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Abstract

This paper presents a hierarchical model framework for the analysis of the stabilizing effect of inventories in multiechelon manufacturing/distribution systems. This framework is used to analyze some recent results presented in the literature which compare the variance of demand to the variance of replenishment orders. These results have generated controversy since they suggest that, in most industries, inventories have a destabilizing effect. We reexamine these conclusions and present empirical observations that indicate that the mechanisms which promote stabilization are dependent on echelon position. These results are also compatible with our normative model framework which defines sufficient conditions for the existence of stabilization due to inventories.

One of the reasons used to justify investment in inventories is the need to absorb demand variability. This need is based on strong economic incentives for reducing fluctuations in production levels (due to fixed setup costs and/or convex manufacturing costs), as well as protection against stockouts. Inventories can act to buffer the production system from changes in product requirements generated by the marketplace. If this stabilizing function is performed effectively, observed variability in production will be less than the observed variability in market demand.

There is considerable empirical evidence that indicates that the variance of retail sales is in fact less than the variance of deliveries from plants to retailers.[1]This data suggests that inventories do not have a stabilizing effect on material flow patterns in spite of the significant investment in finished product inventories in many industries.[2]There is also evidence that the variance of plant production is greater than the variance of manufacturer sales (West(1986), Blanchard(1983)). Taken together these empirical results suggest that the manufacturing/distribution supply chain acts to amplify market variability rather than to dampen it.

Such behavior is inconsistent with our expectations (which are based on the economies and technology of manufacturing). A number of normative models of materials management have been developed which help to explain how various factors (like batch sizing, safety stocks, demand patterns) affect the pattern of replenishment requests. In this paper we develop a normative material management model to analyze the stabilizing effect of inventories in distribution networks. Our model considers the important effects of multiple echelons (wholesale, retail, market) in an environment of nonstationary (correlated) product requirements. Such nonstationarity is a consequence of the stock replenishment procedures commonly used in each echelon of the distribution network.

The model introduced in this paper can shed light on the empirical results on the lack of stabilization. It also provides a basis for developing inventory control policies for multi-echelon systems in a manner which explicitly accounts for the linkage between stocking policies and transferred demand processes. The model indicates that under certain conditions, inventories can have a stabilizing effect and in other cases this effect will not occur. These results suggest that the stabilizing effect of inventories is based on the interaction of cost, technology and market attributes which are peculiar to the industry of interest.

The paper is organized as follows. We begin (section 1) by reviewing the empirical data which describes the effect of inventories on product requirements variance in the manufacturing/distribution supply chain. In section 2 relevant literature is reviewed. We then present our multi-echelon model. We use the model to analyze the relationship between inventory levels at each echelon in the supply chain and, in particular, the variance of input and output demand/order processes which are transmitted from echelon to echelon.

1 Empirical Results

Previously published data on the occurence of stabilization does not reflect the impact of those economic agents which operate between producers and retailers. Such agents are found in the echelons associated with the producer's owned distribution network (comprising Distribution Centers, Warehouses) or in echelons operated by independent wholesalers and distributors who purchase and resell manufactured products. In order to account for this more complex (and more realistic) structure, we must explicitly consider data for the case where the distribution network has multiple echelons.

The "Economic Indicators", published by the Joint Economic Committee of the U.S. Congress contain statistics on shipments from manufacturers, wholesaler sales and retailer sales. This publication summarizes monthly estimates compiled by the Bureau of Census. We use these three time series of sales to represent the production/distribution behavior of the economy with respect to the three major supply chain echelons (manufacturing, distribution, retail). The results are summarized in Table 1. The estimates in each of these time series were developed from representative samples for each industry which contained data from firms of all sizes and in all kinds of businesses. None of these time series represent actual production volumes. In order to estimate production schedules we used the Manufacturer shipments series and corrected it by the change in manufacturer inventory levels . Since the shipments and inventory series reflect different item values (market price versus cost) it is necessary to convert inventories from a cost to market value basis. We used the approach introduced by West(1983) [3]

As a result of the indicated calculations time series on 1) retail sales, 2) wholesale sales, 3) manufacturer sales and 4) production volumes were generated. We computed the mean, standard deviation and coefficient of variation for each of the time series, at both current and 1967 prices, for the period January 1978 to December 1985. The results are presented in Table 1.

Analysis of Table 1 shows that, at current prices, the standard deviation of wholesaler sales is greater than that of retail sales and smaller than that of manufacturer sales. Manufacturer sales standard deviation is itself smaller than the standard deviation of production. We note these results do not take into account the relative size of the material flows in each time series.[4] To take size into consideration, we computed the coefficient of variation. From Table 1 we see that, at current prices, the coefficient of variation increases as we go from the retailer to the wholesaler level, (as noted by Blinder). This measure of variation then decreases in going from the wholesaler to manufacturer level. Finally we note that the coefficient of variation of production is less than the coefficient of variation of retail sales. Similar results are obtained when we consider the 1967 prices version of the series. However in this case the reduction on the coefficient of variation from the wholesaler to manufacturer level is not sufficient to offset the increase from retail to wholesaler level, and the coefficient of variation of production is greater than that of manufacturer sales.

These results clearly support the conclusion of Blinder and others that the variation in deliveries to retailers from wholesalers (.073) is greater than the variation of the market demand (.052) and, West's results which indicate that the variation of production (.066) is greater than the variation of manufacturers shipments (.058). However they also highlight a fact not previously noted. The variability of manufacturers' sales is less than the variability of wholesaler sales and it may even be less than the variability of retail sales (under current prices in our data set). This suggests that wholesalers perform a stabilizing role by smoothing the flow of demands. Without the existence of wholesalers it is clear that the variability of manufacturer shipments would be considerably larger –e. g. equal to the variance of wholesaler shipments.

	Production	Manufacturer Sales [1]	Wholesaler Sales [2]	Retail Sales [3]
Current Prices				
Mean	161294	161308	91590	8603
St. Deviation	22319	22217	17115	15682
Coef. Variation	0.138	0.138	0.187	0.177
1967 Prices [4]				
Mean	62499	62463	35178	33657
St. Deviation	4119	3623	2556	1758
Coef. Variation	0.066	0.058	0.073	0.052

Table 1: OBSERVED VARIABILITY OF PRODUCTION AND SALES

[1]Manufacturer shipments represent net selling values, f.o.b. at the plant, less all discounts and allowances and exclude freight charges and excise taxes. The time series of these shipments reflects the demand for goods and services which are produced by U.S. manufacturers. Manufacturers inventories are at book value of stock on hand at the end of the period (month) and include raw materials and supplies, work in process, and finished goods. All values are seasonally adjusted by the Bureau of Census.

[2]The series on Wholesale sales and inventories is limited to merchant wholesalers. It represents the dollar sales volume of merchandise sold to retailers by wholesalers less returns, allowances, and discounts, and receipts from repairs.

[3]Retail sales include merchandise sold for cash or credit by establishments primarily engaged in retail trade and thus represents the market demand for products.

[4]Since these sales series are expressed in current dollars, it is necessary to deflate the data in order to reflect actual material flows. We used the producer price and consumer price indices (also published by the Joint Economic Committee based on data compiled by the Department of Labor) for this purpose. Producer Price Indices measure average changes in prices received in primary markets of the U.S. by producers of commodities at all stages of processing. Consumer Price Indices provide a measure of price change for fixed market baskets of goods and services purchased by all urban consumers. Both indices have as basis, the year 1967. Since there is no specific index for intermediate transactions(wholesale) we deflated the wholesalers sales series by the Producer Index.

In the remainder of this paper we develop a normative model of distribution system management which can explain the above results. This model specifically considers a production/distribution system with three echelons, the retailer, distribution center and producer. The model will indicate that the observed pattern of material flow is compatible with rational behavior

.

2 Literature Review

Economists have long recognized the relevance of inventory fluctuations in explaining the timing and magnitude of business cycles. The linear decision rule of Holt, Modigliani, Muth and Simon (1960)(HMMS) has been the basis of much theoretical and empirical work and indicates that production smoothing (the production time series has less variance than the demand time series) is associated with cost minimizing behavior. Their model indicates, in particular, the role of buffer inventories in smoothing production in face of random or seasonal sales. In the context of a distribution system the HMMS model tradeoffs would predict the smoothing of replenishment orders from retailers by wholesalers.

Blinder's results, noted above, do not agree with the conclusions suggested by the HMMS model. Blinder(1981a) advanced the use of (s,S) policies by retailers as a possible explanation for the observed behavior pattern. He developed an econometric model, consistent with this class of stock control policies.

The empirical results noted above motivated several studies concerned with the stabilizing effect of inventories. Abel(1985) considered a model in which a firm must decide, each period, on how much to produce and at what price to sell its output (monopolist situation). He assumed that there is a one period production lag. He uses as a measure of stabilization the derivative of production with respect to initial inventory. If the derivative is between -1 and 0 inventories will have a stabilizing effect. Abel showed that if the demand curve is perfectly inelastic, and if there are lost sales, then stabilization occurs. He also developed a relationship between this concept of stabilization and the variance related definition we introduced earlier. Kahn(1985) considered the case where, in each period, a firm must decide on how much product to make available for sale. He showed that the presence of positive serial correlation in the demand process and the use of backlogging for excess demand both lead to a situation where the variance of production is greater than the variance of sales. His results are also valid for the monopolist price setter.

Caplin(1985) showed that for a retailer following an (s,S) continuous review inventory policy, the variance of replenishment orders exceeds the variance of demand. This result is also valid when aggregated over several retailers. Thus the variance of aggregate orders in a distribution network is greater than the variance of aggregate demand. Caplin's result (which pertains to a stationary steady-state model) suggests that the presence of retailers does not act to stablize the aggregate demand faced by the wholesaler who supplies them.

It must be emphasized that Abel(1985) and Kahn(1985) consider only price and production costs (no inventory related costs) while Caplin's results are independent of the firm's cost structure (and hence optimal stocking policies are not considered). Caplin's model is stationary while Abel and Kahn explicitly consider the dynamic aspects of production/inventory requirements. In this paper we present a model which combines the relevant features of the models noted above. It contain an explicit model of inventory stock control at each echelon and location and moreover non-stationary demand and supply processes are considered.

3 The Normative Model

In this section we develop a multiechelon model that can explain the empirical results presented in section 1. This model formulation takes into account the multiechelon linkages. Our main results focus on an explanation of the stabilizing role of wholesalers.

3.1 Model Description

Consider the network structure illustrated in Figure 1. We assume a periodic review system. At the lowest level retailers face a random i.i.d. market demand process. At the beginning of each period, retailers review their inventory position. Depending on their stock control policy and their current status, an order may be issued, which will be delivered after a fixed leadtime. All excess demand is backlogged.

The retailers are all supplied by one distribution center which reviews its inventory position at the beginning of each period. The distribution center will place a replenishment order to the plant according to its inventory control policy (which is to be determined).

3.2 Assumptions

The following assumptions are made in our model.

1. Demand per period at each retailer location is an independent and identically distributed random variable.
2. A periodic review procedure is used and in each period the following sequence of events occurs at each stocking location: order, delivery and sale.
3. Orders are delivered after a fixed leadtime.
4. Excess demand is backlogged at all levels.
5. Fixed plus variable ordering costs are incurred at the retailer level. Variable costs only are charged at the Distribution Center.
6. Holding and shortage costs are charged against expected (end of period) inventory levels.
7. Pipeline holding costs are paid by the receiving location.
8. All costs are stationary.

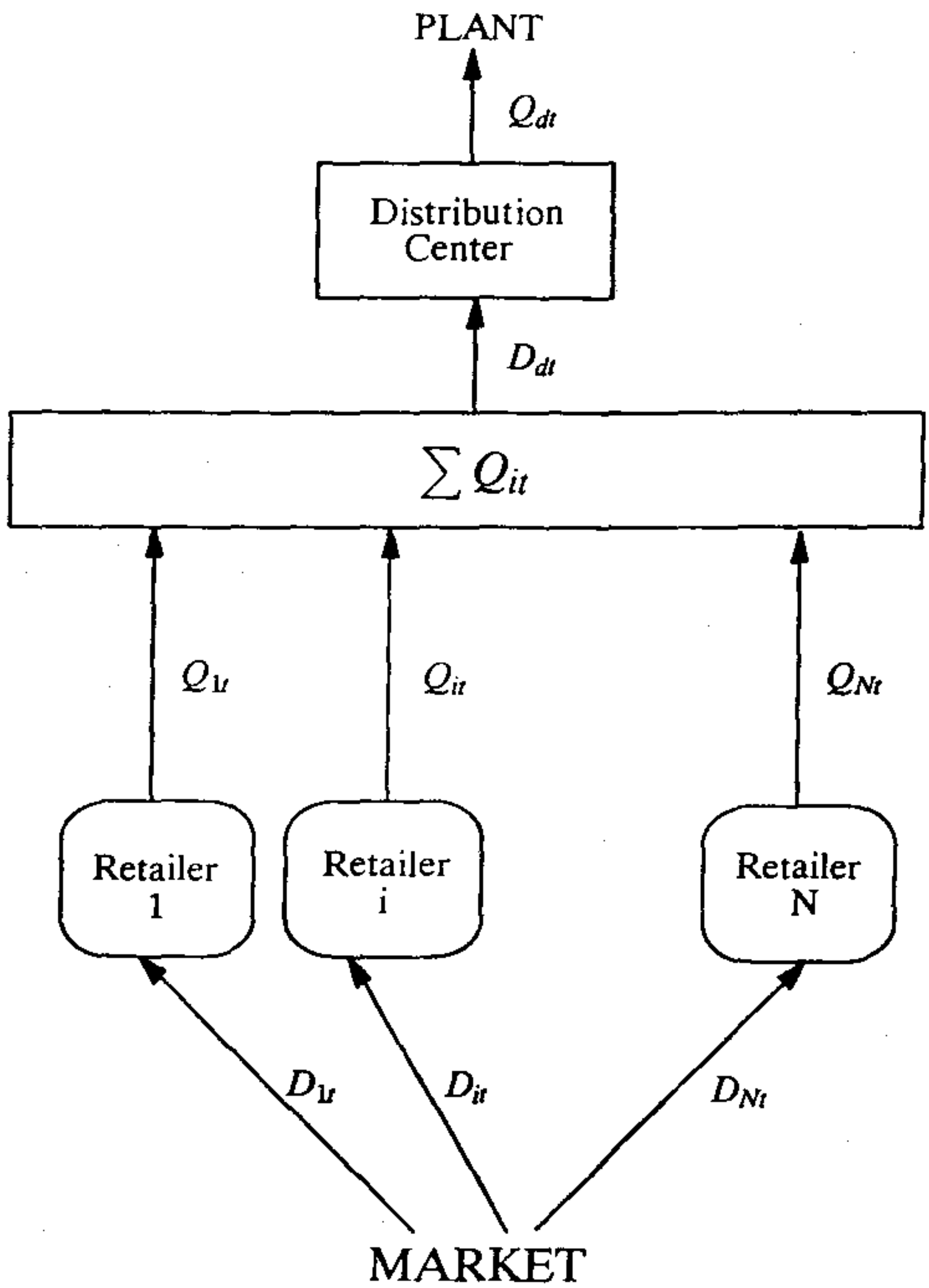

Figure 1: MULTIECHELON STRUCTURE

Assumption 1, concerning stationarity of demand means that the only source of requirements serial correlation is the retailers' ordering decisions. Periodic review and backlogging are common practices in many industries. The assumed ordering cost structure charges fixed replenishment costs to retailers and not to the wholesaler. This situation could arise when wholesalers have adopted efficient order filling technology and/or where the manufacturers absorb the cost of filling orders. In the latter case it could be argued that such costs are passed on to the wholesaler and are reflected in the unit cost (and hence in the holding cost). We have observed such arrangements when wholesalers have long term contracts with their suppliers.

3.3 Notation

We make use of three subscripts in the model. The first subscript refers to the echelon, the second to the location and the third to the time period. Thus X_{ijt} will be the value of variable X in period t, at location j of echelon i. For stationary parameters we omit the time period.

The echelons will be numbered starting from the bottom. (Retailers will be echelon 1).

For each location, echelon and period define,

D : a random variable denoting demand in a period.

Q : the quantity of stock ordered.

I : the amount of inventory on hand at the beginning of a period.

W : the inventory position (on hand plus on order) at the beginning of a period.

c : unit cost of ordering (\$/order/unit).

K : fixed cost of ordering (\$/order).

h : unit holding cost (\$/unit/period).

π : unit shortage cost (\$/unit/period).

v : unit cost of changing production level (\$/unit).

L : leadtime (in periods).

$f(.)$: probability density function for D.

$F(.)$: cumulative distribution function for D.

μ: mean demand.

σ : variance of demand.

β : discount factor. $(0 \leq \beta \leq 1)$

3.4 Formulation

The managerial objective is to determine order quantities for each period and location which minimize total expected discounted cost over an infinite horizon.[5] There are two cost components in the system (retailers- CR and distribution center- CDC). The model statement is as follows.

$$Min \;\; E\left\{\sum_{i=1}^{N} CR_i + CDC\right\} \tag{1}$$

subject to,

$$CR_i = \sum_{t=1}^{\infty} \beta^{t-1}\{K_{1i}\delta(Q_{1it}) + c_{1i}Q_{1it}+ \tag{2}$$

$$\beta^{L_{1i}}h_{1i}(W_{1it} + Q_{1it} - \sum_{l=t}^{t+L_{1i}} D_{1il})^{+} + \beta^{L_{1i}}\pi_{1i}(\sum_{l=t}^{t+L_{1i}} D_{1il} - W_{1it} - Q_{1it})^{+}\}$$

$$CDC = \sum_{t=1}^{\infty} \beta^{t-1}\{c_{21}Q_{21t} + \beta^{L_{21}}h_{21}(W_{21t} + Q_{21t} - \sum_{l=t}^{t+L_{21}} D_{21l})^{+} \tag{3}$$

$$+\beta^{L_{21}}\pi_{21}(\sum_{l=t}^{t+L_{21}} D_{21l} - W_{21t} - Q_{21t})^{+}\}$$

$$W_{1it+1} = W_{1it} + Q_{1it} - D_{1it} \tag{4}$$

$$W_{21t+1} = W_{21t} + Q_{21t} - D_{21t} \tag{5}$$

$$D_{21t} = \sum_{i=1}^{N} Q_{1it} \tag{6}$$

$$\delta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{7}$$

$$0 \leq \beta \leq 1 \tag{8}$$

$$Q_{1it} \geq 0 \quad i = 1, \ldots, N \tag{9}$$

$$Q_{21t} \geq 0 \tag{10}$$

Equation 2 is the total discounted cost over an infinite horizon for retailer i. It includes the usual costs for an inventory system with backlogging, fixed order cost and leaditme. We are assuming that the ordering costs are incurred in the order period and that its variable component includes the holding cost for the pipeline inventory[6]. Equation 3 corresponds to the distribution center cost and is similar to equation 2 without the fixed order cost.

Equations 4 - 6 are the inventory balance equations across time (4 and 5) and locations (6) . The later balance equation capture the demand linkages between echelons. These linkages will, in turn, determine the degree of stabilization which will occur across echelons in the network.

4 Solution Method

We solve the overall optimization problem by a multiechelon decomposition procedure. Each location will determine its optimal policy only after all those locations that it supplies have determined their inventory policy and their asociated ordering policies. The diagram in Figure 2 illustrates this procedure. At the lower level, retailers will determine their optimal stocking policy. Once this is done the demand transmitted to the distribution center can be calculated. This demand acts as an input to the distribution center for the determination of optimal distribution stock control policy. We then move up to the next level. Note that this process is not necessary optimal from the point of view of the overall network– a branch and bound procedure would be required (see Cohen, Kleindorfer and Lee(1986)). However if each location corresponds to a different firm, then this solution approach provides a reasonable approximation to a decentralized control mode. We also assume that there are no interactions between levels (e.g. lateral pooling).

We note that the inventory policy of the distribution center will have an impact on the leadtime offered for replenishment orders made by the retailers. This in turn can impact the retailer inventory policy. The multiechelon decomposition proceeds only once from the bottom up. A more realistic, but computationally demanding approach would cycle through the echelons until policy convergence occurs. Experimentation by Cohen and Lee(1988) for similar manufacturing/distribution supply chain optimization problems indicates fast convergence and good first iteration accuracy for this procedure. Recent results, by Baganha and Cohen (1990), show that the cost performance of policies generated in the first iteration of the multiechelon decomposition procedure is comparable to that attained with the final iteration (fixed point) policy. In the following sub-sections we will solve the optimization problem for each echelon based on the assumption of a one pass procedure.

4.1 The retailer sub- problem

We define the stocking problem at the retailer level in terms of the following mathematical program:

$$Min\ \ E\left\{\sum_{i=1}^{N} CR_i\right\} \tag{11}$$

subject to

$$CR_i = \sum_{t=1}^{\infty} \beta^{t-1}\{K_{1i}\delta(Q_{1it}) + c_{1i}Q_{1it}+ \tag{12}$$

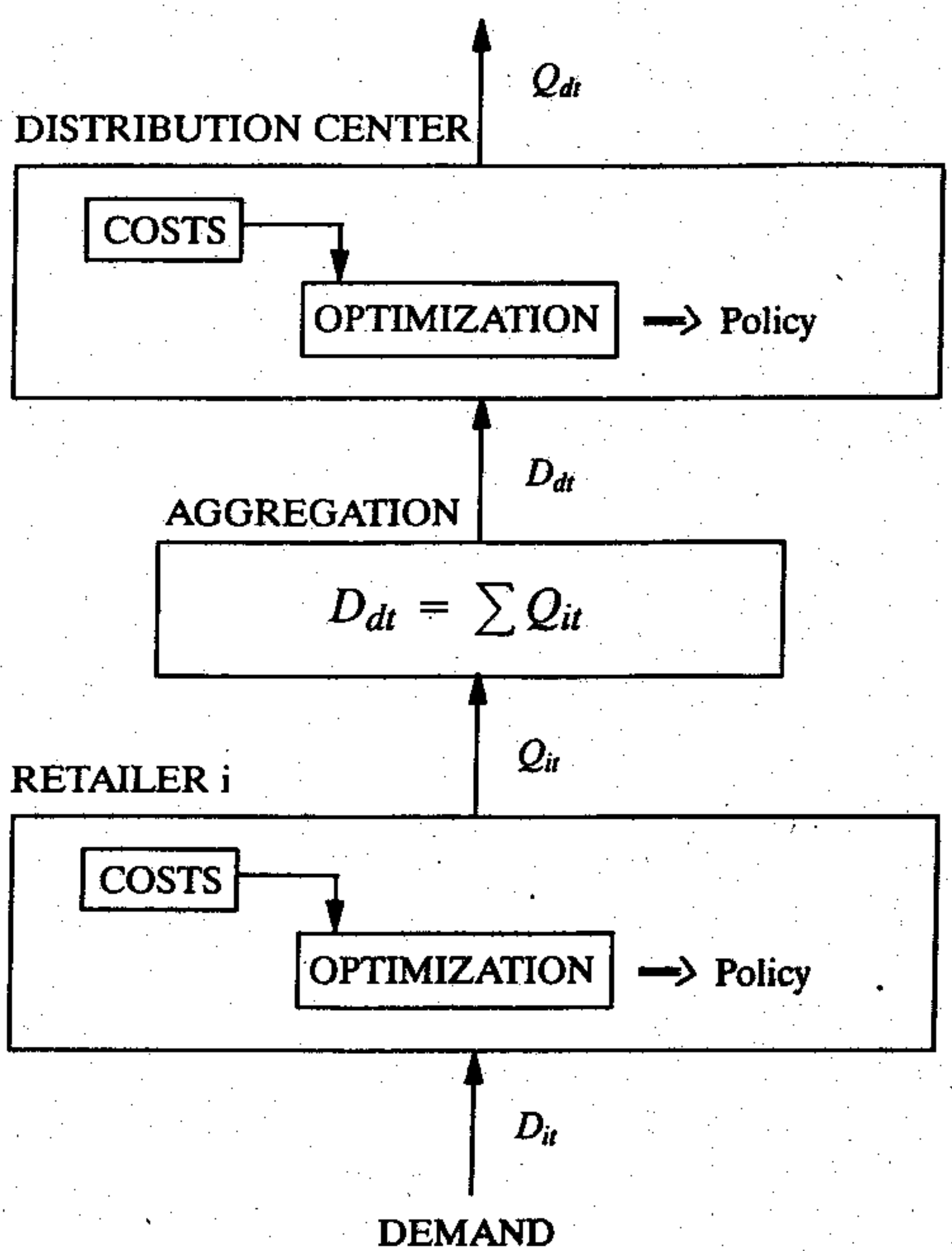

Figure 2: MULTIECHELON DECOMPOSITION

$$\beta^{L_{1i}} h_{1i}(W_{1it} + Q_{1it} - \sum_{l=t}^{t+L_{1i}} D_{1il})^+ + \beta^{L_{1i}} \pi_{1i}(\sum_{l=t}^{t+L_{1i}} D_{1il} - W_{1it} - Q_{1it})^+\}$$

$$W_{1it+1} = W_{1it} + Q_{1it} - D_{1it} \tag{13}$$

$$\delta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{14}$$

$$0 \leq \beta \leq 1 \tag{15}$$

$$Q_{1it} \geq 0 \quad i = 1, \ldots, N \tag{16}$$

Since there are no constraints linking the N retailers, (i.e. we have assumed there is no pooling or coordination among the retailers) the problem decomposes into the solution of N stochastic dynamic inventory problems with backlogging and a fixed order cost. It is well known that the optimal policy for each of these problems is of (s,S) type (e.g. Scarf(1960), Veinott(1966)) and there are algorithms to compute it (Veinott and Wagner(1965), Federgruen and Zipkin(1984), Cohen et al.(1988)).

Define (s_i, S_i) as the parameter values for the optimal policy at location i. Let $r_i(j)$ be the correlation coefficient between the order quantity in period t, Q_{1it}, and the order quantity in period t+j, Q_{1it+j}, i.e. the autocorrelation for the retailer replenishment request time series for a lag of j periods. Based on a renewal approximation introduced by Erhrardt et al.(1981) it can be shown that,

$$Var(Q_{1i*}) = \sigma_i^2 + \frac{2\mu_i \sum_{j=1}^{\infty} m(j)}{1 + M(\Delta - 1)} \tag{17}$$

and

$$r_i(j) = \begin{cases} -E(Q_{1i*}) \sum_{l=0}^{\Delta_i - 1} l f_i(l) / Var(Q_{1i*}) & j = 1 \\ \\ -E(Q_{1i*}) \sum_{l=0}^{\Delta_i - 1} f_i^{(j)}(l) / Var(Q_{1i*}) & \\ - \sum_{l=1}^{j-1} F_i^{(j-1)}(\Delta_i - 1) r_i(l) & j > 1 \end{cases}$$

where

$$\Delta_i = S_i - s_i$$

Q_{1i*} = a random variable denoting the steady state order quantity

$f_i^{(j)}(.)$ = the probability function of the j-fold convolution of demand per period.

$F_i^{(j)}(.)$= the CDF of the j-fold convolution of demand.

and

$$M_i(j) = \sum_{n=1}^{\infty} F_i^{(n)}(j)$$

$$m_i(j) = \sum_{n=1}^{\infty} f_i^{(n)}(j)$$

are, respectively, the renewal function and the renewal density associated with the demand process.

Analysis of values of $r_i(j)$ for different values of j shows that the successive values of Q_{1it} are not independent[7]. It is important to note that this correlation is negative for a lag of one period. For higher order lags it may be either positive or negative. This follows from the batch ordering enforced by (s,S) policies.

In what follows we assume that the reorder process can be described by an autoregressive model with lag parameter equal to the cycle length, which is defined as the number of periods between two consecutive orders and is given by the expression:

Cycle length $p_i = 1/\ P[Q_{1it} > 0]$

Thus for any location i

$$Q_{1it} = \alpha_i + \sum_{j=1}^{p_i} \theta_i(j) Q_{1it-j} + u_{it} \tag{18}$$

Where u_{it} is a random disturbance term that is assumed to be a normal variate with mean 0 and variance σ^2.

To calculate values for the autoregressive model parameters $(\theta_i(j))$ parameters, we use the Yule- Walker equations (the location index i has been supressed) that relate the autocorrelation coefficients to these parameters:

$$\begin{array}{lcllll}
r(1) & = & \theta(1) & + \ \theta(2)r(1) & + \cdots + & \theta(p)r(p-1) \\
r(2) & = & \theta(1)r(1) & + \ \theta(2) & + \cdots + & \theta(p)r(p-2) \\
\vdots & \vdots & \vdots & \vdots \ \vdots & \vdots & \vdots \\
r(p) & = & \theta(1)r(p-1) & + \ \theta(2)r(p-2) & + \cdots + & \theta(p)
\end{array}$$

Since the AR(p) process is stationary there is a solution to the above system of equations (Box and Jenkins(1970)).

Once the θ's are determined we may compute α_i as

$$\alpha_i = E(Q_{1i\cdot})(1 - \theta(1) - \theta(2) - \cdots - \theta(p)) \tag{19}$$

Similarly the variance of u_t is computed from its relation with variance of $Q_{1i\cdot}$.

$$Var(Q_{1i\cdot}) = \frac{Var(u_{it})}{1 - \theta(1) - \cdots - \theta(p)} \tag{20}$$

4.2 The distribution center

The demand at the distribution center is determined by the sum (convolution) of the replenishment orders it receives from each of the retailers. Thus the random variable denoting demand at the distribution center is,

$$D_{21t} = \sum_{i=1}^{N} Q_{1it}$$

where Q_{1it} is defined in (18)

The sum of N autoregressive processes will be an autoregressive moving average process . Since any stationary ARMA process may be converted into an AR process of superior order we represent the distribution center demand by

$$D_{21t} = \gamma + \sum_{j=1}^{p} \phi_j D_{21t-j} + \epsilon_t \tag{21}$$

where γ and ϕ_j are the autoregressive model parameters and ϵ_t is the random disturbance term in period t.

Note that the demand at the Distribution Center will also exhibit negative serial correlation for one period lag.

The stock control optimization problem at the distribution center may now be reformulated as (ignoring location and echelon subscripts):

$$MinE\left[\sum_{t=1}^{\infty}\beta^{t-1}\{cQ_t+\beta^L h(W_t+Q_t-\sum_{l=t}^{t+L}D_l)^+ + \right. \tag{22}$$

$$\left. \beta^L\pi(\sum_{l=t}^{t+L}D_l-W_t-Q_t)^+\}\right]$$

subject to

$$D_t=\gamma+\sum_{j=1}^{p}\phi_j D_{t-j}+\epsilon_t \quad t=1,2,\cdots \tag{23}$$

$$W_{t+1}=W_t+Q_t-D_t \tag{24}$$

where ϵ_t is a normal variate with 0 mean and variance σ_ϵ^2

Baganha(1987) determined conditions for the optimality of myopic policies for this class of problems. These polices are characterized by the fact that ordering decision of period t only takes into account the cost in the period in which the order will be delivered. Two particular cases are summarized in the following Lemma (See Corollaries 3.1 and 3.2 in Baganha(1987) for the proof). The first part of the Lemma pertains to the case where the demand is any autoregressive model and leadtime is one. The second part refers to the situation of general deterministic leadtime with demand represented by an autoregressive model of period one.

Lemma 1 *If in the optimization problem defined by (22)- (24) :*
i. $L=1$ *and* $\phi_1<0$
or
ii. $p=1$ *and* $\phi_1<0$
then a myopic policy is optimal.

Based on this result we assume that the distribution center follows myopic inventory policies.

The following Lemma, also from Baganha(1987), will be useful in the computation of the optimal policy. It defines the demand k periods into the future given the information available now.

Lemma 2 *Let demand in period t be represented by the autoregressive model of order p:*

$$D_t = \gamma + \phi_1 D_{t-1} + \cdots + \phi_p D_{t-p} + \epsilon_t$$

and define

$$\pi_1 = 1 \text{ and } \pi_j = \sum_{i=1}^{j-1} \phi_i \pi_{j-i} \text{ for } j > 2$$

with

$$\pi_{i-j} = 0 \text{ if } i \leq j \text{ and } \phi_i = 0 \text{ if } i > p$$

Then, given the information available at the beginning of period t:
i. the demand in period t+k is

$$D_{t+k} = \sum_{j=1}^{p} D_{t-j} \sum_{i=1}^{k+1} \pi_i \phi_{k+j+1-i} + \sum_{i=1}^{k+1} \gamma \pi_i + \sum_{i=1}^{k+1} \pi_i \epsilon_{t+k+1-i} \tag{25}$$

ii. the expected demand in period t+k is

$$E(D_{t+k}|t-1) = \sum_{j=1}^{p} D_{t-j} \sum_{i=1}^{k+1} \pi_i \phi_{k+j+1-i} + \sum_{i=1}^{k+1} \gamma \pi_i \tag{26}$$

iii. the variance of demand in period t+k is

$$Var(D_{t+k}|t-1) = \sigma_\epsilon^2 \sum_{i=1}^{k+1} \pi_i^2 \tag{27}$$

where $(X|t-1)$ is an operator that conditions the random variable X on the information available at the end of period t-1 (or beginning of t).

The total demand during periods $t \ldots t+L$ will be

$$\sum_{k=0}^{L} D_{t+k} = \sum_{k=0}^{L} \sum_{j=1}^{p} D_{t-j} \sum_{i=1}^{k+1} \pi_i \phi_{k+j+1-i} + \sum_{k=0}^{L} \sum_{i=1}^{k+1} \gamma \pi_i + \sum_{k=0}^{L} \sum_{i=1}^{k+1} \pi_i \epsilon_{t+k+1-i}$$

$$= \sum_{j=1}^{p} D_{t-j} \sum_{k=0}^{L} \sum_{i=1}^{k+1} \pi_i \phi_{k+j+1-i} + \gamma \sum_{k=0}^{L} \sum_{i=1}^{k+1} \pi_i + \sum_{k=0}^{L} \epsilon_{t+k} \sum_{i=1}^{L-k} \pi_i \tag{28}$$

and

$$E(\sum_{k=0}^{L} D_{t+k}|t-1) = \sum_{j=1}^{p} D_{t-j} \sum_{k=0}^{L} \sum_{i=1}^{k+1} \pi_i \phi_{k+j+1-i} + \gamma \sum_{k=0}^{L} \sum_{i=1}^{k+1} \pi_i \tag{29}$$

$$Var(\sum_{k=0}^{L} D_{t+k}|t-1) = \sigma_\epsilon^2 \sum_{k=0}^{L} (\sum_{i=1}^{L-k} \pi_i^2) \tag{30}$$

Note that the variance is independent of the information available. It is only a function of the length of the forecasting period. In the sequel we will use $\sigma^{(L+1)}$ for the standard deviation of the (L+1) period demand. Since $\sum_{k=0}^{L} D_{t+k}$ can be expressed as a linear combination of independent normal variates it is a normal variate.

Using a change of variable we get the equivalent formulation.

$$Min \sum_{t=1}^{\infty} \beta^{t+L-1} E\{cY_t + hE(Y_t - \sum_{l=0}^{L} D_{t+l})^+ + \pi E(\sum_{l=0}^{L} D_{t+l} - Y_t)^+ \tag{31}$$

$$-\beta c E(Y_t - \sum_{l=0}^{L} D_{t+l})^+\}$$

$$D_t = \gamma + \sum_{j=1}^{p} \phi_j D_{t-j} + \epsilon_t t = 1, 2, \cdots \tag{32}$$

$$W_t = Y_{t-1} - D_{t-1} \tag{33}$$

where Y_t is the total inventory position in period t after ordering but before demand.

The myopic solution will result from solving the one period problem,

$$Min c Y_t + hE(Y_t - \sum_{l=0}^{L} D_{t+l})^+ + \pi E(\sum_{l=0}^{L} D_{t+l} - Y_t)^+$$

$$-\beta c E(Y_t - \sum_{l=0}^{L} D_{t+l})^+$$

Compute the derivative, set it to zero and solve for Y_t:

$$Y_t = \Theta\sigma^{(L+1)} + \sum_{j=1}^{p} D_{t-j} \sum_{k=0}^{L} \sum_{i=1}^{k+1} \pi_i \phi_{k+j+1-i} + \gamma \sum_{k=0}^{L} \sum_{i=1}^{k+1} \pi_i \tag{34}$$

where

$$\Theta = Z^{-1}\left(\frac{\pi - c}{h + \pi - \beta c}\right)$$

(34) may also be written as

$$Y_t = \sigma^{(L+1)} + \sum_{k=0}^{L} E(D_{t+k}|t-1) \tag{35}$$

This solution gives rise to the following sufficient condition for distribution center inventory to stabilize requirements passed on to the manufacturer.

Theorem 1 *A sufficient condition for the inventory stabilization effect at the distribution center (i.e. $Var(Q_{21t}) \leq Var(D_{21t})$), is that:*

$$\sum_{i=1}^{L+1} \sum_{j=i+1}^{L+2} \pi_i \pi_j \leq 0 \tag{36}$$

Proof: The quantity ordered in period t+1 (Q_{t+1}) will be given by $Y_{t+1} - (Y_t - D_t)$:

$$Q_{t+1} = \sum_{l=0}^{L} E(D_{t+l+1}|t) - \sum_{l=0}^{L} E(D_{t+l}|t-1) + D_t$$

$$= E(D_{t+L+1}|t) + \sum_{l=1}^{L} [E(D_{t+l}|t) - E(D_{t+l}|t-1)] + D_t - E(D_t|t-1)$$

since $E(D_{t+l}|t) - E(D_{t+l}|t-1) = \pi_{l+1}[D_t - E(D_t|t-1)]$

$$Q_{t+1} = E(D_{t+L+1}|t) + \sum_{l=0}^{L} \pi_{l+1}[D_t - E(D_t|t-1)]$$

$$= \sum_{j=1}^{p} D_{t+1-j} \sum_{i=1}^{L+1} \pi_i \phi_{L+j+1-i} + \gamma \sum_{i=1}^{L+1} \pi_i + \epsilon_t \sum_{i=0}^{L} \pi_{l+1} \tag{37}$$

and

$$Var(Q_{t+1}) = \sum_{j=1}^{p} Var(D_{t+1-j}) \left[\sum_{i=1}^{L+1} \pi_i \phi_{L+1+j-i}\right]^2 + \sigma_\epsilon^2 \left[\sum_{l=0}^{L} \pi_{l+1}\right]^2 +$$

$$2\left(\sum_{i=1}^{L+1} \pi_i \phi_{L+2-i}\right)\left(\sum_{l=0}^{L} \pi_{l+1}\right) cov(D_t, \epsilon_t) \tag{38}$$

since the process is stationary $Var(D_{t+1}) = Var(D_{t+L+1})$. *Substituting L+1 for k in (25) and computing the variance we will get*

$$Var(D_{t+L+1}) = \sum_{j=1}^{p} Var(D_{t+1-j}) \left[\sum_{i=1}^{L+1} \pi_i \phi_{L+1+j-i}\right]^2 + \sigma_\epsilon^2 \left[\sum_{l=0}^{L} \pi_{l+1}\right]^2 \tag{39}$$

Since $cov(D_t, \epsilon_t) = \sigma_\epsilon^2$, $Var(Q_{t+1}) \leq Var(D_{t+L+1})$ *if*

$$\left[\sum_{l=0}^{L} \pi_{l+1}\right]^2 + 2\left(\sum_{i=1}^{L+1} \pi_i \phi_{L+2-i}\right)\left(\sum_{l=1}^{L} \pi_{l+1}\right) \leq \sum_{l=0}^{L} \pi_{l+1}^2$$

Using the fact that $\pi_{L+2} = \sum_{i=1}^{L+1} \pi_i \phi_{L+2-i}$ *we will get, after some computations, the result.*

Corollary 1 *If the leadtime is zero (L=0) and* $\phi_1 < 0$ *then* $Var(Q_{21t}) \leq Var(D_{21t})$.

Proof: If L=0

$$\sum_{i=1}^{L+1} \sum_{j=i+1}^{L+2} \pi_i \pi_j = \pi_1 \pi_2 = \phi_1$$

and the result follows.

Corollary 2 *If demand at the distribution center is represented by an autoregressive model of period 1 (p=1) and* $\phi_1 < 0$ *then* $Var(Q_{21t}) \leq Var(D_{21t})$

Proof: If p=1 then $\pi_k = \phi_1^{k-1}$ $k \geq 1$ *then*

$$\sum_{i=1}^{L+1} \sum_{j=i+1}^{L+2} \pi_i \pi_j = \sum_{i=1}^{L+1} \sum_{j=i+1}^{L+2} \phi_1^{i-1} \phi_1^{j-1} = \sum_{i=1}^{L+1} \phi_1^{i-1} \sum_{j=0}^{L+2-i-1} \phi_1^{i+j}$$

$$= \sum_{i=1}^{L+1} \phi_1^{2i-1} \frac{1 - \phi_1^{L+2-i}}{1 - \phi_1} =$$

$$= \frac{1}{1 - \phi_1} \left[\sum_{i=1}^{L+1} \phi_1^{2i-1} - \sum_{i=1}^{L+1} \phi_1^{L+i+1} \right] =$$

$$= \left[\frac{\phi_1 (1 - \phi_1^{2(L+1)})}{1 - \phi_1^2} - \frac{\phi_1^{L+2} (1 - \phi_1^{L+1}}{1 - \phi_1} \right] \frac{1}{1 - \phi_1} =$$

$$= \frac{\phi_1 (1 - \phi_1^{L+1})(1 - \phi_1^{L+2}}{(1 + \phi_1)(1 - \phi_1)^2} \leq 0$$

The above proposition and corollaries show that the introduction of a Distribution Center between retailers and plant may reduce of variability of the demand faced by the plant.

We note in particular that our analysis of the retailer's stock control problem resulted in a replenishment process which has negative serial correlation for a lag of one period. Consequently the input demand process for the distribution center (which is the aggregation of the replenishment processes over all retailers) will exhibit such negative serial correlation. As a result the conditions required for Corollary 1 or 2 will be satisfied if retailers follow an optimal stocking policy.

If there was no distribution center the retailers would order directly from the plant. Then the plant would face a demand equal to the demand input for the distribution center in our model (which would have variance greater than market demand). Introduction of the distribution center thus has a stabilizing effect.

5 Conclusions

In this paper we reviewed some empirical results concerning the destabilizing effect of inventories. We saw that models available for the study of the impact of inventory policies on the variability of demand throughout the manufacturing/distribution supply chain do not take into account the stochastic linkages associated with the multiechelon structure of such systems.

If the multiechelon structure is taken into account we see that reduction of demand variability is possible. In particular we showed that the variance of the demand faced by a plant will be less when it is filtered through a distribution center than when it is received directly from replenishment orders issued by retailers. If all echelons follow an optimal stocking policy, then such variance reduction will follow.

The analytical models developed in this paper may be extended in several directions. At the distribution center we only considered variable ordering costs. The inclusion of fixed ordering costs would introduce a factor which works against variance reduction. A possible extension would be to see how sensitive the model is to this fixed cost. How high must it be to compensate for the other factors?

The optimization problem was solved by multiechelon decomposition assuming that each location only has local information. This assumption is reasonable if different locations are owned by different firms. When a firm owns the entire multiechelon structure, there are incentives to use more global information. How will this affect the results? Can the system be managed in such a way that inventories absorb the shocks of demand variability?

The main objective of the model developed in this paper is to show how the optimal behavior of firms can lead to a stabilizing effect of inventories. We also highlight a factor not considered in the relevant economics literature — i.e. stochastic linkages between echelons.

6 Notes

[1] Blinder (1981a,1981b) presents results which show that during the period January of 1959 to December 1980 the variance about the trend of retail sales was smaller than the variance of deliveries to retailers.

[2]The term inventory stabilization will be used to characterize the variability of the replenishment process for stocking locations in a distribution network. In general demand fluctuations throughout the manufacturing /distribution chain are generated by the stochastic process associated with the market demand. Stabilization occurs at a location as a result of material control decisions made in response to the product requirements faced by the location. More precisely we say that a location exhibits stabilization when the variance of its inputs requirements is greater than the variance of its output requirements which are determined by its material management decisions. In the context of production inventory systems, we observe stabilization when the variance of demand experienced by a finished goods stockpile is greater than the variance of replenishment orders sent by a stockpile to its supplier location (i.e. a distribution center or the production facility). Blinder's results concerning plant deliveries can be viewed as a proxy for plant demand.

[3] West estimated the ratio of shipments to cost of goods sold for aggegate manufacturing at the two digit SIC code level and for all manufacturing. This ratio is an approximation to the ratio of cost to market price.

[4] The values of the four series differ considerably. We assume that the variability of the samples is representative of the variability of the population.

[5] In this formulation we are assuming that all the locations behave as price takers and will sell the product at the same price. Hence profit and revenue considerations can be ignored.

[6] c_{1i}, for example, includes the holding cost associated with Q_{1it} units during L_{1i} periods, discounted to the ordering period.

[7] To be precise we must say that the correlation is between inventory levels at the end of each period. The correlations between order quantities stem from the inventory correlation. According to our solution procedure, the distribution center only sees orders, so correlation between orders is the relevant factor.

References

[1] ABEL, A. B. , 1985 *Inventories, Stockouts and Production Smoothing,* Review of Economic Studies. pp.283-293.

[2] BAGANHA, M. P. , 1987 *Contributions to the Study of Stochastic Linkages in Multi-Echelon Inventory Systems.* Unpublished Ph.D. dissertation, Department of Decision Sciences, University of Pennsylvania.

[3] BAGANHA, M. P. and M. A. COHEN, 1990 *Decomposition Algorithm Performance in Multi-Echelon Inventory Systems.* ORSA/TIMS Meeting, Philadelphia.

[4] BLANCHARD. O. J. , 1983 *The Production and Inventory Behavior of The American Automobile Industry,* Journal of Political Economy, 91, pp.365-400.

[5] BLINDER, A. S. . 1981a *Retail Inventory Investment and Business Fluctuations,* Brookings Papers on Economic Activity. 2, pp.443-505.

[6] BLINDER, A. S. . 1981b *Inventories an the Structure of Macro Models,* American Economic Review, Vol. 71. May, pp. 11-16.

[7] BLINDER, A. S., 1986c *Can the Production Smoothing Model of Inventory Behavior be Save?* , The Quaterly Journal of Economics, Vol CI, pp. 431-453.

[8] BOX, G. E. P. and G. M. JENKINS, 1970 *Time Series Analysis, Forecasting and Control,* Holden Day.

[9] CAPLIN, A. S., 1985 *The Variability of Aggregate Demand with (s,S) Inventory Policies,* Econometrica, Vol. 53, pp. 1395-1409.

[10] COHEN, M. A. , P. R. KLEINDORFER and H. L. LEE, 1986 *Optimal Stocking Policies for Low Usage Items in Multi-Echelon Inventory Systems,* Naval Research Logistic Quarterly, Vol.33, pp. 17-38.

[12] COHEN, M. A. and H. L. LEE, 1988 ***Strategic Analysis of Integrated Production-Distribution Systems:Models and Methods***, Operations Research, Vol. 36, May-June.

[13] EHRARDT,R. A., C. SCHULTZ and H. WAGNER, 1981 *(s,S) Policies for a Wholesale Inventory System* in Schwartz, L. B. (ed) Multi-Level Production/Inventory Control Systems: Theory and Practice. Amsterdam:North-Holland, pp. 145-161.

[14] FEDERGRUEN, A. and P. ZIPKIN, 1984 *An efficient Algorithm to Compute Optimal (s,S) Inventory Policies*, Operations Research, Vol. 34.

[15] HEYMAN, D. and M. SOBEL, 1984 ***Stochastic Models in Operations Research***, Vol II, McGraw-Hill.

[16] HOLT, C. C., F. MODIGLIANI, J. MUTH and H. SIMON, 1960 *Planning Production, Inventories and the Work Force*, Prentice-Hall, NJ.

[17] JOHNSON, G. D. and H.E. THOMPSON, 1975 *Optimality of Myopic Policies for Certain Dependent Processes*, Management Science, Vol. 21, pp. 1303-1307.

[18] KAHN, J., 1985 *Inventories and the Volatility of Production*. mimeo, MIT.

[19] SCARF, H. E., 1960 ***The optimality of (s,S) policies in the dynamic inventory problem***, Chapter 13 in Arrow, K. J., Karlin, S. and Suppes, P. (Eds), Mathematical Methods in the Social Sciences, Stanford University Press, Stanford, California.

[20] U. S. CONGRESS- Joint Economic Committee ***Economic Indicators*** , years 1978-1985.

[21] VEINOTT, A. F., 1966 *On the Optimality of (s,S) Inventory Policies: New Conditions and A New Proof*, J. SIAM Applied Mathematics, Vol. 14, pp. 1067-1083.

[22] VEINOTT, A. F. and H. M. WAGNER, 1965 *Computing Optimal (s,S) Inventory Policies*. Managament Science, Vol. 11, pp. 525-552.

[23] WEST, K. D., 1983 *A Note on the Econometric Use of Constant Dollar Inventory Series*, Economic Letters, 13, pp. 337-341.

[24] WEST, K. D., 1986 *A Variance Bound Test of the Linear Quadratic Inventory Model*, Journal of Political Economy, Vol. 94, pp. 374-401.