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A SIMPLE PROOF OF THE BROUWER FIXED POINT THEOREM

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ABSTRACT: We present a simple proof of the Brouwer Fixed Point Theorem using only basic concepts of point-set topology.

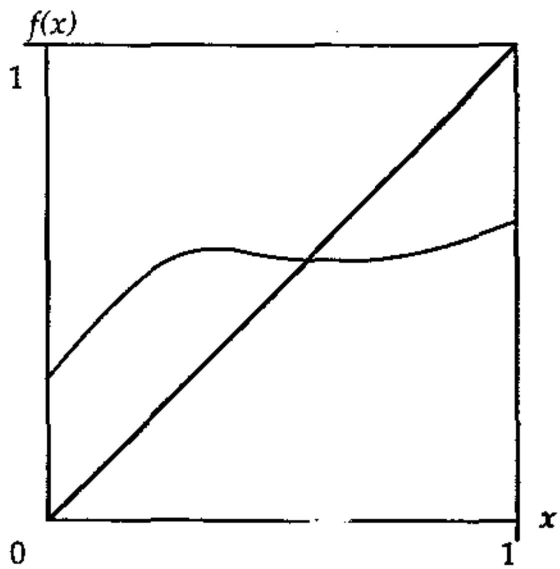
KEYWORDS: Fixed Point Theory, Economics.

* I am grateful to Don Brown, Peter DeMarzo, Michi Kandori, Bill Novshek, and Jon Paul for useful comments and suggestions. I am particularly grateful to Herbert Scarf for an excellent lecture on Sperner's lemma and Fixed Point theory. Responsibility for remaining errors and deficiencies is solely mine.

1. INTRODUCTION

The Brouwer Fixed Point Theorem has become a standard tool in economic theory (Border, 198; Scarf, 1973). Unfortunately, most available proofs of the theorem are rather technical (Boothby, 1971; Hilton and Wylie, 1960; Hirsch, 1963; Scarf, 1973) or somewhat lengthy (Franklin, 1980; Kannai, 1979; Milnor, 1978).

The Brouwer Fixed Point Theorem states that for every continuous map f of a compact, convex set onto itself, there exists at least one fixed point, i.e. a point x such that $f(x) = x$. If f is a function of one variable, then the result is geometrically intuitive, as can be seen from Figure 1: the graph of f must cross the 45^{0} line at least once. Unfortunately, it is difficult to generalize this intuition to the case when f is a function of several variables. In this note, we present a simple geometrical proof of Brouwer's theorem using only basic concepts of point-set topology.



2. THE THEOREM

We begin by proving the No-Retraction Theorem, which is equivalent to the Brouwer Fixed Point Theorem.

THEOREM (No Retraction Theorem): *There exists no continuous map f of the unit disk onto its boundary, such that $f(x) = x$ for all points x in the boundary.*

PROOF: Consider a map f of the unit disk D onto its boundary. (See Figure 2.) Points A and B define a connected section S of the boundary. Since D is bounded and f is continuous, $f^{-1}(S)$ is a compact set (the shaded area in Figure 2). The boundary of $f^{-1}(S)$ consists of k connected components (cf Theorem 5.1 in Massey, 1967). One of these components includes points A and B.[1]

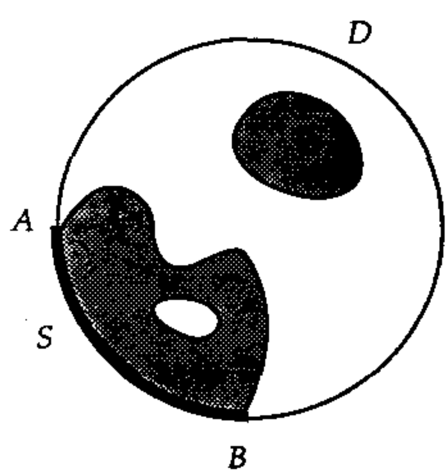


FIGURE 2

Denote by T the set of points along this component of the boundary which are interior to D. One can easily check that T must be connected.[2] Moreover, it can be shown that points in

[1] Note that $A, B \in S$, and S is part of the boundary of $f^{-1}(S)$.

[2] The case when T is not connected is depicted in Figure 3. Consider a sequence of points converging to E (line r). Since $r \notin f^{-1}(S)$, the image of points in r lies outside of S, and thus continuity is violated at E.

T must be mapped to either A or B: if that were not the case, one could find a sequence of points in $D \setminus f^{-1}(S)$ converging to a point in T, and continuity would be violated.[3] But then we conclude that f maps a connected set, T, onto a non-connected set, $\{A, B\}$, which implies that f is not continuous. ∎

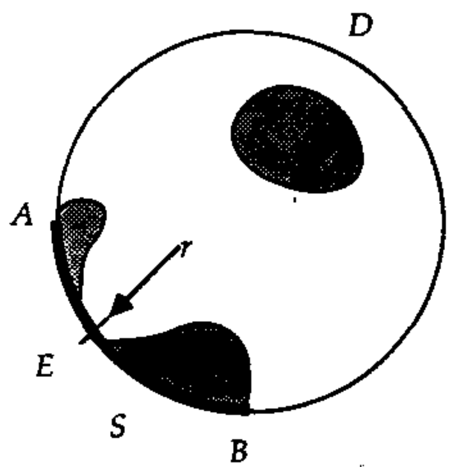


FIGURE 3

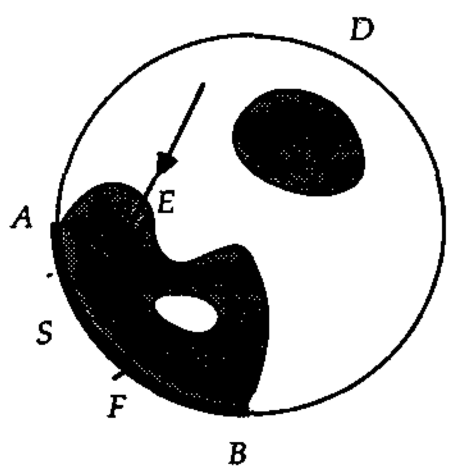


FIGURE 4

[3] See Figure 4. Consider a sequence of points in r which converge to E, a point on the boundary T. If $f(E) \neq A, B$, then it must be $f(E) = F$, where F is some interior point of S. Since ponts in r are mapped to points outside of S, continuity is violated at E.

The Brouwer Fixed Point Theorem follows from the No-Retraction Theorem by an elegant and well-known argument.[4]

THEOREM (Brouwer Fixed Point Theorem): *Every continuous map f of the disk D to itself possesses at least one fixed point.*

PROOF: Suppose that f has no fixed points. For each x, define $h(x)$ as the projection of $f(x)$ on the boundary of D through the pre-image x. (See Figure 5.) Clearly, h is a continuous map of the unit disk onto its boundary, such that $h(x)=x$ for points x in the boundary—a contradiction with the No Retraction Theorem. ∎

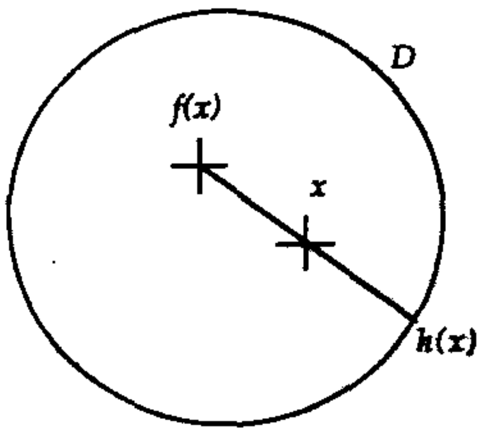


FIGURE 5

[4] The No-Retraction and Brouwer Fixed Point theorems are actually equivalent. See Border (1985, pp. 28-30) for the reverse implication.

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OPTIMAL MATCHING AUCTIONS

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ABSTRACT: Suppose a firm (or a government) wants to buy a given object from one of two possible suppliers. The firm's optimal solution, i.e., the one which minimizes expected payment, has been characterized by Myerson (1981). We present a modified "matching" mechanism which implements Myerson's optimal solution. Supplier 1 first submits a bid, and supplier 2 is then given the option of matching a function of that bid.

KEY WORDS: Optimal auctions, matching auctions, procurement.

* This note is partly based on research done while I was visiting the General Motors Research Laboratories. I am grateful to the Laboratories. in particular the Operating Sciences Department, for their hospitality. I am also grateful to Charlie Bird, Preston McAfee, Paul Milgrom, Mike Riordan, and Marc Robinson for useful comments and suggestions. The usual disclaimer applies.

1. INTRODUCTION

In a seminal paper, Myerson (1981) characterized the optimal solution for auctioning an object which valuation is privately known by each of N bidders. Based on the revelation principle, one can restrict to direct revelation mechanisms, and by optimizing over this set we obtain the characterization of an optimal mechanism. While the problem solved by Myerson (1981) was that of selling an object, an analogous solution applies to the problem of buying an object which production cost is privately known by each of N potential sellers. (In most of this paper, we will consider the latter, which arises frequently in the context of government and industry procurement.)

Direct revelation mechanisms, the method considered by Myerson (1981), are seldom used in practice, if ever. On the other hand, while capable of implementing the optimal solution, they are not the <u>only</u> way of doing it. In fact, if bidders are symmetric, then both the English auction and the first-price sealed bid auction implement the optimal solution. Even if bidders are not symmetric, a modified sealed-bid auction—a mechanism frequently used both in government and in industrial procurement—implements the optimal solution.[1]

In this note, we consider the implementation of the buyer's optimal solution by means of a modified "matching" mechanism similar to the current practice of various corporations. In a (simple) matching mechanism, one of the suppliers first submits a bid. The other supplier is then given the option of matching that bid, or else the first supplier is chosen. The modified matching mechanism works in the same way except that the second supplier (typically the incumbent supplier) has to match a function of the first supplier's bid, the identity function corresponding to the simple matching mechanism. We show that in general the optimal solution can be implemented by means of a modified matching mechanism. We also provide an example where the matching function is explicitly computed.

[1] See McAfee and McMillan (1985). In a simple sealed-bid auction, sellers submit bids simultaneously, and the lowest bid is selected by the buyer. In a modified sealed-bid auction, the buyer credibly commits to select the seller based on a function which compares the various bids, the identity function corresponding to the simple sealed-bid auction.

2. MAIN RESULT

Consider a buyer who wants to acquire a given object, and two possible suppliers, 1 and 2. Each supplier privately knows his own cost. It is common knowledge that costs are independently drawn from the distribution functions $F_i(c_i)$, both continuously differentiable and with a positive derivative $f_i(c_i)$ on $[\alpha_i, \beta_i]$, $i = 1,2$. Define $J_i(c_i) \equiv c_i + \frac{F_i(c_i)}{f_i(c_i)}$, $i = 1, 2$. Throughout the analysis, we will make the regularity assumption that $J_i(c_i)$ is a differentiable increasing function.[2] For simplicity, we will also assume that in-house production is not a viable alternative, i.e. the reservation price is infinity.[3]

The optimal solution from the buyer's point of view, i.e., that which minimizes expected price paid to suppliers, has been characterized in Myerson (1981).[4] It was shown that the buyer ought to select the supplier with the lowest value of $J_i(c_i)$. Furthermore, the solution should be such that a supplier with the highest possible cost (β_i) receives zero expected payoff.

In practice, there are a number of ways by which the optimal solution can be implemented.[5] Here we consider implementation by means of a modified "matching" mechanism. In a (simple) matching mechanism, one of the suppliers first submits a bid. The other supplier is then given the option of matching that bid, or else the first supplier is chosen. The modified matching mechanism works in the same way except that the second supplier has to match a function of the first supplier's bid, the identity function corresponding to the simple matching mechanism. Our main result is that the optimal solution can be implemented by means of a matching mechanism.

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[2] This greatly simplifies the results on optimal mechanisms. In order for the assumption to be satisfied, F must be sufficiently concave, which happens with most distribution functions commonly used (normal and uniform, for example).

[3] If the buyer can produce the good at cost c_B, then in the optimal mechanism the buyer has a reservation price strictly lower than c_B.

[4] This was done under the assumption that both the buyer and the suppliers are rational, risk-neutral players, and that all of the bargaining power is concentrated on the supplier, who is a Stackelberg leader in the sense that he gets to move first by announcing a selection mechanism.

[5] The well-known revenue equivalence theorem (Myerson, 1981, and others) is just an instance of this.

THEOREM: *The optimal solution can be implemented by means of a matching mechanism. The optimal matching function is given by* $J_2^{-1}(J_1(B_1^{-1}(b_1)))$, *where supplier* 1 *is the first to submit a bid. Supplier* 1*'s equilibrium bid function,* $B_1(c_1)$, *is the same as in a symmetric first-price sealed bid auction where costs are drawn form the distribution function* $F_2(J_2^{-1}(J_1(.)))$.

PROOF: Define $Z_2(c_1) \equiv J_2^{-1}(J_1(c_1))$. Suppose seller 1 bids according to an increasing function $B_1(c_1)$. By construction, $J_1(c_1) > J_2(c_2)$ if and only if $Z_2(B_1^{-1}(b_1)) > c_1$. Therefore, if the buyer asks seller 2 to match $Z_2(B_1^{-1}(b_1))$, he will be implementing the optimal solution, for seller 2's optimal strategy is to accept the buyer's offer if and only if $Z_2(B_1^{-1}(b_1)) > c_1$. We now have to show there exists an increasing bidding function $B_1(c_1)$ which is supplier 1's optimal strategy. Supplier 1's expected payoff given that the buyer believes him to bid $B_1(c_1)$ is equal to

$$\Pi_1(c_1) = \max_{b_1} (b_1 - c_1)\left[1 - F_2\left(Z_2(B_1^{-1}(b_1))\right)\right]. \tag{2.1}$$

By the envelope theorem, we get

$$\frac{\partial \Pi_1(c_1)}{\partial c_1} = -\left[1 - F_2\left(Z_2(B_1^{-1}(b_1^*))\right)\right], \tag{2.2}$$

where b_1^* is supplier 1's optimum bid. In equilibrium, b_1^* equals $B_1(c_1)$, and thus

$$\frac{\partial \Pi_1(c_1)}{\partial c_1} = -\left[1 - F_2(Z_2(c_1))\right]. \tag{2.3}$$

Integrating and imposing the boundary condition $\Pi_1(\beta_1) = 0$, we get

$$\Pi_1(c_1) = \int_{c_1}^{\beta_1} \left[1 - F_2(Z_1(\alpha))\right] d\alpha. \tag{2.4}$$

Substituting (2.4) for $\Pi_1(c_1)$ in (2.1) and solving for $B_1(c_1)$ we get

$$B_1(c_1) = c_1 + \frac{\int_{c_1}^{\beta_1} \left[1 - F_2(Z_2(\alpha))\right] d\alpha}{1 - F_2(Z_2(c_1))} \tag{2.5}$$

One can show that this is the equilibrium solution of a first-price sealed bid auction where both bidders' costs are distributed according to $F_2(Z_2(.))$. Finally, we can check that $B_1(c_1)$ is indeed increasing with c_1. In fact,

$$\frac{\partial B_1(c_1)}{\partial c_1} = \frac{\int_{c_1}^{\beta_1} \left[1 - F_2(Z_2(\alpha))\right] d\alpha}{\left[1 - F_2(Z_2(c_1))\right]^2} f_2(Z_2(c_1)) \frac{\partial Z_2(c_1)}{\partial c_1} > 0, \tag{2.6}$$

for $\frac{\partial J_i(c_i)}{\partial c_i} > 0$ implies that $\frac{\partial Z_2(c_1)}{\partial c_1} > 0$. ∎

3. EXAMPLE

Suppose that c_i is uniformly distributed in the interval $[\alpha_i, \beta_i]$, $i = 1, 2$. Since $F_i(c_i) = \frac{c_i - \alpha_i}{\beta_i - \alpha_i}$, we get $J_i(c_i) = 2c_i - \alpha_i$ and thus

$$Z_2(c_1) = \min\left\{\beta_2 ; c_1 - \tfrac{1}{2}(\alpha_1 - \alpha_2)\right\}. \tag{3.1}$$

If we think of 2 as being the incumbent, and 1 the alternative supplier, then it is natural to assume F_2 stochastically dominates F_1 (i.e., the incumbent has a cost advantage). This implies

$$B_1(c_1) = c_1 + \frac{\int_{c_1}^{c_1'} \left[1 - F_2(Z_2(\alpha))\right] d\alpha}{1 - F_2(Z_2(c_1))} \tag{3.2}$$

$$= c_1 + \frac{\int_{c_1}^{c_1'} \left[\beta_2 - \alpha + {}^1/_2(\alpha_1 - \alpha_2)\right] d\alpha}{\beta_2 - c_1 + {}^1/_2(\alpha_1 - \alpha_2)} \tag{3.3}$$

$$= \frac{\beta_2}{2} + {}^1/_4(\alpha_1 - \alpha_2) + \frac{c_1}{2}. \tag{3.4}$$

We can finally derive the buyer's optimal mechanism. He should give the incumbent supplier the option of matching

$$Z_2(b_1) = 2b_1 - \beta_2 - (\alpha_1 - \alpha_2). \tag{3.5}$$

4. FINAL REMARKS

In this note, we considered the implementation of Myerson's optimal auction by means of a modified matching mechanism. There are some ways in which our main result can be extended. First, one often finds that there are more than two potential suppliers, typically one incumbent and several alternative suppliers. In this case, the optimal auction can be implemented by having the rival suppliers first submit bids first (simultaneously), and then giving the incumbent supplier the option of matching a function of the lowest bid so far.[6]

We presented an example in which the optimal matching function is linear. In general, we would expect it to be more complicated. An interesting question is then whether there are simple matching functions (e.g., the first bid minus a fixed percentage) which attain the optimal solution approximately enough.

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[6] Here we assume that prior beliefs about the rival suppliers' costs are identical.

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ON THE ADOPTION OF INNOVATIONS WITH "NETWORK" EXTERNALITIES

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ABSTRACT: We present a simple dynamic model of adoption of an innovation when there are "network" externalities, that is, when one agents' benefit from adoption increases with the number of other adopters. We assume there is a continuum of differentiated potential adopters who are rational agents and have perfect information. Our main conclusion is that if network externalities are strong then the equilibrium adopotion path is discontinuous, even when there is no coordination between potential adopters.

KEYWORDS: Network externalities, adoption and diffusion of innovations.

J.E.L. CLASSIFICATION NUMBERS: 022, 024, 213.

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1. INTRODUCTION

Positive externalities often occur in the diffusion of innovations or standards: the value of adopting an innovation or standard depends positively on how many others adopt the same innovation or standard.

A classical example of positive externalities is a communications network. The utility a subscriber derives from a communications service increases as others join the system. There are, however, many less obvious instances in which "network" externalities are important. As David (1985) argues, "conveniently located repair centers staffed with trained technicians, and adequately stocked spare parts depots, ... constitute a form of 'service network' supporting users of specialized equipment--from trucks to vacuum cleaners." Typically, the size of this 'service network' depends on the number of adopters, thus giving rise to an indirect consumer externalities effect. In the case of standards, such as computer languages or systems of weights and measures, a similar phenomenon occurs. In a sense, one can see a standard as an "intangible device" for communication, and thus apply the "network" argument. Finally, adoption of innovations, such as the R.I.S.C. design for computer architecture, may provide a positive externality on other adopters due to sharing of set-up costs and learning costs.

There is a vast literature on diffusion of innovations. Some authors justify the process of diffusion as the result of incomplete information about the value of an innovation (cf Jensen, 1982). For others, differences in the time of adoption follow from differences between potential adopters (cf David, 1969). Finally, in the case when there are adoption externalities, diffusion can be understood as the equilibrium of a game played by potential adopters (cf Reinganum, 1981).

There is also a vast literature of the issue of network externalities, which includes from this literature is that network externalities likely lead to multiple equilibria, some of which Pareto dominate others.

In this paper, we present a model similar to Ireland and Stoneman's (1983) and David and Olsen's (1984,1986). We assume there is a continuum of heterogeneous potential adopters and that benefits from adopotion depend positively on the measure of adoption (network externalities). We further assume that agents are rational and perfectly informed. As in Ireland and Stoneman (1983) and David and Olsen (1984,1986), we look for equlibrium adoption paths. The main innovation of this paper is the qualitative characterization of the equilibrium adoption path. We argue that in the presence of network externalities there may exist multiple (in fact, a continuum of) equilibrium adoption paths. Assuming there is no exogenous coordination between adopters, we can restrict to a single equilibrium path, the one corresponding to minimum adoption. We finally show that if network externalities are strong then the equilibrium adoption path is discontinuous. This is in sharp contrast with the case of diffusion with no network externalities. In the latter, assuming the primitives of the model (distribution of potential adopters, benefits) are smooth functions, the equilibrium adoption path is also a smooth (and thus continuous) function.

Formally, our analysis is very similar to the study of regular economies and catastrophe theory (e.g., Debreu, 1976; Balasko, 1979). Our main result can thus be restated as follows. If network externalities are strong, then the equilibrium manifold is likely to include singular points. As a consequence, the no coordination equilibrium (the lower envelope of the equilibrium manifold) is described by a path with a catastrophe (i.e., a discontinuous point).

2. BASIC MODEL

Consider a given innovation, with positive "network" externalities, and a measure one of potential adopters.[1] Each potential adopter is characterized by a preference parameter v—the higher v, the higher the net "benefit" of using the technology, other things equal.[2] Specifically, upon adoption, each agent receives a benefit flow given by $B(v, x, t)$, where x is the measure of adopters (at time t) and t is time. It is assumed that B is smooth, all first-order partial derivatives are positive, and v has a smooth c.d.f. $F(v)$.

The assumption that $B_v > 0$ is a matter of convention. $B_x > 0$ corresponds to the idea of "network" externalities. The crucial assumption of the model is that $B_t > 0$. It is agreed among some theorists that the driving force for the diffusion of innovations is an exogenous trend by which adoption benefits increase over time. B_t is supposed to capture such exogenous trend. One must note, however, that B_t also includes the effects of changes in the conditions of supply of the innovation. Therefore, we are assuming that the conditions of supply are sufficiently stable that the increasing trend in benefits is not reversed.[3]

3. EXISTENCE AND UNIQUENESS OF EQUILIBRIUM

The main purpose of the paper is to study the equilibrium adoption path (EAP) for an innovation yielding a net benefit flow $B(v, x, t)$. Since by assumption benefits at time t depend

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[1] For convenience of exposition, we will use the term "network" externalities to represent any kind of positive adoption externalities, which may be due to other factors, e.g., "learning-by-doing" economies. See Cabral (1987).

[2] Different explanations for the heterogeneity of adopters are provided by the literature on the diffusion of innovations. See David and Olsen (1986).

[3] If, for example, the innovation were supplied by a monopolist, one would expect him or her to act strategically and—possibly—set a non-time-stationary price schedule. If this were the case, $B_t > 0$ would be too narrow an assumption. There are two basic situations in which $B_t > 0$ is a reasonable assumption to make. The first one is when supply is competitive, so that changes in price only reflect changes in cost. The second one is when the innovation is an "unsponsored" innovation, i.e., an innovation which is not the property of any firm in particular, so that one cannot talk about "supply" in the usual sense of the word. Examples of "unsponsored" innovations are the QWERTY keyboard standard and the R.I.S.C. design in computer architecture.

only on the measure of adoption at time t, we can begin by looking at the static problem of finding the equilibrium values of x for each value of t.

In equilibrium, all types v for whom B is non-negative must adopt the innovation. Denote by $g(x,t)$ the indifferent adopter's v level, i.e., $B(g(x,t),x,t)=0$. Define $H(x,t) \equiv 1 - F(g(x,t))$. A static equilibrium for a given time t is a value x such that $x = H(x,t)$. Denote by $\phi(t)$ the set of (static) equilibrium measures of adoption for each time t.

Our first result characterizes the graph of the equilibrium correspondence ϕ, which we denote by E. Some additional definitions are needed, however. Points in E where the tangent is vertical are called singular points (e.g., points A and B in Figure 1). Denote by π the natural projection of E onto R (the domain of t). A regular value t is a value such that $\pi^{-1}(t)$ includes no singular points. Finally, denote by $\Re$ the set of regular values.

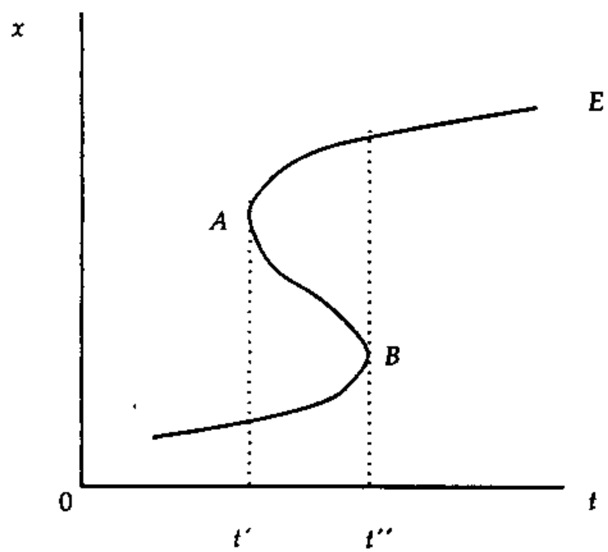


FIGURE 1. Equilibrium manifold.

PROPOSITION 1: (i) E is a one-dimensional smooth manifold. (ii) $\mathfrak{R}$ is an open dense subset of R. (iii) The restriction of E to a connected component of $\mathfrak{R}$ consists of an odd number of functions $f_i(t)$ such that $f_i(t) > f_{i+1}(t)$ and $f_i'(t) > 0$ iff i is odd.

These results are well known from the study of regular economies, catastrophe theory, and standard differential topology (see Balasko, 1979, and references therein). An example of how E may look like is given by Figure 1. The idea of Proposition 1 is that E will in general look like Figure 1.

Our next step is now to find an equilibrium adoption path. This is a function $X(t)$ such that $X(t) \in \phi(t)$ for all t. Inspection of E reveals that genericallly there exist multiple equilibrium adoption paths, in fact, a continuum of equilibrium adoption paths. One natural way of selecting among these is to assume there is no coordination among potential adopters, which seems to be consistent with the assumption that there is a continuum of them. Suppose that each agent makes his or her adoption decision at time t based on the extent of adoption at time $t-\delta$, where δ is arbitrarily small, i.e., each agent assumes that the extent of adoption at time t is very close to what it was at time $t-\delta$. The next result shows that under this assumption there exists a unique equilibrium adoption path, given by the lower envelope of E.

PROPOSITION 2: Suppose there exists a t' such that $\phi(t)$ is single-valued for $t < t'$. Consider an alternative model in which the benefit function is given by $B(v, X(t-\delta), t)$, and denote the (unique) equilibrium adoption path by $EAP(\delta)$. The limit as $\delta \to 0$ of $EAP(\delta)$ is the lower envelope of E.

An example of an equilibrium adoption path corresponding to Proposition 2 is shown in Figure 2. From this we conclude that a necessary condition for the equilibrium adoption path to be discontinuous is that there exist a singular (or catastrophe) point in E. Note, however, that this is not a sufficient condition. Figure 3 depicts a case in which the equilibrium adoption path is continuous even though E includes a singular point. Our final result provides sufficient conditions for the existence of a continuous or discontinuous equilibrium adoption path.

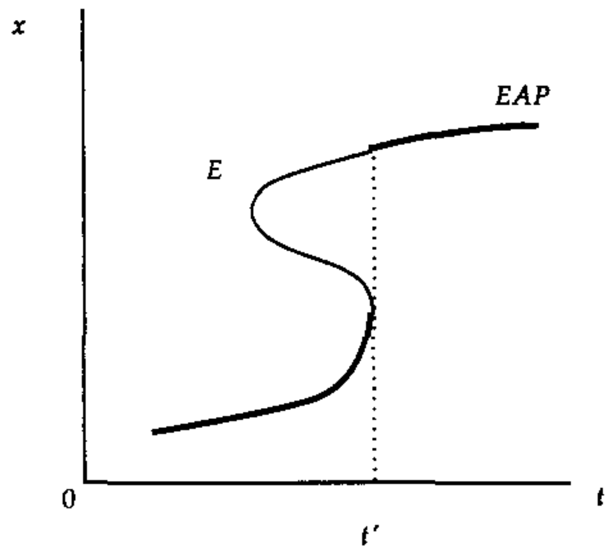


FIGURE 2. Equilibrium adoption path.

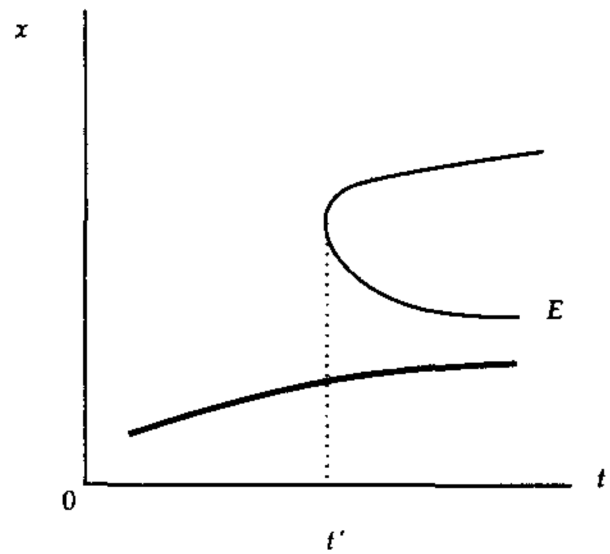
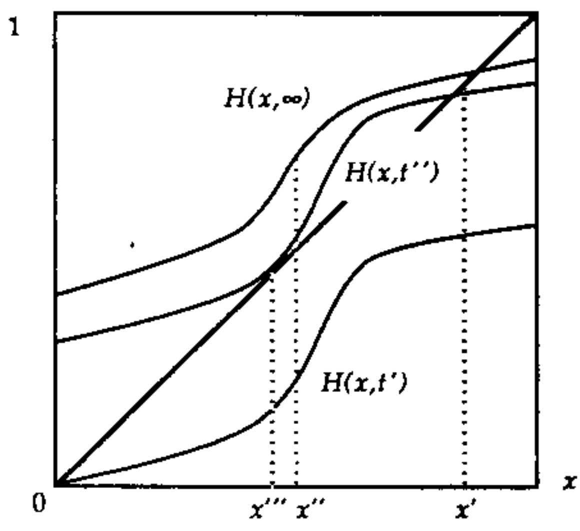


FIGURE 3. Continuous equilibrium adoption path.

PROPOSITION 3: Suppose that $H(0,t)=0$ for all $t<t'$ and denote by x' the lowest fixed point of $\lim_{t\to\infty} H(x,t)$. (i) If for all $x<x'$, $H_x<1$ for all t, then the equilibrium adoption path is continuous. (ii) If there exists an $x''<x'$ such that $H_x(x'',t)>1$ for all t, then the equilibrium adoption path is discontinuous.

The proof of the second part of the proposition can be seen with reference to Figure 4. (The proof of the first part is analogous.) The conditions of the proposition imply that $H(x,t')$ and $H(x,\infty)\equiv \lim_{t\to\infty} H(x,t)$ must be like in Figure 4. If $H_x(x'',t)>1$ for all t, then there must exist a t'' such that (t'',x''') is a singular point, and the equilibrium adoption path is discontinuous at t''.



What are the factors affecting the value of H_x? Straightforward derivation yields

$$H_x = F'(g(x,t)) \frac{B_x}{B_v} = k f B_x \tag{1}$$

where $k \equiv B_v^{-1}$ and $f \equiv g(x,t)$. This finally brings us to the main point of the paper. Even if we assume there is no coordination among potential adopters, we should expect the equilibrium adoption path to be discontinuous in situations of strong network externalities (large B_x) and relative homogeneity among potential adopters (large f).

4. EXAMPLE

In this section we consider a simple example in which F is linear, B is a linear function of $v, t,$ and a concave function of x. Specifically,

$$F(v) = v, \tag{2}$$

$0 \leq v \leq 1$, and

$$B(v,x,t) = v + x^{\alpha} + t - \beta, \tag{3}$$

where $\alpha < 1$, $\beta > 1$, and $t < 1$. From (2)-(3) we get, for $t < 1$,

$$H(x,t) = \max\{0, 1 - \beta + x^{\alpha} + t\}. \tag{4}$$

The restriction of $H(x,t)$ to a particular value of t is depicted in Figure 5. From this, we can adoption path. As can be seen, the equilibrium adoption path is discontinuous at t'', where $t'' = \beta - 1 - (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}$.

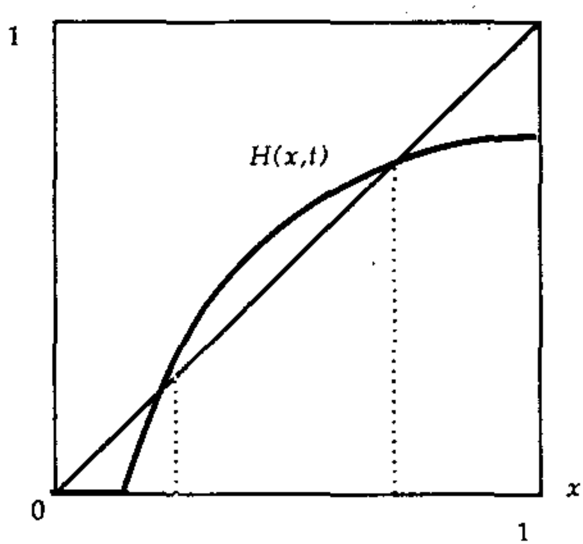
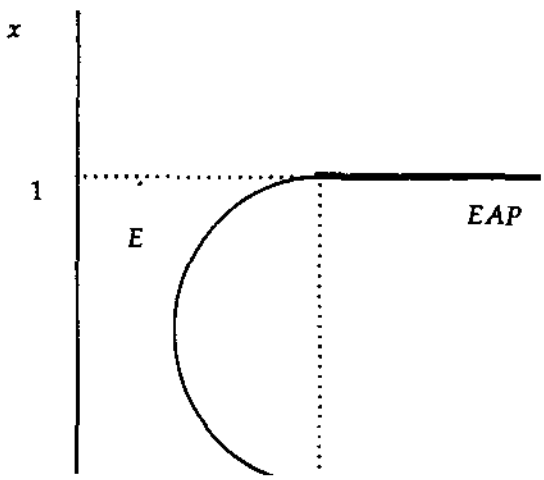


FIGURE 5. Static equilibria for a given t.



One may wonder how realistic the idea of a discontinuous equilibrium adoption path is. There is a vast body of empirical research giving evidence of an S-shaped pattern in the diffusion of innovations. In the remainder of this section we will to argue that a steep S-shaped adoption path can be interpreted as the approximation to a discontinuous point (catastrophe) of the equilibrium adoption path. To do so, we return to the alternative model of adoption with a short observation lag introduced in the previous section. How does the equilibrium adoption path $EAP(\delta)$ behave in the neighborhood of a discontinuous point of $EAP(0)$? We argue that very likely $EAP(\delta)$ looks like an S-shaped diffusion process.

Figure 7 shows how the equilibrium values $X(t)$ can be obtained the neighborhood of a catastrophe point of the underlying process. The thing to notice is that in the interval $[x', x'']$ the function $H(x,t) - x$ is quasi-concave. One can easily check that this implies that the pattern of $EAP(\delta)$ is S-shaped as x goes from x' to x''. Figure 8 depicts the equilibrium values of $X(t)$ for given parameter values (three values of α, $\beta = 1$, and $\delta = .001$).

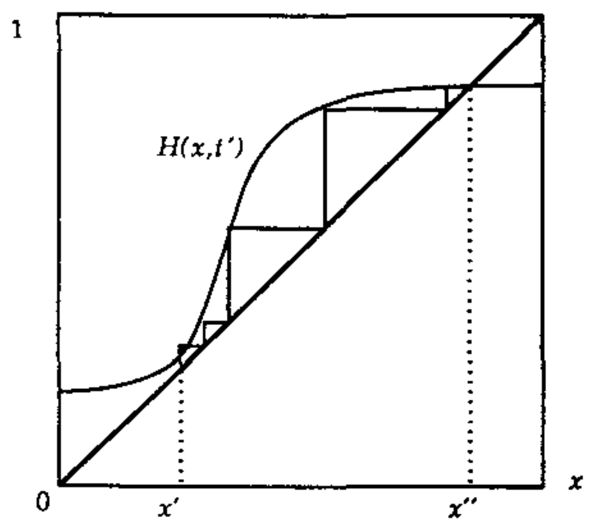


FIGURE 7. Approximate equilibrium adoption path in the neighborhood of a singular point.

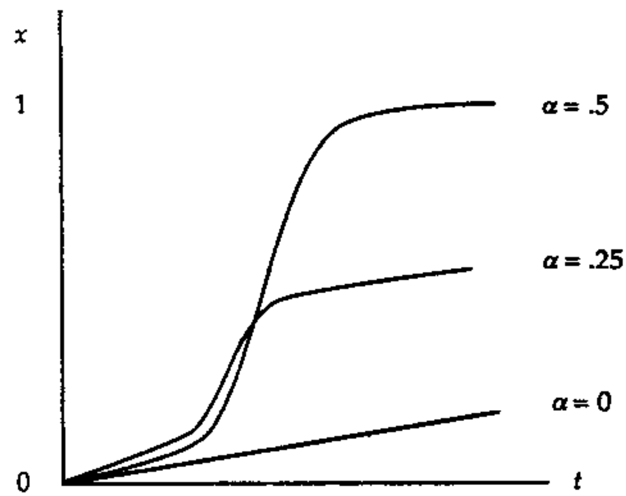


FIGURE 8. Approximate equilibrium adoption path as a function of α.

5. CONCLUSION

The adoption and diffusion of innovations is a far more complex issue than what was described in this paper. For example, agents often behave strategically (e.g., Reinganum, 1981; Fudemberg and Tirole, 1983; Quirmbach, 1986), whereas our assumption of a continuum of adopters abstracts from this possibility. On the other hand, patterns of diffusion are often due to uncertainty regarding the benefits of adoption (cf Jensen, 1982). Finally, there may be several competing innovations (Arthur, 1989; Katz and Shapiro, 1985, 1986a, 1986b), another issue which was not considered in this paper.

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