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GREEDY SOLVABLE KNAPSACKS

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ABSTRACT

The family $\mathbb{K}$ of the feasible solutions for a 0-1 Knapsack with positive coefficients, $\mathbb{K}=\{I\subset\mathcal{N}:\Sigma_{i\in I}a_i\leq b\}$, is an independence system over $\mathcal{N}=\{1,\ldots,n\}$. In some cases, for instance when all the a_i have the same value, this independence system is a matroid over $\mathcal{N}$. We will say then that the knapsack is greedy solvable.

In this paper we study the conditions for a knapsack to be greedy solvable. We present necessary and sufficient conditions, verifiable in polynomial time, for $\mathbb{K}$ to be a member of a finite family of matroids over $\mathcal{N}$.

KEYWORDS: 0-1 Knapsacks, matroids.

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1 INTRODUCTION

Let $\mathcal{X}$ denote the set of feasible solutions for a *o-1 Knapsack* problem with positive coefficients, i.e

$$\mathcal{X} = \{ x\in\{0,1\}^n : \sum_{i=1}^{n} a_i x_i \le b \} ,$$

where $b\in\mathbb{N}$ and $a_i\in\mathbb{N}$ for all i.

We will use the word *knapsack* either to denote $\mathcal{X}$ or to refer the constraint $\sum_{i=1}^{n} a_i x_i \le b$ but the difference will be clear from the context.

Now let $\mathcal{N} = \{1,\dots,n\}$. To each feasible solution $x^j\in \mathcal{X}$ we can uniquely associate the subset I^j of $\mathcal{N}$ for which x^j is the corresponding incidence vector, i.e

$$I^j = \{ i\in\mathcal{N} : x_i^j = 1 \}.$$

Let $\mathcal{F}$ be the family of all these subsets, i.e $\mathcal{F} = \{I^1,I^2,\dots,I^s\}$, $s=|\mathcal{X}|$. Obviously $\mathcal{X}$ and $\mathcal{F}$ are in a one to one correspondence so we will use $\mathcal{X}$ and $\mathcal{F}$ interchangeably.

It is well known that

Proposition 1.1: $\mathcal{F}$ is an independence system over $\mathcal{N}$. ∎

It is also well known that the problem of finding a maximum weight set on an independence system can be solved by the greedy algorithm *iff* the independence system is a matroid over $\mathcal{N}$. We recall now two equivalent definitions of a matroid over $\mathcal{N}$, see White (1986).

Definition 1.2: The independence system $(\mathcal{N},\mathcal{F})$ is a *matroid iff* for any $A\subset\mathcal{N}$, all the maximal (for inclusion) independent sets of A have the same cardinality.

Definition 1.3: The independence system $(\mathcal{N},\mathcal{F})$ is a *matroid iff* for any I_p and $I_{p+1} \in \mathcal{F}$ such that $|I_p|= p$ and $|I_{p+1}|= p+1$, there is an $e \in I_{p+1}-I_p$ for which $I_p + \{e\} \in \mathcal{F}$.

In this paper we shall discuss the problem of finding whether the independence system defined by a knapsack is a matroid over $\mathcal{N}$. When this happens we shall say that the knapsack is *greedy solvable*.

The two following examples illustrate the case where a knapsack is greedy solvable, example 1.4, and where a knapsack is not greedy solvable, example 1.5.

Example 1.4: Let $\mathcal{X} = \{ x \in \{0,1\}^3 : x_1 + 2x_2 + 3x_3 \leq 4 \}$. Then

$$\mathcal{F} = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\} \}$$

and, of course, $\mathcal{F}$ is a matroid over $\mathcal{N}=\{1,2,3\}$.

Example 1.5: Let $\mathcal{X} = \{ x \in \{0,1\}^3 : x_1 + 2x_2 + 3x_3 \leq 3 \}$ Then

$$\mathcal{F} = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\} \}$$

and $\mathcal{F}$ is not a matroid over $\mathcal{N}=\{1,2,3\}$ because $\{3\}$ and $\{1,2\}$ are maximal independent subsets of $\mathcal{N}$ with different cardinalities.

2- CARDINALITY EQUIVALENT KNAPSACKS

We start this section with the following remark: the independence system associated with a cardinality knapsack $\mathbb{C}_k=\{x\in\{0,1\}^n:\Sigma_{i=1}^{n}x_i\le k\}$ exhibits a special structure: the independent sets are the subsets of N with a cardinality not greater than k. Denote by $\mathcal{F}_{o,k}$ the corresponding family of independent sets.

It is easily proved, using definition 1.2, that $\mathbb{C}_k$ is greedy solvable. In this section we try to find other greedy solvable knapsacks that "look like" $\mathbb{C}_k$. We start by giving a definition:

Definition 2.1: Two knapsacks are called *$\mathcal{F}$-equivalent* if they have the same family $\mathcal{F}$ of independent sets. If a knapsack is equivalent to $\mathbb{C}_k$ we will call it $\mathcal{F}_{o,k}$-equivalent.

Example 2.2: The two knapsacks $\mathbb{C}_2=\{x\in\{0,1\}^3:x_1+x_2+x_3\le 2\}$ and $\mathbb{K}=\{x\in\{0,1\}^3:100x_1+101x_2+102x_3\le 204\}$ have the same family of independent sets $\mathcal{F}_{o,2}=\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$ so $\mathbb{K}$ is $\mathcal{F}_{o,2}$-equivalent.

Obviously when a knapsack is $\mathcal{F}_{o,k}$-equivalent it is greedy solvable. Our aim now is to characterize the knapsack instances where this happens.

Let $\mathbb{K}=\{x\in\{0,1\}^n:\Sigma_{i=1}^{n}a_ix_i\le b\}$ be a knapsack instance and assume, with no loss of generality, that $0<a_1\le a_2\le\ldots\le a_n\le b$. We can now state and prove the following

Proposition 2.3: The knapsack $\mathbb{K}=\{x\in\{0,1\}^n:\Sigma_{i=1}^{n}a_ix_i\le b\}$ is $\mathcal{F}_{o,k}$-equivalent, and therefore greedy solvable, iff $b\in[\Sigma_{i=n-k+1}^{n}a_i,\Sigma_{i=1}^{k+1}a_i[$. ∎

Proof: A knapsack is $\mathcal{F}_{o,k}$-equivalent iff no $(k+1)$-set is independent and all the (k)-sets are independent.

Therefore the $(k+1)$-set with the smallest weight must be dependent so we must have $b<\Sigma_{i=1}^{k+1}a_i$. On the other hand all the (k)-sets must be independent so this must happen for the heaviest of those sets and then $b\ge\Sigma_{i=n-k+1}^{n}a_i$. □

<u>Proposition</u> <u>2.4</u>: The condition in proposition 2.3 can be checked in $O(n)$ operations. ■

<u>Proof</u>:

The proof is obvious. □

3. OTHER GREEDY SOLVABLE KNAPSACKS.

In the previous section we learned how to polynomialy identify greedy solvable knapsacks which are $\mathcal{F}_{o,k}$-equivalent. We now address the question of identifying other classes of greedy solvable knapsacks which are no longer $\mathcal{F}_{o,k}$-equivalent but still closely related to cardinality knapsacks.

We shall start by analyzing the case of independence systems where the family of independent sets $\mathcal{F}_{1,k}$ consists of all the sets with cardinality not greater than k and all the $(k+1)$-sets containing $1\in\mathcal{N}$, i.e. the element of $\mathcal{N}$ with the smallest weight a_1.

<u>Proposition 3.1</u>: Let $(\mathcal{N},\mathcal{F}_{1,k})$ be an independence system over $\mathcal{N}$ with a family of independent sets $\mathcal{F}_{1,k}$ such that the following properties hold:

(a) $\mathcal{F}_{1,k} \supset\mathcal{F}_{0,k}$

(b) Let $I\subset\mathcal{N}$ and $|I|=k+1$. Then $I\in\mathcal{F}_{1,k}$ iff $1\in I$

(c) Let $I\subset\mathcal{N}$ and $|I|>k+1$. Then $I\notin\mathcal{F}_{1,k}$.

Then $(\mathcal{N},\mathcal{F}_{1,k})$ is a matroid over $\mathcal{N}$. ■

<u>Proof</u>:

The proof will be given in the more general context of proposition 3.5.□

The fact that $(\mathcal{N},\mathcal{F}_{1,k})$ is a matroid over $\mathcal{N}$ raises the question of identifying, if possible polynomialy, all the $\mathcal{F}_{1,k}$-equivalent knapsacks. We shall start by looking at one example:

<u>Example 3.2</u>: Consider $\mathbb{K}=\{x\in\{0,1\}^4:4x_1+100x_2+101x_3+102x_4\leq 110\}$. One can easily see that the family of independent sets of this knapsack is $\mathcal{F}_{1,1}=\{\emptyset,\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\}\}$. Note that the heaviest set in $\mathcal{F}_{1,1}$, $\{1,4\}$, has weight 106 while the lightest set not in $\mathcal{F}_{1,1}$, $\{2,3\}$, has a weight 201 and the right hand side has a value of 110.

This example suggests that the knapsack $\mathbb{K}=\{x\in\{0,1\}^n:\Sigma^n_{i=1}a_ix_i\leq b\}$ will be $\mathcal{F}_{1,k}$-equivalent iff the two following conditions hold:

(1) The heaviest set in the knapsack will be such that $x_1=1$, $x_i=0$ for $i=2,\dots,(n-k)$ and $x_i=1$ for $i=,(n-k+1),\dots,n$.

(2) The lightest set not in the knapsack will be such that $x_1=0$, $x_i=1$ for $i=2,\dots,(k+2)$ and $x_i=0$ for $i=(k+3),\dots,n$.

Then the following result holds:

Proposition 3.3: The knapsack $\mathbb{K}=\{x\in\{0,1\}^n:\Sigma^n_{i=1}a_ix_i\leq b\}$ will be $\mathcal{F}_{1,k}$-equivalent, and therefore greedy solvable, iff $b\in[a_1+\Sigma^n_{i=n-k+1}a_i,\Sigma^{k+2}_{i=2}a_i[$ ∎

The following result can be easily proved.

Proposition 3.4: The condition in proposition 3.3 can be checked in $O(n)$ operations. ∎

We will now build, recursively, a finite chain of matroids over $\mathcal{N}$, with independence families $\mathcal{F}_{0,k}\subset\mathcal{F}_{1,k}\subset\mathcal{F}_{2,k}\subset\dots\subset\mathcal{F}_{i,k}\subset\dots$, such that checking if a knapsack is $\mathcal{F}_{i,k}$-equivalent can be done in polynomial time.

The family $\mathcal{F}_{2,k}$, for instance, will consist of all the subsets of $\mathcal{N}$ with a cardinality not greater than k and of the $(k+1)$-sets containing at least one of the two lightest elements in $\mathcal{N}$, i.e. containing $1\in\mathcal{N}$ or $2\in\mathcal{N}$.

The following proposition makes this idea more precise.

Proposition 3.5: Let $(\mathcal{N},\mathcal{F}_{i,k})$, $0<i<n-k+1$, be an independence system over $\mathcal{N}$ with a family of independent sets $\mathcal{F}_{i,k}$ such that the following properties hold:

(a) $\mathcal{F}_{i,k}\supset\mathcal{F}_{i-1,k}$

(b) Let $I\subset\mathcal{N}$ and $|I|=k+1$. Then $I\in\mathcal{F}_{i,k}$ iff $\{1,2,\dots,i\}\cap I\neq\emptyset$.

(c) Let $I\subset\mathcal{N}$ and $|I|>k+1$. Then $I\notin\mathcal{F}_{i,k}$.

Then $(\mathcal{N},\mathcal{F}_{i,k})$ is a matroid over $\mathcal{N}$. ∎

Proof:

We will use the second definition of matroid given in section 1. From the definition of $\mathcal{F}_{l,k}$ it is clear that we shall have to worry only with (k)-sets and (k+1)-sets.

Let $I_k, I_{k+1} \in \mathcal{F}_{l,k}$ such that $|I_k|=k$ and $|I_{k+1}|=k+1$.

If $I_k \subset I_{k+1}$ then I_k can be trivially amplified to $I_{k+1} \in \mathcal{F}_{l,k}$.

If I_k is not contained in I_{k+1} but there exists $j \in \{1,2,\ldots,l\}$ such that $j \in I_k$ then, recalling the definition of $\mathcal{F}_{l,k}$, I_k can be amplified using any element of $I_{k+1}-I_k$.

Finally suppose that I_k is not contained in I_{k+1} and there exists no $j \in \{1,2,\ldots,l\}$ such that $j \in I_k$. As $I_{k+1} \in \mathcal{F}_{l,k}$ there exists $e \in \{1,2,\ldots,l\}$ such that $e \in I_{k+1}$. Then $I_k+\{e\} \in \mathcal{F}_{l,k}$ and the proof is complete. □

The two following propositions can be easily proved:

Proposition 3.6: The knapsack $\mathbb{K}=\{x \in \{0,1\}^n : \Sigma_{i=1}^{n} a_i x_i \leq b\}$ will be $\mathcal{F}_{l,k}$-equivalent for $l \in \{1,2,\ldots,n-k\}$, and therefore greedy solvable, iff $b \in [a_l + \Sigma_{j=n-k+1}^{n} a_j, \Sigma_{j=l+1}^{k+l+1} a_j[$. ■

Proposition 3.7: The condition in proposition 3.6 can be checked in $O(n)$ operations. ■

Lets now have a closer look at $\mathcal{F}_{n-k,k}$. This family will consist of all the subsets of N with a cardinality not greater than k and of the (k+1)-sets containing at least one of the elements in $\{1,2,\ldots,n-k\}$. But one can easily see that all the (k+1)-sets of N have this last property so the following holds:

Corollary 3.8: $\mathcal{F}_{n-k,k}=\mathcal{F}_{0,k+1}$. ■

4. CONCLUSIONS

In this paper we have built, for every $k \in \mathbb{N}$, a finite chain of matroids over $\mathbb{N}$ with independence families $\mathcal{F}_{0,k} \subset \mathcal{F}_{1,k} \subset \mathcal{F}_{2,k} \subset \ldots \subset \mathcal{F}_{i,k} \subset \ldots \subset \mathcal{F}_{n-k,k} = \mathcal{F}_{0,k+1}$ such that checking if a knapsack is $\mathcal{F}_{i,k}$-equivalent, and therefore greedy solvable, can be done in polynomial time.

Of course there exist greedy solvable knapsacks which are not equivalent to any of the matroids in the aforementioned chain as in the following example:

Example 4.1: Consider $\mathbb{K}=\{x \in \{0,1\}^4 : 10x_1+11x_2+100x_3+101x_4 \leq 130\}$. One can easily see that the family of independent sets of this knapsack is $\mathcal{F}=\{\emptyset,\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{1,2,3\},\{1,2,4\}\}$.

The knapsack is greedy solvable but $\mathcal{F}$ is not in the chain.

The current research focuses on identifying other classes of matroids for which equivalence to knapsacks can be decided in polynomial time.

REFERENCES

White N. (1986), *Theory of Matroids*, Cambridge University Press, London.