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A NOTE ON THE REPRESENTATION OF INFORMATION STRUCTURE

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A NOTE ON THE REPRESENTATION OF INFORMATION STRUCTURE

1. Economists have always used the idea of a partition to represent an information structure. We show that by imposing this form of representation a price is paid in terms of the number of states of nature needed to describe the environment.

2. A small example will illustrate this point. Take country X who is in the midst of a political row with country Y. Its decisions depend on knowing whether:

1. country Y declares war and has nuclear missiles
2. country Y declares war but has no nuclear missiles
3. country Y does not declare war.

Country X hires one spy which is able to inform whether Y has nuclear missiles or not. The information structure, given this description of the states of nature is $\{\overline{13};\overline{23}\}$. But this is not a partition.

3. The relevant general concept is thus not a partition but a cover. Take the set S of all states of nature, and form subsets $B_i \ldots B_k$ such that (i) $\bigcup_i B_i = S$ and (ii) $B_i \subseteq B_j$ iff $i=j$. A set of such subsets is a cover. We do not require the sets B_i (blocks) to have the empty set as intersection, but require that no block of the cover be a subset of another.

4. There is the possibility of enlarging the state space

in such a way that the information system becomes again representable by a partition. Substituting state 3 by two states:

3': Country Y does not declare war and has missiles

3": Country Y does not declare war and has no missiles.

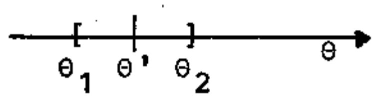
In this state space, $S = \{1,2,3',3''\}$ the spy is an information system represented by a partition $\pi = \{\overline{13'};\overline{23''}\}$. But a price was paid in having too much detail in the description of nature. That extra detail is irrelevant for decision making. The choice is to work with a bigger state space than needed, or generalize the notion of partition and working with covers.

5. It may be advantageous to work with notion of cover when the costs of obtaining information are included in the analysis. Suppose that the value of a parameter θ is relevant for some decision. And that one is interested into knowing whether:

$\theta < \theta_1$: state 1 (s_1)

$\theta_1 < \theta < \theta_2$: state 2 ($s_2$)

$\theta > \theta_2$: state 3 (s_3)



You know "a priori" whether $\theta \in [\theta_1,\theta_2]$ or not. This information can be represented by the partition $\pi = \{\overline{s_1 s_3};\bar{s}_2\}$. Now you can perform an experiment to decide whether $\theta < \theta_1$ (or $\theta > \theta_1$) at a cost C_1 (the experiment to decide whether $\theta > \theta_2$ costs C_2). This experiment would give the partition $\pi_1 = \{\bar{s}_1;\overline{s_2 s_3}\}$, which together with π leads to perfect information. If we take the set of states defined "a priori" and work with partitions the only other alternative to get perfect information is to collect the information in $\pi_3 = \{\overline{s_1 s_2}; \bar{s}_3\}$. But it may be cheaper (it is information on sale in the market, for less than C) to know whether $\theta < \theta'$ (or $\theta > \theta'$). The best strategy is then to buy this information, which must be represented by a cover $\gamma = \{\overline{12}; \overline{23}\}$. Notice that π and γ also generate perfect information. More generally, suppose that the cost of evaluating whether $\theta < \theta'$ is given $C(\theta')$, for $\theta' \varepsilon [\theta_1, \theta_2]$. We should then make the experiment $\theta < \theta^*$, where θ^* is the solution to $\min_{\theta' \varepsilon [\theta_1, \theta_2]} C(\theta')$. One could always expand the space of states of nature and work with partitions. But the way that expansion is made becomes endogeneous, for it depends on the value of θ^*.

Suppose now that the information "a priori" is given by $\pi = \{\bar{s}_1; \overline{s_2 s_3}\}$. Now, if we want perfect information (the cost of an error is big compared to the costs of experiments), the only single experiment that is useful is whether $\theta > \theta_2$. There is no use now for an expansion of the set of states of nature.

The whole point is that if cost considerations are to enter the picture, then it is not clear "a priori" whether it is useful to expand the set of states of nature, and in different circumstances different expansions are useful. If we want to anticipate all these possible expansions we need such detail as to make the solution too costly, albeit impossible. By working with covers we can start with the most economic description possible of nature. Whenever a cover becomes useful we can equivalently expand the set of states in a corresponding concrete way.

6. We say that a partition π is finer than π' if the blocks of π are subsets of a block of π. In general the value of an information system depends on the "a priori" information and preferences of the decision-makers. However if π is finer than π' ($\pi > \pi'$), then (and only then) π is always at least as valuable as π', independently of "a priori" information and préferences. This relation introduces a partial ordering in the set of partitions. This set of partitions forms a lattice, where the greatest lower bound (g.l.b.) and least upper bound (l.u.b.) of two partitions are defined as the operations . and +, as in Lucena (1987).

The concept of "finer than" can be immediately extended to covers. A cover γ is finer than $\gamma'(\gamma \leq \gamma')$, if the blocks of γ are subsets of a block of γ'.

Take now γ and γ' as two covers. We define

$$\gamma + \gamma' = \mathrm{el}\{B \,|\, B \in \gamma_1 \text{ or } B \in \gamma_2\} = \text{l.u.b. } (\gamma_1, \gamma_2)$$

where B is a block and el{.} consists of the operation of elimination of any block that is a proper subset of another. Notice that with this definition the set of partitions is not a closed one under the operation +. Take $\pi_1 = \{\overline{123};\overline{4};\overline{5}\}$ and $\pi_2 = \{\overline{12};\overline{34};\overline{5}\}$. With this definition $\pi_1+\pi_2=\mathrm{el}\{\overline{123};\overline{14};\overline{5};\overline{12};\overline{34};\overline{5}\}=\{\overline{123};\overline{34};\overline{5}\}$ which is a cover.

We recover however our old definition if we look for the smallest partition that is bigger than $\pi_1+\pi_2$, getting $\{\overline{12},\overline{4};\overline{5}\}$.

Define also, with obvious notation:

$$\gamma_1 \cdot \gamma_2 = \text{el}\{B \cap B' \mid B \varepsilon \gamma_1, B' \gamma_2\} = \text{g.l.b. } (\gamma_1 \cdot \gamma_2)$$

This definition, applied to partitions coincides with the definition used before, for π_1, π_2 is now also a partition.

The set of all covers of a set is a lattice under these two operations. Notice also that

$$\gamma_1 \leq \gamma_2 \quad \text{iff} \quad \gamma_1 + \gamma_2 = \gamma_2$$

$$\gamma_1 < \gamma_2 \quad \text{iff} \quad \gamma_1 \cdot \gamma_2 = \gamma_1$$

A simple example illustrates the role of the el{.} operation. Suppose three states of nature and two information structures $\gamma_1=\{\overline{12};\overline{23}\}$ and $\gamma_2=\{\overline{13};\overline{23}\}$. The first one "separates" state one from three and second "separates" states two from three. We can always know whether state one is realized or not. But we cannot <u>guarantee</u> that we "separate" states two and three. In some circumstances that is possible, namely if a are in block 1 of γ_1 and in block two of γ_2:

$$\gamma_1 \cdot \gamma_2 = \text{el}\{1,2,3,23\} = \{\overline{1},\overline{23}\}$$

By eliminating {2} and {3} as blocks of $\gamma_1 \cdot \gamma_2$ we get a cover (in this case is a partition) which gives an equivalent "guaranteed" information structure.

The question now arises whether the results obtained for partitions about the value of information systems generalize to covers. The answer is yes, at least for noiseless information systems.

Teorem: an information system (γ_1) is in general more valuable then another (γ_2) iff $\gamma_1 < \gamma_2$

Dem: Take any two covers, γ_1 and γ_2. Enlarge the state-space in the relevant way, such that the information given by γ_i is now represented by the partition π_i (i=1,2) in the new state-space. It is easy to see that $\gamma_1 \leq \gamma_2$ iff $\pi_1 \leq \pi_2$. The value of γ_i and π_i (i=1,2) is the same, for they are only two different representations of the same information system. The conclusion is immediate from the classical result about the value of information systems mentioned above. For a proof see, e.g. Laffont (1985).

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