CONSTRUCTIVE DUAL METHODS FOR
NONLINEAR DISCRETE PROGRAMMING
PROBLEMS

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ABSTRACT

This paper is primarily concerned with the extension of the bound improving sequence algorithm (Bárcia, 1985, 1987) to nonlinear discrete programming problems. Our attention is then focused to quadratic constrained binary problems. It is shown that for a particular class of these problems, a computationally efficient algorithm is derived from the general approach.
1. INTRODUCTION

This paper is primarily concerned with the extension of the bound improving sequence algorithm (Bárcia, 1985, 1987) to nonlinear discrete programming problems. Our attention is then focused to quadratic constrained binary problems. It is shown that for a particular class of these problems a computationally efficient algorithm is derived from the general approach.

The paper is divided as follows: in section 2 the bound improving sequence algorithm (BISA) for nonlinear discrete programming problems is presented and its insertion into the general class of constructive dual methods for discrete problems is discussed. Section 3 is devoted to the study of quadratic constrained binary problems within the framework developed previously and a computationally efficient version of the general algorithm is obtained. Finally in section 4 some concluding comments are put forward.
2. BOUND IMPROVING SEQUENCE ALGORITHM FOR NONLINEAR DISCRETE PROBLEMS

A class of algorithms for linear discrete problems has been discussed in Barcia (1985, 1987), Barcia and Holm (1986), Barcia and Jörnsten (1986) under the generic designation of BISA, which is a constructive dual approach in the sense that the Lagrangean dual is strengthened iteratively, providing bounds that converge to the optimal value and giving conditions to select the optimal solution.

In this section some ideas presented in Barcia (1984) and Coelho (1986) are developed into a general BISA method for nonlinear discrete problems.

Let us consider the following problem

\[ \begin{align*}
    z^* &= \min f(x) \\
    &\text{subject to } g(x) \leq 0 \\
    &x \in X
\end{align*} \]

where \( f:R^n \rightarrow R \) and \( g:R^n \rightarrow R^m \) are continuous functions and \( X \) is a bounded closed subset of \( R^n \). We shall assume that problem \( (P) \) has a finite optimum value \( z^* \) and that \( x^* \) is an optimal solution. Let \( l_x \) be a lower bound to the optimal value \( z^* \), \( l_x \leq z^* \), which we consider to be known. Under the assumptions above, it is clear that the following problem is equivalent to \( (P) \), in the sense that both produce the same optimal
solutions:

\[ z^* = \min_{x} f(x) \]
\[ g(x) \leq 0 \]
\[ f(x) \geq l_k \]
\[ x \in X \]

(PE)

Consider now the Lagrangean dual of (PE):

\[ (DPE) \quad l_{k+1} = \max_{u \geq 0} \min_{x} f(x) + u g(x) \]
\[ f(x) \geq l_k \]
\[ x \in X \]

Where \( u \) is a vector of Lagrangean multipliers.

In order to simplify we shall denote by \( w(u, l_k) \) the optimal value of the minimisation subproblem in the Lagrangean dual. Then (DPE) becomes

\[ l_{k+1} = F(l_k) = \max_{u \geq 0} w(u, l_k) \quad (1) \]

The formula (1) allows to generate a sequence of values \( l_k \), \( k=0,1,... \), nondecreasing and convergent to a limit \( l \), such that \( l \leq z^* \). In effect, we have:

**Theorem 1**: The sequence \( \{l_k\}, k=0,1,... \), is nondecreasing.

**Proof**: This result is trivial since \( w(0, l_k) \leq l_k \).

**Theorem 2**: The sequence \( \{l_k\}, k=0,1,... \), converges to a limit \( l \), \( l \leq z^* \).

**Proof**: By the weak theorem of Lagrangean duality \( l_k \leq z^* \) for all \( k \), and since \( \{l_k\} \) is a bounded sequence of nondecreasing real values, there is a limit \( l \) as mentioned.
Theorem 3: The optimal value \( z^* \) is a fixed point of the function \( F(\cdot) \), that is

\[ z^* = F(z^*) \]

Proof: By weak duality \( F(z) \leq z^* \) for all \( z \leq z^* \). In particular, \( F(z^*) \leq z^* \). On the other hand, it comes out from Theorem 1 that \( F(z^*) \geq z^* \). 

In order to help understanding how the recursive formula (1) works, we shall consider the geometric interpretation of the Lagrangean dual [see figure 1]. For simplicity we shall take \( m=1 \).

\[ \{g(x), f(x)\} \]

The set \( u \) denotes the image of \( X \) by the mapping \( [g,f] \). The optimal primal and dual values are marked off. We may note that the dual corresponds to maximize the intersections in the \( y \) axis of the support hyperplanes (in this case, straight lines of slope \( u \)) of the set \( u \). Except under the hypothesis of convexity of \( f \) and \( g \) and when a constraint qualification is fulfilled, then a duality gap between primal and dual optimal values will exist, as it is represented in fig. 1. By considering the equivalent primal
A problem (PE), a portion of set is excluded, giving rise to an increase or "strengthening" of the Lagrangean dual. It is also clear from the geometric interpretation that is a fixed point of .

As shown above the sequence will converge to a limit . It is yet open the following question: in which conditions will the sequence converge to ?

Let us now assume that is finite. Under this additional assumption. We have:

**Theorem 4:** is a lower semi-continuous function.

**Proof:** By assumption is a finite set of real values. We shall denote by the ordered real set obtained from adding and .

where and .

For each value , there are belonging to such that . This implies that the set

is constant in the interval and therefore

is constant in the same interval. Hence

\[ W(y) = \max_{u \geq 0} W(u, y) \]

\[ F(y) = \max_{u \geq 0} W(u, y) \]
will also be constant in that interval.

The function \( F(y) \) is a nondecreasing step function constant in every interval \( (y_{p-1}, y_p], \ p=0, \ldots, k \), which implies that the theorem above holds. 

**Theorem 5:** The limit 1 of the sequence \( \{l_k\} \), is a fixed point of the function \( F(.) \).

**Proof:** Since \( \{l_k\} \) is nondecreasing and \( F(.) \) is a lower semi-continuous function, we have

\[
1 = \lim_{k \to \infty} F(l_k) = F(\lim_{k \to \infty} l_k) = F(1). \]

We have just proved that the limit 1 and \( z^* \) are both fixed points of \( F(.) \). If we manage to show that \( z^* \) is the only fixed point of \( F(.) \) in the interval \( (-\infty, z^*] \), then we may conclude that the sequence \( \{l_k\} \) will converge to the optimal value \( z^* \).

First, we prove that \( u=0 \) is an optimal multiplier at any fixed point of \( F(.) \).

**Theorem 6:** If \( y=F(y) \), then \( u=0 \) is an optimal solution of

\[
P(y) = \max_{u \geq 0} w(u, y)
\]

**Proof:** Since \( y \) is a fixed point, then we have

\[
w(u, y) \leq y \text{ for all } u \geq 0
\]
On the other hand, for $u=0$ we have:

\[ y \geq w(0,y) = \min f(x) \geq y \]

\[ f(x) \geq y \]

\[ x \in X \]

Thus $w(0,y)=y$ which shows that $u=0$ is an optimal multiplier. 

The proof that $z^*$ is the only fixed point of $F(.)$, requires an additional assumption which is fulfilled, when the level sets of $f(x)$

\[ L(y) = \{ x \in X : f(x) = y \} \]

are singletons for all $y \in f(X)$.

In many discrete programming problems it will be possible to assume, by considering small perturbations of the objective function, that $L(y)$ are singletons. Therefore the additional assumption will not be too much restrictive in practical terms.

Let $G(y)$ be the image of the level set $L(y)$ by function $g(x)$, that is:

\[ G(y) = \{ g(x) : x \in L(y) \} \]

(A) For every $y(z^* \in X$ and $y \in f(X)$, the following holds:

\[ \exists p \leq 0, \ p \in \text{Conv } G(y) \Rightarrow \exists q \leq 0, \ q \in G(y) \]
It is well known from Lagrangean duality that Conv $G(y)$ is the sub-differential of $w(u,y)$ at $u=0$. The assumption (A) asserts that for every $y \in f(X)$, if the sub-differential of $w(u,y)$ at $u=0$ has a subgradient with all components less or equal to zero, then the set $G(y)$ will contain also a non-positive vector.

We note that if $G(y)$ is a convex set for all $y \leq z^*$, $y \in f(X)$, the assumption A holds trivially. In particular, if the level sets $L(y)$ are singletons, then $G(y)$ are also singletons and therefore convex.

We may now prove the following fundamental result:

**Theorem 7**: If assumption (A) holds for problem (P) with $f(X)$ discrete and finite, then the sequence $(l_k)$ will converge to the optimal value $z^*$.

**Proof**: It is sufficient to show that $z^*$ is the only fixed point of $F(.)$ in $(-\infty, z^*)$. Let us assume that there is another fixed point $y = F(y)$, such that $y < z^*$. By theorem 6, $u=0$ is an optimal multiplier of

$$F(y) = \max_{u \geq 0} w(u,y)$$

and therefore no ascending feasible directions are available for $w(u,y)$ at $u=0$. The feasible directions of $w(u,y)$ at $u=0$ are directions $u \geq 0$. The ascending directions are such that

$$(DA) \quad \min_{u \geq 0} u^p > 0$$

$p \in \text{Conv } G(y)$
i.e., directions that define an angle less than 90° with all
subgradients of \( w(u,y) \) at \( u=0 \). The nonexistence of ascending
feasible directions of \( w(u,y) \) at \( u=0 \) implies

\[
\text{(DAF)} \quad \max \min \ u_p \leq 0 \\
\quad u \geq 0, \quad p \in \text{Conv} \ G(y)
\]

But (DAF) is the Lagrangean dual of the following LP
problem:

\[
\text{(PAF)} \quad \min \ 0_p \\
\quad 0 \leq 0 \\
\quad p \in \text{Conv} \ G(y)
\]

where 0 is the vector of null components.

Since (DAF) is upper bounded, then its dual is feasible
and therefore the set

\[
\{p \leq 0 ; \ p \in \text{Conv} \ G(y)\}
\]

is non-empty.

Assumption (A) will now imply the existence of \( \bar{x} \in L(y) \)
such that \( g(\bar{x}) \leq 0 \). However, this result is absurd since it
corresponds to assume the existence of a feasible solution
of problem (P) with value \( y \) less than the optimal value \( z^* \).

Theorem 8: In the above mentioned conditions, there is an
optimal solution of subproblem \( w(0,z^*) \) that is optimal for
problem (P).
Proof: By theorem 6, \( u=0 \) is an optimal multiplier of

\[
z^* = \max_{u \in \Omega} w(u, z^*) = w(0, z^*) = f(x)
\]

The proof is completed.

Formula (1) that allows to build a sequence \( \{l_k\} \) convergent to the optimal value \( z^* \), requires solving nonlinear knapsack subproblems \( w(u, l_k) \) which are very difficult to solve generally. However, this obstacle will disappear if \( (P) \) is an integer linear programming problem (Barcia, 1985 and 1987) and also, as shown below, for some binary quadratic problems.

Before examining this last particular cases it is convenient to prove a result concerning the number of iterations required to attain the optimal value.

**Theorem 9:** The limit \( l = z^* \) is attained in a finite number of iterations.

Proof: By assumption \( f(x) \) may only assume a finite number of values since \( f(X) \) is finite. This implies, as shown in the proof of Theorem 4, that the same occurs to \( F(y) \).

On the other hand, the assumption A implies that \( z^* \) is the only fixed point of \( F(y) \) for \( y \in [-\infty, z^*] \) and therefore the sequence

\[
l_{k+1} = F(l_k)
\]
is strictly increasing for \( l_k(z^*) \).

This two observations show clearly that the limit \( l = z^* \) is attained in a finite number of iterations.\

3. BINARY QUADRATIC CONSTRAINED PROBLEMS

This section is devoted to the search of a computationally efficient approach derived from formula (1) to a class of binary quadratic constrained problems.

Let us consider a binary quadratic constrained problem as follows:

\[
\begin{align*}
\text{minimize} & \quad c^T x, \\
\text{subject to} & \quad d x + xQx \leq p, \\
& \quad Ax \leq b, \\
& \quad x \in B^n = \{0,1\}^n
\end{align*}
\]

It is clear that a straightforward application of formula (1) would be disastrous since the following subproblems would be obtained:

\[
\begin{align*}
w(u, l_k) & = \min (c + u_d + uA)x + u_Qx - (u_p + ub) \\
& \quad c^T x \geq l_k, \\
& \quad x \in B^n
\end{align*}
\]

These are "quadratic knapsacks" which are computationally very hard to solve at the present state of the art.
However, if we reformulate (BQP) using variable splitting (see Minoux (1986), Jörnsten and Nasberg, 1986; Bárcia and Jörnsten, 1986), which consists in writing down the problem duplicating the variables, as follows:

\[
\begin{align*}
  z &= \min \ c^T x \\
  A x &\leq b \\
  d x + x q x &\leq p \\
  x &= y \\
  x &\in B^n, \ y \in [0,1]^n
\end{align*}
\]

The following equivalent reformulation is obtained

\[
\begin{align*}
  z &= \min \ c^T x \\
  A y &\leq b \\
  d y + y q y &\leq p \\
  (BQP1)
  cx &\geq 1_k \\
  x &= y \\
  x &\in B^n, \ y \in [0,1]^n
\end{align*}
\]

where \(1_k\) is any lower bound on \(z\).

By relaxing constraint \(x=y\) with vector of multipliers \(u\) it comes out the following bound improving dual

\[
1_{k+1} = \max_u \min u (c^T x + u(x-y))
\]

\[
\begin{align*}
  A y &\leq b \\
  d y + y q y &\leq p
\end{align*}
\]
and the subproblem may be decomposed in

\[(SPx) \quad \min (c+u)x \]
\[ \quad cx \leq 1_k \]
\[ \quad x \in B^n \]

and

\[(SPy) \quad \min -uy \]
\[ \quad Ay \leq b \]
\[ \quad dy + yQy \leq p \]
\[ \quad y \in [0,1]^n \]

Now, we note that \((SPy)\) is a continuous nonlinear programming problem. It will be a convex programming problem, and therefore easily solved, if \(Q\) is a positive semi-definite matrix. \((SPx)\) is a 0-1 knapsack problem for which efficient algorithms exist.

In addition, we note that the diagonal elements \(q_{ij}\) may be chosen arbitrarily by compensating in the coefficient \(d_i\), since \(x_i = x_i^2\) for boolean variables. Thus, \(Q\) can always be written with a dominant main diagonal ensuring that it is a positive semi-definite matrix, provided that \(q_{ij} \geq 0\).

In summary, \((SPx)\) as well as \((SPy)\) may be solved easily giving rise to an efficient bound improving sequence algorithm. In order to conclude this section, the conditions in which assumption A is true are investigated. A sufficient condition is provided below:

**Theorem 10:** If \(cx\) is different for every \(x \in B\), then assumption A holds for \((BQP_1)\).
Proof: Because of the hypothesis the level sets of the objective function will be the product of a single point \( x \in \mathbb{B}^n \) by the feasible region for the \( y \) variables, which is convex. Therefore the level sets of the objective function will be convex and so will their image by the relaxed constraints \( x=y \). This fact clearly implies that assumption \( A \) holds.

Now, we may conclude that the bound improving approach for quadratic constrained binary problems is appropriate as long as the following conditions will be satisfied:

(i) Formulation (BQP\( \_1 \)) is adopted;

(ii) The objective function \( c^\top x \) is different for every \( x \in \mathbb{B}^n \) (this can be attained by perturbing the linear term);

(iii) \( Q \) is positive semi-definite.

4. CONCLUSIONS

In this paper we have extended the idea of bound improving sequences to nonlinear discrete programming problems. For a particular class of quadratic constrained boolean problems an efficient implementation appears to be possible.

Computational experience is currently being performed for this class of problems.
5. REFERENCES


