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COMBINING SURROGATE DUALITY WITH BOUND IMPROVING SEQUENCES FOR INTEGER PROGRAMMING

by

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ABSTRACT

Recently a new technique for solving pure integer programming problems has been suggested . It consists on building a sequence of Lagrangean duals that progressively reduces the duality gap and , in a finite number of steps , converges to the optimal value of the original problem . The technique has , however ,a drawback :to produce a new dual ,sometimes, an enumeration step is needed thus deteriorating the performance of the procedure.

In this note we study the conditions under which surrogate duality can play a role in helping on this situation .

By using some results connecting Lagrangean and surrogate duality in integer programming , we will be able to conclude that a combined Lagrangean-surrogate approach may be helpful in avoiding some enumeration . We give a numerical example of a simple problem where such an improvement occurs.

1. INTRODUCTION

Recently a new technique for solving pure integer programming problems has been suggested , Barcia(1987). In section 2 we will briefly sketch this method which consists in building a sequence of Lagrangean duals that progressively reduces the duality gap and , in a finite number of steps , converges to the optimal value of the original problem . The technique has , however ,a drawback :to produce a new dual ,sometimes, an enumeration step is needed thus deteriorating the performance of the procedure.

Some ways of dealing with this situation have already been suggested in Barcia & Hölm(1988).The purpose of the present note is to study the conditions under which surrogate duality , Greenberg & Pierskalla(1970) , can play a role in helping to avoid the enumeration step mentioned above.

Section 3 will be devoted to the discussion of the main results concerning surrogate duality , and its relation with Lagrangean duality , pertaining to this work.

In section 4 we shall use some results connecting Lagrangean and surrogate duality in integer programming , Karwan & Rardin(1979) , are used in order to show that a combined Lagrangean-surrogate approach may be helpful in avoiding some enumeration .Finaly, in section 5 , after giving a numerical example of a simple problem where such an improvement occurs, we shall discuss the computational usefulness of the new approach and provide some concluding remarks.

2. BOUND IMPROVING SEQUENCES

In this section we briefly sketch the basics of the bound improving sequence technique. We shall state only the main results and the reader is referred to Barcia(1985) , Barcia(1987) and Barcia & Hölm(1988) for formal proofs and more details on the subject.

Now consider the following integer linear programming problem:

$$(P) \qquad z = \min cx$$
$$Ax \leq b$$
$$x \in \mathcal{X}$$

where c is an integer n-vector b an integer m-vector and A a matrix of integers of apropriate dimension . For simplicity we shall assume that $\mathcal{X} \subset \mathbb{Z}^n$ is bounded and that problem (P) has always a finite solution.

Suppose now that a lower bound for z, say $\ell_k \leq z$, is known. Defining the set $\mathcal{X}_k = \{ x \in \mathcal{X} : cx \geq \ell_k \}$ problem (P) can now be restated in the following equivalent way:

$$(Pk) \qquad z = \min cx$$
$$Ax \leq b$$
$$x \in \mathcal{X}_k$$

Now take $u \in \mathbb{R}_+^m$ as a vector of non-negative multipliers and consider a Lagragean dual of (Pk) defined as follows:

$$\text{(LD)} \qquad \ell_{k+1} = \max_{u \in \mathbb{R}_+^m} \; \min_{x \in \mathcal{X}_k} cx + u(Ax-b) \qquad k=0,1,...$$

Note first that the usual Lagrangean bound can be obtained from (LD) as $\ell=\ell_1$ by taking $\ell_0=-\infty$. For simplicity we shall assume that ℓ_0 was selected this way and we mention by ℓ the usual Lagrangean bound whenever we need to refer it.

Note also that the inner minimization problem on the right hand side of the equality is a single integer knapsack problem which , although in $\mathcal{NP}$, is one of the easiest non-polinomial problems , see Dudzinsky & Walukiewicz(1987). Since the outer maximization problem can be solved using a subgradient algorithm , see Demianov & Vasilev(1985), (LD) can be computed with a moderate effort.

One can easily prove the following :

Theorem 2.1 : $\ell_{k+1} \geq \ell_k$

Some necessary conditions for equality to hold in theorem 2.1 are easily derived . For instance , one has :

Theorem 2.2 : If $\ell_{k+1}=\ell_k$ then u=0 is an optimal vector of multipliers in (LD).

Of course the interesting question is to be able to characterize the situations such that the above inequality is strict . The answer for this is addressed in the following:

Theorem 2.3 : Consider the level sets of the objective function, $\mathbb{L}(y) = \{ x \in \mathcal{X} : cx = y \}$. Then $\ell_{k+1} > \ell_k$ if and only if $\mathrm{Conv}\ \mathbb{L}(\ell_k) \cap \{ x \in \mathbb{R}^n : Ax \leq b^{-} \} = \emptyset$.

Note now that if the objective function cx is such that its level sets are singletons the first value on the sequence $\{\ell_k\}$ for wich theorem 2.2 does not hold will be z . In this case (LD) provides a device to bridge the duality gap in a finite number of steps.

To bridge the duality gap for a general objective function one must supplement (LD) with an enumeration scheme to deal with the cases for wich $\ell_{k+1} = \ell_k$.

In fact , when equality occurs , one must search the hyperplane $cx = \ell_k$ for a (P)-feasible point. If such a point is found it's optimal . If no such a point can be found we can take $\ell_{k+1} = \ell_k + 1$ and keep using (LD) to generate a non-decreasing sequence of bounds converging to z.

This procedure is interesting only when enumeration is not performed very often because of its time consuming nature.

Barcia & Hölm(1988) report a revised version of the basic algorithm we just sketched enabling some savings in the number of times that the enumeration scheme must be used.

In this note we shall try to avoid enumeration by combining (LD) with a different approach to integer programming duality : surrogate duality.

Let us state a final result in this section which, although not computationaly interesting ,will be very useful when trying to understand geometricaly how (LD) operates:

Theorem 2.4: Computing (LD) is tantamount to convexifying the nonrelaxed constraints in problem (Pk), i.e.

$$\ell_{k+1} = \min cx$$

$$Ax \le b$$

$$x \in \mathrm{Conv}\, \mathcal{X}_k$$

3. Surrogate duality

In this section we recall and comment the main results on surrogate duality and its relation to Lagrangean duality. Our aim is to lay the ground for section 4 where we will use these results to show how some of the enumeration steps referred in the previous section can be avoided.

Again , only the main ideas will be sketched . All proofs will be ommited . The reader is referred to Greenberg & Pierskalla(1970) and Karwan & Rardin(1979) for full proofs and a more detailed treatement of the subject.

Consider now problem (P) as in the previous section . Suppose that we wish to replace the explicit constraints $Ax \le b$ by a single aggregate constraint . Taking non-negative multipliers $v \in \mathbb{R}_+^m$, one gets the following single constrained problem:

$$(P_v) \qquad s(v) = \min cx \quad vAx \le vb \quad x \in X$$

Note first that v can be normalized at 1 without changing (P_v). Obviously , one has the following:

Theorem 3.1: $\forall\, v \in \mathbb{R}_+^m$, $s(v) \le z$.

This fact motivates the wish to push s(v) as far ahead as possible and the surrogate dual of problem (P) is defined as follows:

$$(SD) \qquad s = \max_{v \in \mathbb{R}^m_+} \quad \min_{\substack{vAx \leq vb \\ x \in \mathcal{X}}} cx$$

Note now that (SD) is more difficult to compute than (LD) for k=0 and therefore s is more difficult to obtain than the usual Lagrangean bound ℓ. The inner minimization problem is more difficult because we have the extra constraint $vAx \leq vb$ and , more important , there is no easy way to get the optimal multipliers v because the function s(v) does not have the necessary convexity properties.The reward for these difficulties is summarized in the next theorem:

Theorem 3.2 : $\ell \leq s \leq z$

So we know that surrogate duality bounds , although more difficult to compute , can perform better than the usual Lagrangean bounds.

Of course the interesting issue is to know when do we have a strict improvement $\ell < s$. The answer to this is summarized in the next theorem.

Theorem 3.3 : Either $\ell < s$ or for every optimal Lagrangean multiplier there exists an optimal solution for:

$$\min \{ cx + u(Ax-b) \ , \ x \in \mathcal{X} \}$$

verifying the complementarity condition $u(Ax-b)=0$.

These results provide us with the necessary background to address the key issue of this note : can surrogate duality be of any assistance in avoiding some enumeration required by the bound improving sequence technique?

4. AVOIDING ENUMERATION ?

Suppose now that we are using bound improving sequences , as stated in section 2 , and a situation comes up where we have $\ell_{k+1}=\ell_k$.

The first idea to exploit would be to define a surrogate dual using the set $\mathcal{X}_k$ instead of $\mathcal{X}$:

$$(SD_k) \qquad s_{k+1}= \max_{v\in\mathbb{R}^m_+} \quad \begin{array}{l} \min cx \\ vAx \le vb \\ x \in \mathcal{X}_k \end{array}$$

The inner minimization problem is a two-constrained integer knapsack problem wich is not too hard and , if one could compute the optimal surrogate multipliers v , theorem 3.2 tells us that we may have $s_{k+1}>\ell_k$.

Unfortunately this will not always be the case. In fact one has the following:

Theorem 4.1 : If $\ell_{k+1}=\ell_k$ and u=0 is the unique optimal multiplier in (LD) , then $s_{k+1}=\ell_k$.

Proof : From theorem 2.2 , if $\ell_{k+1}=\ell_k$ then u=0 is an optimal multiplier in (LD). Now , if this is the unique optimal multiplier theorem 3.3 shows that $s_{k+1}=\ell_{k+1}=\ell_k$,because the complementarity condition u(Ax−b)=0 will be always trivialy satisfied. ■

Of course one may argue that (LD) need not have a unique optimal multiplier and then theorem 4.1 does not apply . As we will see this can not be the case. In fact, we can prove a more general result than the previous one, as follows:

Theorem 4.2 : If $\ell_{k+1}=\ell_k$ then $s_{k+1}=\ell_k$.

Proof: Note first that if $\ell_{k+1}=\ell_k$ then , from theorem 2.3, the following occurs:

(i) $\quad \mathrm{Conv}\ \mathbb{L}(\ell_k) \cap \{ x \in \mathbb{R}^n : Ax \le b \} \ne \emptyset$.

On the other hand , if we have $s_{k+1}>\ell_k$ we certainly must have, for any optimal surrogate multiplier v^*:

(ii) $\quad \forall\ x \in \mathbb{L}(\ell_k) \quad v^*Ax > v^*b$

otherwise (SD_k) would imply that $s_{k+1}=\ell_k$. Note now that (ii) remains valid after convexification , i.e., (ii) implies

(iii) $\quad \forall\ x \in \mathrm{Conv}\ \mathbb{L}(\ell_k) \quad v^*Ax > v^*b$

and (iii) contradicts (i) , so $s_{k+1}=\ell_k$ and the proof is terminated. ■

Now we may wish to overcome our difficulties by trying a mixture of Lagrangean and surrogate duality as follows:

$$(MD_k) \qquad m_{k+1}= \max_{\substack{u\in\mathbb{R}^m_+ \\ v\in\mathbb{R}^m_+}} \quad \min \ cx+u(Ax-b) \quad \text{s.t. } vAx \le vb,\ x \in \mathcal{X}_k$$

Of course this new bound would be much more difficult to compute because now one looks for two sets of optimal multipliers. However , an improvement on the bound can eventualy be obtained :

Theorem 4.3 : $m_{k+1} \geq s_{k+1} \geq \ell_{k+1}$.

Proof : To show that the first inequality holds one takes u=0 in the mixed dual . The second inequality is a consequence of theorem 3.2. ∎

As we shall see in next section there are cases where an effective improvement can be obtained . Actually , it is possible to state the conditions under which such an improvement occurs:

Theorem 4.4 : Consider the sets $\mathbb{M}(y,v)$ defined as follows: $\mathbb{M}(y,v) = \{ x \in \mathcal{X} : cx = y \text{ and } vAx \leq vb \}$. We shall have a strict inequality in theorem 4.3 if and only if:

$$\operatorname{Conv} \mathbb{M}(\ell_k,v) \cap \{ x \in \mathbb{R}^n : Ax \leq b \} = \emptyset .$$

for some multiplier $v \in \mathbb{R}_+^m$ on the mixed dual.

Proof: Consider a multiplier $v \in \mathbb{R}_+^m$ on the mixed dual. Consider now the sets $\mathcal{X}(v)=\{ x \in \mathcal{X} : vAx \leq vb \}$.These sets are still bounded subsets of $\mathbb{Z}^n$, so we can use theorem 2.3 with $\mathcal{X}(v)$ instead of $\mathcal{X}$ and the result follows because the $\mathbb{L}(\ell_k)$ sets of theorem 2.3 become the $\mathbb{M}(\ell_k,v)$ sets defined above. ∎

5. An example and conclusions.

We shall now use a very simple numerical example to illustrate how the mixed dual can avoid an enumeration step in (LD) . Consider the following problem:

$$\min \{x+y: x \geq 1/2; y \geq 1/2; x,y \in \{0,1\}\}$$

Now let ℓ_k be any lower bound on the optimal value of the objective function and suppose we compute ℓ_{k+1} using (LD). From theorem 2.4 we know that we have :

$$\begin{aligned} \ell_{k+1} = \min\ & x+y \\ & x \geq 1/2 \ , \ y \geq 1/2 \\ & x,y \in \text{Conv}\ \{ x,y \in \{0,1\}: x+y \geq \ell_k \} \end{aligned}$$

Suppose further that we start with $\ell_0=-\infty$. Then we get $\ell_1=1$, the usual Lagrangean bound , as one can easily see from figure 1. Now if we try to compute ℓ_2 we will get again the value 1 because $\mathbb{L}(1) = \{ (0,1),(1,0) \}$ and

$$\text{Conv}\ \mathbb{L}(1) \cap \{ x,y : x \geq 1/2 \ , \ y \geq 1/2 \} = \{ (1/2,1/2) \} \neq \emptyset$$

as predicted by theorem 2.3. As suggested in section 2 , enumeration would be required to overcome this value.

However we know now that a mixed dual can help us . To see this consider the surrogate multipliers v=(2,4) . The aggregate constraint will then be $2x+4y \geq 3$ and the set $M(1,v)$ is $\{(0,1)\}$. Now, one has:

$$\text{Conv } M(1,v) \cap \{ x,y : x \geq 1/2 , y \geq 1/2 \} = 0$$

and, as predicted in theorem 4.4, we will have $m_2 = 3/2 > 1$ as one can see from figure 1 below.

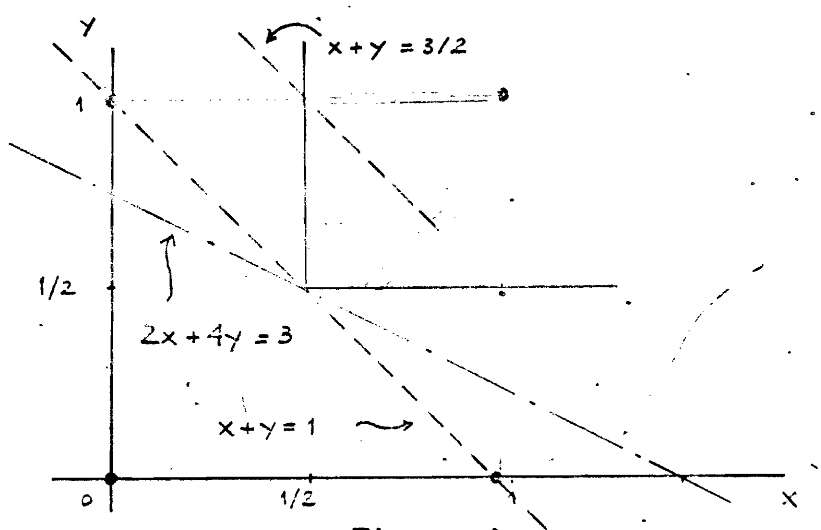

Figure 1

As we have just seen the combined Lagrangean-surrogate approach can help us avoiding enumeration when using bound improving sequences.

The main drawback is the fact that the mixed dual bound is hard to compute because we have a two-constrained knapsack subproblem and there is no 'clean' way to get the optimal values of v. We are presently conducting computational experiments in order to determine the practical usefulness of this approach.

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