opponent's actions through the joint distribution of types and strategies,

whereas in Mas-Colell's (1985) game, payoffs depend only on the marginal distribution of strategies. Green's (1984) game is a very large game: there is a continuum of types and a continuum of players of each type, so that payoffs become well defined, for each play of the game. The assignment games studied in this paper are more attractive in some contexts, such as Bertrand models, where anonymous games would suffer from payoff discontinuities. Examples of Bertrand assignment games satisfying the continuity assumptions in this paper can be found in Pascoa (1986,1993b).

In the last section of the paper we show that for games under uncertainty, with a continuum of states, when the utility functionals are expected utility functionals, the equivalence result applies: behavioral strategies are a perfect substitute for properly defined mixed strategies. It is not known whether this equivalence would still hold in the case of non-expected utility functionals.

The paper is organized as follows. Section 2 reviews the theory of probability in the realizations space, for a continuum of independent random variables. Section 3 describes our methodology. Section 4 presents an assumption on continuity of payoffs. Section 5 uses this assumption to establish some preliminary results that will be used in the main results on existence and purification that are established in section 6. Finally, section 7 relates our approach to Schmeidler's (1973) and establishes that the two approaches are equivalent under a continuity condition, but that this equivalence fails in general. Section 8 addresses games with uncertainty.

2 Probability in the Realizations Space

Consider a nonatomic game defined by the (i) Lebesgue measure space $(I, \mathcal{I}, \lambda)$ of the unit interval $I \equiv [0, 1]$ as the space of players, (ii) a common strategy set S to all players, where S is a compact subset of a finite dimensional space E and (iii) the payoff function of each player i given by $v(i, \cdot) : S \times S^I \rightarrow \mathbb{R}$.

A mixed strategies profile is a mapping from the players' set I into $M(S)$, the set of regular probability measures on the strategy set S . An expected payoff associated with a mixed strategies profile $f \in M(S)^I$ should be computed with respect to the distribution on pure strategy profiles induced by

the continuum of independent randomizations prescribed by f .

More precisely, the product measure $\prod_{i \in I} f(i)$ induces a probability measure μ on the σ -algebra $\mathbf{B}$ of subsets of the realizations set S^I which is generated by the class C of cylinders, that is, sets of the form $\{g \in S^I : g(i) \in (a_i, b_i), i = 1, \dots, m\}$. Clearly, each set in $\mathbf{B}$ is defined by restricting the values of the realizations on at most countably many indices i .

Kolmogorov proposed a definition for the probability measure μ that preserves the independence of the continuum of randomizations. For each cylinder $A = \{g \in S^I : g(i) \in (a_i, b_i), \text{ for } i = 1, \dots, m\}$, let $\mu(A) = \prod_{i=1, \dots, m} f(i)((s \in S : s \in (a_i, b_i)))$. Then μ is extended to $\mathbf{B}$ in the usual way. Clearly, the projections $p_i : S^I \rightarrow S$ defined by $p_i(g) = g(i)$ are independent random variables with distributions $f(i)$. In fact, an infinite set of variables is said to be independent if each finite subset satisfies the independence assumption. Now, for any finite subset J contained in I and given $Z : i \rightarrow Z_i \in \mathbf{B}(S)$, $\mu(\{g \in S^I : p_i(g) \in Z_i, \text{ for } i \in J\}) = \prod_{i \in J} f(i)(Z_i)$.

However, the above definition of probability measure presents some problems for our purpose, as it does in general in the context of any function space. Many interesting subsets of S^I (such as the set of uniformly bounded functions or the set of continuous functions) are not measurable with respect to $\mathbf{B}$, the σ -algebra generated by the cylinders.

Judd (1985) confronted this difficulty when trying to define a sample distribution since the sets of Lebesgue measurable realizations are not $\mathbf{B}$ -measurable. Using the fact that these sets have outer-measure one and inner-measure zero, Judd was able to extend μ in a way that the sample distribution becomes well defined on a full measure set for this extension.

Here we will follow a similar approach. We will extend Kolmogorov's measure on the realization space in a way that will guarantee that almost every realization is Lebesgue measurable, but we will also have to ensure that payoff functions are integrable with respect to the extensions and that the resulting expected payoffs will satisfy a continuity property that guarantees existence and approximate purification of mixed strategies equilibria.

3 Methodology

Our goal is to integrate players' payoffs with respect to a distribution on realizations of the mixed strategies profile in a way that (i) the distribution preserves the independence of players' randomizations and (ii) the resulting expected payoff functionals satisfy Ali-Khan (1986) and Pascoa (1993a) continuity conditions, which guarantee existence and approximate purification (respectively) of a mixed strategies equilibrium.

Let us recall Ali-Khan's (1986) work on abstract games in the version applied by Pascoa (1993a) to games with measure-valued profiles. In Pascoa (1993a) payoff functionals, $V(i, s, \cdot)$ are defined on $M(S)^I$, which is a subset of the dual of the Banach space $L^1(I; C(S))$ of strongly integrable mappings from I into $C(S)$. In fact, the dual of $L^1(I; C(S))$ is the Banach space $L^\infty_W(I; rca(S))$ of equivalence classes of essentially strongly bounded and weak* measurable functions mapping I into regular countably additive signed measures on S .

We say that a sequence (f_n) in $L^\infty_W(I; rca(S))$ converges weak* to $f \in L^\infty_W(I; rca(S))$ iff $\int_I \int_S h(i)(\cdot) d f_m(i) d \lambda \rightarrow \int_I \int_S h(i)(\cdot) d f(i) d \lambda$, for each $h \in L^1(I; C(S))$. If payoff functionals $V(i, \cdot)$ on $S \times M(S)^I$ are continuous (with respect to the weak* topology on $M(S)^I$) and such that $V(\cdot, s, f)$ is $\mathbb{I}$ -measurable, for any $s \in S$, $f \in M(S)^I$, then there exists a measure-valued profile in equilibrium (see Ali-Khan (1986) and Pascoa (1993a)). Moreover, if the functionals $\{V(i, \cdot)\}$ are equicontinuous (for the above topology), then there is a degenerate approximate equilibrium profile $f \in \text{ext } M(S)^I$ (see Pascoa (1993a) Proposition 3) for the game with measure-valued profiles.

To construct expected payoff functionals satisfying the above continuity property we will impose a continuity assumption on the concrete payoffs $v(i, \cdot)$ of the game over pure strategy profiles.

4 A Continuity Assumption

Throughout this paper we will impose assumption (A1) which requires the payoff of any player i , $v(i, \cdot) : S \times S^I \rightarrow \mathbb{R}$, to be continuous on both the

player's own strategy and the pure strategy profile, with respect to the metric topology on (equivalent classes of) measurable profiles defined by the following notion of the convergence: we say that a sequence (g_n) in S^I converges to g if $\int_I h(i)(g_n(i))d\lambda \rightarrow \int_I h(i)(g(i))d\lambda$ for any strongly integrable mapping h from I into $C(S)$ (that is, any element h in the Banach space $L^1(I; C(S))$).

Clearly, the above topology on profiles has subbasic sets of the form $\{g \in L(I; S) : |\int_I (h_\gamma(i)(g(i)) - h_\gamma(i)(x(i)))d\lambda| < \varepsilon; \text{ for } \gamma = 1, \dots, n\}$, for a given $\varepsilon > 0$ and a given finite set $h_1, \dots, h_n$ in $L^1(I; C(S))$. The separability of $L^1(I; C(S))$ implies that the topology can be given a metric described by $\rho(g; g') = \sum_{\gamma=1}^{\infty} 1/2^\gamma |\int_I (h_\gamma(i)(g'(i)) - h_\gamma(i)(g(i)))d\lambda|$ where $(h_\gamma(i))_{\gamma=1}^{\infty}$ is a countable dense set in $L^1(I; C(S))$.

Notice that this topology is stronger than the L^∞ weak* topology, which corresponds to pointwise convergence on linear-valued elements of $L^1(I; C(S))$ and that it is also stronger than the topology of convergence in distribution (contemplated in Mas-Colell(1985)) which corresponds to pointwise convergence on elements of $L^1(I; C(S))$ that are constant across I . Our assumption (A1) includes these two cases and also the case of payoffs that are additively separable on competitors strategies. However, it should be noticed that S^I under the above metric is neither closed nor complete. Its closure is $M(S)^I$, the set of mappings into the set of regular probability measures on S , which we will also work with below.

Assumption (A1) is satisfied by nonatomic games with an indiscriminative structure of strategic information, such as Mas-Colell's (1985) game where payoffs depend only on the distribution on S that are induced by the profiles or Haller's (1986) game where payoffs depend just on the mean strategy. However, the assumption is also relevant for discriminative nonatomic games, where it matters who is playing what, such as the Bertrand games of monopolistic competition where firms' products are not necessarily equally imperfect substitutes of each other and, hence, different firms may want to look at different and more elaborate statistics of the joint distribution of firms and strategies. The Chamberlinian games studied by Pascoa (1986, 1993b) verify assumption A_1 .

5 Preliminary Results

5.1 Measurability of Realizations

By assumption (A1), payoffs depend on the profile $g \in S^I$ only through a ℓ^∞ weak* continuous function F of the sequence $(\int_I (h_m(i)(g(i)d\lambda))_{m=1}^\infty$, where (h_m) is a countable dense set in $L^1(I; C(S))$. Then, the payoff function $v(i, s, \cdot): S^I \rightarrow \mathbb{R}$ can be written as a composite function $V \circ W(i, s, \cdot)$ where $V: S^I \rightarrow X_{m=1}^\infty \mathbb{R}^{\text{supp } h_m}$ is defined by $V_m(g)(i) = h_m(i, g(i))$ and $W(i, s, \cdot): X_{m=1}^\infty \mathbb{R}^{\text{supp } h_m} \rightarrow \mathbb{R}$ is a function which is continuous with respect to the product topology induced by the convergence of integrals. Now each mixed strategies profile f generates a map $f_m: I \rightarrow M(\mathbb{R})$ by $f_m(i)(B) = f(i)(h_m(i)^{-1}(B))$, $B \in \mathcal{B}(\mathbb{R})$.

Let μ_m be Kolmogorov's measure on $Y_m \equiv V_m(S^I)$ induced by f_m . Let N_m be the non-measurable subset of Y_m where the sample distribution fails to exist. That is, $N_m = \{x \in Y_m : x^{-1}(-\infty, c) \text{ is not Lebesgue measurable, for some } c \in \mathbb{R}\}$. The set N_m has nevertheless outer-measure one and inner-measure zero, so μ_m can be extended to $\bar{\mu}_m$ on $A_m = B(Y_m) \cup N_m$ so that $\bar{\mu}_m(N_m) = 0$, as Judd (1985) showed.

So far we have extended Kolmogorov's measures on the realization spaces Y_m so that almost every realization is Lebesgue measurable. Now we will consider a further extension of Kolmogorov's measures that will ensure simultaneously that (i) payoff functionals are integrable with respect to the countable product of the extensions of Kolmogorov's measures (by ensuring measurability of the Borel sets of the topology of continuity) and (ii) the strong law of large numbers holds in each realization space Y_m . While (i) allows us to define correctly what an expected payoff is, (ii) makes the expected payoffs inherit from the payoff functionals a continuity property that guarantees existence and approximate purification of mixed strategies equilibria.

5.2 A Law of Large Numbers

Given a countable dense set (h_m) in $L^1(I; C(S))$, we can extend each of the measures $\bar{\mu}_m$ on the respective realization spaces Y_m so that the sample mean $\int_I h_m(i)(g(i))d\lambda/\lambda(\text{supp } h_m)$ agrees, almost surely, with the theoretical mean $\int_I \int_S h_m(i)(\cdot) df(i)d\lambda/\lambda(\text{supp } h_m)$.

Proposition 1 *Let $L_m = \{x \in Y_m : \int_I x(i)d\lambda \text{ exists and is different from } \int_I \int_S h_m(i)(\cdot)f(i)d\lambda\}$ and let $\bar{\mu}_m^0, \bar{\mu}_m^i$ be the outer and inner measures respectively, generated by $\bar{\mu}_m$. Then $\bar{\mu}_m^0(L_m) = 1, \bar{\mu}_m^i(L_m) = 0$ and there is an extension of $\bar{\mu}_m$ to $A_m \cup L_m$, say $\tilde{\mu}_m$, so that $\tilde{\mu}_m(L_m) = 0$.*

Proof: We will follow the method in the proof of Judd's (1985) theorem 2. To see that $\bar{\mu}_m^0(L_m) = 1$ we will show that L_m intersects any A_m -Borel set of positive $\bar{\mu}_m$ measure. By definition of $\bar{\mu}_m$ as an extension to $\mathbf{B}(Y_m) \cup N'_m$, any Borel set of positive $\bar{\mu}_m$ measure must be of the form $B \cap N'_m$ for some $B \in \mathbf{B}(Y_m)$, $B \neq \emptyset$. Let R_B be the countable set of indices that restrict the realizations in B and choose $x \in B \cap N_m$. If $\int_I x d\lambda \neq \int_I \int_S h_m df d\lambda$, then $x \in L_m$. Otherwise, choose $c \neq \int_I \int_S h_m df d\lambda$ and let w be a realization that coincides with x on R_B and such that $\int_I w d\lambda = c$. By definition $w \in B \cap L_m$ and, moreover, w is measurable, implying that $w \in N'_m$.

To see that $\bar{\mu}_m^i(L_m) = 0$ we will show that L_m does not contain a nonempty A_m -Borel set. In fact, (i) L_m can not contain the nonempty A_m -Borel sets that have a nonmeasurable realization (by definition of L_m) and (ii) L_m can not contain either the A_m -Borel sets whose realizations are measurable because these are of the form $B \cap N'_m$, for some $B \in \mathbf{B}(Y_m)$, and hence contain a realization which is on R'_B identically equal to $\int_I \int_S h_m df d\lambda$.

Now, a measure can be extended to a nonmeasurable set if the inner measure of this set is not decreased and its outer measure not increased. Then there is an extension $\tilde{\mu}_m$ of $\bar{\mu}_m$ to $A_m \cup L_m$ so that $\tilde{\mu}_m^0(L_m) > \tilde{\mu}_m(L_m) = \tilde{\mu}_m^i(L_m) = 0$. ■

5.3 Construction of Expected Payoffs

An immediate consequence of the above proposition is that the Borel sets of the topology of continuity of the payoff functionals on S^I (see assumption (A1)) are measurable with respect to the completion of $X_{m=1}^\infty \tilde{\mu}_m$. In fact, $N_m(z) = \{g \in S^I : |\int_I h_m(i)(g(i))d\lambda - \int_I h_m(i)(z(i))d\lambda| < \varepsilon\}$ is either contained in L_m or contains L'_m . Let $\hat{\mu}_m$ be the completion of $\tilde{\mu}_m$, then $\hat{\mu}_m(N_m(Z))$ is equal to zero in the first case and equal to one in the second case.

Then, given a player with payoff functional $v(i, s, \cdot) = V \circ W(i, s, \cdot)$ defined on S^I , we can define his expected payoff associated with a mixed strategy profile as

$$E(i, s, \cdot) = \int_{X_{m=1}^\infty Y_m} W(i, s, \cdot) d(X_{m=1}^\infty \hat{\mu}_m) = \int_{S^I} v(\cdot) d(X_{m=1}^\infty \hat{\mu}_m \circ V)$$

Here the product measure $X_{m=1}^\infty \hat{\mu}_m$ is defined in the usual way (see Halmos (1950)), not involving any difficulties found in the case of uncountable products.

Moreover, we can regard the above expected payoff as being defined on the quotient set of equivalence classes (for the equality a.e.) of functions in $M(S)^I$. In fact, distinct functions in a same equivalence class may generate distinct Kolmogorov measures and distinct respective extensions, but the respective expectations of a given payoff functional will be the same as a consequence of assumption (A1). To see this, notice that if $\bar{\mu}$ and $\bar{\mu}_0$ are extensions of Kolmogorov measures generated by functions in the same equivalence class, then $\bar{\mu} = \bar{\mu}_0$ on the Borel sets of S^I , for the topology of continuity of the payoff function.

5.4 The Mean Payoff is the Payoff of the Means

Next we present an important result of the paper. Under assumption (A1), payoffs depend on pure strategy profiles only through a functional F defined on countably many sample weighted averages. As a consequence of our strong law of large numbers, expected payoffs will depend on the corresponding countably many theoretical weighted means, through the same functional form F .

Proposition 2 Assume assumption (A1), so that $v(i, s, g) = F(i, s, (\int_I h_m(i)(g(i))d\lambda)_{m=1}^\infty)$, where (h_m) is a countable dense set in $L^1(I; C(S))$ and $F(i, s, \cdot)$ some ℓ^∞ weak* continuous function.

Given a mixed-strategies profile $f \in M(S)^I$, there exist extensions $\hat{\mu}_m$ of Kolmogorov probability measures induced by f on each space Y_m of realizations $i \rightarrow h_m(i)(g(i))$ so that any player's expected payoff satisfies

$$E(i, s, f) \equiv \int_{X_{m=1}^\infty Y_m} F(i, s, (\int_I x_m(i)d\lambda)_{m=1}^\infty) d(X_{m=1}^\infty \hat{\mu}_m(x_m)) = \\ F(i, s, (\int_I \int_S h_m(i)(\cdot) df(i)d\lambda)_{m=1}^\infty)$$

Moreover, the product measure $\prod_{m=1}^\infty \hat{\mu}_m$ preserves players' independent randomizations.

Proof: The equality in the statement follows from proposition 1 and Fubini's theorem. The claim that $X_{m=1}^\infty \hat{\mu}_m$ preserves independence of the continuum of randomizations can be shown by establishing independence for any finite subset of randomizations. Let J be any finite subset of I and (Z_n) a sequence of correspondences with domain I such that $Z_m(i) \in \mathbf{B}(h_m(i)(S))$ and $Z_m(i) = h_m(i)(S)$ for $i \notin J$. Let $T_m = \{x \in Y_m : x(i) \in Z_m(i)\}$. Then $\hat{\mu}_m(T_m) = \prod_{i \in J \cap \text{supp } h_m} f_m(i)(Z_m(i))$ and $(X_{m=1}^\infty \hat{\mu}_m)(X_{m=1}^\infty T_m) = \prod_{m=1}^\infty \prod_{i \in J \cap \text{supp } h_m} f_m(i)(Z_m(i))$.

Notice that the value of $\hat{\mu}_m(T_m)$ depends upon the choice of an element in the class of measures induced by an equivalence class f_m . We have thus shown that independence holds for each representative element of the class of measures and we know that the choice of this particular element does not affect the computation of the expected payoff, due to assumption (A1). ■

5.5 Continuity of the Expected Payoffs

Under the above results a concrete game with payoffs on S^I satisfying assumption (A1) will generate properly constructed expected payoffs on $M(S)^I$ that are weak* continuous as in Ali-Khan (1986) - Pascoa (1993a) existence and purification results.

Corollary 1 *Under assumption (A1) the expected payoff of any player i , $E(i, \dots) : S \times M(S)^I \rightarrow \mathbb{R}$, defined above, is weak* continuous on $M(S)^I$, which is a subset of $L^1(I, C(S))^*$.*

This follows from Proposition 2 and the definition of the weak* topology on $L^1(I, C(S))^*$ (see section 3 above).

6 Equilibria and Purification

Existence of mixed strategies equilibria for the concrete nonatomic game described in section 2, can be easily established now. We say that $f \in M(S)^I$ is a mixed strategies equilibrium profile if $\int_S E(i, \cdot, f) df(i) \geq \int_S E(i, \cdot, \rho) d\rho$, for any $\rho \in M(S)$, and for a.e.i.

Corollary 2 *Under assumption (A1), there exists a mixed strategies equilibrium for the concrete nonatomic game (as defined in section 2).*

This is an immediate consequence of the previous corollary and the results in Ali-Khan (1986) (see also Pascoa (1993a)).

Next we will show that by strengthening assumption (A1) to equicontinuity we can claim that for any mixed strategies equilibrium profile f of a nonatomic concrete game and any $\varepsilon > 0$, there is, appropriately close to f , an ε -equilibrium profile in pure strategies.

Assumption (A2): The family of players' payoff functionals $v(i, \cdot, \cdot)_{i \in I}$ of the concrete nonatomic game is equicontinuous on $S \times S^I$, with respect to the same topology as in assumption (A1).

Proposition 3 *Under assumption (A2), given any mixed strategies equilibrium profile $f \in M(S)^I$ and any $\varepsilon > 0$, there exists a pure strategies ε -equilibrium profile $g \in S^I$ such that $|v(i, g(i), g) - \int_S E(i, \cdot, f) df(i)| < \varepsilon/2$, for a.e.i.*

This result follows from corollary 2 and Pascoa's (1993a) results. By the argument in the proof of Proposition 3 in Pascoa (1993a), $g(i)$ is a pure strategy best response profile to f , for a.e.i, and g is appropriately close, given that pure strategies best response profiles are dense in the set of mixed strategies best response profiles.

7 On Recoverability of Payoffs

Suppose that we had started by defining payoffs on measure-valued profiles, as Schmeidler (1973) did, and that we perform the following experiment: (i) consider first the payoffs associated with profiles mapping into degenerate measures (i.e., pure strategies profiles) and then (ii) compute the expectations according to the method presented in the previous sections. Do we recover Schmeidler's original payoff on $M(S)^I$ as a result of this continuum of independent randomizations?

Pascoa (1993a) presented a counter example where S is defined by $S = \{-1, 0, 1\}$ and Schmeidler's payoff associated with $f \in M(S)$ is given by $-s \sup_{(a,b)} \{ \int_a^b f \, sd\lambda / (b-a) \}$. It was shown that the Bernoulli process $f(i)(-1) = f(i)(1) = 1/2$, for all i , is such that Kolmogorov's expectation of $\sup \{ \int_a^b g(i)d\lambda / (b-a) \}$ is strictly positive whereas Schmeidler's expected payoff is by definition zero. This counter-example showed also that an equilibrium profile in Schmeidler's sense (as the above Bernoulli process) might not be a mixed strategies equilibrium (since players respond to f by choosing $s = -1$).

This counter-example is a consequence of the failure of the strong law of large numbers for a continuum of independent randomizations. In fact, by Feldman and Gilles (1985) argument, for almost every realization g , $\int_a^b g(i)d\lambda / (b-a)$ differs from the theoretical Bernoulli mean zero and is actually positive for some interval (a, b) .

However, the payoff functional considered in this experiment does not satisfy the continuity condition that Schmeidler (1973) assumed to prove existence of equilibria, which was weak continuity in $L^1 \times L^1$ (given that $M(S)$ is the two dimensional simplex in this example). We will show next that the same assumptions on continuity of payoffs on $M(S)^I$ that guarantee existence of equilibria in abstract games over measure-valued profiles will ensure also the recoverability of these functionals as expected payoffs computed with respect to distributions on pure-strategies profiles that preserve the independence of players' randomizations.

Proposition 2 can also be used to establish the condition under which an

abstract game under measure-valued profiles (such as the game in Schmeidler (1973) is a proper representation of the mixed strategy plays of an underlying concrete game over S^I :

Corollary 3 *Given a nonatomic abstract game, over measure-valued profiles, assume players have a common finite-dimensional compact strategies set and payoff functionals $G(i, \dots) : S \times M(S)^I \rightarrow \mathbb{R}$, which are assumed to be continuous, with respect to the weak* topology on $M(S)^I$, and Lebesgue measurable in the players' indices. Then, the underlying concrete game over S^I , with payoff functionals given by $v(i, s, g) = G(i, s, f)$ where $f(i) = \delta_{g(i)}$, is such that its expected payoffs computed using Judd's extensions of Kolmogorov measures on realizations (as in Proposition 2), coincide with the original payoff functionals G .*

8 Games with Uncertainty

To simplify suppose that there are two players with preferences defined on the set of pure strategy profiles, which are now interpreted as actions from a continuum of states T to a compact subset S contained in $\mathbb{R}^n$. Let us consider Savage's (1954) case. Player i preferences on $S^T \times S^T$ are assumed to satisfy the sure thing principle and state independence. Then, these preferences can be represented by an additively separable utility functional U^i given by $U^i(a_1, a_2) = \int_T u^i(a_1(t), a_2(t)) d\lambda^i$, where the bounded function u^i and the probability measure $\lambda^i \in M(T^2)$ are uniquely determined (up to positive affine transformations of u^i).

Now, the approach described in the previous sections can be used to randomize this game. Given the mixed strategy profiles $f_i \in M(S)^T$ of each player, we can find Judd's extensions of Kolmogorov measures on the pure strategy realization spaces, say $\hat{\mu}_1$ and $\hat{\mu}_2$, so that the strong law of large numbers holds (more precisely, so that the sample mean over the entire state set T coincides almost surely with the corresponding theoretical mean). Then, player i 's expected payoff is given by

$$\int_{S^T} \int_{S^T} u^i d\mu_1 d\hat{\mu}_2 =$$

$$\int_T \int_S u^i(s_1, s_2) df_1(t_1) df_2(t_2) d\lambda^i =$$

$$\int_S u^i(s_1, s_2) d(\oint_T f d\lambda^i)$$

where $\oint_T f d\lambda^i \in M(S)^2$ is the Gelfand integral of f (recall that the Gelfand integral of f is an element y of $M(S)^2$ such that $\int_{T_1} \int_{S_2} x df d\lambda^i = \int_{S_2} x dy$ for any $x \in C(S^2)$).

That is, in the randomized game, expected payoffs are the expectations, using Savage's probabilities, of Von-Neumann-Morgenstern functionals on mixed strategies, where the scaling function u^i is just Savage's. In other words, the behavioral strategies are perfect substitutes for the randomization of the game using Kolmogorov's construction of a continuum of independent random variables.

The game over behavioral strategies has an equilibrium and an approximate equilibrium in degenerate behavioral strategies, as shown by Milgrom-Weber (1985). These outcomes can be properly interpreted as a mixed strategies equilibrium and an ϵ -equilibrium in pure strategies which is appropriately close to the former. In other words, the game over behavioral strategies with Von-Neumann-Morgenstern functionals passes a recoverability test analogous to the one described in the previous sections.

9 References

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