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New Bounds for the Asymmetric Travelling Salesman Problem

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Abstract: Recently new Lagrangean techniques for producing lower bounds have appeared in the literature: bound improvement sequences and Lagrangean decomposition. The combination of these approaches yields the Improved Lagrangean decomposition. In this work we will apply this combined approach to the Asymmetric Travelling Salesman Problem. The results obtained using the Improved Lagrangean decomposition on the r-Arborescence formulation of the (ATSP) show that these new bounds often bridge the duality gap without needing valid inequalities or Branch&Bound techniques.
We also present an extended version of the subgradient method developed to speed up the computation of the bounds aforementioned.
Computational experience is reported.

Keywords: Integer programming, duality gap, Lagrangean relaxation, bound improvement sequences, Lagrangean decomposition, integrality property, $\mathfrak{X}$-convexity, subgradient method.

October 90

1. Introduction

Recently new techniques for producing lower bounds have appeared in the literature: bound improvement sequences, *Bárcia* [1985], *Bárcia* [1987] and *Bárcia & Holm* [1988] and Lagrangean decomposition, *Jörnsten & Näsberg* [1986] and *Guignard & Kim* [1987].

Bárcia & Jörnsten [1990] have combined these approaches yielding the Improved Lagrangean decomposition which was applied with success to the Generalized Assignment Problem.

In this work we will follow this path applying this combined approach to the Asymmetric Travelling Salesman Problem, (ATSP).

The results obtained using the Improved Lagrangean decomposition show that these new bounds often bridge the duality gap without needing valid inequalities or Branch&Bound techniques.

This work is organized as follows:

In section 2 we will briefly sketch the bound improving sequence technique which consists in building a sequence of Lagrangean duals that progressively reduces the duality gap. Section 3 deals with the main ideas of Lagrangean decomposition. In section 4 we will see how these two ideas can be combined yielding stronger bounds.

Section 5, which contains the main contributions of this paper, discusses the issues related with the application of the previous ideas to the Travelling Salesman Problem: We start by exploiting the r-Arborescence structure of the Asymmetric TSP and we show that the formulation thus obtained is as strong as the usual one. Then we apply the improved Lagrangean decomposition concept on this formulation obtaining bounds which are able to bridge the duality gap. Finally we present a new extended version of the subgradient method which was developed to speed up the computation of the bounds aforementioned.

We terminate the paper with some computational experience on randomly generated problems and we provide some concluding remarks.

2. Bound improving sequence

We start by presenting a brief sketch of the basic ideas and main results of the bound improving sequence technique.

One way of getting lower bounds for a difficult minimization problem is to relax some of its constraints in order to obtain an easier problem. But if the corresponding Lagrangean dual satisfies the integrality property then its bound does not dominate the bound given by the Linear relaxation, see *Geoffrion* [1974].

One recent and useful technique is to add to the original formulation a redundant constraint which will reduce the set of feasible solutions of the Lagrangean dual. This way we can expect to obtain better lower bounds.

Consider the following integer linear programming problem:

$$(P) \quad z = \min \; cx \qquad Ax \le b \qquad x \in \mathfrak{X}$$

where c is an integer n-vector, b an integer m-vector and A a n×m-matrix of integers.

For simplicity we will assume that $\mathfrak{X}$ is bounded and that (P) has a finite solution.

If a lower bound for z, l_k, is known then we can construct an alternative formulation for (P) using a redundant constraint $cx \ge l_k$:

$$(P_k) \quad z = \min \; cx \qquad Ax \le b \qquad x \in \mathfrak{X}_k = \{ x \in \mathfrak{X} : cx \ge l_k \}$$

Using Lagrangean relaxation we can develop the following Lagrangean dual:

$$(LD_k) \quad l_{k+1} = \underset{\mu \in \Re^m_+}{\text{Max}} \; \underset{x \in \mathfrak{X}_k}{\min} \; cx - \mu(b - Ax) \qquad k=0, 1, \ldots$$

The usual Lagrangean bound, l, can be obtained by solving (LD_0) where $l_0 = -\infty$.

Solving (LD_k) depends mostly on what kind of subproblem turns up. For instance if $\mathfrak{X}=\{0,1\}^n$ then the subproblem consists of a binary Knapsack Problem, which is one of the easiest non-polynomial problems, see *Walukiewicz & Dudzinsky* [1987].

Using this technique we obtain a non-decreasing sequence of lower bounds for (P), $l=l_0\leq l_1\leq l_2\leq\ldots$

Bárcia [1985], *Bárcia* [1987] and *Bárcia & Holm* [1988] present necessary conditions for equality to hold in that sequence. Conditions for a strict increasing sequence of lower bounds are also presented. Nevertheless these conditions are too difficult to verify in practical situations.

Another result pointed out is that to optimize (LD_k) is equivalent to convexifying the non relaxed constraints in (P_k):

$$\begin{aligned} l_{k+1} = \min\ & cx \\ Ax &\leq b \\ x &\in Co(\mathfrak{X}_k) \end{aligned}$$

3. Lagrangean decomposition

In this section we will present the basic ideas and main results of the Lagrangean decomposition technique.

The usual Lagrangean approach is based on the identification of a group of "nice" constraints which define a problem for which we know an efficient algorithm. However it is possible to find for the same problem two or more sets of "nice" constraints. A straightforward possibility is to solve each of these Lagrangean duals and to consider the best Lagrangean bound among them.

Jörnsten & Näsberg [1986] and *Guignard & Kim* [1987] have presented a new Lagrangean relaxation which takes advantage of the whole structure of the problem:

Consider the following integer linear programming problem:

$$\text{(P)}\quad \begin{aligned} z = \min\ & cx \\ Ax &\leq b \\ Bx &\leq d \\ x &\in \mathfrak{X} \end{aligned}$$

where the problems defined by the constraints $\{ x \in \mathfrak{X} : Ax \leq b \}$ and by $\{ x \in \mathfrak{X} : Bx \leq d \}$ are easier to solve than the original problem.

If we duplicate the variables, x, we can restate the problem (P) as:

$$
\begin{array}{lllll}
\text{(DP)} & z = \min [& \alpha cx & + & \beta cy \;] \\
& & Ax \leq b & & \\
& & x \in \mathfrak{X} & & \\
& & & x = y & \\
& & & & By \leq d \\
& & & & y \in \mathfrak{Y}
\end{array}
$$

assuming that $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ and that $\mathfrak{X} \subsetneq \mathfrak{Y}$.

Relaxing the equality constraints we obtain two subproblems, x- and y-subproblems, which are separable.

The associated Lagrangean Dual will be:

$$
d = \underset{\mu \in \Re^n}{\text{Max}} \left[\begin{array}{c} \min \; (\alpha c + \mu)x \\ Ax \leq b \\ x \in \mathfrak{X} \end{array} + \begin{array}{c} \min \; (\beta c - \mu)y \\ By \leq d \\ y \in \mathfrak{Y} \end{array} \right]
$$

This Lagrangean Dual takes full advantage of the problem structure as it explores both substructures at the same time.

Guignard & Kim [1987] have proved that this new lower bound, d, dominates the bounds obtained by relaxing any of the two sets of constraints.

Lagrangean relaxation does not dominate the bound given by the Linear relaxation when its dual has the integrality property, see *Geoffrion* [1974]. This same concept can be generalized to the Lagrangean decomposition, see *Guignard & Kim* [1987]. In order to clarify this point we will describe the primal interpretation of Lagrangean decomposition:

$$
\begin{array}{ll}
d = \min & cx \\
& x \in \mathrm{Co}\{ x : Ax \leq b, x \in \mathfrak{X} \} \\
& x \in \mathrm{Co}\{ x : Bx \leq d, x \in \mathfrak{Y} \}
\end{array}
$$

i.e., computing d is equivalent to optimize the primal objective function on the intersection of the convex hulls of two constraints sets.

Now it is clear that if we have $Co\{x: Ax \le b, x \in \mathfrak{X}\} = \{x: Ax \le b, x \in Co(\mathfrak{X})\}$ (the $\mathfrak{X}$-convexity property) then d will be equal to the Lagrangean bound one gets when constraints $Ax \le b$ are relaxed.

Improved Lagrangean decomposition

Both techniques described seek to obtain stronger lower bounds. *Bárcia & Jörnsten* [1990] have developed an approach which combine these two techniques.

The main idea is to strengthen the Lagrangean decomposition in order to obtain a lower bound which dominates the bounds generated by both techniques individually.

So if we have a lower bound for (P), l_k, we can restate (P) as follows:

$$(DP_k)\ z = \min \left[\alpha cx + \beta cy \right] \quad \begin{array}{l} Ax \le b \\ x \in \mathfrak{X}_k \\ x = y \\ By \le d \\ y \in \mathfrak{Y} \end{array}$$

assuming that $\alpha, \beta > 0$ such that $\alpha+\beta=1$ and that $\mathfrak{X}_k \subseteq \mathfrak{Y}$.

Relaxing the equality constraints we obtain the following Dual:

$$(LD_k)\ d_{k+1} = \max_{\mu \in \Re^n} \left[\min_{\substack{Ax \le b \\ x \in \mathfrak{X}_k}} (\alpha c+\mu)x + \min_{\substack{By \le d \\ y \in \mathfrak{Y}}} (\beta c-\mu)y \right]$$

Each (LD_k) yields a lower bound and so we can construct a non-decreasing sequence of bounds. Now $\mathfrak{X}$ can have a more complicated structure, say $\{0,1\}^n$ with Semi-Assignment constraints but then the x-subproblem consists of a binary Multiple Choice Knapsack Problem which is as easy as the binary Knapsack Problems, see *Walukiewicz & Dudzinsky* [1987].

The Improved Lagrangean decomposition yields bounds that dominate the ones obtained when applying each technique individually:

- $d_{k+1} \ge l_{k+1}$
- $d_{k+1} \ge d_k \ge d$

The following primal interpretation of this dual helps to understand the cases where we have strict dominance of the sequence of bounds.

$$d_{k+1} = \min_{x \in Co\{ x \in \mathfrak{X} : cx \geq d_k, Ax \leq b \} \cap Co\{ x \in \mathcal{Y} : Bx \leq d \}} cx$$

5. Bounds for the Asymmetric Travelling Salesman Problem

5.1 Formulations for the Travelling Salesman Problem

The Travelling Salesman Problem (TSP) is the problem of finding a minimum cost tour so that each of n cities is visited exactly once.

One of the most well-known mathematical formulations for the Asymmetric Travelling Salesman Problem is due to *Dantzing, Fulkerson & Johnson* [1954].

Considering a direct and weighted graph $\mathcal{G} = (N, A, C)$ with node set $N=\{1, 2, \dots, n\}$ and an arc set $A = N \times N \setminus \{(i,i)\}$, and weights $c_{ij} \geq 0$, $(i,j) \in A$, and let x_{ij} $(i,j) \in A$:

$$x_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ belongs to the problem solution} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{array}{llll}
\text{(ATSP)} \quad \min & \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij} & & (0) \\
\text{s.t.:} & \sum_{j \in N} x_{ij} = 1 & i \in N & (1) \\
 & \sum_{i \in N} x_{ij} = 1 & j \in N & (2) \\
 & \sum_{i \in S} \sum_{j \notin S} x_{ij} \geq 1 & \forall S \subset N: 1 \leq |S| < n & (3) \\
 & x_{ij} \in \{0, 1\} & i, j \in N & (4)
\end{array}$$

This formulation is redundant. The assignment constraints, (1) and (2), ensure that the solution has disjoint subtours. If we impose r-connectivity, i.e. from a specific node r all the others can be reached, then only tours are feasible and so constraints (3) can be halved:

$$(\overline{\text{ATSP}})_r \qquad \min \sum_{i\in N}\sum_{j\in N} c_{ij}\, x_{ij} \qquad (0)$$

$$\text{s.t.:} \quad \sum_{j\in N} x_{ij} = 1 \qquad i \in N \qquad (1)$$

$$\sum_{i\in N} x_{ij} = 1 \qquad j \in N \qquad (2)$$

$$\sum_{i\in S}\sum_{j\notin S} x_{ij} \geq 1 \qquad \forall S \subset N : r \in S,\ |S| < n \qquad (3')$$

$$x_{ij} \in \{0, 1\} \qquad i, j \in N \qquad (4)$$

These two formulations are obviously identical but the last one yields the possibility of developing a new procedure to achieve lower bounds for the (ATSP).

The Linear relaxation of the (ATSP) associated with the first formulation, $(\overline{\text{ATSP}})$, is known for producing good bounds, see *Christofides* [1971]. So it is interesting to find out if the bound produced by $(\overline{\text{ATSP}})_r$ is as good.

The following results are an answer to this question.

<u>Theorem 1</u>: For all $r \in N$ the set of feasible solutions of $(\overline{\text{ATSP}})_r$ and the set of feasible solutions of $(\overline{\text{ATSP}})$ are the same.

Proof:

Let

$$\mathcal{A} = \{ x : x_{ij} \in [0, 1],\ \sum_{j\in N} x_{ij} = 1,\ \sum_{i\in N} x_{ij} = 1 \ (i, j \in N) \}$$

$$\mathcal{C} = \{ x : x \in \mathcal{A},\ \sum_{i\in S}\sum_{j\notin S} x_{ij} \geq 1,\ \forall S \subset N : 1 \leq |S| < n \}$$

$$\mathcal{C}_r = \{ x : x \in \mathcal{A},\ \sum_{i\in S}\sum_{j\notin S} x_{ij} \geq 1,\ \forall S \subset N : r \in S,\ |S| < n \}$$

$\mathcal{C}_r \supseteq \mathcal{C}$.

$\mathcal{C} \overset{?}{\supseteq} \mathcal{C}_r$. By contradiction, suppose that exists $x \in \mathcal{C}_r$ such that $x \notin \mathcal{C}$ then:

$\exists\, S \subset N,\ 1 \leq |S| : \sum_{i\in S}\sum_{j\notin S} x_{ij} < 1$ and such that $r \notin S$, so we have $\sum_{i\notin S}\sum_{j\in S} x_{ij} \geq 1$ and then we get: $\sum_{i\in S}\sum_{j\notin S} x_{ij} - \sum_{i\notin S}\sum_{j\in S} x_{ij} < 0 \quad (*)$

As $x \in \mathcal{A}$: $\sum_{i\in N} x_{ij} = \sum_{i\in S} x_{ij} + \sum_{i\notin S} x_{ij} = 1$ and $\sum_{j\in N} x_{ij} = \sum_{j\in S} x_{ij} + \sum_{j\notin S} x_{ij} = 1$ we can rewrite (*) as:

$$\sum_{i\in S}\sum_{j\in S} \qquad\qquad \sum_{i\notin S}\sum_{j\notin S} \quad (*)$$

and similarly as:

$$\sum_{i\in S}\sum_{j\notin S} x_{ij} - \sum_{i\notin S}\sum_{j\in S} x_{ij} = n - |S| - \sum_{i\notin S}\sum_{j\notin S} x_{ij} - |S| + \sum_{i\in S}\sum_{j\in S} x_{ij} \underset{(*)}{<} 0$$

The above inequalities are not compatible. This contradiction shows that $\mathbf{C} \supseteq \mathbf{C}_r$.

$\therefore$

The following result arises naturally:

<u>Corolary 1</u>: $V(\overline{ATSP})_r = V(\overline{ATSP})$ for all $r \in N$

are good in the asymmetric case, see *Balas & Toth* [1985].

If we drop constraints (1) we will have the Min-sum r-Arborescence Problem. This problem can be solved in $O(n^2)$ time by finding the Shortest Spanning Arborescence rooted at node **r** (see *Fischetti & Toth* [1989], *Edmonds* [1967]) and by adding the minimum-cost arc entering node **r**. However the Lagrangean dual with the Min-sum r-Arborescence as it subproblem satisfies the integrality property so its optimum value is $V(\overline{ATSP})_r = V(\overline{ATSP})$.

5.2 New Bounds

We can identify in the $(ATSP)_r$ formulation two sets of "nice" constraints: (1) the Semi-Assignment constraints and (2), (3') the r-Arborescence constraints.

By considering the Lagrangean decomposition we can not expect to obtain a better bound than the bound given by $(\overline{ATSP})$ because both x- and y- subproblems have the $\mathfrak{X}$-convexity property.

Trying to break the $\mathfrak{X}$-convexity property we can introduce in the $(\text{ATSP})_r$ formulation the redundant constraint $\sum_{i\in N}\sum_{j\in N} c_{ij}\, x_{ij} \geq d_k$ where d_k is a lower bound for (ATSP).

The Improved Lagrangean decomposition reformulation will be (letting α=1 and β=0):

$$
\begin{array}{llll}
\min & \sum_{i\in N}\sum_{j\in N} c_{ij}\, x_{ij} & & \\
\text{s.t:} & \sum_{j\in N} x_{ij} = 1 & & i\in N \\
& \sum_{i\in N}\sum_{j\in N} c_{ij}\, x_{ij} \geq d_k & & \\
& x_{ij} \in \{0,1\} & & i,j\in N \\
& \quad x_{ij} = y_{ij} & & i,j\in N \\
& & \sum_{i\in N} y_{ij} = 1 & j\in N \\
& & \sum_{i\in S}\sum_{j\notin S} y_{ij} \geq 1 & \forall S\subset N: r\in S,\ |S|<n \\
& & y_{ij} \in \{0,1\} & i,j\in N
\end{array}
$$

Relaxing now the equality constraints we get:

$$
d_{k+1} = \max_{\mu_{ij}\in \Re} \left[\begin{array}{lll} \min & \sum_{i\in N}\sum_{j\in N} (c_{ij} - \mu_{ij})\, x_{ij} & \\ \text{s.t:} & \sum_{j\in N} x_{ij} = 1 & i\in N \\ & \sum_{i\in N}\sum_{j\in N} c_{ij}\, x_{ij} \geq d_k & \\ & x_{ij} \in \{0,1\} & i,j\in N \end{array} + \begin{array}{lll} \min & \sum_{i\in N}\sum_{j\in N} \mu_{ij}\, y_{ij} & \\ \text{s.t:} & \sum_{i\in N} y_{ij} = 1 & j\in N \\ & \sum_{i\in S}\sum_{j\notin S} y_{ij} \geq 1 & \forall S\subset N: r\in S, |S|<n \\ & y_{ij} \in \{0,1\} & i,j\in N \end{array} \right]
$$

Thus x-subproblem is now a Multiple Choice Knapsack Problem which does not have the $\mathfrak{X}$-convexity property and so we can hope that the bounds given will bridge the duality gap.

To solve each of these duals we use the subgradient method, see *Shapiro* [1979]. In fact we can guarantee that the first bound obtained is greater or equal to the bound given by the resolution of $(\overline{\text{ATSP}})$ using:

$$\mu_{ij}^0 = \frac{c_{ij} - \bar{\lambda}_i}{2} \ (i,j\in N) \text{ and } d_0 = \lceil V(\overline{\text{ATSP}}) \rceil$$

where $\bar{\lambda}_i$ are the optimum values of the linear dual variables associated with constraints (1).

This practical result may seem too demanding in the sense that it asks for the optimization of $(\overline{ATSP})$. However, if we use the Lagrangean relaxation with the Min-sum r-arborescence as its subproblem we can replace $\bar{\lambda}_i$ by the best Lagrangean multipliers, λ_i^*, found during the optimization of the associated dual.

Once solved each dual if $d_{k+1}>d_k$ then a new dual is produced otherwise we reached the optimal dual solution.

5.3 Extended subgradient method

The method described is computationally expensive due to the optimization of the x-subproblems.

However considering only a fixed root **r** we are not taking full advantage of the y structure. Considering all the n min-sum r-Arborescence problems, with $r \in N$, we can obtain the best bound in only $O(n^3)$ time.

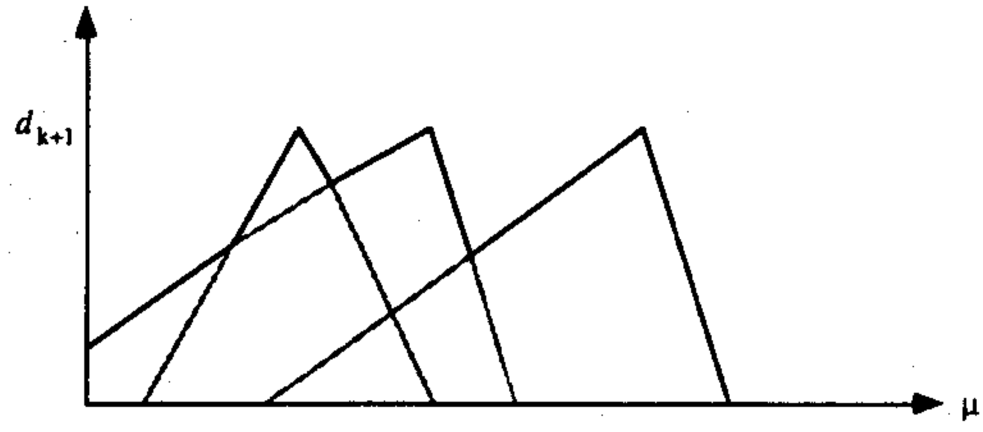
In this way we obtain the best possible bound of each set of multipliers and so we can expect to reduce the number of iterations of the subgradient method and so the number of binary Multiple Choice Knapsacks Problems to be solved.

So we developed the following extended dual, (ED_{k+1}):

$$d_{k+1} = \underset{\mu_{ij} \in \Re}{\mathrm{Max}} \left[\min \sum_{i \in N} \sum_{j \in N} (c_{ij} - \mu_{ij})\, x_{ij} + \underset{r \in N}{\mathrm{Max}} \left[\min \sum_{i \in N} \sum_{j \in N} \mu_{ij}\, y_{ij} \right] \right]$$

$$\begin{array}{llll} \text{s.t:} \sum_{j \in N} x_{ij} = 1 & i \in N & \text{s.t:} \sum_{i \in N} y_{ij} = 1 & j \in N \\ \sum_{i \in N} \sum_{j \in N} c_{ij}\, x_{ij} \geq d_k & & \sum_{i \in S} \sum_{j \notin S} y_{ij} \geq 1 & \forall S \subset N: r \in S, |S| < n \\ x_{ij} \in \{0, 1\} & i,j \in N & y_{ij} \in \{0, 1\} & i,j \in N \end{array}$$

Note that the function to be maximized is no longer concave as we can see on the following bidimensional example, where the bold line represents the $(ED_{k+1},(\mu))$:



To optimize (ED_{k+1}) we adapted the subgradient method.

The two procedures, though very similar, differ in one aspect. Instead of using the subgradient of the function to be maximized on each iteration, which does not exist, we use the subgradient of the subfunction which gave the best bound for the current set of multipliers.

Next theorem shows that the extended subgradient method will converge to the optimum value of (ED_{k+1}), i.e. d_{k+1}:

<u>Theorem 2</u>: Consider a finite number of concave functions, $f_i(y)$ $i \in \mathbf{I}$ and for any $y \in \Re^n$, and denote by $\partial f_i(y)$ its subdiferential. Assume also that all these functions share the same maximal value F^*.

Define $F(y) = \underset{i \in \mathbf{I}}{\mathrm{Max}}\, f_i(y) = f_k(y)$ and $\gamma(y) \in \partial f_k(y)$.

Define $y^{t+1} = y^t + \theta_t\, \gamma(y^t)$ with $\theta_t = p_t \dfrac{F^* - F(y^t)}{\|\gamma(y^t)\|^2}$ and $\varepsilon_1 < p_t < 2 - \varepsilon_2$

If $\exists\, M>0 : \|\gamma(y^t)\| < M$ then there exists a sub-sequence $\{y^t\}_{t \in \mathbf{L}}$, $\mathbf{L} \subsetneq \mathbf{N}$, such that: $\lim\limits_{t \in \mathbf{L}} F(y^t) = F^*$

Proof:

Notice that $F^* = f_t(y_t^*) = \underset{y}{\mathrm{Max}}\, f_t(y)$.

$$\|y^{t+1} - y_t^*\|^2 = \|y^t + \theta_t \gamma(y^t) - y_t^*\|^2 = \|y^t - y_t^*\|^2 + \theta_t^2 \|\gamma(y^t)\|^2 - 2\theta_t (y^t - y_t^*)\gamma(y^t) \le$$

By the concavity of f_t:

$$\le \|y^t - y_t^*\|^2 + \theta_t^2 \|\gamma(y^t)\|^2 - 2\theta_t [f_t(y_t^*) - f_t(y^t)] =$$

$$= \|y^t - y_t^*\|^2 + \frac{(p_t^2 - 2p_t)\,[F^* - F(y^t)]^2}{\|\gamma(y^t)\|^2}$$

Let $\Delta_t = \frac{[F^* - F(y^t)]^2}{\|\gamma(y^t)\|^2} > 0$ then $(p_t^2 - 2p_t)\Delta_t = p_t(p_t - 2)\Delta_t \le -\varepsilon_1\varepsilon_2\Delta_t = -\varepsilon\Delta_t \ (\varepsilon > 0)$

And finally $\| y^{t+1} - y_t^* \|^2 \le \| y^t - y_t^* \|^2 - \varepsilon\Delta_t \ \ \forall\, t \in \mathbb{N}$

As y_t^* is a maximizer of one of the functions $f_i(y)$, and as there is only a finite number of those functions, there exists at least one subsequence of indexes, $t \in \mathbb{L} \subseteq \mathbb{N}$, for which the maximizer y_t^* is always taken on the same function f_i. Thus y_t^* will not depend on the choice of $i \in \mathbb{L}$ and so we will write y^* instead of y_t^*.

Then on this subsequence we shall have $\| y^{t+1} - y^* \|^2 \le \| y^t - y^* \|^2 - \varepsilon\Delta_t \ \forall\, t \in \mathbb{L}$ (1)
This inequality shows that the sequence $\| y^t - y^* \|^2$, $t \in \mathbb{L}$, is non-increasing and bounded by zero which implies that it has a limit. Now if we take the limit of (1) along the subsequence $t \in \mathbb{L}$ we shall have:

$\lim\limits_{t \in \mathbb{L}} \Delta_t = \lim\limits_{t \in \mathbb{L}} \frac{[F^* - F(y^t)]^2}{\|\gamma(y^t)\|^2} = 0$ and as F is continuous and $\|\gamma(y^t)\| < M$ then:

$\lim\limits_{t \in \mathbb{L}} F(y^t) = F^*$

$\therefore$

6. Computational Results

For the computational tests we used a PC-IBM compatible, Unisys PW-800 with a CPU 80386 at 20 MHz. MICROSOFT FORTRAN and Pascal compilers were used.

The cost matrixes of the test problems were randomly generated. Two classes of test problems were considered: class I with costs in the set {0, 1,...,50} and class II with costs in the set {0, 1,...,500}. Notice that as the costs, c_{ij}, are integer the value of the objective function will be integer. In order to obtain the optimum value of the test problems we used a Branch&Bound algorithm, see *Gromicho* [1990].

The computational experience reported about the improved Lagrangean decomposition is limited by the code for the Multiple Choice Knapsack Problem. The code used does not take any advantage of the characteristics of our subproblem.

We use the Lagrangean relaxation with the Min-sum r-Arborescence problem to approximate the bound given by the Linear relaxation and we initialized the extend subgradient method multipliers accordingly.

On the following results we have on column **Opt** the optimum value, column **LP** the bound given by the linear programme, followed by the column **Gap** which represents the gap percentage. The other results concern the improved Lagrangean decomposition: column **ED** is the bound given by the improved Lagrangean decomposition; column **Gap** is the gap percentage; column **BIt** is the iteration where the best (integer) bound was achieved; column **TIt** is the total number of iterations performed and 0^{+} denotes the cases where the bounds given by the improved Lagrangean dual did not manage to increase the Linear bound in one unity.

6.1 Class I

The following examples were taken from a sample of a hundred instances with five nodes:

Opt	LP	Gap%	ED	Gap%	BIt	TIt
82	81.00	1.2	81.11	*	7	7
45	42.50	5.1	106.38	1.5	52	151
56	52.50	6.3	55.01	*	97	97
69	68.00	1.4	68.07	*	5	5
114	108.50	4.8	113.29	*	25	25
88	86.67	1.5	87.10	*	5	5
111	107.00	3.6	108.92	1.9	13	132
61	59.50	2.5	60.36	*	3	3
84	75.00	10.7	80.42	4.3	118	151
61	52.00	14.8	57.56	5.6	58	151
78	73.50	5.8	75.80	2.8	46	151
		5.3		1.3	37	75

The following examples were taken from a sample of fifty instances with ten nodes:

Opt	LP	Gap%	ED	Gap%	BIt	TIt
64	62.99	1.6	63.15	*	10	10
121	116.95	3.3	117.99	2.5	33	151
76	75.00	1.3	75.00	1.3	0	151
92	89.99	2.2	90.99	1.1	15	151
92	87.46	4.9	88.80	3.5	41	151
92	88.98	3.3	88.99	3.3	0	151
53	51.00	3.8	51.00	3.8	0	151
79	74.48	5.7	77.18	2.3	116	151
97	96.00	1.0	96.00	1.0	0	151
93	91.64	1.5	92.06	*	27	27
90	83.41	7.3	83.97	6.7	0^{+}	151
109	102.36	6.1	102.95	5.5	0^{+}	151
72	67.94	5.6	69.92	2.9	65	151
		3.7		2.6	24	131

The following examples were taken from a sample of twenty-five instances with fifteen nodes:

Opt	LP	Gap%	ED	Gap%	BIt	TIt
105	101.20	3.6	101.60	3.2	0^+	151
66	65.00	1.5	65.00	1.5	0	151
93	92.00	1.1	92.00	1.1	0	151
77	75.86	1.5	75.93	1.4	0^+	151
72	70.50	2.1	71.04	*	100	100
127	125.14	1.5	125.57	1.1	0^+	151
94	90.75	3.5	90.87	3.3	0^+	151
70	68.50	2.1	68.75	1.8	0^+	151
		2.1		1.7	12	151

6.2 Class II

The following examples were taken from a sample of a hundred instances with five nodes:

Opt	LP	Gap%	ED	Gap%	BIt	TIt
697	614.50	11.8	656.84	5.8	108	151
947	934.00	1.4	946.26	*	8	8
617	575.50	6.7	612.53	0.7	75	151
847	730.50	13.8	835.14	1.4	150	151
724	705.50	2.6	721.71	0.3	51	151
782	753.50	3.6	781.03	*	108	108
894	884.50	1.1	893.35	*	5	5
973	963.00	1.0	972.05	*	16	16
829	808.50	2.5	828.25	*	20	20
820	750.50	8.5	776.38	5.3	135	151
793	735.00	7.3	783.58	1.2	120	151
482	439.00	8.9	478.53	0.7	104	151
		5.8		1.3	75	102

The following examples were taken from a sample of fifteen instances with ten nodes:

Opt	LP	Gap%	ED	Gap%	BIt	TIt
696	665.40	4.4	679.64	2.4	139	151
752	748.50	0.5	751.08	*	68	68
595	539.30	9.4	552.34	7.2	133	151
608	571.3	6.0	580.73	4.5	118	151
		5.1		3.5	115	130

The following examples were taken from a sample of ten instances with fifteen nodes:

Opt	LP	Gap%	ED	Gap%	BIt	TIt
885	876.00	1.0	876.00	1.0	0	151
762	749.70	1.6	750.06	1.6	151	151
971	935.50	3.7	936.64	3.5	118	151
		2.1		2.0	90	151

The following examples were taken from a sample of ten instances with twenty nodes:

Opt	LP	Gap%	ED	Gap%	BIt	TIt
659	658.00	0.2	658.14.	*	25	25
720	694.50	3.5	698.49	3.0	136	151
861	854.00	0.8	854.64	0.7	91	151
834	823.50	1.3	823.74	1.2	0^+	151
		1.5		1.2	63	120

6.3. Coments

The bounds given by the improved Lagrangean dual do manage to bridge the duality gap for both classes of test problems as we can see on the results above.

7. Conclusions

Using the improved Lagrangean decomposition we manage to bridge the duality gap without using valid inequalities or a Branch&Bound method. We can expect to achieve the same kind of results, or better ones, with other problems.

The extension of the subgradient method presented may also be applied to another class of problems with the same characteristics.

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