DETERMINACY OF EQUILIBRIA IN NONSMOOTH ECONOMIES

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Abstract

Concavifiable preferences are representable by a function which is twice differentiable almost everywhere, by Alexandroff's [1] theorem. We show that if the bordered hessian determinant of a concave utility representation vanishes only a null set of bundles, then demand is countably rectifiable, that is, except for a null set of bundles, it is a countable union of C^1 manifolds. Given this property, equilibrium prices will be locally unique, for almost every endowment. We given an example of an economy satisfying these conditions but not the Katzner [6] - Debreu [2], [3] smoothness conditions.

1 Introduction

Smoothness of consumer demand, as required in Debreu's [2] seminal work on determinacy of equilibria, is a strong condition. In general, even for C^2 preferences, demand will be smooth only on a generic set of prices and incomes, which is the preimage of the set of bundles where indifference surfaces have nonzero curvature (see Debreu [3],[4] and Katzner[6]). Differentiability of demand in almost every price and income is the most that can be expected in general and Rader [9] showed that this property together with Lusin's condition on demand (that the function maps null sets into null sets) are enough to guarantee determinacy of equilibrium prices, for almost every endowment. Rader [9] [10] showed that the first requirement is satisfied by concavifiable preferences with income - Lipschitzian demand or by D^2 utility, whereas the second requirement holds for analytic utility.

Contemporary is the geometric measure theory approach of Kleinberg [7], who established an alternative sufficient condition which is the approximate differentiability of demand, outside of a countable set. This condition was shown to hold for C^1 utility functions such that the normalized gradient admits, except on a countable set, an approximate derivative bounded from below.

Our paper examines the contribution of concavifiability of preferences to the issue of determinacy of equilibria, using also a geometric measure theory approach. We dispense with Kleinberg's [7] assumption that utility is of class C^1 and appeal to a theorem by Alexandroff [1] establishing that concave functions are twice differentiable almost everywhere. We assume that the bordered hessian matrix of the concave utility representation is nonsingular almost averywhere and show that, except for a null set of bundles, demand is a countable union of C^1 manifolds. This rectifiability property is weaker than Rader's [9], since it does not imply differentiablity almost everywhere, but it suffices for determinacy of equilibria, in almost every endowment.

where $\epsilon(s)/s$ converges to zero as $s \to 0$, uniformly on the directions y and also independently of the choice of the extension u_i. Moreover, the matrix $H(x) = [\Delta u_1(x)' \ldots \Delta u_n(x)']$ is uniquely determined, independent of the choice of the extension and is a symmetric negative semidefinitive matrix.

Actually, the matrix $H(x)$ exists only at points where u is at least once differentiable (see Alexandroff [1] pp. 5 and 6) but not necessarily at any such point. Furthermore, any concave function admits at almost every point x a second-order Taylor expansion and the matrix of the quadratic form in this expansion is the matrix $H(x)$.

2.2. Rectifiability

Recall that a function $f : A \subseteq \Re^n \to \Re^m$ is said to be (locally) Lipschitzian if for any $a \in A$ there exist an open ball $B_\epsilon(a)$ and a constant M such that $\| f(x) - f(y) \| \leq M \| x - y \|$, for any $x, y \in B_\epsilon(a)$. Denote by L^n the Lebesgue measure in $\Re^n$. We say that a set $E \subseteq \Re^n$ is l - rectifiable if $L^n(E) < \infty$ and L^n - almost all of E is contained in the union of the images of countably many locally Lipschitzian functions from $\Re^l$ to $\Re^n$. A set $E \subseteq \Re^n$ is said to be countably l - rectifiable if L^n - almost all of E is countained in the union of countably many rectifiable sets. From the point of view of geometric measure theory, rectifiable sets behave like C^1 manifolds. In fact, in the definition of a rectifiable set E one can take the Lipschitzian functions to be C^1 diffomorphisms on compact domains with disjoint images whose union coincides with E almost everywhere (see Federer [5] 3.2. 18 and 3.2.29).

A measurable subset of $\Re^n$ is said to have density zero at $a \in \Re^n$ if $\forall_{\epsilon > 0}, \exists \bar{\delta} > 0$ such that $L^n(B_\delta(a) \bigcap A) < \epsilon L^n(B_\delta(a))$ for any $\delta < \bar{\delta}$. Now, consider an extended real valued measurable function f defined on a measurable set $A \subseteq \Re_n$. The approximate lim sup of f at $a \in \Re_n$ is defined as *ap lim* $sup_{x \to a} f(x) = inf\ \{t \in \Re : \{x \in A : f(x) > t\}$ has density zero at a$\}$. Similarly, ap $lim\ inf\ f(x)_{x \to a} = sup\ \{t \in \Re : \{x \in A : f(x) < t\}$ has density zero at a$\}$.

2 Rectifiability of Demand

2.1. Preliminarities

The preference relation π is a strictly convex monotone continuous complete preording on $\Re^n_+$. The relation π is said to be locally concafiable if for any compact convex set $K \subseteq \Re^n_+$ there is a concave utility representation u_K for π. Local concavifiability was already implied by Katzner's [6] conditions, as Mas-Colell [8] showed.

For technical reasons we want to think of all concave functions as defined throughout $\Re^n$ and taking the value $-\infty$ outside of the effective domain. The extended real-valued function obtained this way is called a proper concave function. Denote by *dom u* the effective domain of the function u and by int(dom u) the respective interior. A proper concave function u on $\Re^n$ is differentiable on a dense subset D of int(dom u) and the complement of D in int(dom u) is a set of measure zero; moreover, u is actually continuously differentiable on D (see Rockafeller [11] 25.5). Alexandroff [1] established that a proper concave function u on $\Re^n$ is also twice-differentiable almost everywhere on int(dom u). Since the domain D of the first derivative may have an empty interior, we should be more precide and recall the exact statement of Alexandroff's theorem.

Let $u(\cdot;\bar{x}_{-1}) : \Re \to \Re \cup \{-\infty\}$ be the proper concave function given by $u(x_i;\bar{x}_{-1}) = u(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_n)$. The right derivative $u'_+(\cdot;\bar{x}_{-1})$ and the left derivative $u'_-(\cdot;\bar{x}_{-i})$ are well defined throughout the effective domain of u and $u'_-(\cdot;\bar{x}_{-1}) \leq u'_+(\cdot;\bar{x}_{-1})$. Alexandroff [1] defined an extended partial derivative u_i of u as any function satisfying the inequality $u'_-(x_i;\bar{x}_{-i}) \leq u_i(x) \leq u^+(x_i;\bar{x}_{-i})$ where $x = (x;\bar{x}_{-i})$. Note that $u_i(x)$ coincides with the partial derivative $\frac{\partial u}{\partial x_i}(x)$ when it exists.

Proposition 1 (Alexandroff):*The extend partial derivative u_i is differentiable almost everywhere on int (dom u) and at any point x of differentiability, for any direction $y \in \Re^n$, we have $| u_i(x+sy) - u(x) - \Delta u_i(x) \cdot sy | \leq \epsilon(s)$,* From the definition, if the domain A has density zero at a then *ap lim* $sup_{x\to a} f(x) = -\infty$. It is immediate to see that if *lim sup* exists then it is equal to the approximate *lim sup*.

Recall that a function f from $A \subseteq \Re^n$ to $\Re^m$ is said to be pontwise Lipschitzian at $a \in A$ if $lim\, sup_{x\to a} \parallel f(x) - f(a) \parallel / \parallel x - a \parallel < \infty$. Similary, a function $f : A \subseteq \Re^n \to \Re^m$ is said to be approximately pointwise Lipschitzian at $a \in A$ if $ap\, lim\, sup_{x\to a} \parallel f(x) - f(a) \parallel / \parallel x - a \parallel < \infty$. By lemma 3.1.8 in Federer [5], if a function $f : A \subseteq \Re^n \to \Re^m$ is approximately pointwise Lipschitzian then A is a countable union of measurable sets such that the restriction of f to each set is Lipschitzian. Then, the range of f is a countably rectifiable set. An immediate consequence is that f satisfies Lusin's condition.

Now we will establish a result on the rectifiability of the inverse of a differentiable function.

Proposition 2 *Let f and φ be functions defined on open subsets of $\Re^n$ and taking values in $\Re^n$; suppose that the restriction of f to a measurable subset D admits the restriction of φ to $f(D)$ as an inverse function. Let a be a limit point of D and $b = f(a)$. If f is differentiable at a and $f'(a)$ is nonsingular, then* $ap\, lim\, sup_{s\to 0, \varphi|_{f(D)}} \frac{\|\varphi(b+s)-\varphi(b)\|}{\|s\|} \leq \parallel (f'(a))^{-1} \parallel$.

Moreover, if D is a full measure subset of $dom\, f$ then the above inequality holds as an equality.

Proof: Inverting the incremental ratio we have, $ap\, lim\, sup_{s\to 0} \parallel \varphi(b+s) - \varphi(b) \parallel / \parallel s \parallel = inf\, \{t \in \Re : \{x \in (D-a) : \parallel f(a+x) - f(a) \parallel / \parallel x \parallel < 1/t\}$ has density zero at a$\} = 1/sup\, \{t \in \Re : \{x \in (D-a) : \parallel f(a+x) - f(a) \parallel / \parallel x \parallel < t\}$ has density zero at a $\} = 1/apliminf_{x\to 0, \varphi|f(D)} \parallel f(a+x) - f(a) \parallel / \parallel x \parallel \leq 1/liminf_{x\to 0} \parallel f(a+x) - f(a) \parallel / \parallel x \parallel$ where the last inequality follows from the following set inclusion: $\{t \in \Re : \{x \in (D-a) : \parallel f(a+x) - f(a) \parallel / \parallel x \parallel < t\}$ has density zero at a$\} \supseteq \{t \in \Re : \{x \in (dom\, f - a) : \parallel f(a+x) - f(a) \parallel$

$/ \parallel x \parallel < t\}$ has density zero at a}.

If D has full measure in *dom* f then the two sets in the above set inclusion are equal and the above inequality holds as an equality. To finish the proof recall that for a mapping h with nonsingular derivative h' at a point a, the lim inf of the incremental ratio of h at a is equal to $\parallel (h'(a))^{-1} \parallel^{-1}$ (see Federer [5] p. 209). ■

2.3. The Result

Theorem 1 *If a continuous, monotone, strictly convex preference relation* π *on* $\Re^n_+$ *is locally concavifiable and for each utility representation the bordered hessian matrix is singular only in a set of measure zero, then the demand function* $d :$ *int* $\Delta^{n-1} \times \Re_{++} \rightarrow \Re^n_+$ *is approximately pointwise Lipschitzian for almost every value.*

In other words, demand is countably rectifiable and, except for a null set of bundles, it is the union of countably many pairwise disjoint compact C^1 *submanifolds.*

Proof: Consider a countable cover of $\Re^n_+$ by closed cubes. For each closed cube $K \subseteq \Re^n_+, \pi$ is representable by a concave utility function u with no critical point, by monotonicity of π. For simplicity, we assume that u is a proper concave function with effective domain K. We will consider only the optimal interior solutions in K, since the boundary of K is a n-null set.

Let N be the null subset of int K where u is not twice-differentiable. Let $\delta(x)$ be the bordered hessian determinant of u at $x \in$ *int* $K \backslash N$, (that is the determinant of the $(n+1) \cdot (n+1)$ matrix which is formed by bordering $D^2u(x)$ with $[\nabla u(x) 0]$ as a last row and its transpose as a last column). Let CP be the null subset of int $K \backslash N$ where the bordered hessian determinant δ of u vanishes and let RP be the complement of CP on int $K \backslash N$.

Recall Debreu's [3] decomposition of the demand function and adapt it to the case where *dom* Du is just a full measure subset of int K. Let $g : x \to \nabla u(x)/ \parallel \nabla u(x) \parallel$ and $\hat{g}(x)$ be the vector of the first $n-1$ components of $g(x)$. The restriction $\tilde{d}$ of the demand function d to $d^{-1}(dom\ Du)$ is the inverse of the map $x \to (p, y) \equiv (\hat{g}(x), g(x) \cdot x)$, which is the composition of $\alpha : x \to (\hat{g}(x), u(x))$ and $\beta : (\hat{p}, v) \to (\hat{p}, m(v))$, where $m(v) = min\{p \cdot z : u(z) \geq v\}$.

Let $\tilde{g}$ and $\tilde{\alpha}$ be any extensions of g and α, respectively, obtained using Alexandroff's extended partial derivatives. The Jacobian determinant of $\tilde{\alpha}$ at $x \in$ *int* $K \backslash N$ is $J\tilde{\alpha}(x) = -\delta(x)g^n(x)/ \parallel \nabla u(x) \parallel^n$, where $g^n(x)$ is the n^{th} component of $g(x)$. Then $J(\alpha)(x) \neq 0$ on RP. The function m has the same Jacobian determinant as the inverse of the function v given by $s \to max\{u(z) : p \cdot z \leq s\}$. Now $\nabla v(s) = \parallel \nabla u(x) \parallel$, for $x = d(p, s)$. Therefore $J(\beta o \tilde{\alpha}) = -\delta(x)u_n(x)/ \parallel \nabla u(x) \parallel^{n+2} \neq 0$ if and only if $x \in RP$.

Now, by Proposition 1 above, for $x^0 \in RP$, let $(p^0, y^0) = (\beta o \alpha)(x^0)$ and we have $ap\, lim\, sup_{(p,y) \to (p^0,y^0)} \parallel \tilde{d}(p, y) - \tilde{d}(p^0, y^0) \parallel / \parallel (p, y) - (p^0, y^0) \parallel < \infty$, where $\tilde{d} = d \mid_{d^{-1}(dom\ u)}$. Since dom Du has full measure int K we also have $ap\ lim\ sup_{(p,y) \to (p^0,y^0)} \parallel \tilde{d}(p, y) - \tilde{d}(p^0, y^0) \parallel / \parallel (p, y) - (p^0, y^0) \parallel < \infty$, for $(p^0, y^0) = (\beta o \alpha)(x^0), x^0 \in RP$. We have proven that the demand function is approximately pointwise Lipschitzian on the inverse image of RP and RP is a full measure subset on int K. To complete the proof, notice that the countable union of the null complements of each set RP in the respective cube K is a null subset of $\Re^n_+$. ∎

3 Local Uniqueness

Consider an exchange economy with m consumers and n goods. Let d_j : *int* $\Delta^{n-1} \times \Re_{++} \to \Re^n_+$ be the demand function and $w_j \in \Re^n_{++}$ be the endowment of the j^{th} consumer $j = 1, \ldots, m$. We keep the demand functions fixed and parameterize an economy by an endowment vector $w = (w_1, \ldots, w_m)$. Given $w \in \Re^{nm}_{++}$, an element $p \in \Delta^{n-1}$ is an equilibrium price vector of the economy w if $\sum_{j=1}^{m} d_j(p, p \cdot w) = \sum_{j=1}^{m} w_j$. Let $E(w)$ be the set of equilibrium price vectors of the economy w. We say that equilibrium prices are locally unique if all elements in $E(w)$ are isolated points of this set.

Theorem 2 *If the demand functions $d_1, \ldots, d_m$ of all consumers are approximately pointwise Lipschitzian for almost every demand bundle, then the equilibrium prices are locally unique, for almost every endowment vector.*

Proof: Let $U = int\ \Delta^{n-1} \times \Re_{++} \times \Re_{++}^{n(m+1)}$ and define the function $F : U \rightarrow \Re^{nm}$ as in Debreu [2] by $e = (p, y_1, w_2, \ldots, w_m) \rightarrow F(e) = (F_1(e), \ldots, F_m(e))$, where $F_1(e) = d_1(p, y_1) + \sum_{i=2}^{m} d_i(p, p \cdot w) - \sum_{i=2}^{m} w_i$ and $F_j(e) = w_j$ for $j \neq 1$. For $e \in U, p \cdot F_1(e) = y_1$. Also p is an equilibrium price vector iff $(p, y_1, w_2, \ldots, w_m) \in F^{-1}(w)$. We want to show that for *a.e.*w, every e in $F^{-1}(w)$ is isolated. Let us start by claiming that the range of F is a m.n-countably rectifiable set (through Lipschitzian restrictions of F).

Let M_j be the full measure subset of $\Re_+^n$ which is the countable union $U_{k=1}^{\infty} M_{jk}$ of disjoints images of C^1 diffeomorphic restrictions of d_j. Define $\tilde{M}_1 = \{w \in \Re^n \times \Re_+^{n(m-1)} : w = F(e), e = (p, y_1, w_2, \ldots, w_m)$ and $d_1(p, y_1) \in M_1\}$ and $\tilde{M}_j = \{w \in \Re^n \times \Re_+^{n(m-1)} : w = F(e), e = (p, y_1, w_2, \ldots, w_m)$ and $d_j(p, p \cdot w_j) \in M_j\}, j \neq 1$. We want to show that $U_{k=1}^{\infty} \tilde{M}_{jk}$ is of full measure in the range of F.

Let $\rho : \Re_+^{nm} \rightarrow \Re_+^{nm}$ be defined by $\rho_1(x_1, \ldots, x_m) = x_1 + \ldots + x_m$ and $\rho_j(x_1, \ldots, x_m) = x_j$ $(j = 1, \ldots, m)$. Now ρ is linear and invertible and therefore maps full measure sets into full measure sets. Notice that $\rho(S)$ is of full measure in $\Re_+^{nm}$ iff $\rho_1(S)$ is full measure in $\Re_+^n$ for $S \subseteq \Re_+^{nm}$. The set $M_1 \times \ldots \times M_m$ is full measure in $\Re_+^n$ and by the above argument $\rho_1(M_1 \times \ldots \times M_m)$ is of full measure in $\Re_+^n$. Now $F_1(e) = \rho_1(d(e)) - e_{-1} \cdot 1$. Let ψ be defined on $\Re_+^n \times \Re_+^{n(m-1)}$ by $\psi_1(y_1, y_2) = y_1 - y_2 \cdot 1$ and $\psi_2(y_1, y_2) = y_2$. Clearly, ψ maps null sets into null sets. Let $A = \rho_1(d(U_{j=1}^{m} d_j^{-1}(M_j))) \times \Re_+^{n(m+1)}$ and notice that A is of full measure in $\Re_+^{nm}$, since $A = \rho_j(M_1 \times \ldots \times M_m) \times \Re_+^{n(m-1)}$. Then $\psi(\Re_+^{nm} \backslash A)$ is of zero measure in $\Re^n \times \Re_+^{n(m-1)}$. Therefore, $U_{j=1}^{m} \tilde{M}_j$ is of full measure in the range of F. That is, the range of F is n.m. - countably rectifiable through F.

To complete the proof of the theorem, notice that the set $F(U)$ can regarded as the union of countably many disjoint m.n. - rectifiable sets R_K by taking appropriate set differences). Now, one can take the countably many Lipschitzian functions in the definition of each rectifiable set R_K to be C^1 diffeomorphisms D_{ik} on compact domains with disjoint images, whose union coincides with the rectifiable set almost everywhere. Let $N = F(U) \backslash U_{K=1}^{\infty} U_{i=1}^{\infty}$ range D_{ik}, which is a m.n. - null set.

Now any endowment vector w in $F(U)\backslash N$ is such that $E(w)$ has only isolated points. In fact, by the C^1 inverse function theorem, any element $w \in F(U)\backslash N$ and any element $e \in F^{-1}(w)$ have neighborhoods O_w and 0_e, respectively, that are homeomorphic under the restriction to 0_e of some C^1 map D_{ik}. Moreover, for each $w \in F(U)\backslash N$, this map D_{ik} is uniquely determined, because the rectifiabe sets are disjoint and the D_{ik} maps have disjoint images. Then, for any $w \in F(U)\backslash N$, any price vector p in $E(w)$ has a neighborhood 0_p where there are no other elements of $E(w)$ (this neighborhood 0_p is induced by O_e through the one-to-one correspondence between $E(w)$ and $F^{-1}(w)$). ■

4 An Example

Now, we will give an example of a pure exchange economy where utility functions are concave, but not differentiable, and we still have finiteness of equilibria, for almost every endowment, although the set of endowments generating infinite equilibria is dense. Consider an economy with two consumers with the same concave utility function on $\Re_+^2$, which is constructed so that an indifference curve is differentiable only at irrational points.

For each $q_n \in Q$, let f_n be characteristic function of the set $\{y \in \Re_+ : y \geq$

$q_n\}$. Notice that f_n is nondecreasing and jumps at q_n. Let $f = \sum_{n=1}^{\infty} \frac{1}{2^n} f_{q_n}$; now f is nondecreasing, at any $q_n \in Q$ it has a jump of magnitude $\frac{1}{2^n}$ and it is differentiable except at the rationals, with zero derivative.

Now integrate f to obtain a convex increasing function. Since f has a countable set of discountinuities it is Riemann integrable. Let $g : \Re \to \Re$ be such that $g(x) = \int_0^x f(t)\, dt$. The derivative g' exists at every continuity point of f, that is, at any irrational x. Furthermore, at any rational q_n we have $g'_-(q_n) = f(q_n^-)$ and $g'_+(q_n) = f(q_n)$. Notice that g is strictly convex: for $\lambda \in (0,1)$ we have $g(\lambda x + (1-\lambda)y) = \int_0^{\lambda x} f(t)dt \;+\; \int_{\lambda x}^{\lambda x+(1-\lambda)y} f(t)dt \;<\; \int_0^{\lambda x} f(t)dt \;+\; \int_0^{(1-\lambda)y} f(t)dt = \lambda g(x) \;+\; (1-\lambda)g(y)$.

Now let $h : \Re_{++} \to \Re_{++}$ be defined by $h(x) = g(1/x)$. The function h is decreasing, convex, differentiable only at irrational points: it will be used as an indifference curve. Let the utility function be $u : \Re^2_{++} \to \Re$ given by $u(x_1, x_2) = x_2 - h(x_1)$.

The preferences in this example satisfy the assumptions of theorems 1 and 2 but not the smoothness conditions of Katzner [6] or Debreu [3],[4]. In fact, the bordered hessian determinant of u at $(x_1, x_2) \in (\Re \backslash Q) \times \Re$ is equal to $h''(x_1) = \frac{1}{x_1^2} g''(1/x_1) - \frac{2}{x_1^3}\, g'(1/x_1)$. Here g' is the derivative of Alexandroff's extend partial derivative $g'_+ = f$ and therefore g'' vanishes identically on its domain $\Re \backslash Q$. Then $h''(x_1) = -\frac{2}{x_1^3} f(1/x_1) > 0$, for any $x_1 \in \Re \backslash Q$ and the bordered hessian determinant is nonzero on the full measure set $(\Re \backslash Q) \times \Re$. Let us examine the demand functions and the equilibria of this economy. If $(x_1, x_2) \in (\Re_{++} \backslash Q) \times \Re_{++}$ then $-h'(x_1) = p_1/p_2$, implying $x_1 = (h')^{-1}(-p_1/p_2)$ except at prices associated with points where h is not differentiable and these prices are elements of the subdifferentials $\partial h(q_n) = \{p_1/p_2 \in \Re_{++} : -h'_+(q_n) \leq p_1/p_2 \leq -h'_-(q_n)\}$ and generate demand $x_2 = (y - p_1 q_n)/p_2$. We have determined completely the form of the demand function.

It is easy to see that when the endowment vector (w_1, w_2) is such that w_1

is irrational, then the equilibrium price ratio is equal to $-h'(w_1/2)$. For w_1 rational, the set of equilibrium price ratios is the interval $[-h'_+(w_1/2), -h'_-(w_1/2)]$. That is, almost every endowment generates finite equilibrium prices but the set of endowments giving rise to infinite equilibrium prices is dense.

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