*UNIVERSIDADE NOVA DE LISBOA*
***Faculdade de Economia***

# THE MARXIAN TRANSFORMATION PROBLEM: a fundamental result or an 'unnecessary detour' ?

Margarida Mateus

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UNIVERSIDADE NOVA DE LISBOA
FACULDADE DE ECONOMIA
Campo Grande, 185
1700 LISBOA

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## 1. Introduction

The transformation problem between Marxian values and competitive prices has been at the center of one of the longest and liveliest debates in the history of economic thought. The metric of values was created by Marx in order to measure the rate of exploitation intrinsic to a capitalist system, although the system is driven by the competitive model, which means that firms would maximize profits and the outcome would be an equal profit rate across sectors. The debate started with Engels, and Böhm-Bawerk (1898) and continued with Bortkirwicz (1907), Hilferding (1904), ...

In the Appendix we give a historical perspective of the solution to the problem. The first mathematically correct solution was given by Sèton (1957). In his tradition, Section 2 presents the algorithm developed by Okishio (1972)-Morishima (1973) to transform values into prices in the context of the Leontief-Sraffa model. However, as was recognized by several economists, there are important limitations to this model[1]. In order to have positive values (or prices) in a model with multiple techniques of production for the same good, fixed capital, joint production or heterogeneous labor, the previous simultaneous equations models had to be generalized. Section 3 discusses several solutions for the heterogeneous labor problem. One solution has been akin to the human capital theory, which reduces the different types of labor to a homogeneous type. The other solution establishes

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(1) The theory of general equilibrium had showed similar problems.

conversion coefficients. However, this raises the problem that there are different classes of labor and different rates of exploitation. Section 4 discusses the other problems that can be solved only in the context of a von Neumann model. This generalization is not achieved without a cost: values are no longer additive, and the rate of exploitation can be defined in several ways. These various concepts are used in Section 5 that discusses the so-called Fundamental Marxian Theorem. We show that the positiveness of the rate of profit or growth rate of the economy depends on technological conditions and Marxian concepts ae an 'unnecessary detour'. Finally, we conclude with Section 6 that discusses briefly in what ways modern Marxian theory can be extended. In a positive context the Marxian concepts would need to be integrated in a broader field of Sociology of Economics. In a normative context, the recast of the Marxian value system as a minimization of labor utilization is unsatisfactory from a social welfare point of view.

## 2. The solution of the Transformation Problem

In spite of criticisms of a long string of economists, starting with Böhm-Bawerk and ending up in Samuelson (ilustrated in his famous 'eraser theorem'), the Marxian transformation problem of values into prices and vice-versa has been solved by Seton-Okishio-Morishima. Seton's contribution is presented and discussed in the Appendix (pp 28-31) and we will focus here on Okishio-Morishima's[2] analysis within the context of an iteration approach.

We assume that each product is associated with only one activity, there is no joint production, no fixed capital (only circulating capital) and that labor is homogeneous. Eliminating any of these assumptions causes the problem to change in a non-trivial way.

Let A, D, L and Y be the matrices of physical input coefficients column-vector of the subsistence-consumption of the wage-good per worker per hour, row-vector of the quantity of labor required to produce one unit of commodities[3], and column-vector of quantities of outputs. All these are measured in convencional units. To produce any specified amounts of commodities, Yo, commodities are required in the amounts (A+DL) Yo, so that we obtain surplus outputs of $Y_o - My_o$, where M=A+DL. The vector of values, $\Lambda$, would be defined by $\Lambda = \hat{A} + L$ and the vector of prices, p, by $p = (1+r)pM$[4].

The solution algoritm proposed by Morishima consists in the following iterative procedure:

$$Yt+1 = \frac{\Lambda\, Yt}{\Lambda\, Myt}\; My_t \qquad t=0,\ 1,\ 2\ldots \qquad (1)$$

$$Pt+1 = (1+\bar{r})P_t M \qquad t=0,\ 1,\ 2\ldots \qquad (2)$$

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(2) See N.Okishio (1972) and M.Morishima (1973,1978b) for more details

(3) Consequently, DL will be the matrix of labor-feeding input coefficients.

(4) See in the Appendix below (p 28-9) for more details.

where $t$ represents time. Assuming that $M$ is primitive[5], and $\Lambda$ is known, the limit of the sequence $y_t$, $\bar{y}$, is associated with the caracteristic root of largest absolute value - the Frobenius-Perron root - $\bar{q}$, such that:

$$\bar{q}\,\bar{y} = M\,\bar{y} \qquad (3)$$

and $\left(\frac{1}{\bar{q}} - 1\right)$ is the total surplus value for all basic sectors[6] divided by the total capital for the same sectors, that is $\bar{r} = \frac{1}{\bar{g}} - 1$. After calculating $\bar{r}$ by standardizing the scale of the sectors of production (i.e., after determining the specific units in terms of which Marx measured the quantities of the commodities when he dealt with the transformation problem), the interaction formula (2) would allow to arrive at the final solution of the problem.

Regarding $\bar{M} = (1+\bar{r})M$ as given, $\bar{M}$ is a Markov matrix, i. e. a non-negative matrix whose largest positive characteristic is unity. Provided $\bar{M}$ is primitive, its eigenvector, $\bar{p}$, associated with the largest root is given by $\bar{p} = \lim_{t \to x} p_t$ and, accordingly from (2) we would get

$$\bar{p} = \bar{p}\,\bar{M} = (1+\bar{r})\;\bar{p}\,M \qquad (4)$$

where $\bar{p}$ is the 'production price' vector. The transformation into prices would be most effective as closer the initial price, $p_o$, is of $\bar{p}$.

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(5) See M. Morishima (1964, pp 14) for the definition of primitiveness of a matrix.

(6) For more details, see page 29 below.

Morishima (1978, pp 165-6) concludes that Marx's prices are a first approximation to the true equilibrium 'production prices', that Marx truncated the iteration process and calculated $p_1 = (1+\bar{r})\Lambda M$, where $p_o = \Lambda$. Morishima (1973, pp 77-9; 1978, pp 170-1) and Samuelson (1971, pp 414-6) point out the conditions under which Marx's $p_1$ equals $\bar{p}$, thus vindicating Marx's approach.

Okishio's (1972) transformation formula

$$p_{t+1} = \frac{p_t c}{p_t Mc} p_t M \qquad (5)$$

where $c$ is an arbitrary positive vector (particularly, $c$ may be taken as the actual output vector), uses outputs and inputs as measured in conventional units. He showed that, if $p_o = \Lambda$, $p$ converges to the long-run equilibrium price vector $\bar{p}$ and $\frac{p_t c}{p_t Mc}$ converges to $(1 + \bar{r})$. This approach has some advantages over Morishima's.

## 3. Extensions of the basic transformation problem

Three types of problems have been raised concerning the traditional simultaneous equation approach to the transformation problem: i) case of alternative techniques of production; ii) case of fixed capital, and iii) case of heterogeneous labor. The first problem causes the matrix of input coefficients to be rectangular (there will be more activities than products), the second leads to an output matrix that has been assumed identity so far, and the third problem is related with the weighting system needed to aggregate differenttypes of labor.

In this Section we will quickly state problems i) and ii) (whose solution is presented in Setion 4) and carefully present and discuss problem iii).

An economy in which alternative techniques, fixed capital and joint-production are allowed without any restriction is called a von Neumann economy. In this economy the basic department is the activity that uses several inputs to produce several outputs at a certain level. Fixed capital (as machines) is considered as any other input and, as a joint output we obtain machines one-year older. There will be as many capital goods as necessary to characterize the process of production.

Morishima (1973, pp 179-96) and Steedman (1975) discuss the difficulties that Marx's theory of value may encounter in the von Neumann economy. The former shows the circumstances under which value may be negative, the latter points out also the possibility of negative surplus value and positive profit.

One of the most long-standing criticisms of Marx's theory of value concerns the heterogeneity of labor (Böhm-Bawerk (1898) seems to be the first). Several authors, starting with Hilferding (1904), namely Okishio (1963), Bowles and Gintis (1977) and Morishima (1973, 1978a) have proposed a solution.

Okishio's solution, inspired in . ilferding, is akin to the concept of human capital in neoclassical economics. In this case, the value system must be replaced by the following

$$\lambda_i = \sum_j a_{ij} \lambda_j + \sum_k l_{ik} \theta_k \qquad i,j = 1,2,..n \quad k = 1,...m \qquad (6)$$

where $a_{ij}$, $l_{ik}$, $\lambda_i$ and $\theta_k$ is the input-output coefficient, the labour (of type k) - output coefficient, the value of commodity i and the conversion rate of k-th labor, respectively. If we choose the first labor as standard, then

$$\theta_1 = 1 \qquad (7)$$

Now, in order to produce skilled labor, additional inputs for educating and training are necessary. Considering ( $s_{k1} \ldots s_{kn}$), ($\tau_{k1} \ldots \tau_{km}$ ) and $\Gamma_k$ the n commodity inputs, the m labor coefficient inputs used for educating and training and the normal amount of labor during his lifetime (raw labor), respectively, then we can write

$$\Gamma_k \theta_k = \Gamma_k + \sum_i s_{ki} \lambda_i + \sum_j \tau_{kj} \theta_j \qquad k = 2, 3, \ldots n \quad (8)$$

and conclude that skilled labor is overvalued by the amount of additional inputs for education and training. The simultaneous solution of equations (6)-(8) determines the equilibrium values of the $\lambda$ t's and $\theta$ 's.

Morishima (1973, pp 179-96) raises a fundamental problem when there is heterogeneous labor. For each type of skilled labor there should be defined a rate of exploitation and consequently, as Bowles and Gintis (1977) have shown, skilled labor could 'exploit' unskilled labor. However, Bowles-Gintis' model is very different from the one just exposed, since they do not try to explain the conversion factors. For each type of labor, say type k, they define a rate of exploitation as follows:

$$e_k\ (y) = (L_k y - \Lambda_k\ DLy)\ / \Lambda_k\ DLy \qquad (9)$$

where $L_k$ and $\Lambda_k$ are the k-th rows of matrices L and $\Lambda$, respectively[7]. Then the central theorem (theorem 2)[8] states that in an economy in which some of each type of labor appears directly or indirectly in each non-luxury good, if all rates of exploitation are positive, the profit rate is positive, and if the profit rate is positive, at least one rate of exploitation is positive. At some point they use relative wages to define conversion factors.

Bowles-Gintis' thesis is frontally opposed to Morishima's. While Morishima contends that with heterogeneous labor and unequal rates of exploitation the dialectical characteristic of class structure in capitalism (working class versus capitalist class) is no longer sustainable, Bowles and Gintis argue that the capitalists divide the working class in segments ('divide to conquer') and that the assumption of equal rates of exploitation represents a serious obstacle to the understanding of the internal structure of the working class and the possibilities for alliances between it and other segments of the population.

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(7) y and DL are defined above (pp 3).

(8) See S. Bowles and H. Gintis (1977, pp 189-90) for more details.

In the same vein, they question Hilferding-Okishio's approach since segmentation may not be based on differences of education and training(9).

Thus, we conclude that if we take a 'human capital' approach to labor heterogeneity, it is possible to identify the conversion factors and since different types of labor reflect different human capital endowments, it is still possible to define a common rate of exploitation in 'standardized labor' and maintain the dual class structure. If we take the labor market segmentation approach the dual class approach ceases to be applicable and, moreover, the conversion factors remain to be identifiable.

4. <u>The Marx(?) - von Neumann model</u>

To solve problems i) and ii) of Section 3, Morishima (1974, 1976, 1978b) has recast the Marxian theory of value in the more general von Neumann framework.

Let us start by defining the following primal problem:

$$\text{Min} \left\{ Lx \mid Bx \geqslant Ax + DL\, x^a , \quad x \geqslant o^n \right\} \tag{10}$$

where A, B, x and $x^a$ are the input-coefficient matrix, the output-coefficient matrix, the intensity-of-production vector and the actual intensity vector, respectively. This linear

(9) This reflects the results obtained in U.S modern Labor Economics that there exists racial or sex discrimination.

programming problem defines the minimum necessary labor. Then, its dual defined as follows:

$$\text{Max}\left\{ \Lambda\, DLx^a \;\middle|\; \Lambda B \leqq \Lambda A + L \quad , \Lambda \geqq o^m \right\} \tag{11}$$

gives as a solution the system of values, $\Lambda^*$, which is a generalization of the Marxian value system. The rate of surplus labor can be defined as $\eta = (Lx^a - Lx^*)/Lx^*$, where $x^*$ is the solution of the primal problem. As $\Lambda^*$ may not be unique, there may be many (in fact, infinitely many) optimum value systems, which is a difficulty for the generalization, the paid-unpaid definition of the rate of exploitation in the generalized value system can now be defined as $e = (Lx^a - \Lambda^* DLx^a)/\Lambda^* DLx^a$. Notice that $\Lambda^* DLx^a$ is unique since it is the maximum of the dual problem.

To formulate Morishima's extension of the von Neumann model, let $g$ be the rate of growth, $r$ the rate of profit, $w$ the level of consumption or the real wage rate and $c$ the consumption-coefficient vector (a row vector). The workers' consumption is made explicit in the von Neumann model of balanced growth equilibrium: workers do not save, while capitalists do not consume. The economy can be modelled as follows[10]:

$$(1 + g)\; x\left[A + wLc\right] \leqq x\,B \tag{12}$$

$$(1 + r)\left[A + wLc\right]p \geqq Bp \tag{13}$$

$$(1 + g)\left[xA + wLc\right] p = xBp \tag{14}$$

$$(1 + r)\left[xA + wLc\right] p = xBp \tag{15}$$

(10) Following closely M. Morishima (1976, pp 244-5).

$$xBp > o \tag{16}$$

$$xLcp > o \tag{17}$$

$$x \geqslant o, \neq o \tag{18}$$

$$p \geqslant o, \neq o \tag{19}$$

(12) implies that in the state of balanced growth it is impossible to use more of a good, either as an input of production or a consumption good, than is available as a result of production in the previous period: (13) that, in equilibrium, a uniform rate of profit prevails and no proces can earn profits at a higher rate; (14) that zero prices are charged for those goods which are overprodeuced; (15) that no process is utilized if its profit rate is less than the maximum rate; (16) that the value of output is positive; (17) that the value of consumption is positive; (18) that at least one process is used in production; and (19) that all goods cannot be free.

Under the assumptions that each process uses some physical inputs, each commodity is produced by some processes, that labor is indispensable for reproduction and that the minimum limit of the real wage rate is feasible, Morishima has demonstrated that there exists a set of solutions (g, r, x, p) satisfying the conditions for balanced growth (12) - (19).

However, so far, no typical Marxian notion has been introduced in the (12)-(19) system, which is very similar to the traditional von Neumann model as explained in, e.g., Burmeister and Doboll (1970, pp 207-15). Notice that in the von Neumann model, the wage rate is given and that thist is compatible with the existence of surplus labor. Morishima (1978, pp 95-118) argues that neoclassical economists either assume the so-called Inada condition or a Malthusian hypothesis for the population growth in order to determine the wage rate. In the Marxian view, the equilibrium wage rate would be below the mimimum limit which justifies the existence of the'industrial reserve army'. Against this, we would argue that the von Neumann model only defines a mapping in the wage-profit frontier between wage rate and the maximal and feasible growth rate, it does not formulate a labor market adjustment or a theory of distribution.

## 5. About the 'irrelevance' of the Fundamental Marxian theorem

The so-called Fundamental Marxian theorem states that an economy is profitable if and only if the rate of surplus value, or rate of exploitation, is positive and that there exists a positive value system. The theorem has been proved by Okishio (1963) and Morishima (1973) for the Leontief-Sraffa model and by Morishima (1974) for the Marx-von Neumann model. Roemer (1981) has proved it in a general equilibrium framework. The Marxian interpretation of this theorem is that the source of profit is the surplus value. Namely, surplus value created by labor is appropriated by capital as profit: capital exploits labor. This would expain the most significant qualitative feature of the capitalist economy. But, is exploitation restricted to a capitalist system?

Formulated for the Leontief-Sraffa system, the theorem asserts that the rate of profit, $r$ , determined by

$$(1+r)\ w\ D\ L\left[I-(1+r)\ A\right]^{-1}=1 \tag{20a}$$

is positive if and only if

$$w\ D\ L\left[I-A\right]^{-1}<1 \tag{21a}$$

If we introduce the working hours per day, $T$ , and make $\frac{B}{T}=w\ D$, the two equations above can be rewritten as

$$(1+r)\ \frac{B}{T}\ L\left[I-(1+r)\ A\right]^{-1}=1 \tag{20b}$$

$$T^{*}<T \quad \text{, where} \quad T^{*}=B\ L\left[I-A\right]^{-1} \tag{21b}$$

For the value system $\Lambda=\Lambda A+L$, we can conclude that $T^{*}$ equals the value of the subsistence commodity bundle, $B\Lambda$ , that is the labor-time socially necessary for producing $B$ with the use of technology (A, L) actually prevailing in the economy[11]. And, finally, $T^{*}<T$ means that the working time expended by the workers is greater than the socially necessary time, that is there exists exploitation.

Alas! If $L$ is positive, the only necessary and sufficient condition for $\Lambda$ to be positive and, consequently for $r$ to be positive, is that A satisfies the well-known Hawkins-Simon condition, i.e., that the technology be productive. Thus, we do not need the recourse to any Marxian concept to define the existence of positive profits.

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(11) See Morishima (1974a, pp 71) for more details.

The generalization of the Fundamental Marxian theorem to the Marx-von Neumann model has been accomplished by Morishima (1974b, pp 617-621). He assumes that: i) labor is indispensable to produce the consumption-subsistence basket, ii) for the economy to grow at the capacity growth rate, it is necessary that $r > 0$, and iii) when $w=0$, $r > 0$ for a non-zero $p$. Then, Morishima proves the Generalized Fundamental Marxian Theorem: positive exploitation is necessary and sufficient for the system to have positive growth capacity as well as to garantee capitalists positive profits. Three observations can be made: a) the theorem expresses only an equivalence, not an esencial assumption, b) in the proof of the theorem, assumption ii) already refers to profitability; and c) the recource to Marxian concepts is an 'unnecessary detour' for the von Neumann model, since more fundamental technological assumptions on productiveness and necessity of labor would accomplish that role(12).

From the Leontief-Sraffa or von Neumann models we can deduct a wage-profit frontier (or its dual, a consumption possibility frontier)(13) that is always downward sloping. Using the technological data (A, L) of the models above in the context of the Marxian value system we can deduct an exploitation

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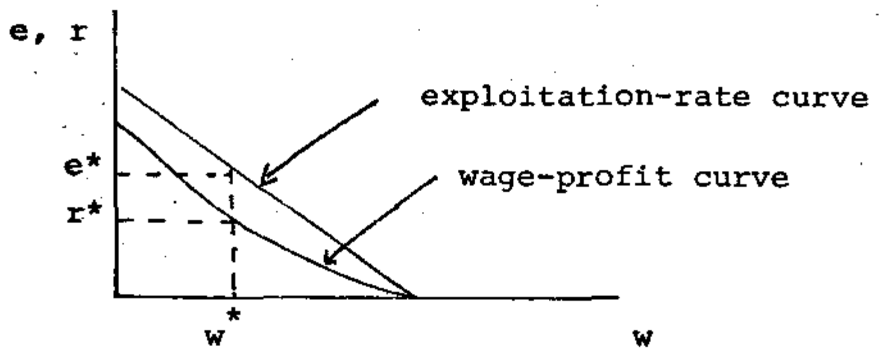
(12) See Burmeister and Dobell (1970, pp 210-226) for the traditional von Neumann model.

(13) See Burmeister et all (op. cit. pp 228-307) for a development of the concept.

rate curve that likewise is also downward sloping(14) as the following diagram shows(15).




It is also possible to prove that $e > r$ for all salary rates (except for $e=r=o$). It is now clear that any of these neoclassical models only provide a curve for the factor prices, a theory of income distribution needs to be added to determine a point on the curve.

As an aside but directly related to this topic, we may briefly mention the controversy between the two Cambridges about capital theory. The neoclassical school (US-Cambridge) sustains that the marginalist theory of income distribution is valid at both the micro and macro level. On the other hand, UK-Cambridge has argued that that theory is invalid at the macro level, due to the difficulties involved with aggregation. The answer to this problem was given by Champernowne (1954 ) by introducing the concept of a linear index for aggregation, but the hypotheses necessary for its validity are still somewhat questionable.

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(14) This demonstrates that the Marxian laws of falling rate of profit and pauperization are theoretically incompatible, even if technological progress is introduced.

(15) See M. Morishima (1973, pp 64).

Morishima (1973) has proposed another theory of aggregation based on the Marx-Keynes' theory of labor value. It is, however, easy to see that the same problem may occur in this context, since when we have at least three exploitation-rate curves we can have a phenomenon akin to the 'reswitching' of the wage-profit curves.

If now we take a certain consumption-subsistence level, $w^*$, we can use the wage-profit curve to determine the profit rate, $r^*$, without recource to the exploitation rate, and mutatis-mutantis for the exploitation rate curve. This shows also that a theory of income distribution can be constructed with the Marxian concepts if the consumption-subsistence level is well defined, but the Marxist theory is still far from providing a satisfactory answer.

## 6. From positive to normative economics: can the Marxian value system be useful?

So far we have demonstrated two things: a) that the Marxian value system as modernly defined is internally coherent(16), within the assumptions specified(17), and b) that it is an 'unnecessary detour' to the competitive or von Neumann models constructed above.

(16) This is Rogin's validity I of an economic theory, that of internal consistency.

(17) In pages 3 and 10 above.

This last conclusion, previously defended by Samuelson (1971, 1974, 1982 ), Weizsäcker (1973a,1971 ) Steedman (1981) and to a certain extent by Morishima (1973) himself, seems to result from the fact that the Marxian theory has not been considered in all its dimensions. In fact, nothing in the Marxian models reviewed here is intrinsic to a capitalist system. The same models could be applied to a state capitalism, a burocratic socialism, or whatever system.

An important achievment of neoclassical economics[18] has been the proof that, under fairly general assumptions, a competitive equilibrium is Pareto optimal and that a Pareto optimum can be decentralized through a competitive equilibrium. However, as e.g. Arrow and Hahn (1971) have emphasized, this theorem does not apply only to capitalism. Its application is entirely valid to a perfectly decentralized socialist system of the type proposed by Lerner and O.Lange. Notice that this applies both to a static and dynamic economy. Morishima (1978b) attributes normative value to the Marx-von Neumann model when he defines values as the dual to the labor-input minimization problem. According to him, although a capitalist economy would try to maximize profits, a socialist economy would try to minimize labor input. However, a labor input minimization would be consistent with quite a large number of outcomes, and at the

(18) Of course, neoclassical economics has its weaknesses, that is it suffers from barter illusion, ...

same time would be quite far from a feasible optimum either defined by a social welfare function or by a Pareto optimum, if we take the individual welfare as the building block. It seems to us that even the Marxian model cannot avoid the concepts of individual or social welfare. In fact, the concept of subsistence-consumption is ill-defined. In a modern world we certainly are not talking about minimum nutritional levels. But, then, how to define and combine the different needs of the working class? The problem seems more accute when we consider non-homogeneous labor, or introduce several generations. Morishima (1978b) e.g., suggests that in a socialist state there is still exploitation, since the 'managerial class' (burocracy) can appropriate part of the surplus or produce public goods without social utility (e.g. over-production of armament). In our view, this problem cannot be sensibly discussed without recource to the concepts of social and individual utility (whichever the political system).

If we are basically interested in a Marxian behavioral model, a Marxian type of analysis could be developed with the use of a socio-economic model integrating Marxian concepts. This seems to be only possible in a broader context, that of Sociology of Economics and much has yet to be done.

Finally, we would raise a simple, although misplaced question: in the context of the simple models that Marxian economic analysis has elaborated, what institutional system will allow 'decentralization' of a 'Marxian optimal model'?

## 7. Appendix

In volume I of 'Capital', Marx developed his labor theory of value analysing the social relations of production of the capitalist mode of production. From this theory of value he constructed a theory of exploitation in which exploitation, the expropriation of surplus value, produces profit. Using an objective way of assigning a value (i.e. embodied labor time) to a commodity, assuming a 'fair exchange'[19] between commodities and a subsistence real wage or subsistence bundle for workers (quantitatively identical to the socially necessary labor time embodied in the commodity labor-power[20]), Marx devised the concept of surplus value as the difference between a day's labor time and the value of the day's labor-power, from which the idea of exploitation could be easily derived.

According to Marx, under the assumptions of equal rate of exploitation or rate of surplus value (approximately equal to the ratio of profits to wages) and identical organic composition of capital[21] (approximately equal to the capital--labor ratio) for every branch of production, all commodities are sold at their values. In the real world, however, the organic composition of capital is not the same in all industries,

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(19) An exchange between equivalents.

(20) Labor-power is man's mental and physical capabilities that as a commodity is exchanged in the market.

(21) Marx used implicitly this assumption at the beginning of 'Capital'.

consequently commodities would be exchanged at prices proportional to their embodied labor time. So, following Marx's law of value, relative prices correspond to relative labor values given different organic compositions of capital and a uniform rate of surplus value across industries.

But, is this compatible with a uniform rate of profit on capital (defined as the ratio of profits to the whole amount of capital - constant and variable - laid out by the capitalists) that competition between capitalists does in fact establish regardless the composition of capital? To answer this question, let $c_i$, $v_i$, $s_i$, $r_i = \frac{s_i}{c_i+v_i}$, $q_i = \frac{c_i}{v_i}$, $e = e_i = \frac{s_i}{v_i}$ be the sectoral constant capital, variable capital, surplus value, rate of profit (calculated on the basis of the values of commodities), organic composition of capital ( $i$ standing for the index of the sector) and the rate of surplus value (uniform across sectors by Marx's assumption), respectively. Then, by definition $r_i = \frac{e}{qi + 1}$ and a <u>contradiction</u>[22] in Marx's analysis emerges: with a uniform $r$ (assumed to prevail with perfect competition) and different $q$'s, we cannot logically have a uniform $e$. In Morishima's (1978, p 149) words, 'the price of a commodity (the 'production price'), $p_i$, calculated by adding to the

(22) The 'Great Contradiction' according to Böhm-Bawerk.

total capital outlay ($c_i + v_i$, the 'cost price') an amount proportional to it according to the rate of profit, i.e., $(1+r)(c_i+v_i)$ will, in general, be different from the corresponding value of the commodity, $c_i+v_i+s_i = w_i = c_i + (1+e)v_i$. But, then, commodities will not, in fact, exchange according to their labor values, while value is a concept which plays the most fundamental role in Marx's economic analysis and critique'.

Blaug (1968, p 23)) writes, 'it would seem that we must abandon the labor theory of value as a theory of relative prices: the pressures that equalize the rate of profit necessarily produce different rates of surplus value between industries ... the amount of surplus value that a worker produces is apparently influenced by the amount of capital with which he is furnished: surplus value is not simply 'expropriated labor'.

But Marx had to prove that in capitalism exchange is made at 'production prices', exploring to the limits his theory of value. This is what he does in chapter IX, volume III of 'Capital' in which he transforms 'values' into 'prices' trying to find out a solution to the 'great contradiction'.

We will present Marx's own proposal and a historic perspective of other proposals of solution to the so-called transformation problem, within the context of a simultaneous equation approach.

## 7.1. Marx's solution

Marx constructs an example where he uses 5 industries and assumes that none of the products of these industries enters into the production of any other (case where mutual interdependence is abstracted from). We will present both the value and price systems in a generic form and use only 3 sectors in an economy with simple reproduction (stationary economy). By hypothesis, sector 1 produces means of production, sector 2 workers' consumption goods (wage goods) and sector 3 capitalists' consumption goods (luxury goods).

The value system is

$$c_i+v_i+s_i = w_i \qquad i = 1, 2, 3 \tag{22}$$

where $\sum_1^3 c_i = C$, $\sum_1^3 v_i = V$, $\sum_1^3 s_i = S$ and $\sum_1^3 w_i = W$.

Defining the average rate of profit as $r = \dfrac{S}{C + V}$, the price system will be

$$c_i+v_i+r(c_i+v_i) = p_i \qquad i = 1, 2, 3 \tag{23}$$

and $\sum_1^3 p_i = P$. Intuitively we see that total surplus value is identical with total profit, $S=r(C+V)$ and that total price equals total value, $P=W$. These are Marx's two 'invariant characteristics' of the value system[23]. In general, however, the majority of commodities are not sold 'at their values' but 'at prices of production' which normally diverge from their

(23) An 'invariant characteristic' is necessary to determine absolute prices and refers essentially to characteristics of the value system that are said to remain invariant to the transformation into prices.

values, that is individual prices and values differ[24].

The procedure consists in calculating $r$ from the value system, applying it to the sectoral 'cost prices' and then calculating $p_i=(1+r)(c_i+v_i)$. Consequently, we have all the industries sharing in a pool of surplus value, not proportionately to their variable capital but to their quotient of the total capital invested in the economy (C+V).

It can be shown that the deviations of individual prices from individual values are uniquely related to the organic composition of capital in each industry ($q_i$). In fact, if an industry has a high degree of mechanization relative to the average, $q_o$ (say $q_i > q_o$) its output will be sold at prices above value by apropriating surplus value from other industries since it must earn the average rate of profit.

Therefore, although the Marxian labor theory of value seems true at the aggregate level, it only applies accidently at the micro level of individual commodities, as Marx himself could conclude. He went further in his analysis and considered his solution to the transformation problem a confirmation of the capitalist system where $r$ instead of $e$ is relevant, giving the impression that surplus value is attributable to both labor-power and capital.

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(24) As referred above (page 20).

But a closer examination of Marx's procedure allows us to say that he failed to solve the problem maintaining the contradiction between his value system and his 'price-of-production' system, between his volume I and volume III of 'Capital'. In fact, Marx's solution is untenable because of its logical inconsistency, since in reality there exists mutual interdependence between departments: in his price model the capitalists' outlays on total capital, $c_i+v_i$, appear in value terms and outputs are expressed in price terms. Of course, in an economy in which price calculation is universal, that is in a 'bourgeois' economy with a Walrasian accounting system both capital used in production and final outputs must be expressed in price terms. Marx was aware of this inconsistency and in a second stage of his argument he drops the assumption of non-interdependence.

Marx's procedure is also criticizable because he maintained the real wage constant when transforming 'values' into 'prices'.

## 7.2. Bortkiewicz's solution

Bortkiewicz (1907) presented an alternative solution starting out with Marx's simple-reproduction equilibrium:

$$\begin{aligned} c_1+v_1+s_1 &= c_1+c_2+c_3 \\ c_2+v_2+s_2 &= v_1+v_2+v_3 \qquad (24) \\ c_3+v_3+s_3 &= s_1+s_2+s_3 \end{aligned}$$

where the left- and right-hand sides of system (24) represent outputs of the three sectors and demands for their outputs, respectively. Bortkiewicz fully consistently transformed values into prices by calculating the ratio of price to value of sectors 1, 2 and 3 (respectively x, y and z) and using those ratios as conversion coefficients. He concluded that inputs, outputs and the rate of profit should be determined simultaneously by the following system

$$\begin{aligned}(1+r)\ (c_1x+v_1y) &= (c_1+c_2+c_3)x \\ (1+r)\ (c_2x+v_2y) &= (v_1+v_2+v_3)y \\ (1+r)\ (c_3x+v_3y) &= (s_1+s_2+s_3)z\end{aligned} \tag{25}$$

in four unknowns (x, y, z and r), where the right-hand side of (25) gives us the values of the stationary equilibrium outputs in terms of prices. Bortkiewicz transformed (25) into a determinate system by assuming that: i) the value model was expressed in terms of money, and ii) gold is the money-commodity produced by sector 3, in which case it could be stated that $z \equiv 1$ , implying no deviation of the price from the value in department 3. This was Bortkiewicz's 'invariance postulate'. As Blaug (1968, p 236) says 'taking a leaf out of Ricardo's book, he identified luxury goods with gold and thus ensured that money prices were expressed in terms of the labor value of gold'.

The solution to (25), taking into account the conditions for a stationary equilibrium (system (24)), allowed Bortkiewcz to conclude that: i) a system of prices and rate of profit can be consistently derived from a value model, that is unique solutions for relative prices in terms of any one commodity can be obtained, ii) in general, it is not true (as opposed to Marx) that the two Marxian 'invariant characteristics' of the value model are satisfied simultaneously, depending upon the level of the organic composition of capital in the gold industry relative to the one of the total social cpital before the transformation to price terms, and iii) the rate of profit does not depend on characteristics of sector 3 unless conditions existing in industrie(s) related to capitalists'consumption do in fact influence conditions in the wage-goods industrie(s).

As to conclusion ii), according to Sweezy (1968, pp 122-3) the divergence of total value from total price is simply a question of choice of the unit of account (i.e., choice of numeraire linking prices with labor-values) and is not a relevant theoretical issue since the proportions of the price scheme (ratio of total profit to total price, of output of constant capital to output of wage goods, ...) come out the same.

## 7.3. Winternitz's solution

Winternitz (1948, p 276) criticized Bortkiewcz's solution as too restrictive to solve the transformation problem since it used the equilibrium conditions appropriate to a Marxian simple reproduction scheme. Without need to determine the distribution of outputs among the departments, but assuming that when $w_1$ varies by $x$ then $c_1$ also varies by $x$ and when $w_2$ varies by $y$ $v_1$ also varies by $y$, he replaced system (25) by

$$
\begin{aligned}
(1+r)\ (c_1x+v_1y) &= w_1\ x \\
(1+r)\ (c_2x+v_2y) &= w_2\ y \qquad\qquad (26) \\
(1+r)\ (c_3x+v_3y) &= w_3\ z
\end{aligned}
$$

where $w_i$ represents the value of output of department $i$, as in equations (22). Equations (22) and (26) determine uniquely r, x, y and z. To get absolute prices, Winternitz used an additional constraint, by which total price should equal total value: $w_1x + w_2y + w_3z = w_1 + w_2 + w_3$. The other Marxian 'postulate of invariance' could not be met(25).

Winternitz's solution is very similar to Bortkiewcz's one, although much simpler. As Meek (1967, p 153) refers 'it is the special merit of Winternitz to have exposed the triviality of the whole problem as so posed - a triviality which tended to be hidden by Bortkiewicz's over-elaborated and rather confusing method'. Meek (1967, pp 153-4) shows that, although Winternitz's approach proves that it is formally possible to transform values into prices when elements of

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(25) See M.Dobb (1955, p 276) for an interpretation of Winternitz's solution in what regards the effect on the profit rate of a transformation from values to prices.

input as well as output are involved, it is in general insatisfactory as to invalidate Marx in what he considers 'Marx's essential point: after aggregate surplus value had been converted into profit, and values consequently transformed into prices, is the ratio $\frac{\Sigma w_i}{\Sigma v_i}$ (ratio of the value of commodities in general to the value of the commodity labor-power) equal to the ratio $\frac{\Sigma w_i^p}{\Sigma v_i^p}$ (same ratio but after $v_i$ and $w_i$ have been transformed from values into prices)?'

Another criticism to Winternitz's procedure (consequently, to Bortkiewcz's too) is that the number of departments must be three, while Marx's original system had 5 sectors. This implies that conditions of aggregation have to be met, namely the restrictive condition that in a given department all sectors have the same organic composition of capital[26].

7.4. Seton's solution

Seton (1957) was the first to formulate the problem in a form which is valid for any number of sectors.

Following Morishima's[27] presentation we can generalize for $n$ sectors, after converting von Bortkiewcz-Winternitz

(26) See F. Seton (1957, pp 149-60) and N. Okishio (1963, pp 289-99).

(27) See M. Morishima (1978, pp 156-8).

model into Seton's form.

Let current outputs of commodities be taken as their unit of measurement and labor be measured in man-hours. Let $a_{1i}$, $l_i$, $\lambda_i$, $d_2$ and $p_i$ be the quantity of the capital good and the quantity of labor required to produce one unit of good $i$, the value of commodity $i$, the subsistence-consumption of the wage-good per worker per hour and the price of good $i$ ($l = 1, 2, 3$), respectively. Then by definition

$$c_i = \lambda_1 a_{1i} \quad , \quad v_i = \lambda_2 d_2 l_i \quad , \quad w_i = \lambda_i \qquad (27)$$

$$x = p_1 / \lambda_1 \quad , \quad y = p_2 / \lambda_2 \quad , \quad z = p_3 / \lambda_3 \qquad (28)$$

that substituted into (22) and (26) give us the price-determination system

$$\begin{aligned} (1+r)\ (p_1 a_{11} + p_2 d_2 l_1) &= p_1 \\ (1+r)\ (p_1 a_{12} + p_2 d_2 l_2) &= p_2 \\ (1+r)\ (p_1 a_{13} + p_2 d_2 l_3) &= p_3 \end{aligned} \qquad (29)$$

and the value-determination system

$$\begin{aligned} \lambda_1 a_{11} + l_1 &= \lambda_1 \\ \lambda_1 a_{12} + l_2 &= \lambda_2 \\ \lambda_1 a_{13} + l_3 &= \lambda_3 \end{aligned} \qquad (30)$$

respectively. Systems (29) and (30) can be generalized by creating 3 groups: I including sectors that produce means of production, II that produce wage-goods and III that produce luxury-goods. In matrix form, these systems of $n$ simultaneous equations can be rewritten as

$$(1+r)\ p\ (A+DL) = p \tag{31}$$

$$\Lambda A + L = \Lambda \tag{32}$$

where $A = \begin{bmatrix} A_I & A_{II} & A_{III} \\ o & o & o \\ o & o & o \end{bmatrix}$, $D = \begin{bmatrix} o \\ D_{II} \\ o \end{bmatrix}$, $L = (L_I, L_{II}, L_{III})$

$$p = p_I, p_{II}, p_{III}), \quad \Lambda = (\Lambda_I, \Lambda_{II}, \Lambda_{III})$$

From (31) we see that (1+r) is the reciprocal of a characteristic root p ($p = \frac{1}{1+r}$) of the matrix of augmented-input coefficients, M = A+DL and p is the eigenvector associated to p. It can be shown that: i) if we assume indecomposability of a submatrix of M (by deleting rows and columns related to sector III), the solutions to (31) which have economically meaningful signs are uniquely determined; ii) r is positive if and only if it is productive , iii) the necessary and sufficient condition for a positive equilibrium rate of profit is that the rate of exploitation is positive (this is Morishima's 'Fundamental Marxian theorem') and, iv) the principle of equal profitability across sectors is compatible only with any one normalization condition (either invariance of total output or of total surplus output), unless the coefficients of the equations satisfy some restrictive conditions. According to Seton those conditions are: i) state of simple reproduction, ii) $q_3 = \bar{q}_o$, and iii) non distorting aggregation of n sectors into 3 sectors

To solve the transformation problem, Seton defined $\hat{\Lambda}$ as the diagonal matrix of values and rewrote the price model as

$$(1+r)\ (p\,\hat{\Lambda}^{-1})\ (\hat{\Lambda}\,M\,\hat{\Lambda}^{-1}) = (p\,\hat{\Lambda}^{-1}) \qquad (33)$$

where $p\,\hat{\Lambda}^{-1}$ is a row vector of prices relative to labor values (conversion factors), while $(\hat{\Lambda} M \hat{\Lambda}^{-1})$ is an nxn matrix in which the jth column shows the values of the n commodity quantities used, both as material inputs and as wage-goods, per unit of value of gross output(29). The column sum of column j is thus $(c_j+v_j)$ using the initial notation. $(1+r)^{-1}$ is the Frobenius-Perrou root of $(\hat{\Lambda}\,M\,\hat{\Lambda}^{-1})$.

To obtain prices from known values we would only have to apply the conversion factors to the latter.

As a result of conclusion i) above(30), Seton re-stated Borkiewcz's point that the equilibrium (average) rate of profit, r, is completely independent from the coefficients of luxury-goods sectors, that is only sectors I and II are accounted for in the calculation of r. These are called the 'basic sectors'.

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(28) A non-negative matrix M is productive if there is a positive x vector such that, either $x > Mx$, or $x > xM$.

(29) Notice that any matrix $\hat{B}$ can be substituted in equation (33) and we could still state that (1+r) is the characteristic root associated with $\hat{B}\,M\,\hat{B}^{-1}$, as Seton (1957) had already referred to. Samuelson (1982, p 13) gives the example of a matrix B called humoristically the 'Napoleonic' system.

(30) See page 27.

## List of References


1. ARROW, K.J. & MAHN, F.M. (1971), General Competitive Analysis, Mathematical Economic Texts (6), Oliver & Boyd.
2. BAUMOL, W.J. (1974), 'The transformation of values: what Marx 'really' meant (An Interpretation)', JEL, March, 51-62.
3. BLAUG, M. (1968), Economic Theory in Retrospet, Richard Irwin, Inc, Revised ed.
4. BÖHM-BAWERK, E. von (1898), Karl Marx and the close of his system, in Sweezy, P., ed., Karl Marx and the close of his system by Eugen von Böhm-Bawerk and Böhm-Bawerk's criticism of Marx by Rudolf Milferding with an appendix by L. von Bortkiewicz, N.Y.: Augustus M. Kelley, 1949.
5. BORTKIEWICZ, L. von (19)7), 'On the correction of Marx's Fundamental theoretical construction in the third volume of Capital, in Sweezy, P., ed., Karl Marx ... (see reference 4).
6. BOWLES, S. & GINTIS, M. (1977), 'The Marxian theory of value and heterogeneous labour: a critique and reformulation', Cambridge Journal of Eco, 1, 173-192.
7. ________________________ (1978), 'Professor Morishima on heterogeneous labour and Marxian value theory', Cambridge Journal of Econs, 2, 311-314.
8. BURMEISTER, E. & DOBELL, A.R. (1970), Mathematical theories of Economic Growth, The Macmillan Co.
9. CHAMPERNOWNE, D.G. (1954), 'The Production Function and the theory of capital: a comment', RES, XXI, 112-135.

10. DOBB, M. (1955), 'A note on the Transformation Problem', in: On economic theory and socialism, London: Routledge and Paul, chap 17, pp 273-81.

11. FUJIMORI, Y. (1982), Modern Analysis of value theory, Lecture notes in Economics and Mathematical Systems, No 207, N.Y.: Springer-Verlag.

12. HILFERDING, R. (1904), 'Böhm-Bawerks Marx Kritik', in: Marx-studien, Blätter zur theorie und Politik des wissenschaftlichen Socialismus, von Max Adler und Rudolf Hilferding, Bd. I, Wien.

13. MARX, K., Capital, Vol I, II, III, International Publishers, N.Y., 1967.

14. MAY, K. (1948), 'Value and Price of Production: a Note on Winternitz's solution', EJ, Dec, 58, 596-99.

15. MEEK, R. (1956), 'Some Notes on the Transformation Problem', EJ, March, 66, 94-107.

16. MORISHIMA, M. (1964), Equilibrium, Stability and Growth, Oxford.

17. ______________ (1973), Marx's Economics, Cambridge

18. ______________ (1974a), 'The Fundamental Marxian theorem: a Reply to Samuelson', JEL, March, 71-74.

19. ______________ (1974b), 'Marx in the light of Modern Economic theory', Econometrica, Vol 42, No 4, July, 611-32.

20. ______________ (1974c), 'Marx' s Economics: a Comment on C.C. von Weizsäcker's Article', EJ, 84, 387-91.

21. ____________________ ,(1976a), 'Marx from a von Neumann viewpoint', in: M. Brown, K.Sato & P. Zarembka (eds), Essays in Modern Capital theory, North Holland Pub. Co.

22. ____________________ (1976b), 'Positive profits with negative surplus value - A Comment' EJ 86, 599-603.

23. MORISHIMA, M. (1978a), 'S.Bowles and M. Gintis - on the Marxian theory of value and heterogeneous labour', Cambridge Journal of Econs, 2, 305-309.

24. ______________ (1978b), Value, Exploitation and Growth: Marx in the light of Modern Economic theory, McGraw-Hill, London.

25 ___________ & SETON, F. (1961), 'Aggregation in Leontief Matrices and the labour theory of value', Econometrica, April, 29, 203-20.

26. NGUYEN, D. (1982), 'Notes on Professor Samuelson's Analysis of the Marxian Transformation Problem', Southern Econ. Journal, July, 1-10.

27. NUTI, D.M. (1977), 'The transformation of labour values intp Production Prices and the Marxian theory of Exploitation', in: Schwartz, J., ed, the subtle Anatomy of Capitalism, Goodyear Pub. Co, Chapt 6, 88-105.

28. OKISHIO, N. (1963), 'A Mathematical Note on Marxian theorems', Weltwirtschaftliches Archiv, 91, 287-99, Metz.

29. ______________ (1972), 'On Marx's Production Prices', Keisaigaku Kenkyu (in Japanese).

30. ROEMER, J.E. (1981), Analytical foundations of Marxian Economic theory, Cambridge University Press.

31. ROGIN, L. (1956), The Meaning and Validity of Economic theory, New York: Harper & Brothers.

32. SAMUELSON, P. (1970), 'The Transformation from Marxian 'Values' to Competitive'Prices': A Process of Rejection and Replacement', Proceedigs of the National Academy of Sciences, Sept, 67(1), 423-5.

33. SAMUELSON, P. (1971), 'Understanding the Marxian Notion of Exploitation: a Summary of the so-called Transformation Problem between Marxian Values and Competitive Prices, JEL, 9, 399-431.

34. ____________ (1974), 'Insight and Detour in the theory of Exploitation: a Reply to Baumol', JEL , 12(1), 62-70.

35. ____________ (1982), 'The Normative and Positivistic Inferiority of Marx's 'values' Paradigm', Southern Econ Journal, July, 11-18.

36. ____________ (1983), Foundations of Economic Analysis, Harvard Univ. Press, Enlarged ed.

37. SCHWARTZ, J. ed (1977), The subtle Anatomy of Capitalism, Goodyear Pub. Co Inc.

38. SHAIKH, A. (1977), 'Marx's theory of value and the 'Transformation Problem'', in: Schwartz, J. ed, The subtle .... (see reference 37.), chap 7, 106-139.

39. SETON, F. (1957), 'The 'Transformation Problem'', RES, June, 149-60.

40. STEEDMAN, I. (1975), 'Positive profits with negative surplus value', EJ, 85, 114-23.

41. ________________ (1976a), 'Positive profits with negative surplus value: a reply to Morishima', EJ, 86, 604-6.

42. ________________ (1976b), 'Positive profits with negative surplus value: a reply to Wolfstetter', EJ, 86, 873-6.

43. ________________ (1981), Marx after Sraffa, verso

44. SWEEZY, P.M. (1942), The theory of Capitalist Development, Modern Reader Paperbacks, New York.

45. SWEEZY, P.M. ed (1949), Karl Marx and the close of his system ... (see reference 4.), New York: A.M. Kelley.

46. WEIZSÄCKER, C.C. von & SAMUELSON, P.A. (1971), 'A new labour theory of value for rational planning through use of the bourgois profit rate', Proceedings of the National Academy of Sciences 68(6), 1192-4.

47 ________________ (1973a), 'Modern Capital theory and the Concept of Exploitation', Kyklos, 245-79.

48. ________________(1973b), 'Morishima on Marx', EJ, Dec, 1245-54.

49. WINTERNITZ, J. (1948), 'Values and Prices: a solution of the so-called Transformation Problem', EJ, June, 58, 276-80.

50. WOLFSTETTER, E. (1976), 'Positive profits with negative surplus value: a comment', EJ, 86, 864-72.