*UNIVERSIDADE NOVA DE LISBOA*

***Faculdade de Economia***

# TWO NOTES ON ADJUSTMENT COSTS TECHNOLOGIES

Alfredo M. Pereira*

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* Department of Economics, Stanford University,
Stanford, Ca 94305.

UNIVERSIDADE NOVA DE LISBOA
Faculdade de Economia
Campo grande, 185
1700 LISBOA

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# ON THE EXISTENCE OF OPTIMAL OUTPUT PATH FOR CRTS FIRMS IN ADJUSTMENT COSTS TECHNOLOGIES *


ABSTRACT/HEADNOTE

In this note a formal proof of the non-indeterminacy of the optimal intertemporal output path for a firm with CRTS and strictly convex adjustment costs is presented. This proof provides a rationale for some widely assumed properties on the adjustment cost functions, in that it highlights these properties as sufficient conditions for non-indeterminacy.



(*) Helpful comments on an earlier draft of this note were provided by António Borges, Peter Hammond, Lawrence Lau and John Shoven. The usual disclaimers apply.

# ON THE EXISTENCE OF OPTIMAL OUTPUT PATH FOR CATS FIRMS IN ADJUSTMENT COSTS TECHNOLOGIES

## 1. INTRODUCTION

It is well known that the static optimization problem of a price-taking firm with constant returns to scale (CATS) technology lacks enough concavity for the optimal production plan to be determined. One way to solve this problem is to adequately re-specify the firm's economic environment.

This note will consider the dynamic behavior of CATS firms in the presence of adjustment costs (AC). Intuitively, for conveniently defined AC functions, enough concavity might be present such that, the intertemporal maximization of the net cash flows is well-defined and therefore the indeterminacy result does not hold. A close look at the relevant literature supports this intuition. In fact, such non-indeterminacy result has been sometimes either implicitly assumed [see Junankar (1972,p.43), and Nickell (1978,p.41)] or indirectly shown to be true for quadratic AC functions [see Gould (1968)]. However, no general result is available.

On the other hand, a set of properties on the mathematical specification of AC has been traditionaly assumed in the literature [see for example, Gould (1968), Abel (1980), and Blanchard (1983)]. Such properties however, are essentially rooted on intuitive considerations, rather than on well established theoretical or empirical justifications.

This note develops a formal proof of the non-indeterminacy of the optimal output path for a price-taking CRTS firm in the dynamic framework induced by the existence of AC. Such a proof provides a rationale for the set of properties on AC functions, since these are sufficient conditions for the non-indeterminacy result to hold.

## 2. THE MODEL

Consider a price-taking firm that produces the intertemporal output sequence $Y=\{y_1,\ldots,y_T\}$, using the vector of factor inputs $X=\{x_1,\ldots,x_T\}$, where $x_t=(x_{t1},\ldots x_{tn})$. The respective prices are $P=\{p_1,\ldots,p_T\}$, and $W=\{w_1,\ldots,w_T\}$, where $w_t=(w_{t1},\ldots w_{tn})$. Let $x_j$'s with $1\leq j\leq m$, be flow inputs, and $x_i$'s with $m+1\leq i\leq n$, be stock inputs, whose gross increase in period t is given by $\Delta x_{ti}$. The production technology at each t is represented by a twice continuously differentiable CRTS production function, $y_t=F(x_t)$.

We will further assume, that adjustments in the stock inputs i, towards their optimal levels are not costless. This idea is captured by the existence of input specific adjustment cost functions defined over gross stock accumulation, $\Delta x_{ti}$. The adjustment cost function for input $x_i$, $AC_i(\Delta x_{ti})$ is twice continuously differentiable. As it is standard in the literature, it will be assumed that $AC_i(\Delta x_{ti})$ is characterized by:

$$AC_i(0)=0,\ AC_i(\Delta x_{ti})>0 \text{ for } \Delta x_{ti}\neq 0;$$

$$\partial AC_i(\Delta x_{ti})/\partial\Delta x_{ti}>0 \text{ for } \Delta x_{ti}>0, \text{ and } \partial AC_i(\Delta x_{ti})/\partial\Delta x_{ti}<0, \text{ for } \Delta x_{ti}<0;$$

$$\partial^2 AC_i(\Delta x_{ti})/\partial\Delta x_{ti}^2>0 \text{ for } \Delta x_{ti}\neq 0. \qquad (1)$$

Consider now, the intertemporal problem of the CRTS, price-taking firm, the maximization of the present value of all the future net cash flows $CF_t$, specified as follows:

for $1 \le t \le T$, $1 \le s \le t$,

$$\mathrm{Max}_{\{x\}} \Sigma_t CF_t =$$

$$= \Sigma_t [\Pi_s (1+r_s)^{-1}][p_t f(x_t) - \Sigma_j w_{tj} x_{tj} - \Sigma_i w_{ti} \Delta x_{ti} - \Sigma_i AC_i(\Delta x_{ti})] \quad (2)$$

subject to: for all $i$, $x_{oi}=0$ (w.l.o.g.)

for all $i,j,t$, $x_{ti} \ge 0$, $x_{tj} \ge 0$

for all $i,t$ $\Delta x_{ti} = x_{t+1,i} - (1-\theta_i) x_{ti}$, (3)

where, $r_s \ge 0$, is the one period objective discount rate from s to s-1, and, $0 \le \theta_i < 1$, is the exogenous depreciation rate of stock input i.

## 3. A NON-INDETERMINACY THEOREM

A non-indeterminacy theorem will now be stated and proved.

Theorem. Under assumptions (1), the intertemporal problem of the price-taking firm as in (2)-(3), is well-defined. Therefore, the optimal intertemporal output path is well determined.

Proof.

In the first part of the proof it will be shown that for each and every feasible $\{X;Y\}$ for which $\Sigma CF_t > 0$, there is an optimal re-scaling of production, for which (2) is maximized. In fact, suppose that for all t, the use of inputs is made h-times higher than the original level. Then, by CRTS, (2) can be rewritten as:

$$\Sigma_t \{[\Pi_s (1+r_s)^{-1}][p_t h f(x_t) - h\Sigma_j w_{tj} x_{tj} - \Sigma_i w_{ti} \Delta^* x_{ti} - \Sigma_i AC(\Delta^* x_{ti})]\} \quad (4)$$

where $\Delta^* x_{ti} = hx_{t+1,i} - (1-\theta_{ti})hx_{ti} = h\Delta x_{ti}$.

The existence of an optimal h for each and every feasible {X;Y} is guaranteed by (5)-(7) below.

$$\partial\{\Sigma_t CF_t(h)\}/\partial h = \qquad (5)$$

$$= \Sigma_t[\Pi_s(1+r_s)^{-1}]\{[p_t f(x_t) - \Sigma_j w_{tj} x_{tj} - \Sigma_i w_{ti}\Delta x_{ti} - \Sigma_i \Delta x_{ti}\partial AC/\partial\Delta x_i(\Delta^* x_{ti})]\}$$

Notice that $\partial\{\Sigma_t CF_t(h)\}/\partial h$ is decreasing. In fact,

$$\partial^2\{\Sigma_t CF_t(h)\}/\partial h^2 =$$

$$= -\Sigma_t[\Pi_s(1+r_s)^{-1}]\Sigma_i(\Delta x_{ti})^2 \partial^2 AC/\partial\Delta x_i^2(.) < 0, \qquad (6)$$

where the inequality is due to the strict convexity of AC(.).

It will now be shown that there exists a critical point $h^*$, which by virtue of (6) is a maximum, i.e.

$$\partial\{\Sigma_t CF_t(h^*)\}/\partial h = 0. \qquad (7)$$

In fact, let h go to $+\infty$. The fact that $\partial AC(\Delta x_{ti})/\partial\Delta x_{ti}$ and $\Delta x_{ti}$ have the same sign coupled with $\partial^2 AC(\Delta x_{ti})/\partial\Delta x_{ti}^2 > 0$ for $\Delta x_{ti} \neq 0$, imply that the limit of $\Delta x_{ti}\partial AC(\Delta^* x_{ti})/\partial h$ as h goes to $+\infty$ is $+\infty$. Therefore, as h goes to $+\infty$, $\lim \partial\{\Sigma_t CF_t(h)\}/\partial h = -\infty$. On the other hand, as h goes to zero, $\partial AC(\Delta x_{ti})/\partial\Delta x_{ti}$ approaches zero too. As a consequence, $\partial\{\Sigma_t CF_t(h)\}/\partial h > \Sigma_t CF_t > 0$. Finally, by continuity of $\partial\{\Sigma_t CF_t(h)/\partial h$ and by the Intermediate Value Theorem, there exists h* which satisfies (7). From this we may conclude, that problem (2)-(3) is well defined for each initial input path yielding positive intertemporal net cash flow.

In the second part of this proof, it will be shown that an optimal input/output trajectory actually exists in the set of all feasible input/output paths. Notice that, the range of our objective (2) has a lower bound at zero. Also, by construction the set of values $\Sigma_t CF_t(\{h^*X;Y(h^*)\})$ for all feasible {X;Y} is bounded above and closed. Then, by Weierstrass's theorem, a rescaling factor $h^{**}$ of some feasible {X;Y} exists for which

$$h^{**}X=X^{**}=\arg\ \mathrm{Max}_{\{x\}}\Sigma_t CF_t(.;.) \qquad (8)$$

The corresponding optimal intertemporal output path is well defined and is given by $Y=\{F(x_1^{**}),\ldots,F(x_T^{**})\}$. QED

REFERENCES

ABEL, A. (1980), 'Empirical Investment Equations' Carnagie-Rochester Conference Series in Public Policy, vol. 12, 39-91.

BLANCHARD, O. (1983), 'Dynamic Effects of a Shift in Savings: the Role of Firms' Econometrica, vol. 51 (5), 1593-91.

GOULD, J. (1968), 'Adjustment Costs in the Theory of Investment of the Firm' Review of Economic Studies, vol. 35 (1), 47-55.

JUNANKAR, T. (1972), Investment: Theories and evidence. Macmillan Press LTD.

NICKELL, S. (1978). The Investment Decisions of Firms. Cambridge University Press

# A NOTE ON THE COMPUTATION OF THE USER'S COST OF STOCKS *


ABSTRACT

This note derives the exact modified formula for the user's cost of stock inputs for a firm in a dynamic environment induced by the presence of adjustment costs. Also, in the particular case where no adjustment costs exist, the exact and usual formulas are contrasted and their relative adequacy for empirical work is discussed.



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# A NOTE ON THE COMPUTATION OF THE USERS' COST OF STOCKS

Alfredo M. Pereira

In the empirical work in economics, the usual formula for the user's cost of stock i, $UC_{it}$, is of the form

$$UC_{it}=w_{it}(r_t+\theta_{it}-\Delta w_{it}/w_{it}) \qquad (1)$$

where $w_{it}$ is the price of the investment good i, $\Delta w_{it}/w_{it}$ is its proportional rate of price increase, $r_t$ is the one period objective discount rate for the firm from t to t-1, and $\theta_{it}$ is the depreciation rate of stock i at t. See, for example Christensen and Jorgenson (1969), Field and Grebenstein (1980), Hall and Jorgenson (1967), and Loeff and Harkema (1976).

This note will consider a price-taking firm in a discrete time dynamic environment induced by the existence of adjustment costs. The exact modified formula for the user's costs of stock inputs, accounting for acquisition as well as adjustment costs, is obtained as a derived concept: it is the rental, when imputed to stocks, that makes the static profit maximization at each period consistent with the maximization of the present value of the intertemporal net cash flow. In the particular case in which there are no adjustment costs, the exact discrete time formula for the user's costs of stock inputs is obtained and contrasted with the usual continuous time formula (1). The relative adequacy of the exact discrete time formula for empirical work is then discussed.

I. The exact modified user's cost of stocks in the presence of adjustment costs.

Consider a price-taking firm that produces the intertemporal output sequence $Y=\{y_1,\ldots,y_T\}$, using the vector of factor inputs, $X=\{x_1,\ldots,x_T\}$, where $x_t=(x_{1t},\ldots x_{nt})'$. The respective prices are $P=\{p_1,\ldots,p_T\}$ for outputs and $W=\{w_1,\ldots,w_T\}$ for inputs, where $w_t=(w_{1t},\ldots w_{nt})$. Let $x_j$'s with $1\le j\le m$, be flow inputs, and $x_i$'s with $m+1\le i\le n$, be stock inputs, whose net increase in period t is given by $\Delta x_{ti}$. The production technology at each t is represented by a $C^2$ production function, $y_t=F(x_t)$, which is strictly increasing in every input and strictly concave.

We will further assume that adjusting stock inputs towards their optimal levels is not costless. This idea is captured by the existence of input-specific cost functions 'à la Gould' (1968), defined over gross stock accumulation, which include both acquisition and adjustment costs. The $C^2$ cost function for input $x_i$, $C_i(\Delta x_{it})=w_{it}\Delta x_{it}+AC(\Delta x_{it})$, is such that

$$C_i(0)=0;$$

$$C_i(\Delta x_{it})>0 \text{ and } \partial C_i(\Delta x_{it})/\partial\Delta x_{it}>0 \text{ for } \Delta x_{it}>0; \qquad (2)$$

$$\text{and} \quad \partial^2 C_i(\Delta x_{it})/\partial\Delta x_{it}^2>0 \text{ for } \Delta x_{it}\neq 0.$$

Consider now, the intertemporal problem of the price-taking firm. The firm wishes to maximize the present value of the future net cash flows, specified as follows:

for $1 \le t \le T$, $1 \le s \le t$,

$$Max_{\{x\}}\Sigma_t CF_t = \Sigma_t[\Pi_s(1+r_s)^{-1}][p_t F(x_t) - \Sigma_j w_{jt}x_{jt} - \Sigma_i C_i(\Delta x_{it})] \quad (3)$$

subject to: for all $i,j,t$, $x_{it} \ge 0$, $x_{jt} \ge 0$

$$\text{for all } i,t \quad \Delta x_{it} = x_{it} - (1-\theta_{it})x_{i,t-1}. \quad (4)$$

The necessary conditions for an interior maximum solution to problem (3)-(4), are:

for all t

$$w_{jt} = p_t \partial F(.)/\partial x_{jt}, \quad \text{for all } j, \quad (5)$$

$$\partial C_i(\Delta x_{it})/\partial \Delta x_{it} - (1-\theta_{it})(1+r_{t+1})^{-1}\partial C_i(\Delta x_{i,t+1})/\partial \Delta x_{i,t+1} =$$

$$= p_t \partial F(.)/\partial x_{it}, \quad \text{for all } i. \quad (6)$$

Because of the assumptions on the production and input-specific cost functions, these conditions are also sufficient.

From (5)-(6), it can be seen that the static profit maximization problem is consistent with the intertemporal maximization of the discounted net cash flow if and only if the exact modified users' cost of stock input $i$, $UC^m_{it}$, is given by the left-hand side of (6).

After some manipulations $UC^m_{it}$ can be written as

$$UC^m_{it} = [(\partial C_i(\Delta x_{it})/\partial \Delta x_{it})/(1+r_{t+1})]* \quad (7)$$

$$\{1+r_{t+1} - (1-\theta_{it})[1+\Delta(\partial C_i(\Delta x_{it})/\partial \Delta x_{it})/(\partial C_i(\Delta x_{it})/\partial \Delta x_{it})]\}.$$

Accordingly, the modified users' cost of i can be interpreted as the real present value at t of the current marginal cost of the firm's investment in i at t+1. To compute this real value, the marginal cost is adjusted by the real interest rate, which in turn is obtained from the nominal interest rate and the chain marginal cost index of $x_i$ per unit of still undepreciated $\Delta x_{it}$.

## II. Exact versus usual formulas for the user's cost of stocks in the absence of adjustment costs

In the absence of adjustment costs, (7) reduces to

$$UC^{e}_{it}=[w_{it}/(1+r_{t+1})][1+r_{t+1}-(1-\theta_{it})(1+\Delta w_{it}/w_{it})], \qquad (8)$$

which after some straightforward manipulations, can be rewritten as

$$UC^{e}_{it}=[w_{it}/(1+r_{t+1})][r_{t+1}+\theta_{it}w_{it}/w_{it}-\Delta w_{it}/w_{it}], \qquad (9)$$

It is apparent (9) resembles (1). However, there are three essential differences, which arise due to the distinction between discrete and continuous time. These differences are: $UC^{e}_{it}$ should be regarded as a discounted value; in (9) the relevant interest rate is $r_{t+1}$ and not $r_t$; and, also in (9), the depreciation rate is adjusted by the relative price of investment in stock $x_i$ in t and t+1. These differences are such that in general no proportionality exists between (1) and (9). Thus (1) and (9) yield two different and in general uncompatible time series for the user's cost of stock inputs. In fact, (1) and (9) are compatible only when the rate of interest is assumed to be constant and firms hold static expectations.

As a consequence, the usual formula (1) is not accurate and should not be used in empirical work whenever the analysis is pursued in a discrete time framework and/or when discrete time data is used. Instead, (1) should be replaced by expressions of the form (9), which provides the exact formula for the user's cost of stock inputs to be used in empirical work - in the sense that (9) is derived from the discrete rather than continuous time specification of the firm's intertemporal behavior.

## III. Summary

This note first derives a modified exact formula for the user's cost of stocks. This modified formula is specially designed to deal with empirical situations in which the adjustment of the stock inputs towards the desired optimal levels is not costless and instantaneous. This modified formula is exact in the sense of being derived from the discrete rather than continuous time specification of the firm's intertemporal behavior.

This note also discusses the particular case in which there are no adjustment costs. It is argued that the usual formula for the user's cost of stocks is not accurate for empirical work and should accordingly be replaced by the exact formula obtained in this note.

# REFERENCES

Christensen, Laurits R., and Dale W. Jorgenson, "The Measurement of U.S. Real Capital Input, 1929-1967", *Review of Income and Wealth,* series 15 (Dec. 1969), 293-320.

Field, Barry C., and Charles Grebenstein, "Capital-Energy Substitution in U.S. Manufacturing," *The Review of Economics and Statistics* 70 (May 1980), 207-212.

Gould, J. D., "Adjustment Costs in the Theory of Investment of the Firm," *Review of Economic Studies* 35, (1968), 47-55.

Hall, Robert E., and Dale W. Jorgenson, "Tax Policy and Investment Behavior," *The American Economic Review* 57 (1967) 391-414.

Loeff, S. Schim van der, and R. Harkema, "Three Models of Firm Behavior," *The Review of Economics and Statistics* 58 (Feb. 1976) 13-21.